Thermal field theory (M. Laine / U. Bern) [for more, see 1701.01554]

Outline:

1. Basics on "fields" in cosmology
2. Basics on "particles" in cosmology
3. Resummation and effective field theories
4. Example: thermal production of gravitational waves

1. Basics on "fields" in cosmology

Motivation: Many important cosmological degrees of freedom are usually described by fields: inflaton, Higgs vev, gravitational waves,... We would like to learn how to compute their thermodynamic properties (energy density, pressure,...) as wall as the coefficients characterizing their real-time behaviour (damping / equilibration rate,...). To get going, we will start with a harmonic oscillator.

Harmonic oscillator: Let us first compute its partition function.

$$Z = \mathrm{Tr}(e^{-\beta\hat{H}}) , \quad \beta \equiv \frac{1}{T} , \quad \hat{H} = \text{Hamiltonian}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} = \underbrace{\hbar\omega}_{\equiv \varepsilon} \left(\hat{a}^+\hat{a} + \frac{1}{2} \right)$$

$$\Rightarrow Z = \sum_{n=0}^{\infty} \langle n | e^{-\beta\hat{H}} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta\varepsilon(n+\frac{1}{2})} = \frac{e^{-\beta\varepsilon/2}}{1-e^{-\beta\varepsilon}} .$$

From the partition function we get the (canonical) free energy:

$$Z \equiv e^{-\beta F} \Leftrightarrow F = -T\ln Z = \frac{\varepsilon}{2} + T\ln\left(1-e^{-\beta\varepsilon}\right) .$$

We also need a "propagator" ($\hat{x} \leftrightarrow \hat{\phi}$):

$$\langle \hat{x}^2 \rangle \equiv \frac{\mathrm{Tr}(\hat{x}^2 e^{-\beta\hat{H}})}{\mathrm{Tr}(e^{-\beta\hat{H}})} = (-T)\frac{2}{m}\frac{\partial}{\partial\omega^2} \ln \mathrm{Tr}(e^{-\beta\hat{H}})$$

$$= \frac{2\hbar^2}{m}\frac{\partial}{\partial\varepsilon^2} F = \frac{\hbar^2}{m}\frac{1}{\varepsilon}\frac{\partial F}{\partial\varepsilon}$$

$$= \frac{\hbar^2}{m}\cdot\frac{1}{\varepsilon}\left[\frac{1}{2} + T\frac{\beta e^{-\beta\varepsilon}}{1-e^{-\beta\varepsilon}}\right]$$

$$= \frac{\hbar^2}{m}\frac{1}{\varepsilon}\left[\underbrace{\frac{1}{2}}_{\text{"vacuum contribution" (independent of } T)} + \underbrace{\frac{1}{e^{\beta\varepsilon}-1}}_{\equiv n_B(\varepsilon) \,=\, \text{Bose distribution}}\right]$$

Scalar field theory: In momentum space, a field can be viewed as a collection of oscillators: $\varepsilon \to \varepsilon_k \equiv \sqrt{k^2+m^2}$. But the system is extensive, so it is useful to consider free energy density:

$$Z_\phi = \prod_{\vec{k}} \exp\left\{-\frac{1}{T}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right]\right\},$$

$$f_\phi \equiv -\frac{T\ln Z_\phi}{V} = \frac{1}{V}\sum_{\vec{k}}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right],$$

$$\langle\phi(t,\vec{x})\phi(t,\vec{y})\rangle = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{1}{\varepsilon_k}\left[\frac{1}{2} + n_B(\varepsilon_k)\right].$$

Remarks:

* infinite-volume limit:  $e^{ik_x(x+L)} = e^{ik_x x} \Rightarrow k_x = \frac{2\pi}{L}\cdot n_x$

$$\Rightarrow \frac{1}{L^3}\sum_{\vec{k}} = \sum_{\vec{n}}\frac{|\Delta\vec{k}|^3}{(2\pi)^3} \xrightarrow{V\to\infty} \int\frac{d^3\vec{k}}{(2\pi)^3} \equiv \int_{\vec{k}}.$$

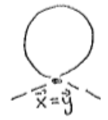
* vacuum part: $f_\phi \supset \int_{\vec{k}}\frac{\varepsilon_k}{2}$ $\Rightarrow$ requires regularization and renormalization

* thermal part: $f_\phi \supset \int_{\vec{k}} T\ln\left(1-e^{-\beta\varepsilon_k}\right)$ $\Rightarrow$ exponentially convergent

* low-T limit, $T \ll m$: $\varepsilon_k \geq m$, $\beta\varepsilon_k \geq \frac{m}{T} \gg 1 \Rightarrow e^{-\beta\varepsilon_k} \ll 1$

$$\Rightarrow f_\phi \supset T\int_{\vec{k}}\left(-e^{-\beta\varepsilon_k}\right) \approx -T\left(\frac{mT}{2\pi}\right)^{3/2}e^{-m/T}.$$

* high-T limit, $T \gg m$: let us look at the propagator with $\vec{x}=\vec{y}$:



$$\langle\phi(t,\vec{x})\phi(t,\vec{x})\rangle \supset \int_{\vec{k}}\frac{1}{\varepsilon_k}\cdot\frac{1}{e^{\varepsilon_k/T}-1} \overset{\varepsilon_k\approx k}{=} \int_0^\infty\frac{dk\,k^2}{2\pi^2}\cdot\frac{1}{k}\cdot\frac{e^{-k/T}}{1-e^{-k/T}}$$

$$= \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\int_0^\infty dk\,k\,e^{-\frac{kn}{T}}$$

$\left(x = \frac{kn}{T}\right)$

$$= \frac{T^2}{2\pi^2}\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\zeta(2)=\frac{\pi^2}{6}}\underbrace{\int_0^\infty dx\,x\,e^{-x}}_{\Gamma(2)=1!=1} = \underbrace{\frac{T^2}{12}}_{\text{"thermal mass correction"}}.$$

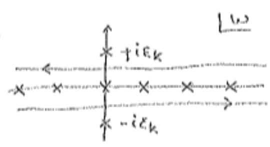
Exercise: Prove the following representation:

$$\frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{1}{e^{\varepsilon_k/T}-1}\right] \overset{!}{=} T\sum_{\omega_n}\frac{1}{\omega_n^2+\varepsilon_k^2}, \quad \omega_n = 2\pi T n,\ n\in\mathbb{Z}.$$

The ω_n are called Matsubara frequencies, and $\omega_n = 0$ is the "Matsubara zero mode".

Hint: consider the contour integral

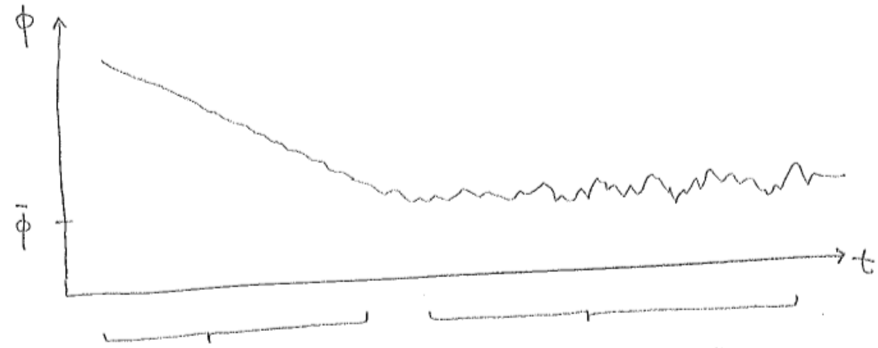
$$\int_{\leftrightarrows}\frac{d\omega}{2\pi i}\frac{1}{\omega^2+\varepsilon_k^2}\,i n_B(i\omega)$$



and close the contour in two possible ways.

Real time: Above we assumed (i) thermal equilibrium and (ii) flat spacetime (Minkowski metric). Both restrictions can be lifted, whereby many important new phenomena are revealed.

