Going beyond NNLO QCD corrections to SIDIS

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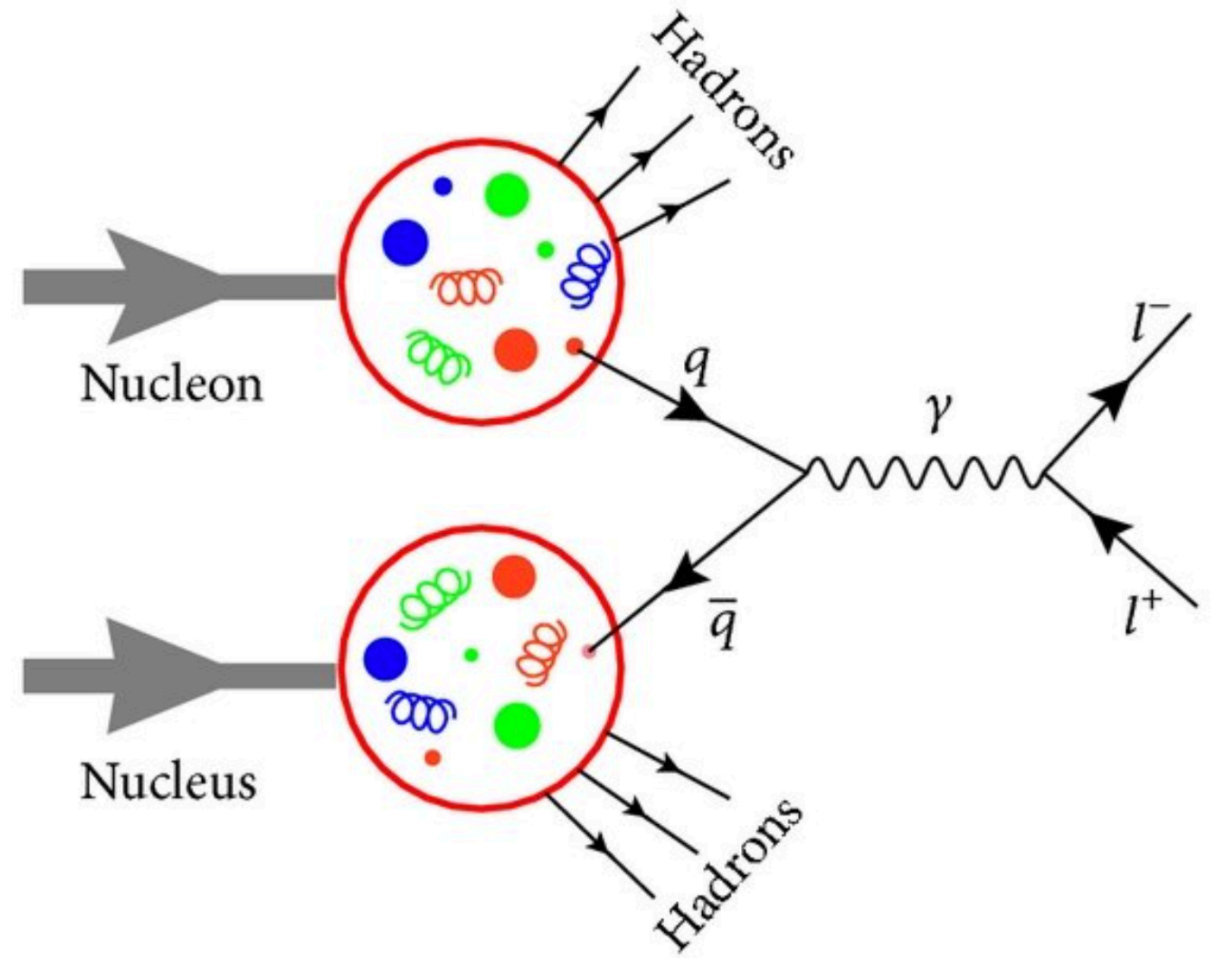
MITP

25th October 2024

- Introduction to SIDIS
- Hadronic Cross section
- QCD improved Parton Model
- NNLO QCD effects
- Resuming soft gluons
- Conclusion

Drell-Yan Production at the LHC

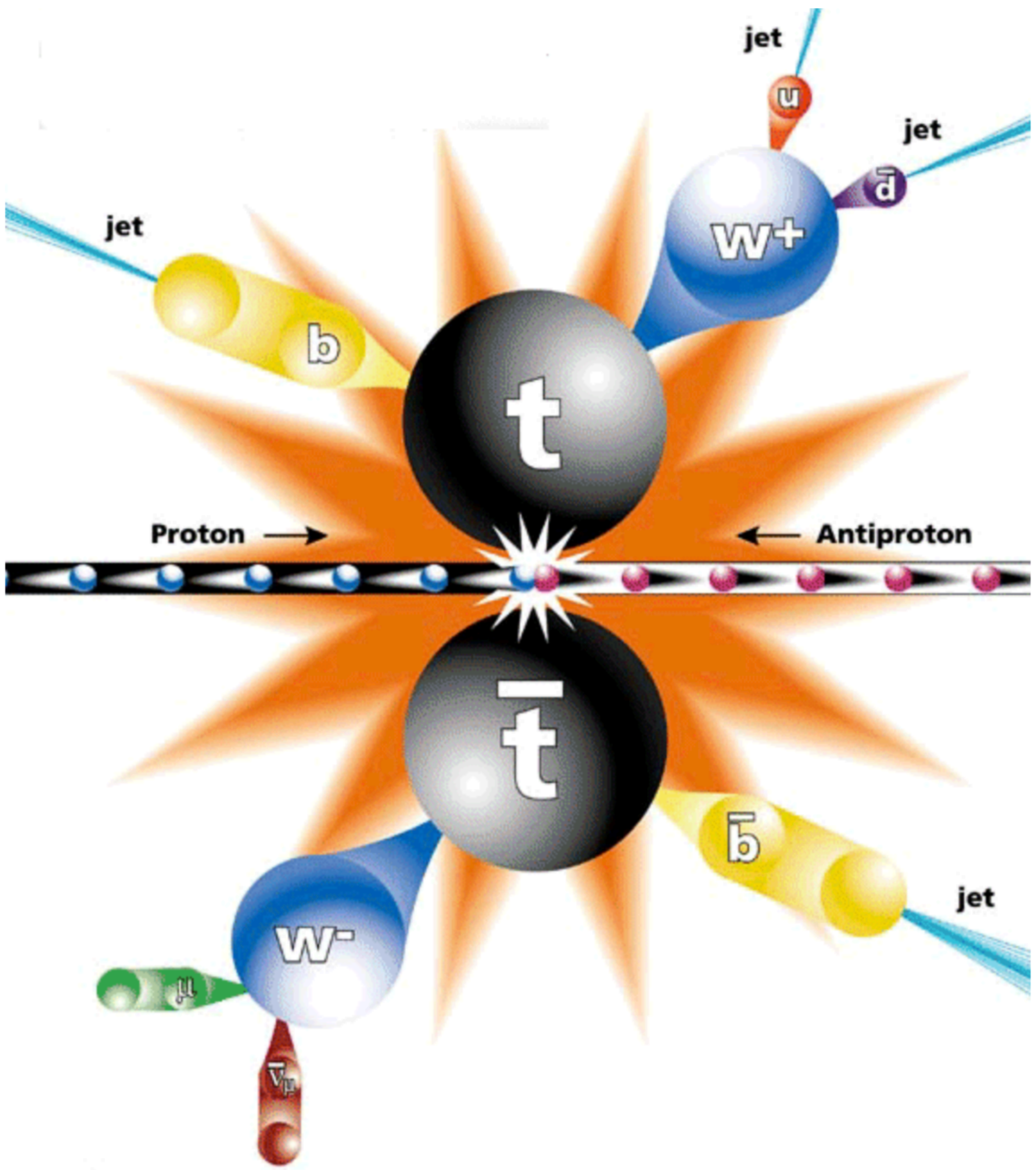
$$\sigma(q^2,\tau) = \sigma_0(\mu_R^2)\int \frac{dz}{z}\Phi_{ab}\left(\frac{\tau}{z},\mu_F^2\right)\Delta_{ab}(q^2,\mu_F^2,z)$$



Parton Distribution Function

$$\Phi_{ab}(\mu_F^2,z) = \int \frac{dy}{y} f_a(y,\mu_F^2) f_b\left(\frac{z}{y},\mu_F^2\right)$$

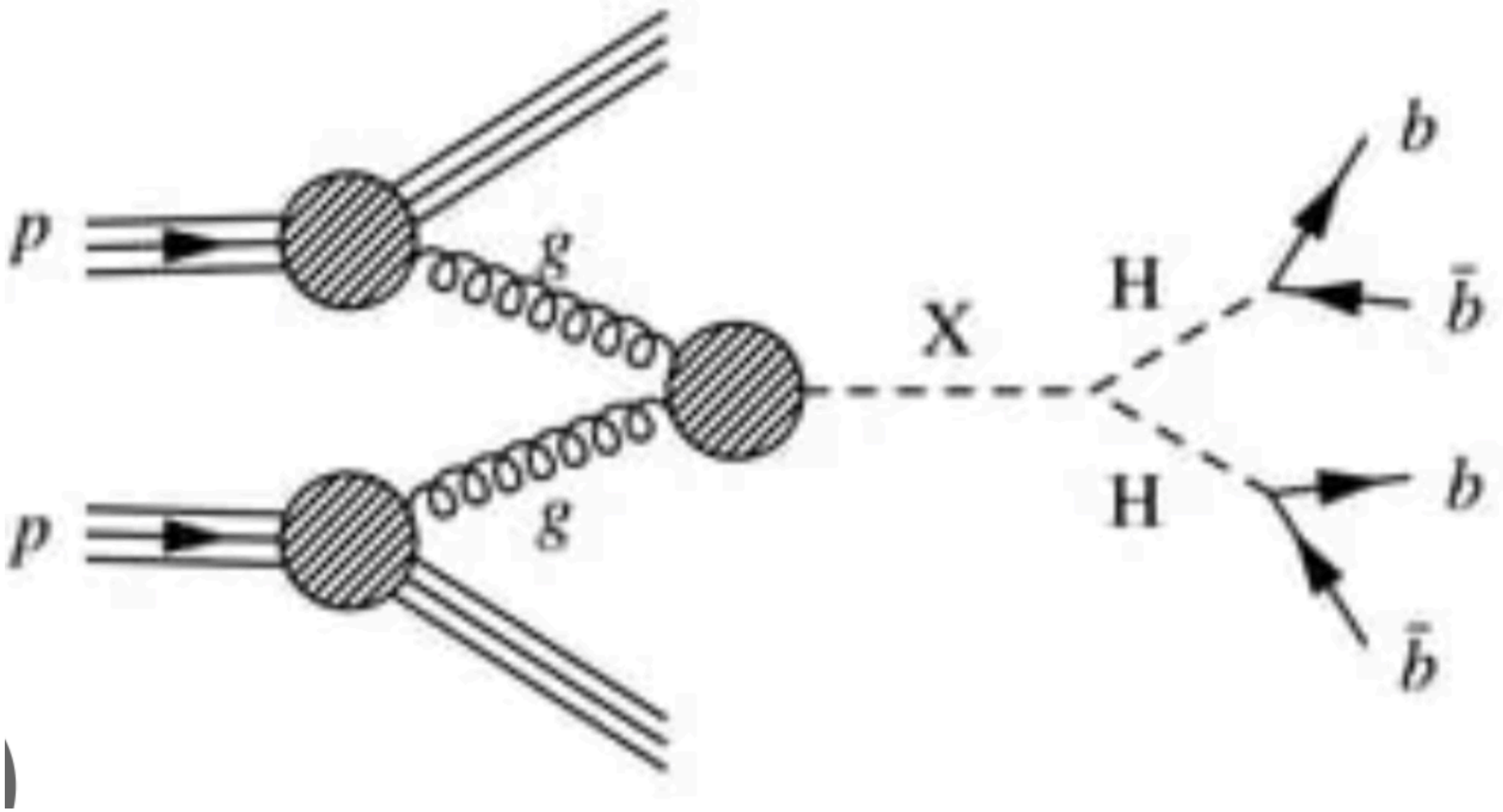
Top pair Production at the LHC



Parton Distribution Function

$$\Phi_{ab}(\mu_F^2, z) = \int \frac{dy}{y} f_a(y, \mu_F^2) f_b\left(\frac{z}{y}, \mu_F^2\right)$$