Basics of equilibration: As our everyday experience suggests, most multiparticle systems lose memory of initial conditions and approach the state of maximal entropy (minimal information). Often this dynamics can be described by the "Landau theory of thermodynamical fluctuations." A useful sketch to keep in mind:



"dissipation", or approach to equilibrium with an equilibration rate Γ:

$$\phi \sim \bar{\phi} + [\phi(t=0) - \bar{\phi}]\, e^{-\Gamma t}$$

"fluctuations", or random kicks (~Brownian motion) transferring energy from the heat bath back to ϕ

A prototypical system displaying this dynamics is given by the Langevin equation:

$$\partial^\mu \partial_\mu \phi + \Gamma u^\mu \partial_\mu \phi + V'(\phi) = \xi \quad , \quad \langle \xi(X)\, \xi(Y) \rangle = \Omega\, \delta^{(4)}(X-Y).$$

- u^μ: plasma four-velocity
- $V'(\phi)$: effective potential
- $\partial^\mu \partial_\mu$: we assume metric signature $(+ - - -)$
- $\Gamma u^\mu \partial_\mu \phi$: friction that transfers energy from ϕ to plasma
- ξ: random kicks that transfer energy from plasma to ϕ

Expanding background: Consider $ds^2 = dt^2 - a^2(t)\, d\vec{x}^2$ ← "comoving coordinates"

Let us write down an action in local Minkowskian coordinates and transform then to comoving ones:

$$S = \int dt\, d^3\vec{y}\ \left[\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla_{\vec{y}}\phi)^2 - V(\phi) \right]$$
$$= \int dt\, d^3\vec{x}\ \underbrace{a^3(t) \left[\tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2a^2}(\nabla_{\vec{x}}\phi)^2 - V(\phi) \right]}_{\equiv \mathcal{L}}$$

Euler–Lagrange:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = a^3(t)\,\dot{\phi} \ ; \ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = a^3\left(\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}\right) ; \ \frac{\partial \mathcal{L}}{\partial \phi} = a^3\left[\frac{\nabla_{\vec{x}}^2\phi}{a^2} - V'(\phi)\right]$$

($\frac{\dot{a}}{a} \equiv H$ = Hubble rate)

$$\Rightarrow \left(\partial_t^2 + 3H\partial_t - \frac{\nabla_{\vec{x}}^2}{a^2}\right)\phi + \ldots + V'(\phi) = \ldots$$

- $3H\partial_t$: "Hubble friction"
- $\ldots$: friction
- $\ldots$: noise

Equal-time correlator: In order to make contact with the equilibrium formalism from p. 1-2, let us compute $\langle \phi(t,\vec{x})\phi(t,\vec{y})\rangle$ from the Langevin equation. To simplify the task, we set $u^\mu \to (1,\vec{0})$, $V'(\phi) \approx m^2\phi$, and ignore expansion

$$\Longrightarrow \left(\partial_t^2 - \nabla^2 + \Gamma\partial_t + m^2\right)\phi = \xi \,.$$

A special solution can be found with a Green's function:

$$\phi(X) = \int_Y G_R(X-Y)\,\xi(Y)\,, \qquad X \equiv (t,\vec{x})\,,$$

$$G_R(X) = \underbrace{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{\vec{k}}}_{\equiv \int_K,\ K\equiv(\omega,\vec{k})} e^{-i\omega t + i\vec{k}\cdot\vec{x}}\ \underbrace{\frac{1}{-\omega^2+k^2-i\Gamma\omega+m^2}}_{\equiv G_R(K)}\,.$$

"retarded"

More generally, if the "propagator" is $\dfrac{1}{-\omega^2+k^2+m^2-\Pi(\omega,k)}$, then $\Gamma = \dfrac{\mathrm{Im}\,\Pi}{\omega}$.

Subsequently we can take an average over the noise ξ:

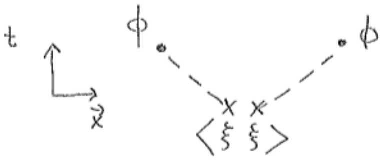
$$\langle\phi(X)\phi(Y)\rangle = \int_{Z,W} G_R(X-Z)\,G_R(Y-W)\,\langle\xi(Z)\xi(W)\rangle$$

$$= \Omega\int_Z G_R(X-Z)\,G_R(Y-Z)$$

$$= \Omega\int_{K,Q}\underbrace{\int_Z e^{-iK\cdot(X-Z)-iQ\cdot(Y-Z)}}_{(2\pi)^4\delta^{(4)}(K+Q)\,e^{-iK\cdot(X-Y)}} G_R(K)\,G_R(Q)$$

equal time $x^0=y^0$

$$= \Omega\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\ \frac{1}{-\omega^2-i\Gamma\omega+\epsilon_k^2}\ \frac{1}{-\omega^2+i\Gamma\omega+\epsilon_k^2}\,.$$



The integral can be carried out with the residue theorem:

$$\omega^2+i\Gamma\omega-\epsilon_k^2 = \left(\omega+\frac{i\Gamma}{2}-\sqrt{\epsilon_k^2-\frac{\Gamma^2}{4}}\right)\left(\omega+\frac{i\Gamma}{2}+\sqrt{\epsilon_k^2-\frac{\Gamma^2}{4}}\right) \Longrightarrow \text{poles in lower half-plane}$$

$$\omega^2-i\Gamma\omega-\epsilon_k^2 = \left(\omega-\frac{i\Gamma}{2}-\sqrt{}\right)\left(\omega-\frac{i\Gamma}{2}+\sqrt{}\right) \Longrightarrow \text{poles in upper half-plane}$$

If we close the contour in the upper half-plane, this leads to

$$\langle\phi(X)\phi(Y)\rangle = \Omega\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\frac{2\pi i}{2\pi}\left\{\frac{1}{i\Gamma(i\Gamma+2\sqrt{})\,2\sqrt{}} + \frac{1}{(i\Gamma-2\sqrt{})\,i\Gamma(-2\sqrt{})}\right\}$$

equal time $x^0=y^0$; pole at $\omega = \frac{i\Gamma}{2}+\sqrt{}$; pole at $\omega = \frac{i\Gamma}{2}-\sqrt{}$

$$= \frac{\Omega}{\Gamma}\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left\{\frac{i\Gamma-2\sqrt{}}{-\Gamma^2-4(\epsilon_k^2-\frac{\Gamma^2}{4})} - \frac{i\Gamma+2\sqrt{}}{-\Gamma^2-4(\epsilon_k^2-\frac{\Gamma^2}{4})}\right\}\frac{1}{2\sqrt{}}$$

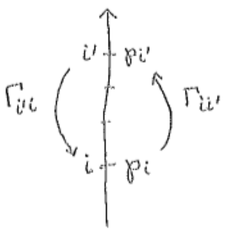
$$= \frac{\Omega}{\Gamma}\int_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\,\frac{1}{2\epsilon_k^2}\,.$$

Comparison with p. 2: this agrees with the contribution from $\omega_n = 0$, if $\Omega = 2\Gamma T$, which is known as a fluctuation-dissipation relation.

2. Basics on "particles" in cosmology

Motivation: Many cosmological relics are normally thought of as particles: dark matter, neutrinos (left-handed and right-handed), baryon asymmetry. They are regularly described by Boltzmann equations, so we should recall the basics thereof.

H-theorem: To motivate Boltzmann equations, consider first a more general setting, with "states" $\{i\}$, occupied with probabilities $\{p_i\}$, and with transition rates $\Gamma_{ii'}$ between them. Time reversal invariance: $\Gamma_{ii'} \stackrel{!}{=} \Gamma_{i'i}$.



$\hat{\rho} = \sum_i |i\rangle p_i \langle i|$

The probabilities change according to a "master equation":

$$\frac{dp_i}{dt} = \underbrace{\sum_{i'} p_{i'}\Gamma_{i'i}}_{\text{"gain": from } i' \text{ to } i} - \underbrace{\sum_{i'} p_i \Gamma_{ii'}}_{\text{"loss": from } i \text{ to } i'} = \sum_{i'} (p_{i'} - p_i)\Gamma_{ii'} .$$

Consider now $H \equiv \sum_i p_i \ln p_i = -\frac{S}{k_B}$ (Gibbs entropy).

$$\Rightarrow \frac{dH}{dt} = \sum_i \{ \dot{p}_i \ln p_i + \dot{p}_i \}$$

$$= \sum_{i,i'} \Gamma_{ii'} \{ p_{i'}\ln p_i - p_i \ln p_i + p_{i'} - p_i \}$$

(Symmetrize in $i \leftrightarrow i'$)

$$= \frac{1}{2}\sum_{i,i'} \Gamma_{ii'} \{ p_{i'}\ln p_i + p_i \ln p_{i'} - p_i \ln p_i - p_{i'}\ln p_{i'} \}$$

$$= \frac{1}{2}\sum_{i,i'} \underbrace{\Gamma_{ii'}}_{\geq 0}\, \underbrace{p_i}_{\geq 0} \underbrace{\left(1 - \frac{p_{i'}}{p_i}\right) \ln\left(\frac{p_{i'}}{p_i}\right)}_{(1-x)\ln x \,\leq\, 0 \;\forall x}$$

$$\leq 0 .$$

So, entropy can increase with time, $\frac{dS}{dt} \geq 0$.

Equilibrium: In equilibrium, $\frac{dH}{dt} = 0 = \frac{dS}{dt}$.

This requires that $\frac{p_{i'}}{p_i} = 1$ if $\Gamma_{ii'} \neq 0$, i.e. "ergodicity".

As a result, $\frac{dp_i}{dt} = 0$, even if $\sum_{i'} p_{i'}\Gamma_{i'i} \neq 0$.

This is known as "detailed balance": the gain and loss terms can be fast, but they cancel each other.

The domain $\dot{H} < 0$ corresponds to the dissipation regime of the Landau theory (p. 3), the domain $\dot{H} = 0$ to the fluctuation regime.

Boltzmann equation: Now we replace p_i through a phase space distribution $f_j(t,\vec{x},\vec{p})$, where "j" enumerates different particle species. For simplicity we assume translational invariance, i.e. no $\vec{x}$, and isotropy, i.e. dependence only on $p \equiv |\vec{p}|$.

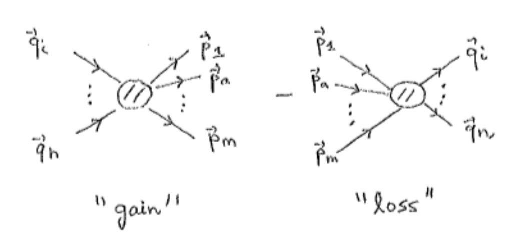
Left-hand side: In an expanding background (p.3), momenta redshift:

$$f_j(t,p) = f_j\left(t, p_0 \frac{a(t_0)}{a(t)}\right)$$

$$\Rightarrow \frac{df_j}{dt} = \frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial p}\left[-p_0 \frac{a(t_0)}{a^2(t)}\dot{a}(t)\right]$$

$$= (\partial_t - pH\partial_p) f_j \, .$$

Right-hand side: The "collision term", $\left(\frac{\partial f_j}{\partial t}\right)_{coll}$, involves transitions. Setting $j \to 1$ it reads



$$\left(\frac{\partial f_1}{\partial t}\right)_{coll} = \frac{c}{2\epsilon_1}\sum_{m,n}\int d\Phi_{1+m\to n}\, |\mathcal{M}|^2_{1+m\to n}$$

$$\times \Big\{ f_i \cdots f_n (1 \pm f_1)(1 \pm f_a)\cdots(1\pm f_m) \quad \text{"gain"}$$

$$- f_1 f_a \cdots f_m (1\pm f_i)\cdots(1\pm f_n) \Big\}, \quad \text{"loss"}$$

$$c = \frac{1}{i_a! \, i_i!} = \text{symmetry factor for identical particles}$$

$$d\Phi_{1+m\to n} = \int \prod_{a=1}^{m} \frac{d^3\vec{p}_a}{(2\pi)^3 2\epsilon_a} \prod_{i=1}^{n} \frac{d^3\vec{q}_i}{(2\pi)^3 2\epsilon_i} (2\pi)^4 \delta^{(4)}\Big(P_1 + \sum_a P_a - \sum_i Q_i\Big)$$

$|\mathcal{M}|^2_{1+m\to n}$ = transition matrix element squared

$1+f$ = Bose enhancement for bosons

$1-f$ = Pauli blocking for fermions

Detailed balance: In equilibrium,

$$1 + f_a \to 1 + n_B(\epsilon_a) = \frac{e^{\beta\epsilon_a} - 1 + 1}{e^{\beta\epsilon_a} - 1} = e^{\beta\epsilon_a} f_a$$

$$1 - f_a \to 1 - n_F(\epsilon_a) = \frac{e^{\beta\epsilon_a} + 1 - 1}{e^{\beta\epsilon_a} + 1} = e^{\beta\epsilon_a} f_a$$

$$\Rightarrow \{\ldots\} = f_i \cdots f_n f_1 f_a \cdots f_m \left[e^{\beta(\epsilon_1 + \epsilon_a + \ldots + \epsilon_m)} - e^{\beta(\epsilon_i + \ldots + \epsilon_n)}\right]$$

"gain" "loss"

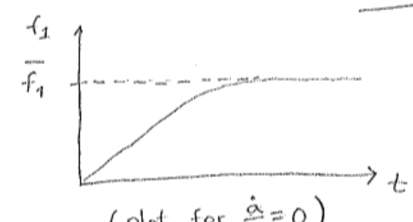
(vanishes by energy conservation!)

Linear response: Write $f_1 = \bar{f}_1 + \delta f_1$, $\bar{f}_1 \in \{n_B, n_F\}$, and expand in δf_1. The leading term vanishes by detailed balance.

$$\Rightarrow (\partial_t - pH\partial_p) f_1 = -\Gamma \delta f_1 + \mathcal{O}(\delta f_1)^2$$

$$= \Gamma(\bar{f}_1 - f_1) + \mathcal{O}(\delta f_1)^2$$

("gain" or "particle production") ("loss")



(plot for $\frac{\dot{a}}{a} = 0$)

A derivation: The goal now is to demonstrate how Boltzmann equations arise from quantum field theory. For this we compute a self-energy $\Pi(\omega,k)$ like on p. 4; extract $\Gamma = \frac{\mathrm{Im}\,\Pi}{\omega}$; and compare with p. 6. As a tool, we use the Matsubara ("imaginary-time") formalism, cf. p. 2.

Imaginary time: Recall Feynman's path integral:

$$\langle \phi_b | e^{-i\hat{H}(t_b - t_a)} | \phi_a \rangle = \int_{\phi(t_a)=\phi_a}^{\phi(t_b)=\phi_b} \mathcal{D}\phi \; \exp\left\{ i \int_{t_a}^{t_b} dt \int_V d^3\vec{x} \left[\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi) \right] \right\}.$$

What interests us:

$$Z_\phi = \mathrm{Tr}\,(e^{-\beta\hat{H}}) = \int \mathcal{D}\phi_a \, \langle \phi_a | e^{-\beta \hat{H}} | \phi_a \rangle .$$

The two are related by a Wick rotation: $t_a \to 0$, $it_b \to \beta$, $it \to \tau$

$$\Rightarrow Z_\phi = \int_{\phi(\beta)=\phi(0)} \mathcal{D}\phi \; \exp\Big\{ \underbrace{\int_0^\beta d\tau \int_V d^3\vec{x} \left[-\tfrac{1}{2}\partial_\tau\phi\,\partial_\tau\phi - \tfrac{1}{2}\partial_i\phi\,\partial_i\phi - V(\phi) \right]}_{\equiv -L_E} \Big\} .$$

Periodicity: p. 2: $\frac{1}{L}\sum_{k_n} \leftrightarrow \int \frac{dk}{2\pi}$, $k_n = \frac{2\pi}{L}\cdot n$

Now: $L \leftrightarrow \beta = \frac{1}{T}$.

So: $\int \frac{d\omega}{2\pi} \to T \sum_{\omega_n}$, $\omega_n = 2\pi T n$.

Momentum space: $\phi(\tau,\vec{x}) = \sum\!\!\!\!\!\!\!\int_K \phi(K)\, e^{iK\cdot X}$, $K = (\omega_n, \vec{k})$, $X = (\tau, \vec{x})$

$$\Rightarrow \int_X \tfrac{1}{2}\left(\partial_\tau\phi\,\partial_\tau\phi + \partial_i\phi\,\partial_i\phi + m^2\phi^2 \right) = \sum\!\!\!\!\!\!\!\int_K \tfrac{1}{2}\, \phi(-K)\left(\omega_n^2 + k^2 + m^2 \right)\phi(K).$$

Propagator: $\underbrace{\phi(K)\, \sum\!\!\!\!\!\!\!\int_Q \phi(Q)}\; = \frac{1}{\omega_n^2 + \epsilon_k^2}$.