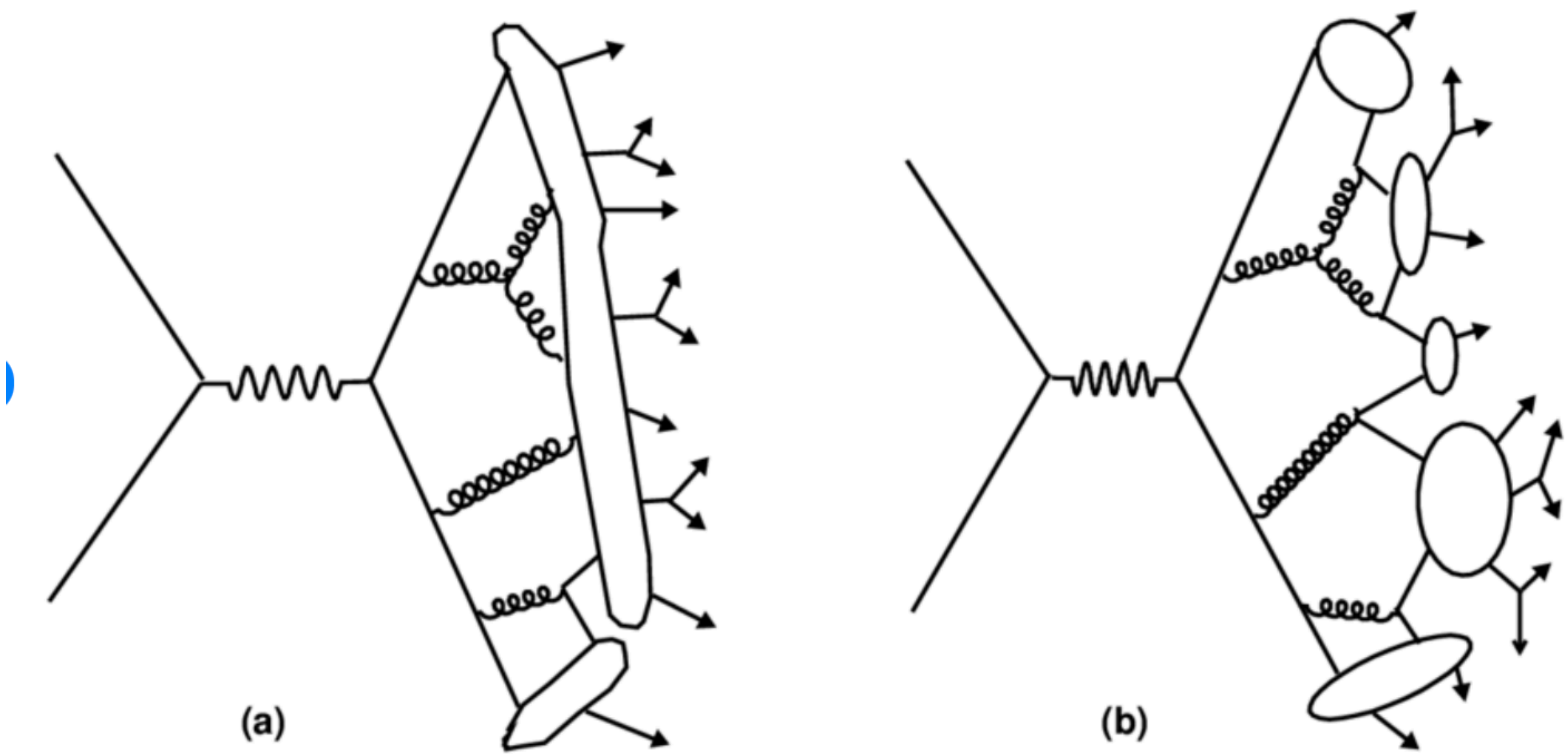
Higgs Production at the LHC



Gluon flux from Parton Distribution Function

$$\Phi_{ab}(\mu_F^2, z) = \int \frac{dy}{y} f_a(y, \mu_F^2) f_b\left(\frac{z}{y}, \mu_F^2\right)$$

Hadronization:



Fragmentation Function:

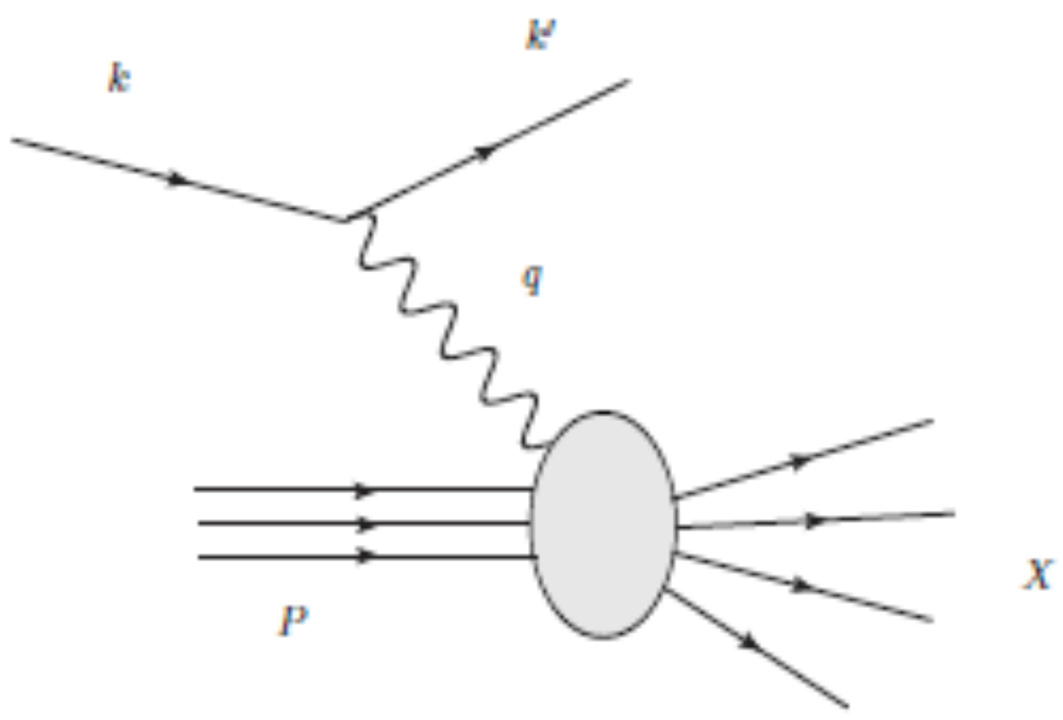
Probability of a Parton converting to Hadron

what is DIS?

Inclusive DIS (Deep Inelastic Scattering),

lepton + hadron -> lepton + X

one sums up all the particles in the final state, except the scattered lepton



Depends on Parton Distribution Function (PDF) of the incoming hadron.

- HERA: deep structure of proton at highest Q^2 and smallest x

PDF extraction

GRV, GJR ...

MRST, MSTW ...

CTEQ, CT# ...

NNPDF

ABM, ABKM

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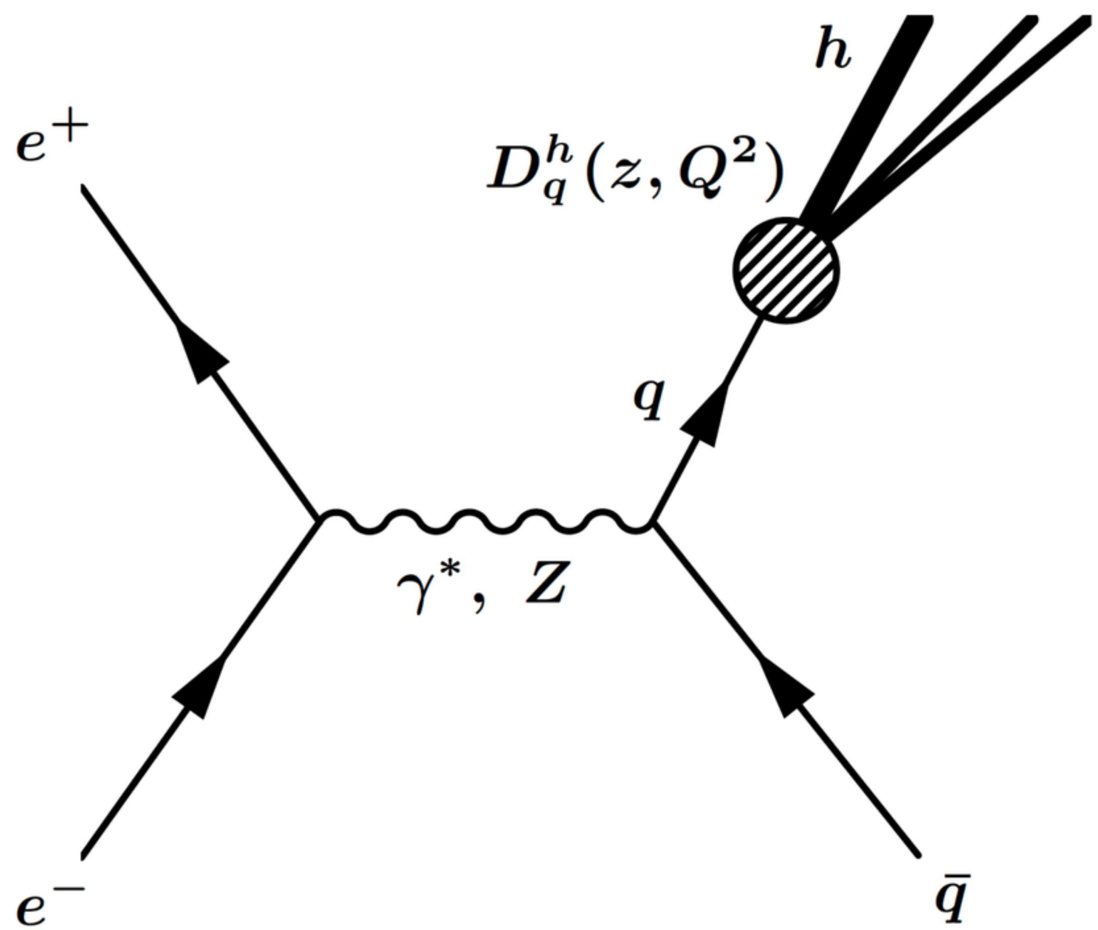
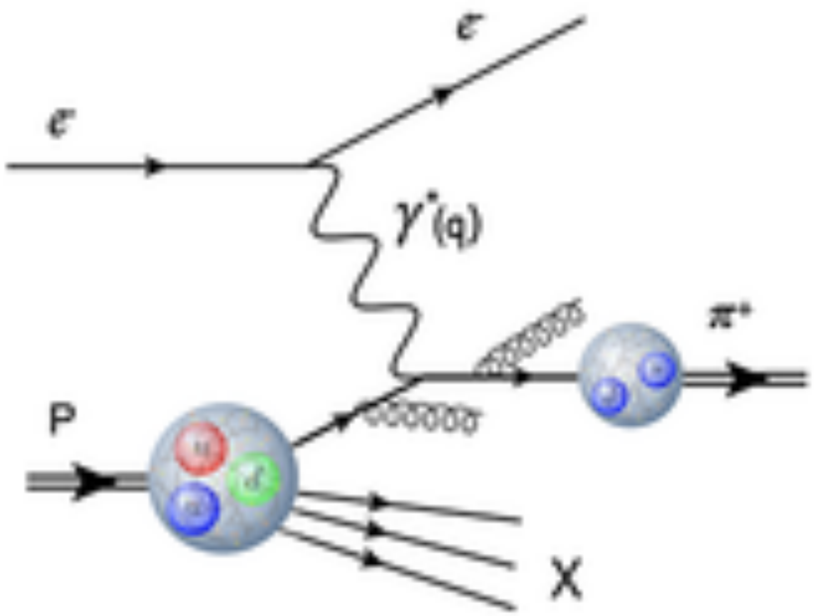
Fragmentation Function:



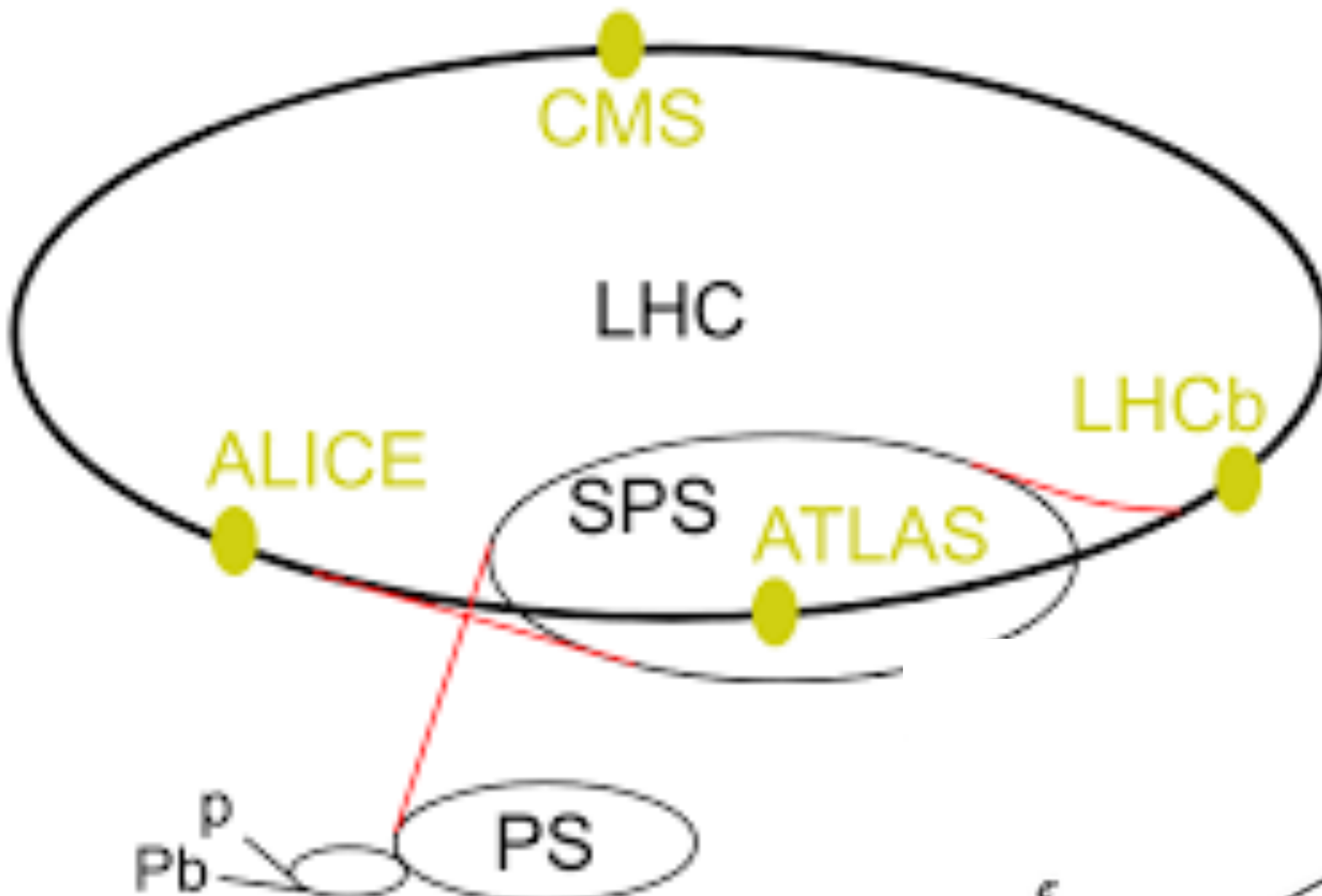
SIDIS?