Theory: Consider a toy model of the form

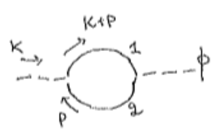
$$\mathcal{L} = \tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi) + \sum_{j=1}^{2}\left[\tfrac{1}{2}\partial^\mu\chi_j\,\partial_\mu\chi_j - U(\chi) \right] - g\,\phi\,\chi_1\chi_2 .$$

$\phi \,\text{-}\text{-}_g\!\!<^{\chi_1}_{\chi_2}$

Self-energy:

$$\int_X g\,\phi\,\chi_1\chi_2 = \int_X \sum\!\!\!\!\!\!\!\int_{KQP} e^{iX\cdot(K+Q+P)}\, g\,\phi(K)\,\chi_1(Q)\,\chi_2(P)$$

$$= g \sum\!\!\!\!\!\!\!\int_K \phi(K) \sum\!\!\!\!\!\!\!\int_P \chi_1(-K-P)\,\chi_2(P) \quad , \quad P = (p_n, \vec{p})$$



$$\Rightarrow \Pi(K) = g^2 \sum\!\!\!\!\!\!\!\int_P \underbrace{\frac{1}{(\omega_n + p_n)^2 + (\vec{k}+\vec{p})^2 + m_1^2}}_{\text{denominator} \,\equiv\, \epsilon_{kp}^2} \cdot \underbrace{\frac{1}{p_n^2 + p^2 + m_2^2}}_{\text{denominator} \,\equiv\, \epsilon_p^2} .$$

Carrying out the sum: Luckily the Matsubara sum is doable!

$$T\sum_{p_n} \frac{1}{(\omega_n+p_n)^2+\varepsilon_{kp}^2}\cdot\frac{1}{p_n^2+\varepsilon_p^2}$$

$$= T\sum_{p_n} T\sum_{q_n} \frac{\beta\,\delta_{\omega_n+p_n-q_n,0}}{(q_n^2+\varepsilon_{kp}^2)(p_n^2+\varepsilon_p^2)}$$

$$= \int_0^\beta d\tau\, e^{i\omega_n\tau}\underbrace{\left\{T\sum_{q_n}\frac{e^{iq_n(\beta-\tau)}}{q_n^2+\varepsilon_{kp}^2}\right\}}_{\frac{n_B(\varepsilon_{kp})}{2\varepsilon_{kp}}\left[e^{(\beta-\tau)\varepsilon_{kp}}+e^{\tau\varepsilon_{kp}}\right]}\underbrace{\left\{T\sum_{p_n}\frac{e^{ip_n\tau}}{p_n^2+\varepsilon_p^2}\right\}}_{\frac{n_B(\varepsilon_p)}{2\varepsilon_p}\left[e^{(\beta-\tau)\varepsilon_p}+e^{\tau\varepsilon_p}\right]}$$

$$= \frac{n_B(\varepsilon_{kp})n_B(\varepsilon_p)}{4\varepsilon_{kp}\varepsilon_p}\int_0^\beta d\tau\, e^{i\omega_n\tau}\Big[e^{(\beta-\tau)(\varepsilon_{kp}+\varepsilon_p)} \to \frac{1-e^{\beta(\varepsilon_{kp}+\varepsilon_p)}}{i\omega_n-\varepsilon_{kp}-\varepsilon_p}$$

$$+\,e^{(\beta-\tau)\varepsilon_{kp}+\tau\varepsilon_p} \to \frac{e^{\beta\varepsilon_p}-e^{\beta\varepsilon_{kp}}}{i\omega_n-\varepsilon_{kp}+\varepsilon_p}$$

$$+\,e^{\tau\varepsilon_{kp}+(\beta-\tau)\varepsilon_p} \to \frac{e^{\beta\varepsilon_{kp}}-e^{\beta\varepsilon_p}}{i\omega_n+\varepsilon_{kp}-\varepsilon_p}$$

$$+\,e^{\tau(\varepsilon_{kp}+\varepsilon_p)}\Big] \to \frac{e^{\beta(\varepsilon_{kp}+\varepsilon_p)}-1}{i\omega_n+\varepsilon_{kp}+\varepsilon_p}$$

$e^{\beta(\varepsilon_{kp}+\varepsilon_p)}-1 = [n_B^{-1}(\varepsilon_{kp})+1][n_B^{-1}(\varepsilon_p)+1]-1$

$e^{\beta\varepsilon_{kp}}-e^{\beta\varepsilon_p} = n_B^{-1}(\varepsilon_{kp})+1-[n_B^{-1}(\varepsilon_p)+1]$

$$= \frac{1}{4\varepsilon_{kp}\varepsilon_p}\Big\{[1+n_B(\varepsilon_{kp})][1+n_B(\varepsilon_p)]-n_B(\varepsilon_{kp})n_B(\varepsilon_p)\Big\}\Big\{\frac{1}{i\omega_n+\varepsilon_{kp}+\varepsilon_p}-\frac{1}{i\omega_n-\varepsilon_{kp}-\varepsilon_p}\Big\}$$

$$+\frac{1}{4\varepsilon_{kp}\varepsilon_p}\Big\{[1+n_B(\varepsilon_{kp})]n_B(\varepsilon_p)-[1+n_B(\varepsilon_p)]n_B(\varepsilon_{kp})\Big\}\Big\{\frac{1}{i\omega_n+\varepsilon_{kp}-\varepsilon_p}-\frac{1}{i\omega_n-\varepsilon_{kp}+\varepsilon_p}\Big\}.$$

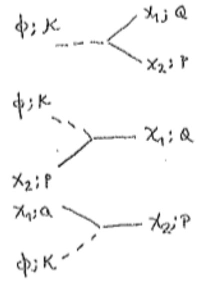
Analytic continuation: We carry out the Wick rotation $\omega_n \to -i[\omega+i0^+]$, and make use of

$$\frac{1}{\omega+\Delta+i0^+} = \mathbb{P}\left(\frac{1}{\omega+\Delta}\right)-i\pi\,\delta(\omega+\Delta)$$

$$\Rightarrow \operatorname{Im}\Pi = g^2\int_p \frac{\pi}{4\varepsilon_{kp}\varepsilon_p}\Big\{\big[(1+n_{kp})(1+n_p)-n_{kp}n_p\big]\big[\delta(\omega-\varepsilon_{kp}-\varepsilon_p)-\underbrace{\delta(\omega+\varepsilon_{kp}+\varepsilon_p)}_{\text{not realized}}\big]$$

$$+\big[(1+n_{kp})n_p-(1+n_p)n_{kp}\big]\big[\delta(\omega+\varepsilon_p-\varepsilon_{kp})-\delta(\omega+\varepsilon_{kp}-\varepsilon_p)\big]$$

$n_p \equiv n_B(\varepsilon_p)$



$$\Rightarrow \Gamma = \frac{g^2}{2\omega}\int_{p,q}\frac{1}{4\varepsilon_p\varepsilon_q}\Big\{(2\pi)^4\delta^{(4)}(K-Q-P)\big[(1+n_q)(1+n_p)-n_qn_p\big]$$

$$+(2\pi)^4\delta^{(4)}(K+P-Q)\big[n_p(1+n_q)-n_q(1+n_p)\big]$$

$$+\underbrace{(2\pi)^4\delta^{(4)}(K+Q-P)}_{\int d\Phi_{1\to2}\ \text{or}\ \int d\Phi_{2\to1}}\underbrace{\big[n_q(1+n_p)-n_p(1+n_q)\big]}_{\text{what is left from phase space distributions after expansion to first order in } \delta f_1}\Big\}$$

Summary: Normal Boltzmann equations can be recovered by computing 1-loop diagrams in thermal field theory and extracting their imaginary part.

——— o ———

3. Resummation and effective field theories

Motivation: In secs. 1 and 2, the Bose distribution, $n_B(\epsilon_k) = \frac{1}{e^{\beta\epsilon_k}-1}$, appeared frequently. If we consider energies $\epsilon_k \ll T$, then $n_B \gg 1$. This implies that there are many quanta, and naive perturbation theory with few legs can break down. In such a situation, advanced tools are needed.