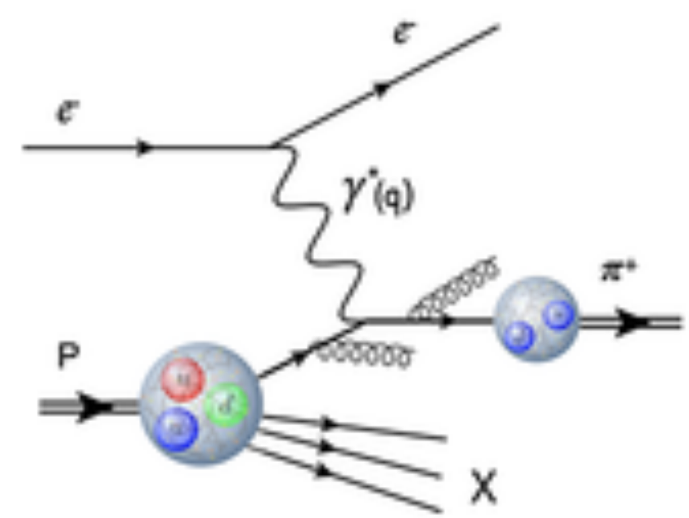
In SIDIS, in addition to the scattered lepton, we tag one of the fi



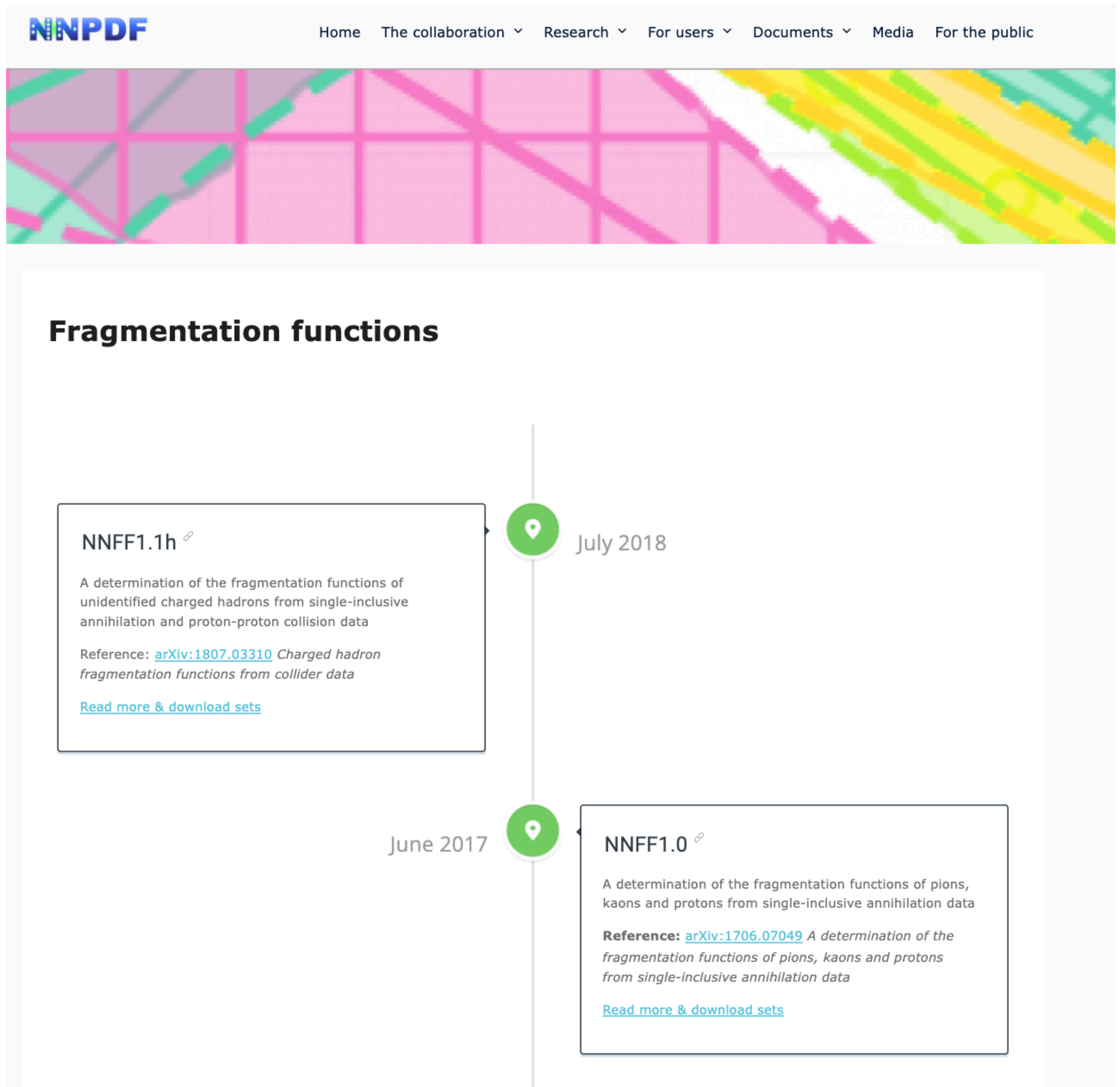
SIDIS depends on Parton Distribution Function (PDF) of the incoming hadron and Parton Fragmentation (FF) of the final state hadron.

COMPASS collaboration for FF

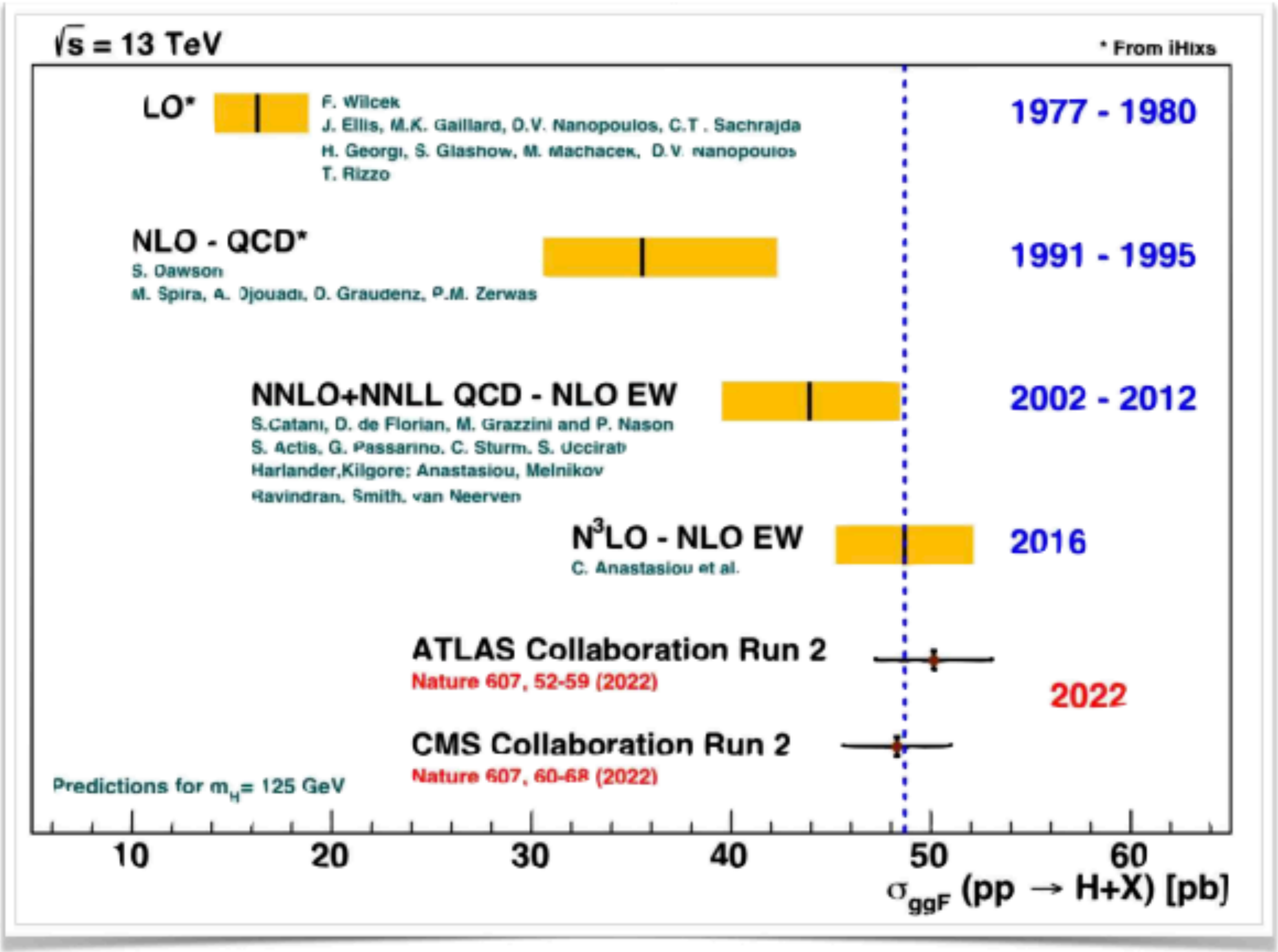




Fragmentation Function:

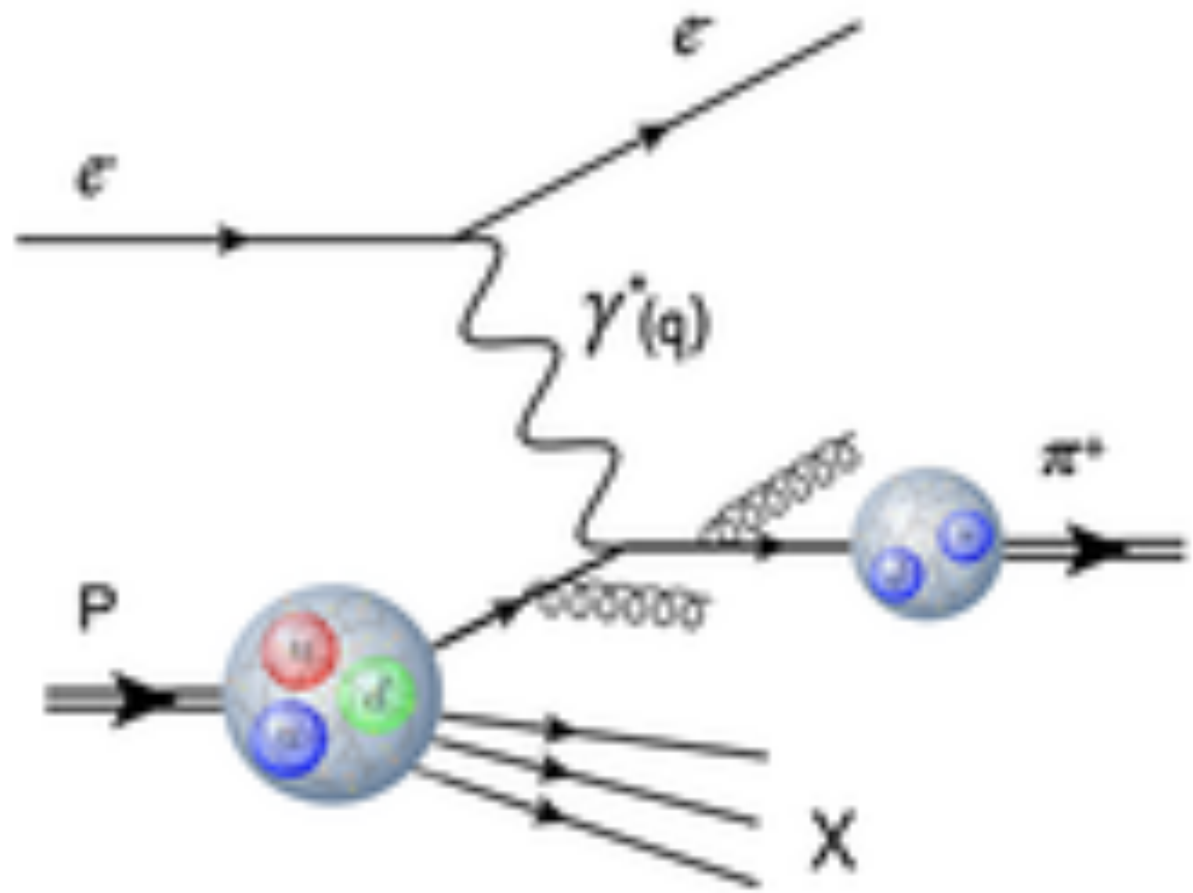


Why "Second order QCD corrections?"



Cross-section for inclusive Higgs production in gluon-gluon fusion

Why "Semi-Inclusive DIS (SIDIS) ?"

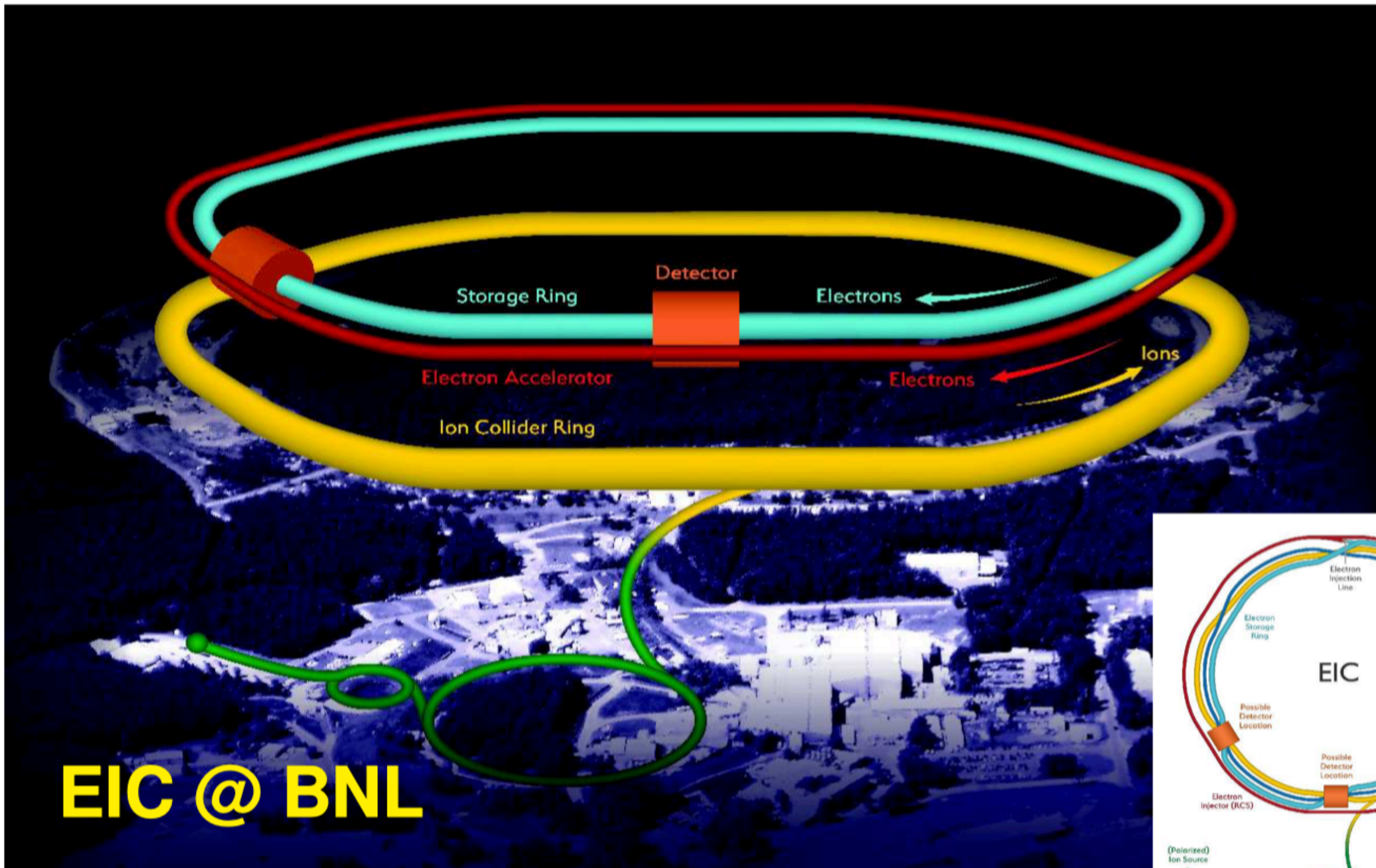


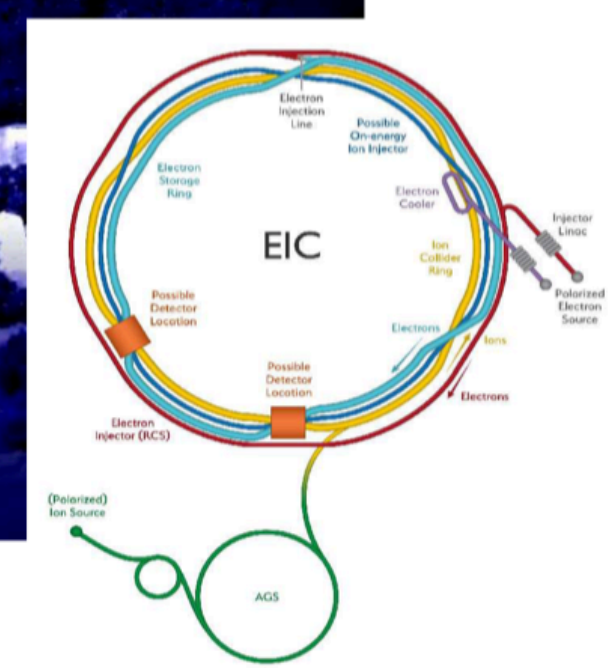
Sensitive to Parton Distribution Function

Fragmentation Function - mechanism for Hadronisation

- Electron-Ion Collider

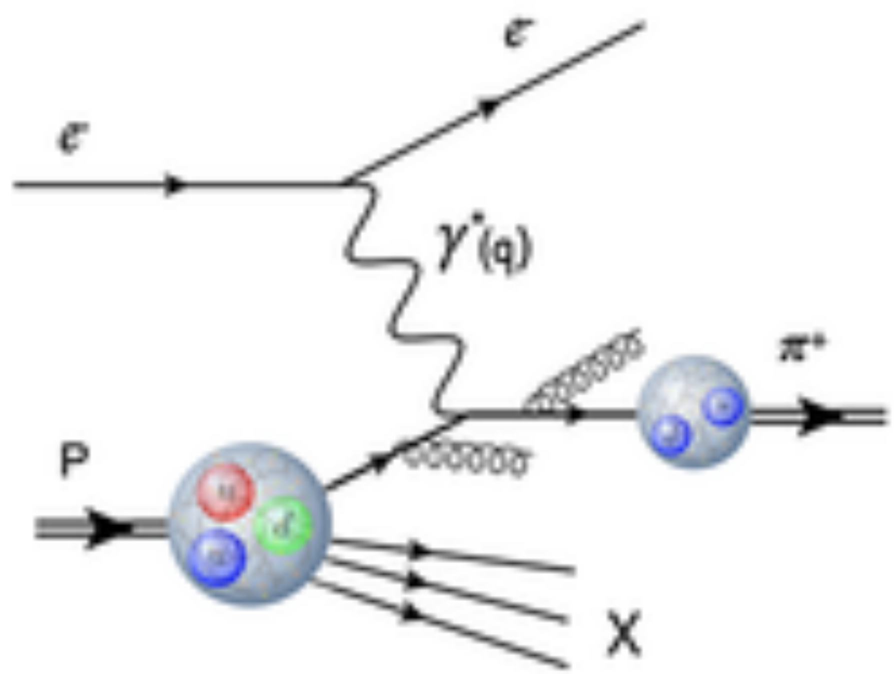
  A machine that will unlock the secrets of the strongest force in Nature





what is SIDIS?

In SIDIS, in addition to the scattered lepton, we tag one of the final-state hadrons.



SIDIS depends on Parton Distribution Function (PDF) of the incoming hadron and Parton Fragmentation (FF) of the final state hadron.

Predictions for SIDIS

Semi Inclusive Deep Inelastic Scattering (SIDIS) helps to study hadron structure both incoming hadron as well as the hadron that fragments in the final state.

Perturbative QCD provides framework to compute SIDIS cross sections order by order in strong coupling constant.

Leading order results are sensitive to theoretical uncertainty

1. Renormalisation scale dependence
2. Factorisation scale dependence from PDFs anfd FFs
3. Choice of PDFs and FFs

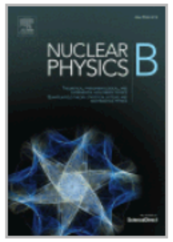
Higher order predictions are essential to resolve them

NLO Altarelli et al 1979

ELSEVIER

Nuclear Physics B

Volume 160, Issue 2, 3 December 1979, Pages 301-329



Processes involving fragmentation functions beyond the leading order in QCD ☆

G. Altarelli, R.K. Ellis, G. Martinelli, So-Young Pi

S+V NNLO

Vogelsang et al 2022

Threshold resummation at N^{3}LL accuracy and approximate N^{3}LO corrections to semi-inclusive DIS

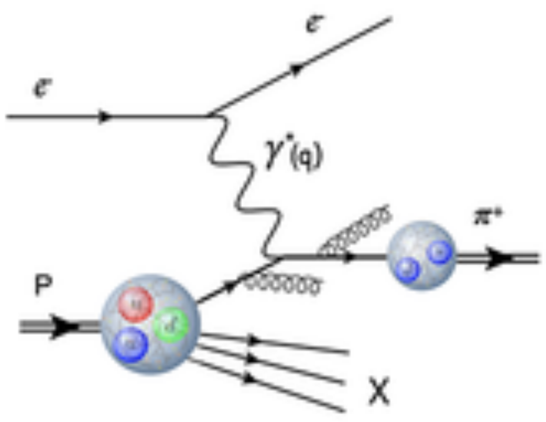
March 16, 2022

Maurizio Abele[a], Daniel de Florian[b], Werner Vogelsang[a]

Approximate NNLO QCD corrections to semi-inclusive DIS

March 16, 2022

Semi-Inclusive DIS - Second order QCD effects



NNLO QCD corrections to polarized semi-inclusive DIS #2

Saurav Goyal, Roman N. Lee, Sven-Olaf Moch, Vaibhav Pathak, Narayan Rana et al. (Apr 15, 2024)

e-Print: 2404.09959 [hep-ph]

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Next-to-Next-to-Leading Order QCD Corrections to Semi-Inclusive Deep-Inelastic Scattering #3

Saurav Goyal (IMSc, Chennai and HBNI, Mumbai), Sven-Olaf Moch (Hamburg U., Inst. Theor. Phys. II), Vaibhav Pathak (IMSc, Chennai and HBNI, Mumbai), Narayan Rana (NISER, Jatni), V. Ravindran (IMSc, Chennai and HBNI, Mumbai) (Dec 29, 2023)

Published in: *Phys.Rev.Lett.* 132 (2024) 25, 251902 • e-Print: 2312.17711 [hep-ph]

pdf DOI cite claim reference search 13 citations

Scattering Process:

$$e^-(k_l) + H(P) \to e^-(k_l^{'}) + H'(P_H) + X'$$

Factorises as

$$\frac{d^2\sigma_{e^-H}}{dE_l' d\Omega dz} = \frac{E_l'}{E_l}\frac{\alpha_e^2}{Q^4} L^{\mu\nu}(k_l, k'_l, q) W_{\mu\nu}(q, P, P_H).$$

Leptonic Tensor

$$L_{\mu\nu} = 2\Big[k_\mu k_\nu' + k_\mu' k_\nu - \frac{Q^2}{2} g_{\mu\nu}\Big]$$

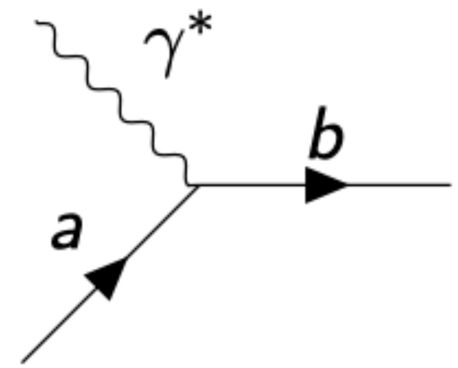
Hadronic Tensor

$$W^{\mu\nu} = F_1\Big[-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big] + F_2\Big[\frac{1}{P.q}(P^\mu - \frac{P.q}{q^2}q^\mu)(P^\nu - \frac{P.q}{q^2}q^\nu)\Big]$$

Paron Model for SIDIS

$$F_I = x^{I-1} \sum_{a,b} \int_x^1 \frac{dx_1}{x_1} f_a(x_1, \mu_F^2) \int_z^1 \frac{dz_1}{z_1} D_b(z_1, \mu_F^2)$$
$$\times \mathcal{F}_{I,ab}\left(\frac{x}{x_1}, \frac{z}{z_1}, Q^2, \mu_F^2\right).$$

- $f_a dx_1$: The probability of finding a parton of type ‘a’ which carries a momentum fraction x_1 of the parent hadron H.
- $D_b dz_1$: The probability that a parton of type ‘b’ will fragment into hadron H' which carries a momentum fraction z_1 of the parton.
- $\mathcal{F}_{I,ab}$ are the finite coefficient functions (CFs) that can be computed perturbatively, it is related to partonic cross section.