Infrared problem: To understand the situation from different perspectives, let us inspect the scalar field propagator from p. 2:

$$\frac{1}{\epsilon_k}\left[\frac{1}{2} + \frac{1}{e^{\epsilon_k/T}-1}\right] = T\sum_{\omega_n} \frac{1}{\omega_n^2+\epsilon_k^2}, \qquad \omega_n = 2\pi T n,\ n\in\mathbb{Z}$$

for $\epsilon_k \ll T$:

$$\frac{1}{\epsilon_k}\left[\frac{1}{2} + \frac{1}{\frac{\epsilon_k}{T}+\frac{1}{2}\frac{\epsilon_k^2}{T^2}+\frac{1}{6}\frac{\epsilon_k^3}{T^3}+\ldots}\right]$$

$$= \frac{1}{2\epsilon_k} + \frac{T}{\epsilon_k^2}\left(1+\frac{\epsilon_k}{2T}+\frac{\epsilon_k^2}{6T^2}+\ldots\right)^{-1}$$

$$= \frac{T}{\epsilon_k^2} + \frac{1}{\epsilon_k}\left(\frac{1}{2}-\frac{1}{2}\right) + \frac{1}{T}\left(\frac{1}{4}-\frac{1}{6}\right)+\ldots$$

for $\epsilon_k \ll T$:

$$\frac{T}{\epsilon_k^2} + 2T\sum_{n=1}^{\infty}\frac{1}{(2\pi n T)^2} + \mathcal{O}\left(\frac{\epsilon_k^2}{T^3}\right)$$

$$\frac{1}{2\pi^2 T}\cdot\frac{\pi^2}{6} = \frac{1}{12T}$$

(Matsubara zero mode) → $\frac{T}{\epsilon_k^2}$

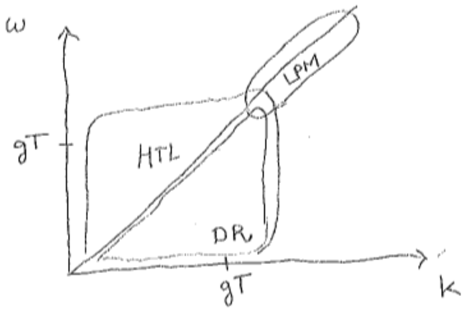
Either way, the leading term is $\frac{T}{\epsilon_k^2} \gg \frac{1}{\epsilon_k}$, i.e. much larger than the vacuum contribution.

This implies that if we look at the observables sensitive to small energies, the perturbative expansion fails:

$$\sim \frac{1}{\epsilon_k^2} \qquad \sim\left(\frac{1}{\epsilon_k^2}\right)^2\frac{g^2T^2}{12} \qquad \sim\left(\frac{1}{\epsilon_k^2}\right)^3\left(\frac{g^2T^2}{12}\right)^2$$

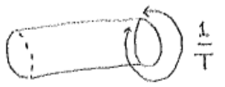
All of these are of the same order, if $\epsilon_k \sim gT$.

Resummation: In order to fix the issue, we need to re-organise or "resum" the perturbative series. Ideally, this can be systematized by making use of effective-theory methods. Some common frameworks and their domains of validity:



ω	k	ω^2-k^2	name
spacelike	gT	$-g^2T^2$	dimensional reduction (DR)
gT	gT	$\pm g^2T^2$	hard thermal loops (HTL)
$\frac{g^4T}{\pi^3}$	$\frac{g^2T}{\pi}$	$-\frac{g^4T^2}{\pi^2}$	non-Abelian Langevin dynamics
πT	πT	g^2T^2	Landau-Pomeranchuk-Migdal (LPM)

DR:



T large → $\frac{1}{T}$ small

The basic idea is to consider the "problematic" Matsubara zero modes as the key degree of freedom. Since they do not depend on τ, the fields live in $d=3$ dimensions. The unproblematic modes $\omega_n \neq 0$ are "integrated out".

Key concepts:

* scale hierarchy: $k \sim m \ll 2\pi T = \min\{|\omega_n|, n \neq 0\}$.
* factorization: treat different scales with different methods.
* IR side: write down most general L_E for zero modes.
* UV side: determine parameters of L_E by "matching", so that the contributions from $\omega_n \neq 0$ are also included.

Scalar field:

$$S_{eff} = \frac{1}{T} \int_V d^3\vec{x} \left\{ \frac{1}{2}(\partial_i \phi_3)^2 + \frac{1}{2} m_3^2 \phi_3^2 + \frac{1}{4}\lambda_3 \phi_3^4 + \ldots \right\}.$$

- $\frac{1}{T}$: from $\int_0^\beta d\tau$
- $\frac{1}{2}(\partial_i \phi_3)^2$: only spatial derivatives, for $\partial_\tau \phi_3 = 0$
- $m_3^2 = m^2|_{T=0} + \frac{g^2 T^2}{12}$
- $\ldots$: truncated higher-dimensional operators

Gauge fields:

The presence of a gauge symmetry,

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1},$$

makes the construction even more interesting. Since we restrict to τ-independent fields, U should not depend on τ, either:

$$A'_i = U A_i U^{-1} + \frac{i}{g} U \partial_i U^{-1},$$

$$A'_0 = U A_0 U^{-1}.$$

Therefore A_0 has turned into a scalar field in the adjoint representation

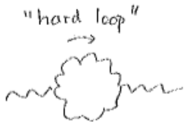
$$\Rightarrow S_{eff} = \frac{1}{T} \int_V d^3\vec{x} \left\{ \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} D_i^{ab} A_0^b D_i^{ac} A_0^c + \frac{1}{2} m_E^2 A_0^a A_0^a + \ldots \right\}$$

- D_i^{ab}: covariant derivative
- m_E^2: "Debye mass squared", $m_E^2 \sim g^2 T^2 + \mathcal{O}(g^4 T^2)$.

Fermions:

If we write a path integral for $Z_\psi = \mathrm{Tr}(e^{-\beta \hat{H}})$, we find Grassmann fields $\psi, \bar{\psi}$ which have to be antiperiodic over β $\Rightarrow e^{i\beta\omega_n} = -1 \Rightarrow \omega_n = 2\pi T(n + \frac{1}{2})$.
So there are no zero modes, and fermions do not appear in S_{eff}.

HTL:



Now $\omega_n \neq 0$ is allowed, as we subsequently want to analytically continue to real frequencies. Restricting to the quadratic part, we expect that

$$S_{\rm eff} \supset \frac{1}{2} \sum\!\!\!\!\!\!\int_P A^a_\mu(-P) \left\{ P^2 \delta_{\mu\nu} - P_\mu P_\nu + \Pi_{\mu\nu}(P) \right\} A^a_\nu(P) ,$$

where $P_\mu \Pi_{\mu\nu}(P) = 0$, because of a Ward identity. However, at $T \neq 0$, there is no Lorentz invariance. In fact we can define two independent projectors:

$$\mathbb{P}^T_{\mu\nu} \equiv \delta_{\mu i}\delta_{\nu j}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right) , \quad P = (p_n, \vec{p}) , \; p = |\vec{p}| ,$$

$$\mathbb{P}^E_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2} - \mathbb{P}^T_{\mu\nu} .$$

Then

$$\Pi_{\mu\nu}(P) = \mathbb{P}^T_{\mu\nu}\Pi^T_P + \mathbb{P}^E_{\mu\nu}\Pi^E_P ,$$

$$\Pi^T_P = \frac{m_E^2}{2}\left\{ -\frac{p_n^2}{p^2} + \frac{i p_n}{2p}\left(1 + \frac{p_n^2}{p^2}\right) \ln\frac{i p_n + p}{i p_n - p} \right\} ,$$

$$\Pi^E_P = m_E^2 \left(1 + \frac{p_n^2}{p^2}\right)\left\{ 1 - \frac{i p_n}{2p} \ln \frac{i p_n + p}{i p_n - p} \right\} .$$

Static limit:

Let us check what happens when $p_n \to 0$:

$\mathbb{P}^T_{00}$	$\mathbb{P}^T_{0i}$	$\mathbb{P}^T_{ij}$	$\mathbb{P}^E_{00}$	$\mathbb{P}^E_{0i}$	$\mathbb{P}^E_{ij}$	Π^T_P	Π^E_P
0	0	$\delta_{ij} - \frac{p_i p_j}{p^2}$	1	0	0	0	m_E^2

So we go back to the DR theory from p. 10!