Partonic Cross sections

$$\hat{\sigma}_{I,ab} = \frac{\mathcal{P}_I^{\mu\nu}}{4\pi} \int \mathrm{dPS}_{X'+b}\, \overline{\Sigma} |M_{ab}|^2_{\mu\nu}\, \delta\Big(\frac{z}{z_1} - \frac{p_a \cdot p_b}{p_a \cdot q}\Big)$$

Fragmentaion

$|M_{ab}|^2$ is the squared amplitude for the process

$$a(p_a) + \gamma^*(q) \to \text{“}b\text{”}(p_b) + X'$$

$\mathcal{P}_I^{\mu\nu}$ are the projectors to project out CFs

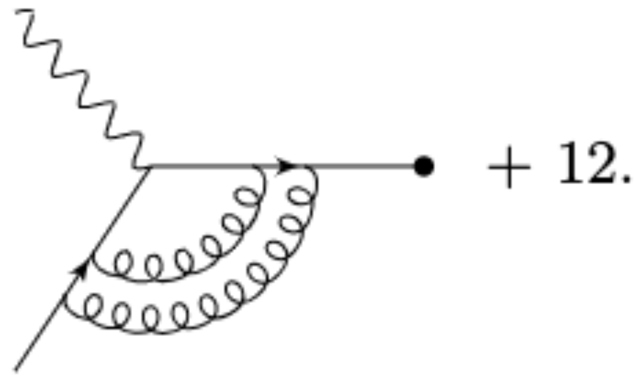
$$\mathcal{P}_1^{\mu\nu} = \frac{1}{(D-2)}\big(T_1^{\mu\nu} + 2x T_2^{\mu\nu}\big)$$

$$\mathcal{P}_2^{\mu\nu} = \frac{2x}{(D-2)x_1}\big(T_1^{\mu\nu} + 2x(D-1)T_2^{\mu\nu}\big).$$

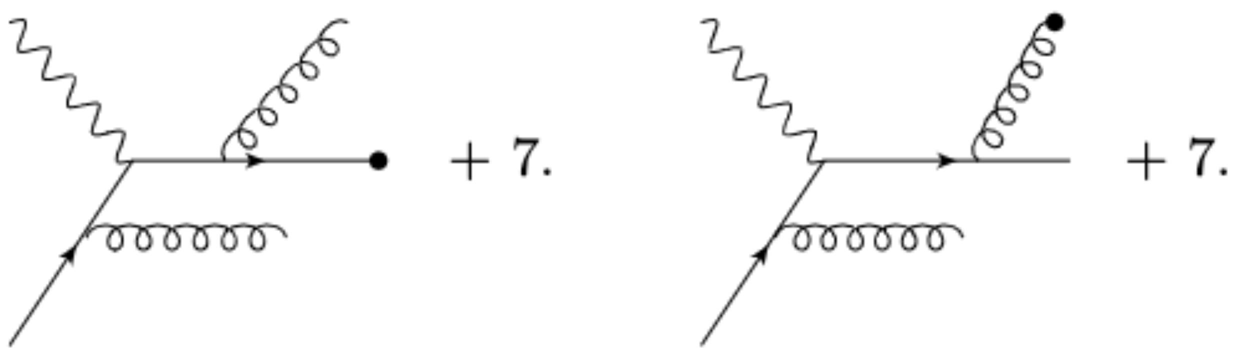
Partonic Subprocesses:

LO		$\gamma^* q \to q$
NLO	1 Loop:(V)	$\gamma^* q \to q$
		$\gamma^* q \to q + g$
		$\gamma^* g \to q + \overline{q}$
	2 Loop:(VV)	$\gamma^* q \to q$
NNLO	1 Loop:(RV)	$\gamma^* q \to q + g$
		$\gamma^* q \to q + g + g$
		$\gamma^* q \to q + q_i + \overline{q}_i$
	1 Loop:(RV)	$\gamma^* g \to q + \overline{q}$
		$\gamma^* g \to q + \overline{q} + g$

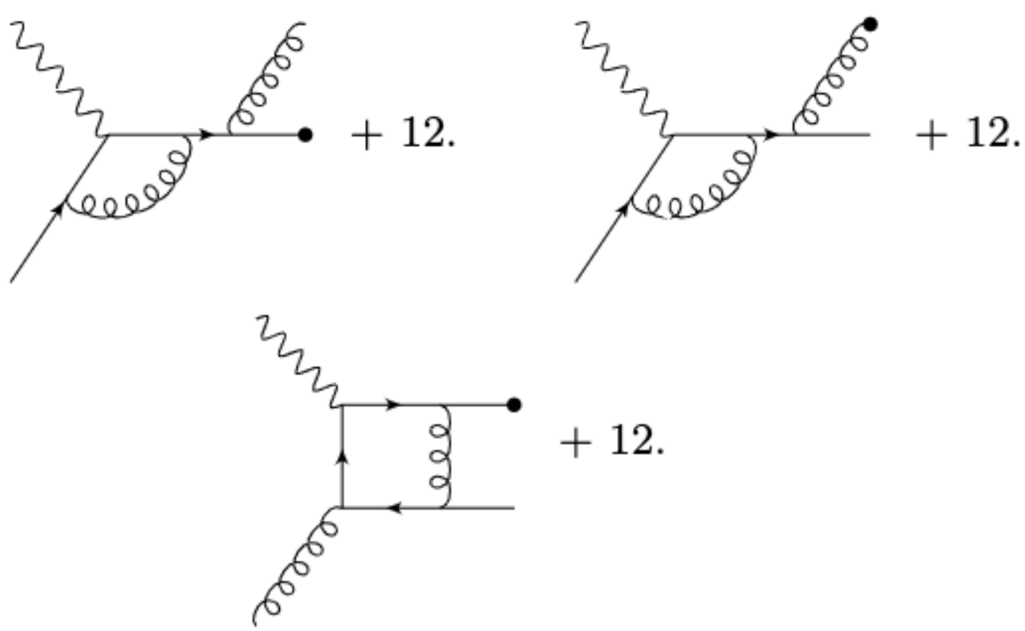
Partonic Subprocesses:



Pure Virtual



Pure Real



Mixed Real-Virtual

Loop Integrals:

We encounter a large number of loop integrals:

Integration-by-parts identities reduce them to fewer Master Integrals (MIs) .

$$\int d^D l \frac{\partial}{\partial l^\mu} [\frac{l^\mu, p^\mu}{D_1^{\nu_1} D_2^{\nu_2} \ldots D_n^{\nu_n}}] = 0$$

We choose a convenient set of families

Maping the loop integrals onto these Integral families is done by shifting of momenta ('Reduze') .

We used 'LiteRed' package perform IBP reduction to obtain MIs

Legs

Phase Space Integrals:

3-Body Phase Space, $p_a + q \rightarrow$ "p_b" $+ k_1 + k_2$,

$$\int [dPS]_3 =$$

$$\frac{1}{(2\pi)^{2D-3}} \int d^D k_1 \int d^D k_2 \int d^D p_b \delta(k_1^2)\delta(k_2^2)\delta(p_b^2)\delta^D(p_a + q - p_b - k_1 - k_2)\delta(z' - \frac{p_a.p_b}{p_a.q})$$

Reverse Unitarity method :

$$\delta(p^2 - m^2) \rightarrow \frac{i}{p^2 - m^2} - \quad c.c.$$

can be almost forget

We get total '21' MIs in phase space calculation.

Next task: Solving the Master Integrals

Master Integrals

Generalization with set of MIs

$$\vec{I} = \Big(I_1, I_2, \cdot, \cdot, \cdot, I_N\Big)$$

$\{I_i(\vec{x})\}$ depend on Scaling variables

$$x_i = f_i\left(\frac{s_{ij}}{Q^2}\right)$$

$$\vec{x} = \Big(x_1, x_2, \cdot, \cdot, \cdot, x_M\Big)$$

Differential equation:

$$d\vec{I} = \sum_{i=1}^{M} \mathbf{A}_i dx_i \; \vec{I}$$

$$\frac{\partial}{\partial x_i} \begin{bmatrix} I_1 \\ \cdot \\ I_N \end{bmatrix} = \begin{bmatrix} \mathbf{A_{11}} & \cdots & \mathbf{A_{1N}} \\ \cdot & \cdots & \cdot \\ \mathbf{A_{N1}} & \cdots & \mathbf{A_{NN}} \end{bmatrix} \begin{bmatrix} I_1 \\ \cdot \\ I_N \end{bmatrix}$$

$$i = 1, 2, \cdot, \cdot, \cdot M$$

Canonical/Henn's Basis

Consider Diff equation:

$$d\vec{I}(\vec{x},n) = \sum_i \mathbf{A}_i(\vec{x},n)dx_i\ \vec{I}(\vec{x},n)$$

Choose U Transformation such that

$$U^{-1}\mathbf{A}(\vec{x},n)U - U^{-1}dU = (n-4)\ \overline{\mathbf{A}}(\vec{x})$$

Diff equation contains ‚n' independent A

$$d\vec{\overline{I}}(\vec{x},n) = (n-4)\sum_i \overline{\mathbf{A}}_i(\vec{x})dx_i\ \vec{\overline{I}}(\vec{x},n)$$

Solution $\vec{\overline{I}}(\vec{x},n) = \vec{\overline{I}}(\vec{x}_0,n)\ \mathbf{P}\exp\left((n-4)\int \frac{d\lambda}{\lambda}\overline{\mathbf{A}}(\lambda)\right)$

$\mathbf{P}$ - Path Ordered exponential

Computation

Dimensionally regulated integrals contain functions that require correct analytic continuation:

$$(z' - x')^{a\epsilon - b}, (1 - z' - x')^{c\epsilon - d}$$

Feynman $+i\varepsilon$ prescription of propagators

$$x' \equiv x' - i\varepsilon \quad \text{and} \quad z' \equiv z' - i\varepsilon.$$

Partial fractioning and theta function to separate different sectors.

$$\left(\frac{z'-x'}{1-x'}\right)^{\varepsilon} = \left|\frac{z'-x'}{1-x'}\right|^{\varepsilon} \Big(\theta(z' - x') + (-1 + i\varepsilon)^{\varepsilon} \theta(x' - z')\Big)$$

$$\left(\frac{1-z'}{1-z'-x'}\right)^{\varepsilon} = \left|\frac{1-z'}{1-z'-x'}\right|^{\varepsilon} \Big(\theta(1 - z' - x') + (-1 - i\varepsilon)^{\varepsilon} \theta(z' + x' - 1)\Big)$$

`+` Distributions

Dimensionally regulated integrals contain divergences as x ->1 and z->1

$$\frac{(1-x)^\epsilon}{1-x} = \frac{1}{\epsilon}\delta(1-x) + \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!}\left[\frac{\ln^i(1-x)}{1-x}\right]_+$$

$$= \frac{1}{\epsilon}\delta(1-x) + \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} D_i(x)$$

`+` distributions

$$\int_0^1 dx f(x)[g(x)]_+ = \int_0^1 dx \left(f(x) - f(1)\right) g(x)$$

Double distribution,

$$\int_0^1 dx \int_0^1 dz \frac{f(x,z)}{[1-x]_+[1-z]_+} =$$
$$\int_0^1 dx \int_0^1 dz \frac{f(x,z) - f(1,z) - f(x,1) + f(1,1)}{(1-x)(1-z)}$$

Mass factorization

Soft divergences cancels among virtual and real emission processes,

The collinear divergences related to the a and b partons in the initial state and the final fragmentation state remain.

These divergences can be factored out into Altarelli-Parisi (AP) kernels (mass factorisation) at factorisation scale,

$$\frac{\hat{\sigma}_{I,ab}(\epsilon)}{x'^{I-1}} = \Gamma_{c\leftarrow a}(\mu_F^2,\epsilon) \otimes \mathcal{F}_{I,cd}(\mu_F^2,\epsilon) \tilde{\otimes} \tilde{\Gamma}_{b\leftarrow d}(\mu_F^2,\epsilon),$$

Convolution: $[f \otimes g](x) = \int_x^1 \frac{dt}{t} f(t) g(\frac{x}{t})$

NNLO results:

Dominant contribution:

1. Quark Initiated processes and
2. Quarks fragmenting to hadrons

The MIs were computed using two different methods

Initial and final collinear singularities were removed using mass factorization

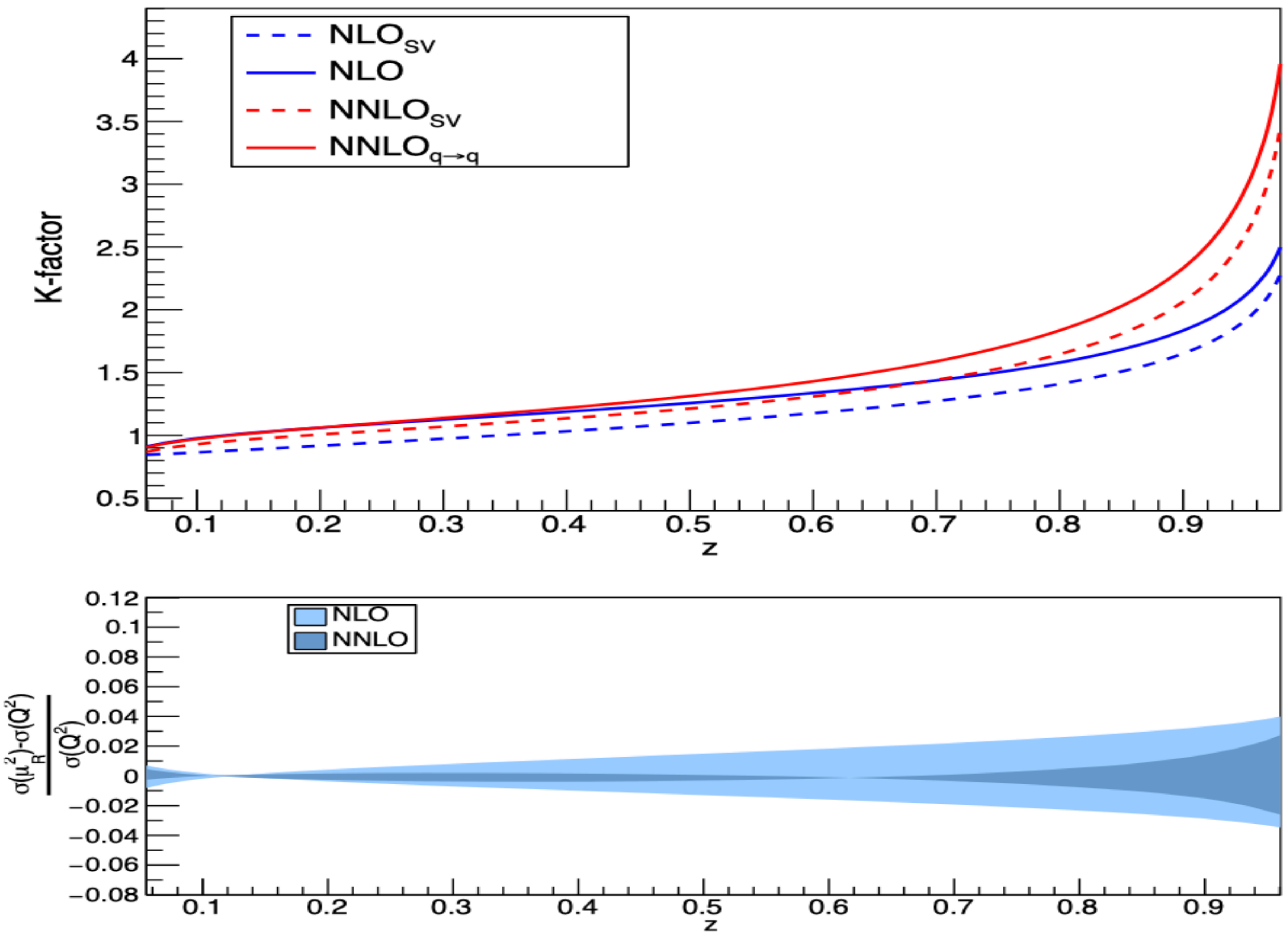
We confirmed the + distributions and Dirac delta contributions, called Soft-Virtual part in the literature

We obtain regular contributions for the first time

Numerical Impact:

Spin-independent

$$d\sigma = \frac{1}{4}\sum_{s,S,s',S_H} d\sigma^{s',S}_{s,S}$$



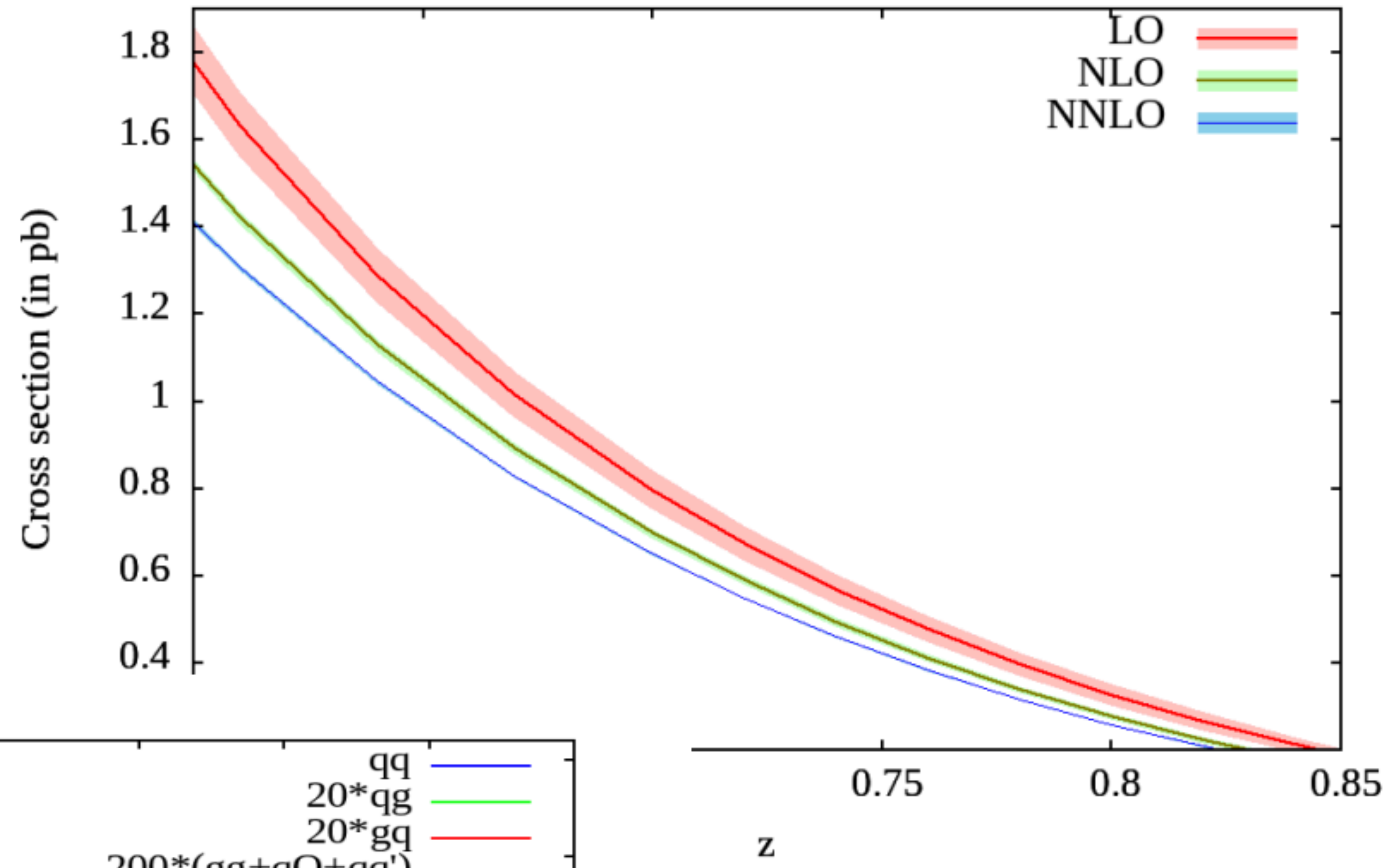
[5] S. Moch et.al.{arXiv:0404111}

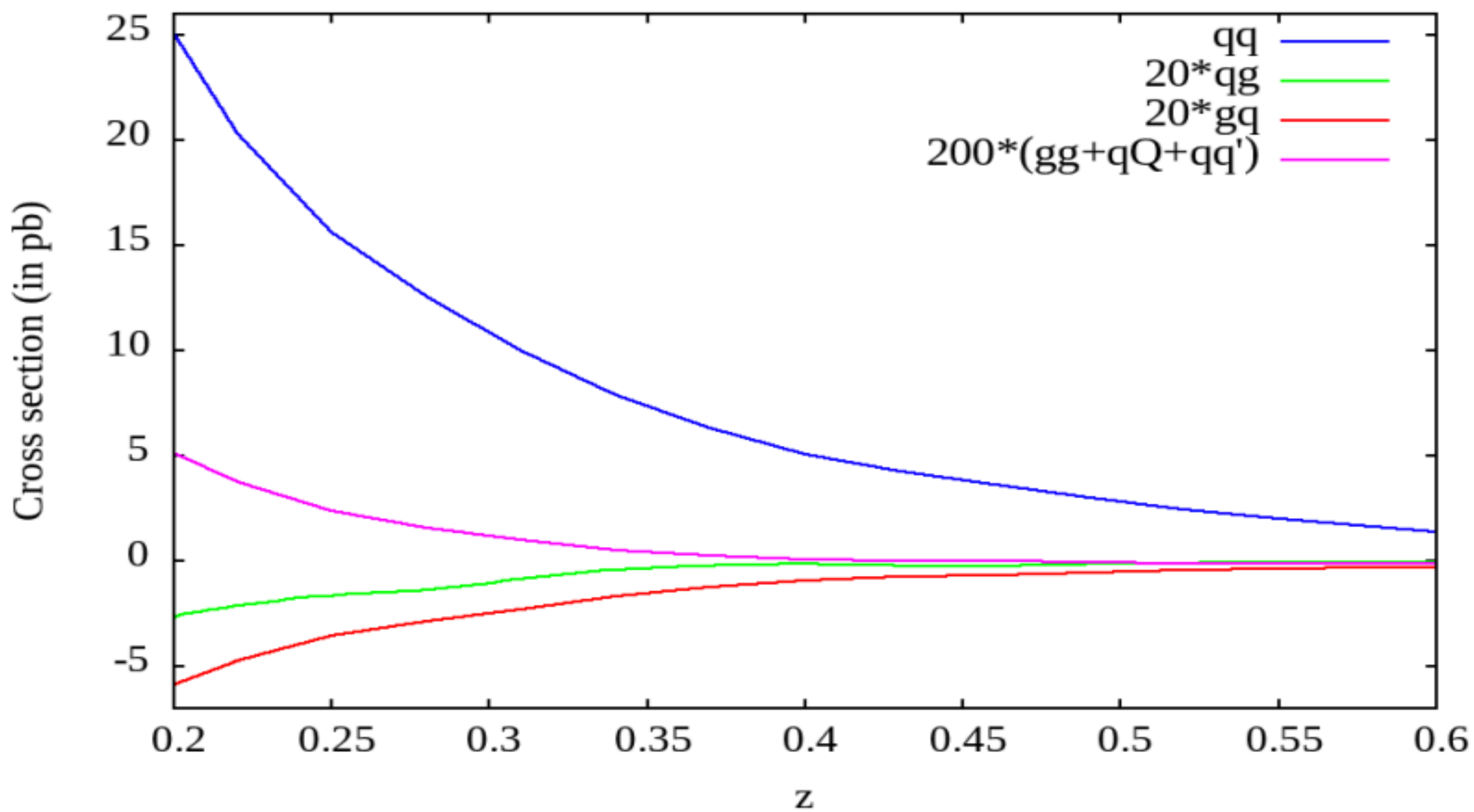
$$xq(x,\mu_F^2) = 0.6x^{-0.3}(1-x)^{3.5}(1+5.0x^{0.8}),$$
$$xg(x,\mu_F^2) = 1.6x^{-0.3}(1-x)^{4.5}(1-0.6x^{0.3}).$$

Numerical Impact:

Spin-dependent

$$d\Delta\sigma = \frac{1}{2}\sum_{s',S_H}\left(d\sigma^{s',S_H}_{s=\frac{1}{2},S=\frac{1}{2}} - d\sigma^{s',S_H}_{s=\frac{1}{2},S=\cdot}\right.$$





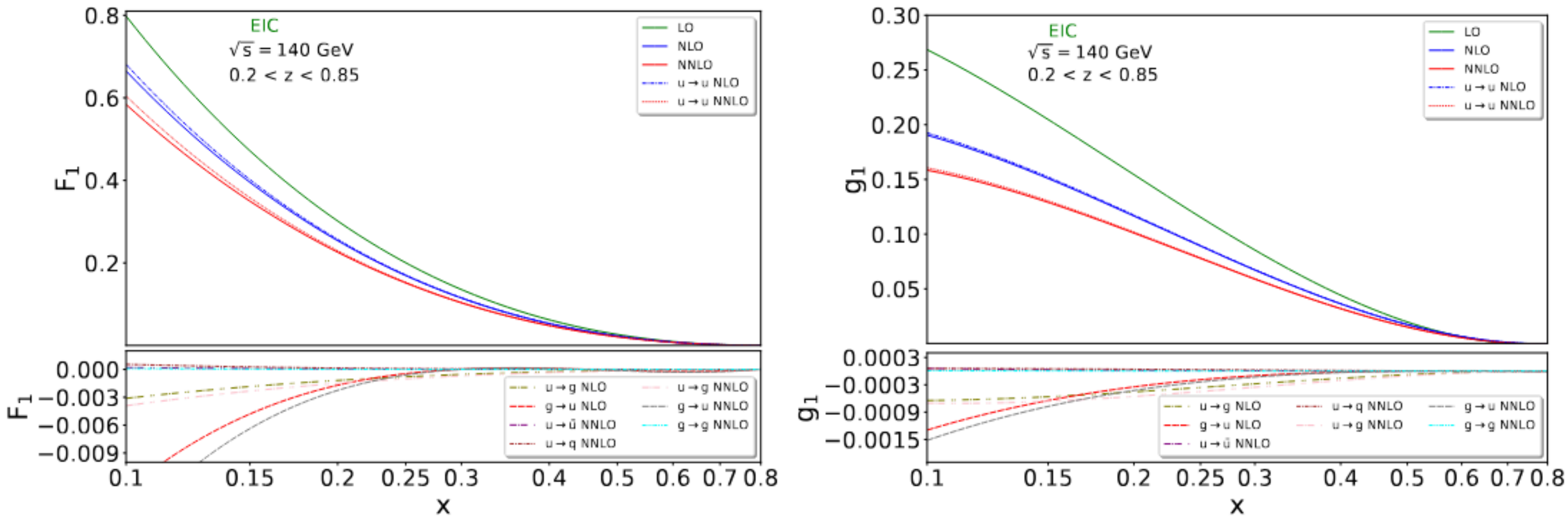


FIG. 11: Contributions from all partonic channels to SF F_1 (left panel) and g_1 (right panel) as a function of x for the EIC energy $\sqrt{s} = 140$ GeV.