Spectral function:

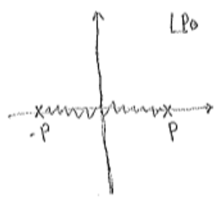
Like the self-energy, we can also represent the propagator as a sum of the T and E parts (and a gauge fixing part). From the propagator, we obtain an important quantity, a <u>spectral function</u>, similarly to how we got Γ on p. 8:

$$\rho^{T,E}_{\mathcal{P}} \equiv \mathrm{Im}\left\{ \frac{1}{P^2 + \Pi^{T,E}_P} \right\}_{p_n \to -i(p_0 + i0^+)} , \quad \mathcal{P} = (p_0, \vec{p}) .$$

Two important properties are worth noting:

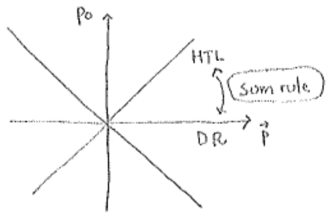
* "spectral representation": $\displaystyle \frac{1}{P^2 + \Pi^{T,E}_P} = \int_{-\infty}^{\infty} \frac{dp_0}{\pi} \, \frac{\rho^{T,E}_{\mathcal{P}}}{p_0 - i p_n}$.

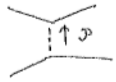
* $\displaystyle \ln \frac{p_0 + p + i0^+}{p_0 - p + i0^+} = \ln\left|\frac{p_0 + p}{p_0 - p}\right| - i\pi\,\Theta(p - |p_0|)$.



Therefore there is a large imaginary part in the "spacelike" domain $|p_0| < p$. This is known as <u>Landau damping</u>, and leads to many important physical consequences.

Sum rule: Remarkably, the HTL theory has a relationship to the DR theory not only at $p_n = 0$, but also for "light-like" kinematics, $\vec{p} = \vec{p}_\perp + p_\parallel \vec{e}_\parallel$, $p_0 = p_\parallel$, where we are just slightly spacelike! This turns out to correspond to "t-channel" scattering:





Claim:

$$I \equiv \int_{-\infty}^{\infty} \frac{dp_\parallel}{\pi} \left[\frac{\rho^T(p_\parallel, \vec{p}_\perp + p_\parallel \vec{e}_\parallel)}{p_\parallel} - \frac{\rho^E(p_\parallel, \vec{p}_\perp + p_\parallel \vec{e}_\parallel)}{p_\parallel} \right] \frac{p_\perp^2}{p_\perp^2 + p_\parallel^2} = \frac{1}{p_\perp^2} - \frac{1}{p_\perp^2 + m_E^2} .$$

Proof:

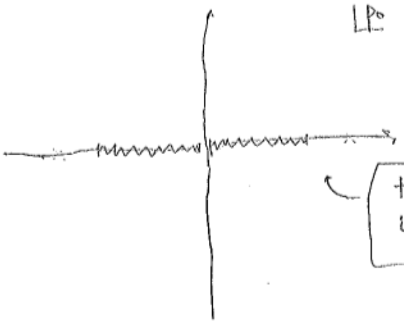
$$\rho^T(p_\parallel, \vec{p}_\perp + p_\parallel \vec{e}_\parallel) = \mathrm{Im} \left\{ \underbrace{\frac{1}{-(p_\parallel + i0^+)^2 + p_\perp^2 + p_\parallel^2 + \Pi^T(-i(p_\parallel + i0^+), \vec{p}_\perp + p_\parallel \vec{e}_\parallel)}}_{\equiv \Delta_E^T(-i(p_\parallel + i0^+), \vec{p})} \right\}$$

Important: $\Pi^T(-p_n, \vec{p}) \overset{(1)}{=} \Pi^T(p_n, \vec{p})$

$$\Rightarrow \Pi^T(-i[p_0 + i0^+], \vec{p})^* = \Pi^T(-ip_0 + 0^+, \vec{p})^* = \Pi^T(ip_0 + 0^+, \vec{p})$$

$$\overset{(1)}{=} \Pi^T(-ip_0 - 0^+, \vec{p}) = \Pi^T(-i[p_0 - i0^+], \vec{p})$$

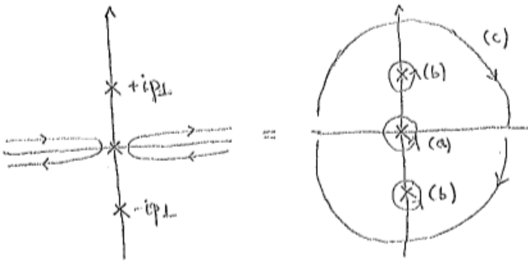
$$\Rightarrow \mathrm{Im}\, \Pi^T(-i[p_0 + i0^+], \vec{p}) = \frac{\Pi^T - \Pi^{T*}}{2i} = \underbrace{\frac{\Pi^T(-i[p_0 + i0^+], \vec{p}) - \Pi^T(-i[p_0 - i0^+], \vec{p})}{2i}}_{\text{"cut"}}$$



there are non-analycities in $\pi^{T,E}$ on the real axis

So, denoting $\Delta_R^T(p_0, \vec{p}) \equiv \Delta_E^T(-ip_0, \vec{p})$, we can write

$$I = \int \frac{dp_\parallel}{2\pi i} \cdot \frac{1}{p_\parallel} \cdot \frac{p_\perp^2}{p_\perp^2 + p_\parallel^2} \cdot \left[\Delta_R^T(p_\parallel, \vec{p}_\perp + p_\parallel \vec{e}_\parallel) - \Delta_R^E(p_\parallel, \vec{p}_\perp + p_\parallel \vec{e}_\parallel) \right]$$



The contour can be deformed with the Cauchy theorem.

Contributions: (a) pole at $p_\parallel = 0$

$$\Rightarrow \frac{p_\perp^2}{p_\perp^2} \cdot \left[\Delta_R^T(0, \vec{p}_\perp) - \Delta_R^E(0, \vec{p}_\perp) \right]$$

$$\overset{\text{p.11}}{=} \frac{1}{p_\perp^2} - \frac{1}{p_\perp^2 + m_E^2} \Rightarrow \square .$$

(b) poles at $p_\parallel = \pm i p_\perp \Rightarrow p = \sqrt{p_\parallel^2 + p_\perp^2} = 0$.

It can be verified (expand to 3rd order in p!) that $\Pi^T(p_n, \vec{0}) = \Pi^E(p_n, \vec{0}) = \frac{m_E^2}{3}$.

Therefore, the two contributions cancel.

(c) even if $\Delta_R^{T,E}$ were constant at large $|p_\parallel|$, the prefactor $\frac{1}{p_\parallel} \cdot \frac{p_\perp^2}{p_\perp^2 + p_\parallel^2}$ guarantees that the arcs do not contribute.

4. Example: thermal production of gravitational waves

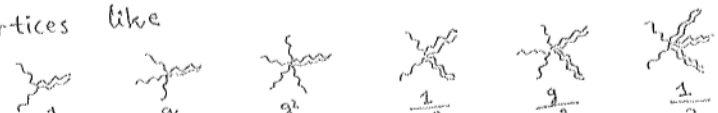
Motivation: We now want to apply the tools from secs. 1-3 to a physically relevant example, the production rate of gravitational waves / gravitons from a Standard Model plasma. This gravitational wave background is analogous to the cosmic microwave background and the cosmic neutrino background, but it was produced much earlier, at a higher temperature.

Qualitative picture: Consider a heat bath made of gluons (〰), interacting via cubic (g) and quartic (g^2) vertices, where $\alpha_s = \frac{g^2}{4\pi}$.

Gravitons (≈≈) couple to the transverse & traceless part of the energy-momentum tensor,

$$T^{TT}_{\mu\nu} \supset F^a_{\mu\varsigma} F^{a\varsigma}_{\nu} \, .$$

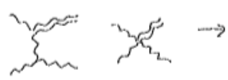
According to the Einstein equations, the coupling is suppressed by $8\pi G \sim \frac{8\pi}{m_{pl}^2}$. After a rescaling to give graviton fields the canonical dimension, this gives rise to vertices like



$\frac{1}{m_{pl}}$ $\quad \frac{g}{m_{pl}}$ $\quad \frac{g^2}{m_{pl}}$ $\quad \frac{1}{m_{pl}^2}$ $\quad \frac{g}{m_{pl}^2}$ $\quad \frac{1}{m_{pl}^3}$ $\quad \ldots$

What is the leading process?