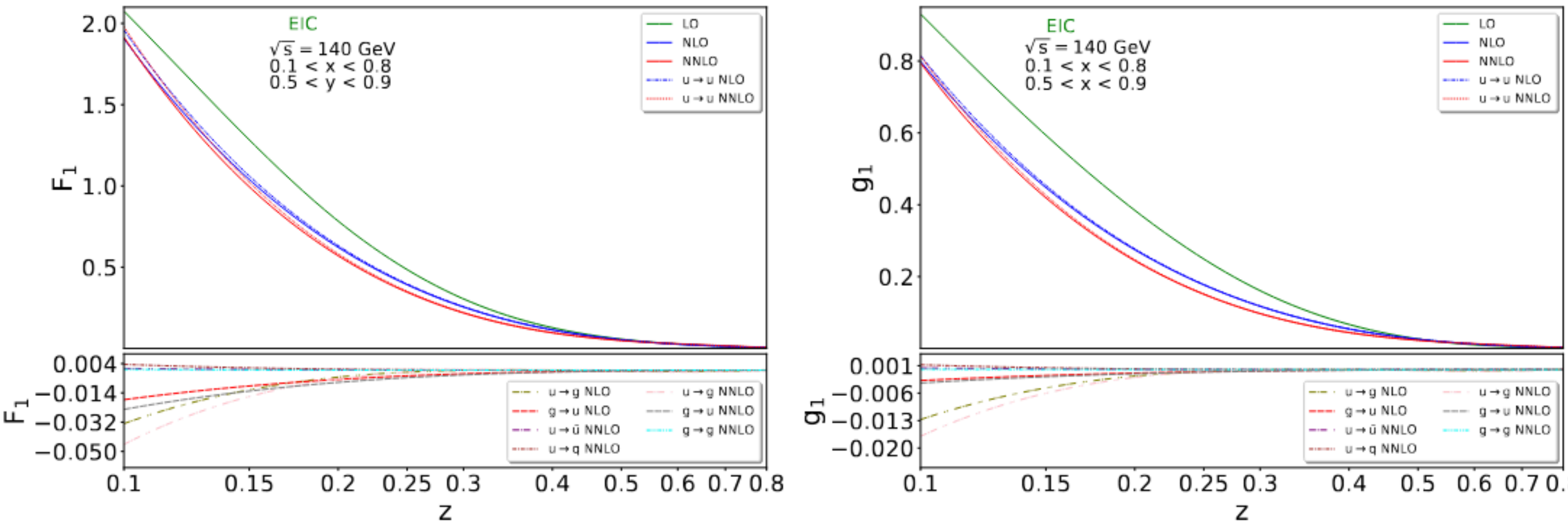


FIG. 12: Contributions from various partonic channels to F_1 (left panel) and g_1 (right panel) as a function of z for the EIC energy $\sqrt{s} = 140$ GeV.

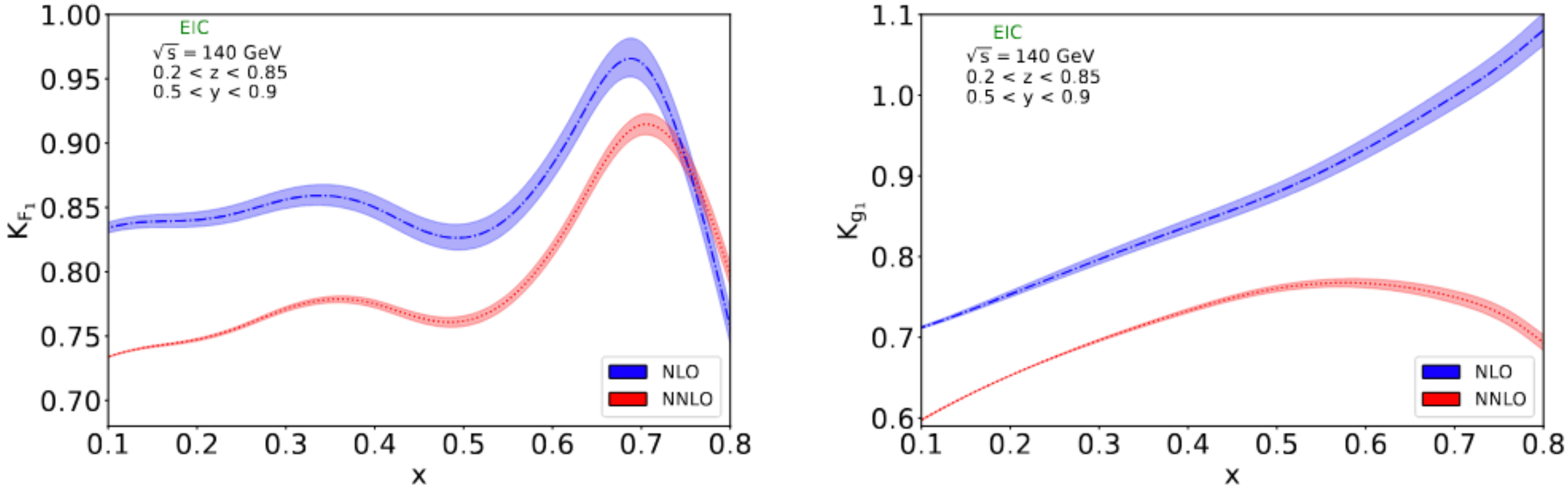


FIG. 13: K-Factor for F_1 and g_1 as a function of x at EIC energy $\sqrt{s} = 140$ GeV.

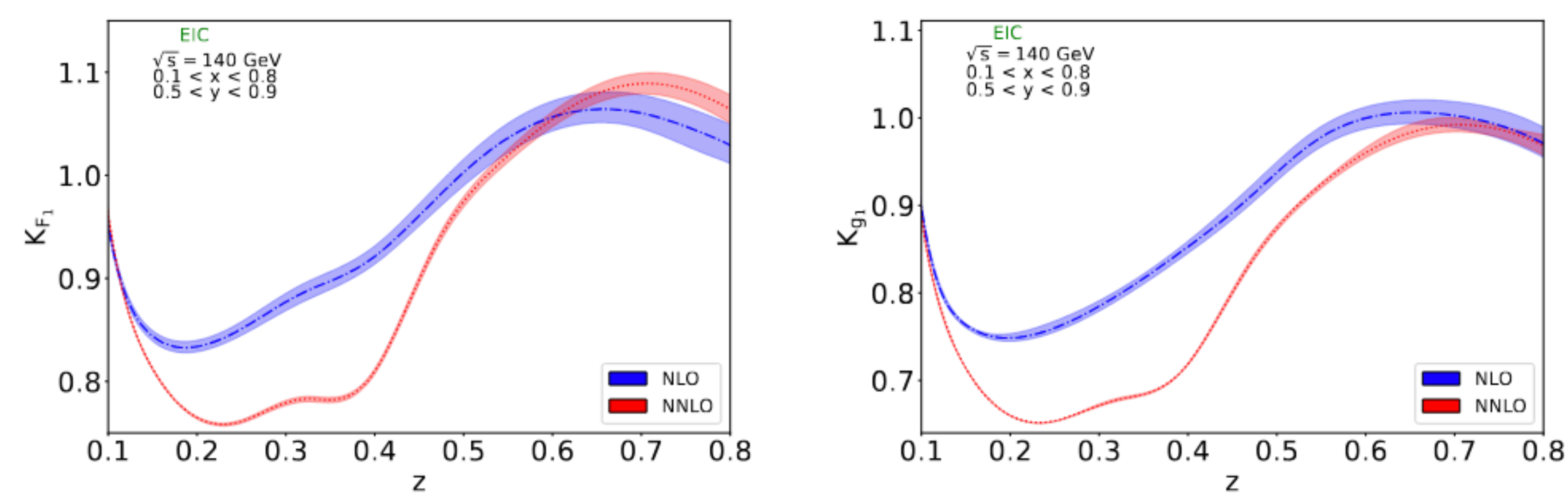


FIG. 14: K-Factor for F_1 andg_1 as a function of z at EIC energy $\sqrt{s} = 140$ GeV.

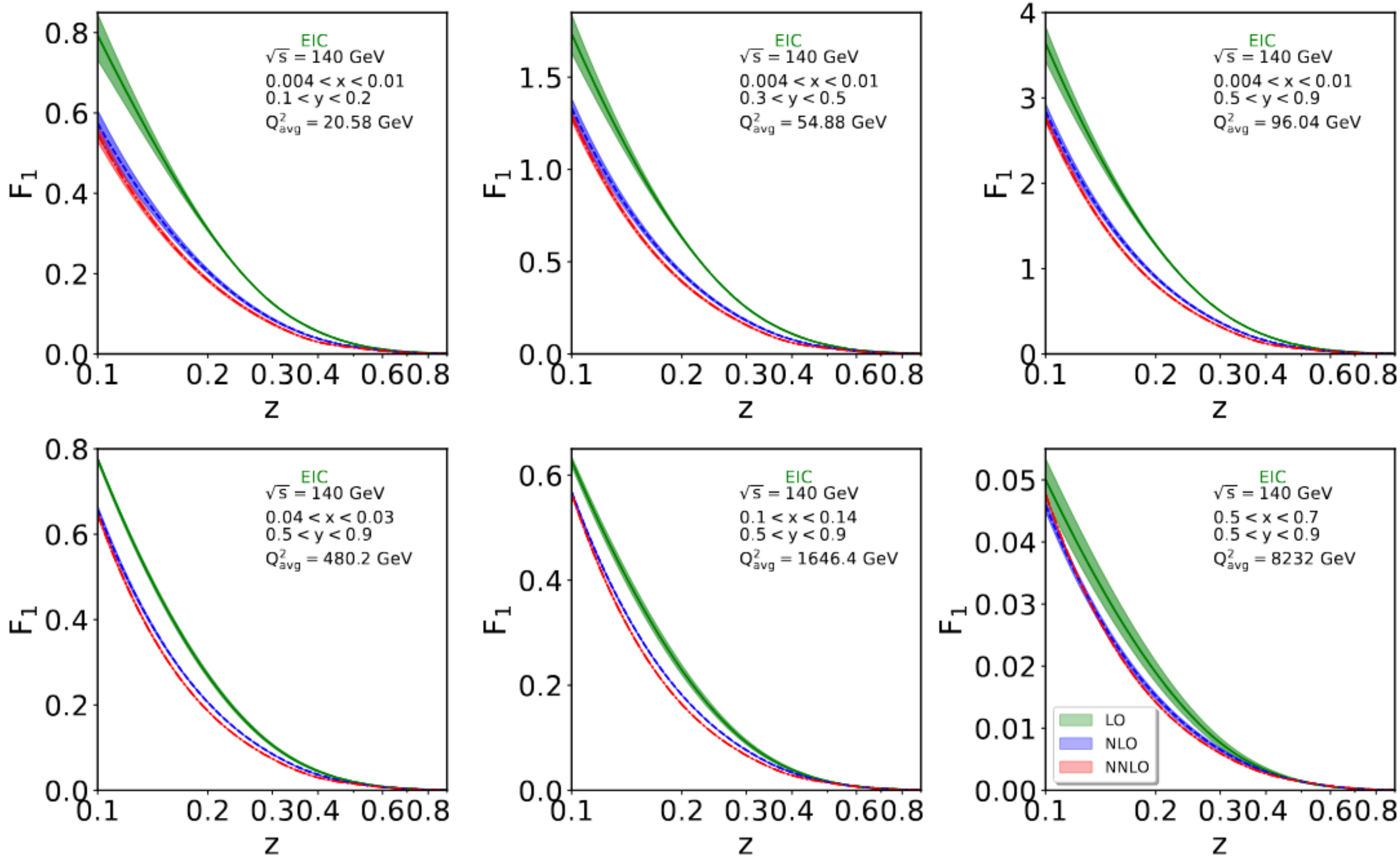


FIG. 17: Plots for the scale variation of $\mathrm{F}_1$with respect to z for 6 different energies Q^2.