 → not kinematically allowed!

 → $\mathcal{M} \sim \frac{g}{m_{pl}} \times (\text{momenta})$.

Then we may estimate a production rate in the sense of p.6:

$$(\partial_t - kH\partial_k) f_{gw} = \Gamma_{gw}\left(n_B(k) - f_{gw}\right) + \ldots \, ,$$

$$\Gamma_{gw} \sim \frac{1}{2k} \int d\Phi_{1\,1\to 2} \left[n_B n_B (1+n_B) - n_B(1+n_B)(1+n_B) \right] |\mathcal{M}|^2 .$$

Suppose that integrate over k, to get $n_{gw} \equiv \int_{\vec{k}} f_{gw}$, and omit f_{gw} on the right-hand side (maximal production):

$$\overset{(*)}{\Rightarrow} \underbrace{(\partial_t + 3H)}_{\text{GeV}} \underbrace{n_{gw}}_{\text{GeV}^3} \sim \underbrace{\frac{\alpha_s}{m_{pl}^2}}_{1/\text{GeV}^2} \cdot T^6$$

Integrate over Hubble time $t \sim \frac{m_{pl}}{T^2}$

$$\Rightarrow n_{gw} \sim \frac{\alpha_s T}{m_{pl}} \cdot T^3 \sim \frac{\alpha_s T}{m_{pl}} \, \bar{n}_{gw}$$

So gravitons do <u>not</u> equilibrate, unless $T_{max} \sim \frac{m_{pl}}{\alpha_s}$!

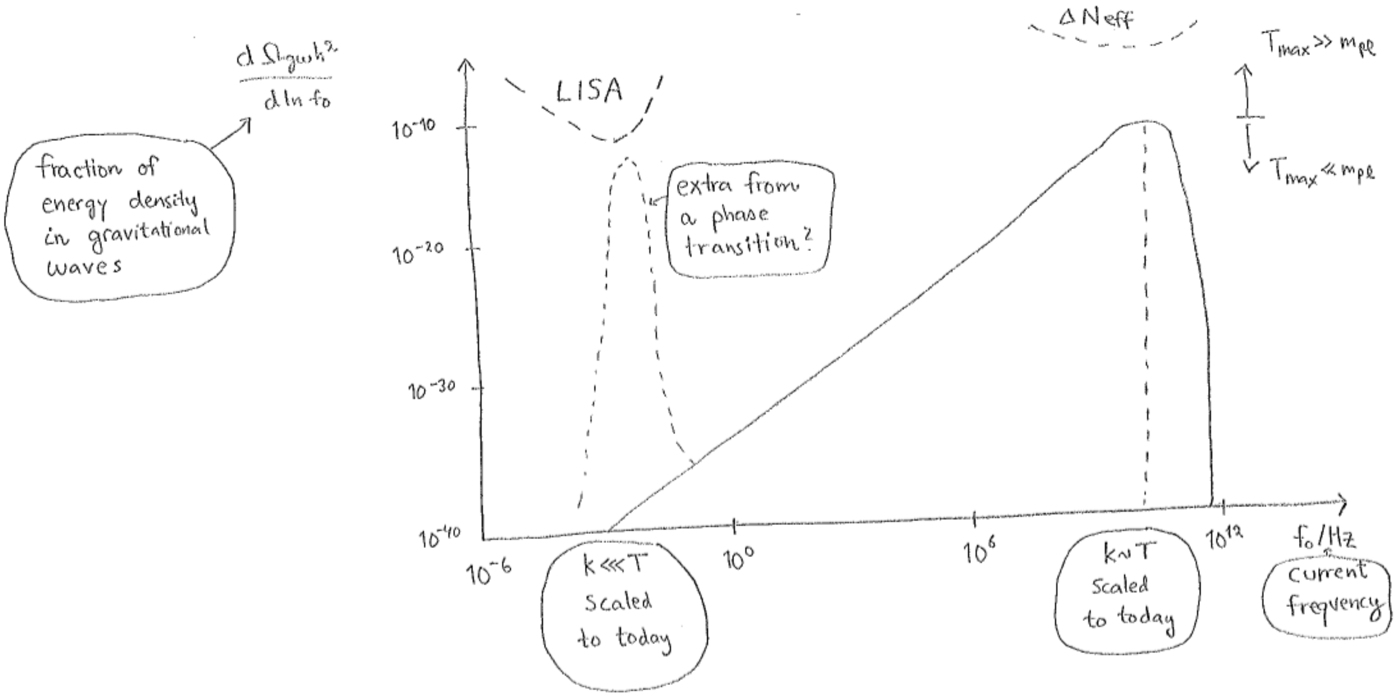
Details for (*):

$$\int_{\vec{k}} (\partial_t - kH\partial_k) f_{gw}$$

$$= \partial_t n_{gw} - H \int_0^\infty \frac{dk\,k^2}{2\pi^2} \, k\, \partial_k f_{gw} \quad \text{(partial integration)}$$

$$= (\partial_t + 3H)\, n_{gw}$$

Empirical side: Even if gravitons do not equilibrate, there is a distribution of them around:



The high-frequency part could be tested with:

* future tabletop experiments $\left(c\cdot\Delta t \sim \frac{c}{f_0} \sim \frac{3\cdot 10^8 \frac{m}{s}}{10^{11}\frac{1}{s}} \sim \mathrm{cm} \right)$

* the total energy density carried by gravitational waves:

$$\frac{e_\nu}{e_\gamma} = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{eff} \; , \qquad \frac{e_{gw}}{e_\gamma} = \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \Delta N_{eff} \; .$$

Standard Model neutrinos: $N_{eff} = 3.044 \pm 0.001$.
Standard Model gw's: $\Delta N_{eff} \lesssim 0.001$ for $T_{max} \lesssim 2\times 10^{17}\,\mathrm{GeV}$.

Some numbers: * for reference:

$T_{CMB} = 2.7\,\mathrm{K}$; $1\,\mathrm{eV} = 11600\,\mathrm{K}$; $\hbar = 6.6\times 10^{-16}\,\mathrm{eV\,s}$

$$\Rightarrow f_0 = \frac{\omega}{2\pi} = \frac{\hbar\omega}{2\pi\hbar} = \frac{2.7\,\mathrm{K}}{2\pi\hbar} = \frac{2.7}{11600}\cdot\frac{1}{2\pi\times 6.6\times 10^{-16}\,\mathrm{s}} \sim 10^{11}\,\frac{1}{s}$$

($\hbar\omega \sim T$ (Planck)) (similar to above!)

* why do thermally produced gravitational waves have similar maximal frequency as photons?

Their energy comes from thermal scatterings, so $k_{max} \sim T$ at the time of production.
Afterwards k redshifts, $k(t) = k(t_{max})\frac{a(t_{max})}{a(t)}$.

But entropy conservation guarantees that $sa^3 = \mathrm{const}$, and $s = g_s(T)\frac{2\pi^2}{45}T^3$, where $g_s(T)$ is slowly varying.

$$\Rightarrow \frac{k(t_{CMB})}{T_{CMB}} = \frac{k(t_{max})}{T_{max}}\,\frac{T_{max}\,a(t_{max})}{T_{CMB}\,a(t_{CMB})} = \frac{k(t_{max})}{T_{max}}\Bigg[\underbrace{\frac{g_s(T_{CMB})}{g_s(T_{max})}}_{\sim 10^{-2}}\cdot\underbrace{\frac{s_{max}\,a^3_{max}}{s_{CMB}\,a^3_{CMB}}}_{1}\Bigg]^{\frac{1}{3}} .$$

$\underbrace{\qquad}_{10^{-2/3} \approx 0.2}$

———→———

Need for resummation: Let us now look a bit more precisely at the determination of the rate Γ_{gw}. We focus on the "t-channel" diagram that turns out to be the most important one: The goal is to estimate the "leading logarithm".

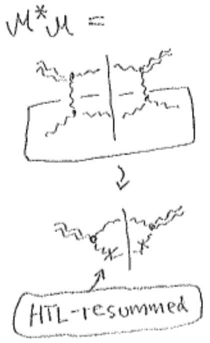
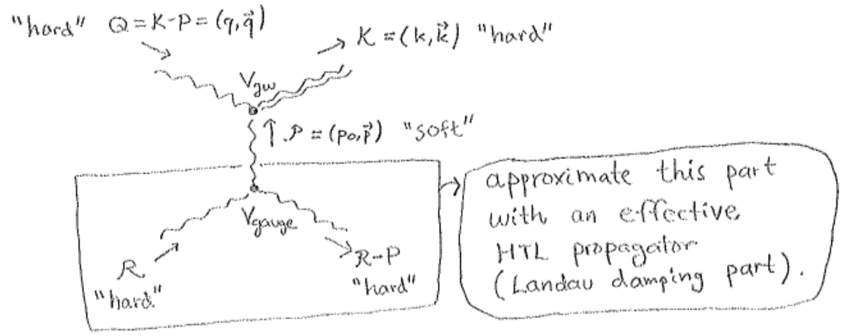


Diagram:



Kinematics: Since the line with P is resummed, it's not a sharp pole like in naive Boltzmann equations. The phase-space reads

$$\int\frac{dp_0}{2\pi}\int\frac{d^3\vec p}{(2\pi)^3}\int\frac{d^3\vec q}{(2\pi)^3}\frac{1}{2\epsilon_q}\,2\pi\delta(p_0+q-k)\,(2\pi)^3\delta^{(3)}(\vec p+\vec q-\vec k)$$

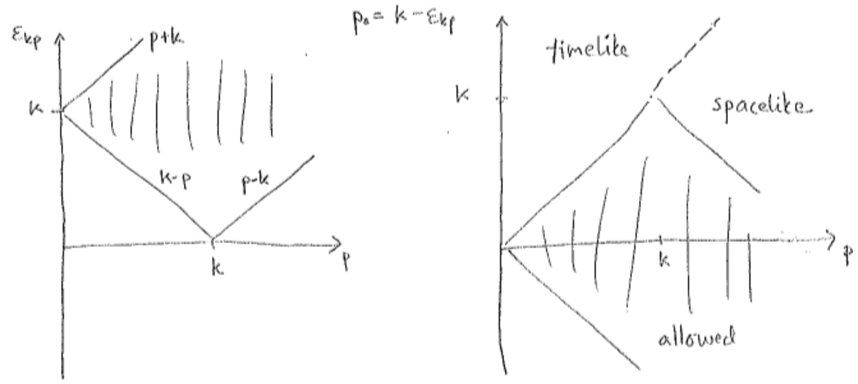
$$=\int dp_0\int\frac{d^3\vec p}{(2\pi)^3}\frac{1}{2\epsilon_{kp}}\,\delta(p_0+\epsilon_{kp}-k),\quad\text{with}\ \begin{cases}\epsilon_{kp}=\sqrt{p^2+k^2-2pkz},\\ z=\cos\theta_{\vec p,\vec k}.\end{cases}$$

Carry out the integral over z:

$$\int\frac{d^3\vec p}{(2\pi)^3}=\int_0^\infty\frac{dp\,p^2}{4\pi^2}\int_{-1}^{+1}dz=\int_0^\infty\frac{dp\,p^2}{4\pi^2}\int_{\epsilon^+_{kp}}^{\epsilon^-_{kp}}d\epsilon_{kp}\,\frac{dz}{d\epsilon_{kp}}\qquad\Big|\ \frac{d\epsilon_{kp}}{dz}=\frac{-pk}{\epsilon_{kp}},\quad \epsilon^{\pm}_{kp}=|p\pm k|$$

$$\Rightarrow\int\frac{d^3\vec p}{(2\pi)^3}\frac{1}{2\epsilon_{kp}}=\int_0^\infty\frac{dp\,p}{8\pi^2k}\,\Theta(|p-k|<k-p_0<p+k).$$

Determine the allowed domain:



Lightcone variables: Let us consider the contribution from small momenta, $p < k \sim \pi T$. Given that $p \geq |p_0|$, we write $\vec p =: \vec p_\perp + p_0\vec e_\parallel$.

Then $p^2 - p_0^2 = p_\perp^2$.

Measure (fixed p_0): $dp\,p = dp_\perp\,p_\perp$

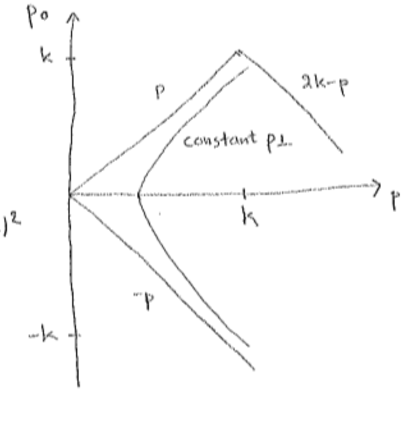
Integration domain:

($p_0 > 0$:) $p_0 < p < 2k - p_0$

$\Rightarrow 0 < p_\perp^2 < 4k(k-p_0) \overset{p_0 \ll k}{\approx} (2k)^2$

($p_0 < 0$:) $-p_0 < p < 2k - p_0$

$\Rightarrow$ same domain.



Phase space distributions: Consider the structure familiar from Boltzmann equations:

$\varnothing$ (p.13) $\quad \varnothing$

$$\text{P. 6}: \quad n_B(q_0)\, n_B(p_0) \left[1 + f_{gw}(k)\right] - f_{gw}(k) \left[1 + n_B(p_0)\right]\left[1 + n_B(q_0)\right]$$

$$= n_B(k - p_0)\, n_B(p_0)$$

$p_0 \ll k \sim \pi T$

$$\approx n_B(k)\, \frac{T}{p_0} \;.$$

Vertices: The production rate contains $\sim |\mathcal{M}|^2$, and $\mathcal{M}$ originates from V_{gw}. In turn, V_{gw} comes from $T^{TT}_{\mu\nu} \supset F^a_{\mu\rho} F^a_{\nu\rho}$, and is quadratic in momenta. One of the momenta must be $\sim \vec{p}_\perp$, so that we get a transverse structure; the other could be $\sim K$.

Soft propagator: For the gauge field line carrying the soft momentum P, we need the HTL-propagator from p.11. Only the part $\mathbb{P}^T_{\mu\nu} \equiv \delta_{\mu i}\delta_{\nu j}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)$ is large:

$$K^\mu K^\nu\, \mathbb{P}^T_{\mu\nu} = k^2 - \frac{(\vec{k}\cdot\vec{p})^2}{p^2} = \frac{k^2 p_\perp^2}{p^2} \overset{k\sim\pi T}{\sim} (\pi T)^2 \,.$$

Putting together:

$$\dot{f}_{gw} \sim \frac{1}{2k} \cdot \frac{8\pi}{m_{pl}^2} \cdot \int dp_0\, \frac{n_B(k) T}{p_0} \int_0^{(2k)^2} \frac{dp_\perp^2}{16\pi^2 k} \cdot \left[g^T - g^E \right](p_0, \vec{p}_\perp + p_0 \vec{e}_\parallel)\, \frac{k^2 p_\perp^4}{p^2}$$

$$= \frac{n_B(k) T}{4 m_{pl}^2} \int_0^{(2k)^2} dp_\perp^2 \int \frac{dp_0}{\pi}\, \frac{g^T - g^E}{p_0}\, \frac{p_\perp^2}{p_\perp^2 + p_0^2}\, p_\perp^2$$

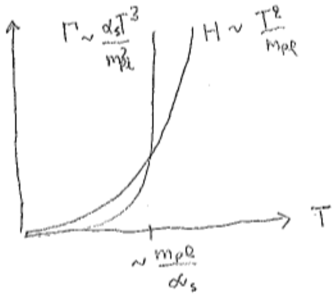
$$\text{p.12}: \quad \frac{1}{p_\perp^2} - \frac{1}{p_\perp^2 + m_E^2}$$

$$\int_0^{(2k)^2} dp_\perp^2\, \frac{m_E^2}{p_\perp^2 + m_E^2} = m_E^2 \ln\left(\frac{m_E^2 + (2k)^2}{m_E^2}\right)$$

$f_{gw} \ll n_B(k)$

If we write $(\partial_t - kH\partial_k) f_{gw} \approx \Gamma_{gw}\, n_B(k)$ like on p.13, then

$$\Gamma_{gw} \sim \frac{T m_E^2}{4 m_{pl}^2} \ln\left(\frac{m_E^2 + 4k^2}{m_E^2}\right) .$$



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