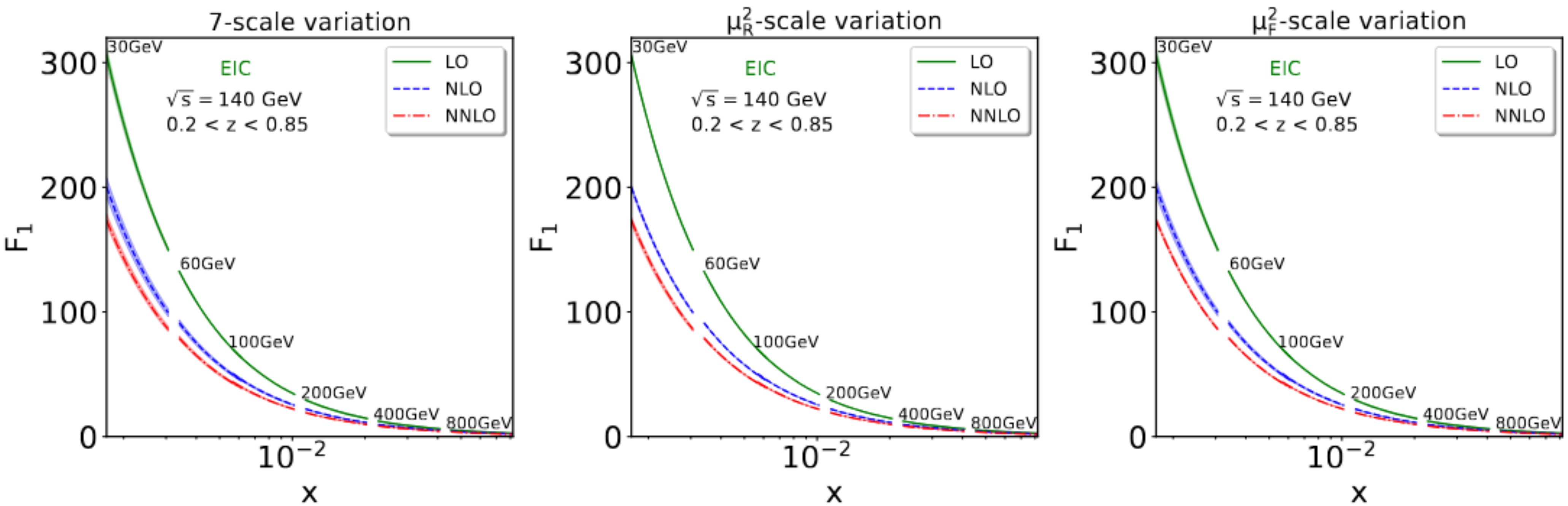


FIG. 15: Plots for the scale variation of F_1 with respect to x for 6 different energies Q^2.

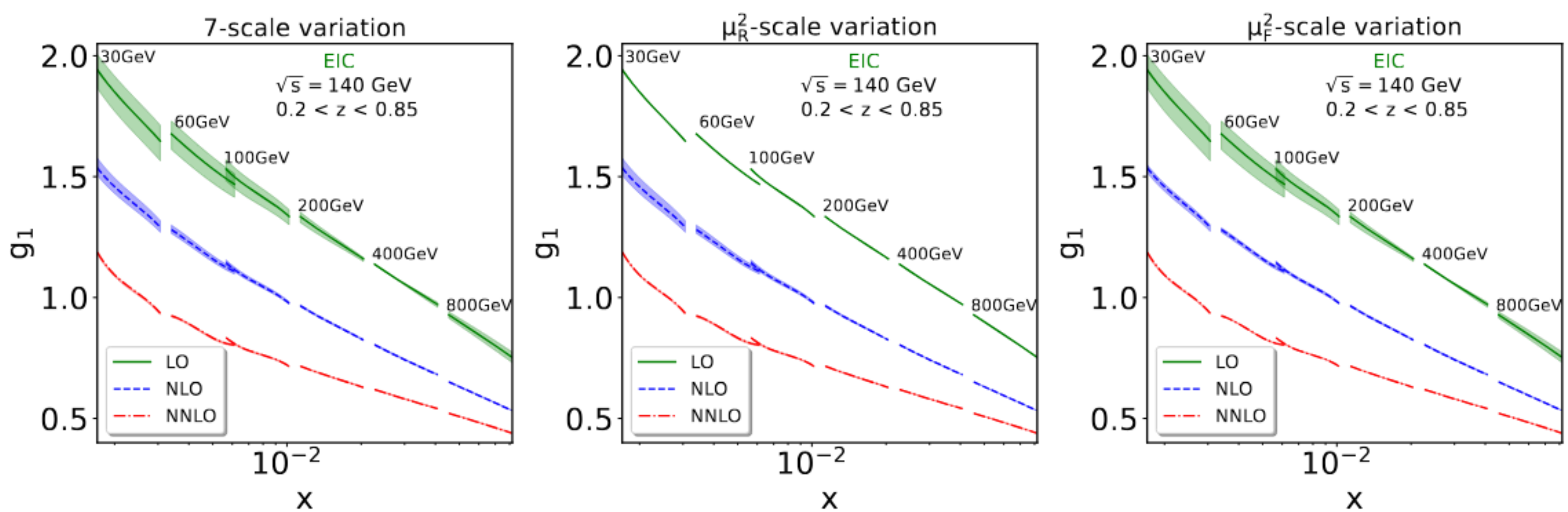


FIG. 16: Plots for the scale variation of g_1 with respect to x for 6 different energies Q^2.

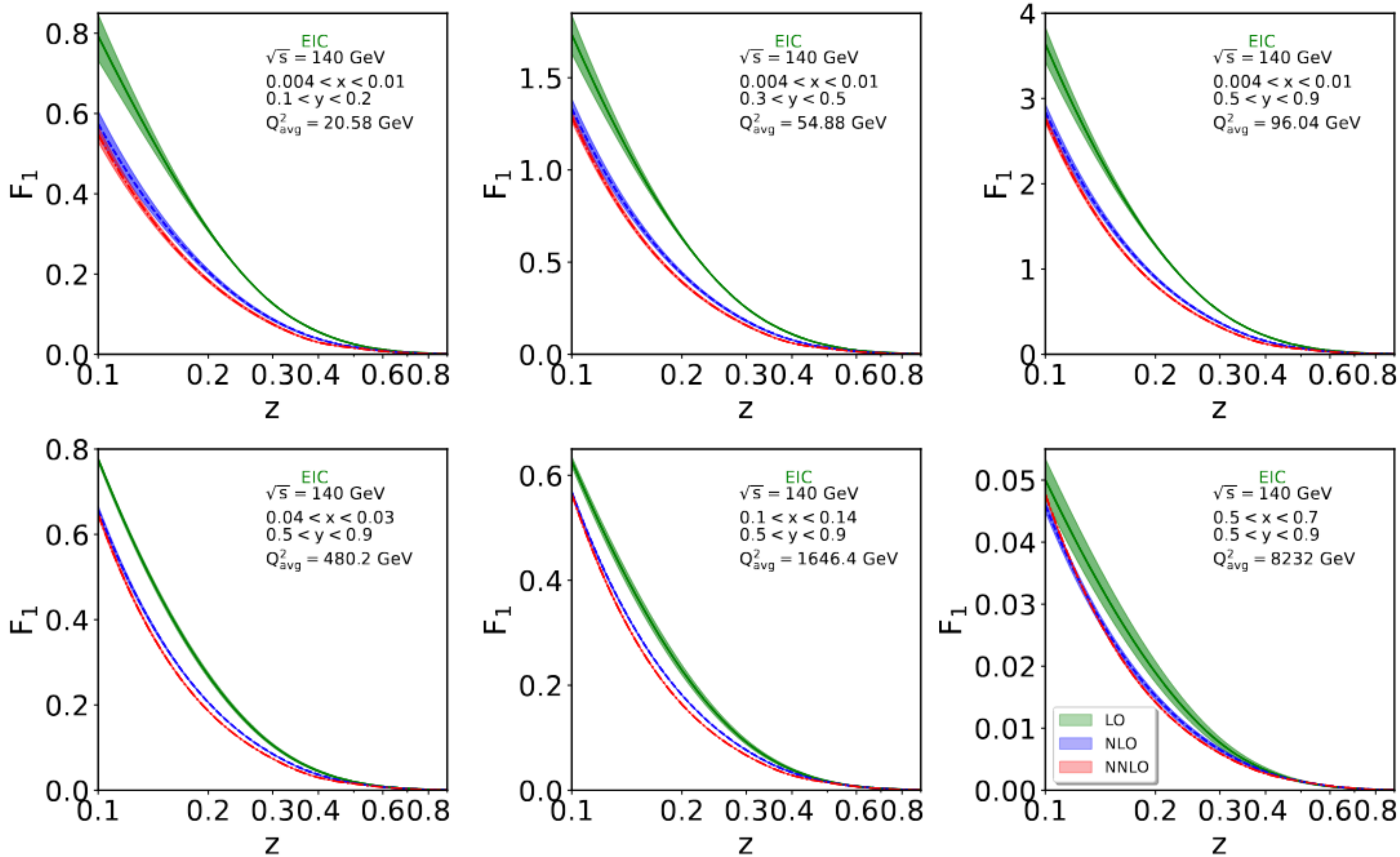


FIG. 17: Plots for the scale variation of $F_1$with respect to z for 6 different energies Q^2.

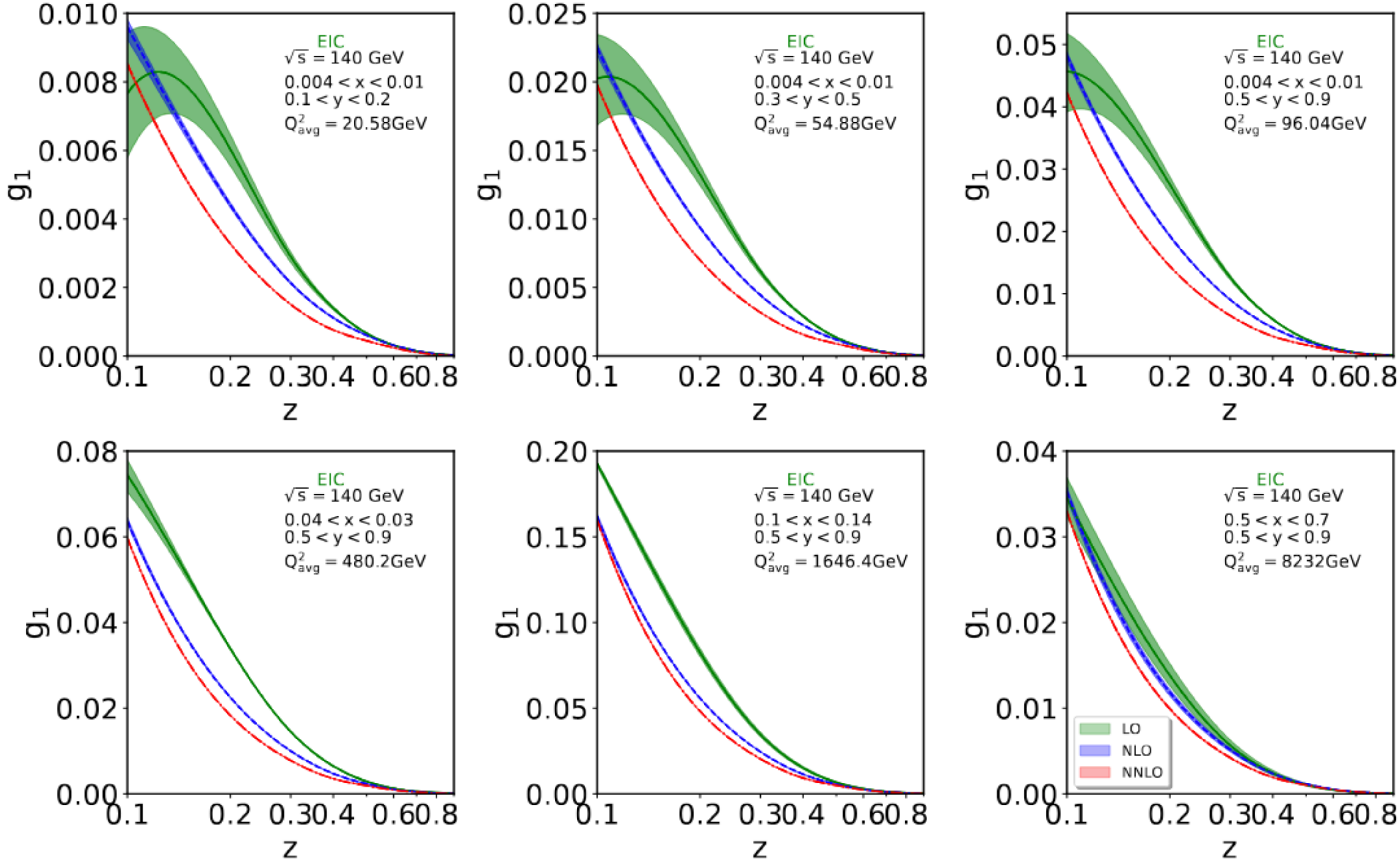


FIG. 18: Plots for the scale variation of g_1 with respect to z for 6 different energies Q^2.

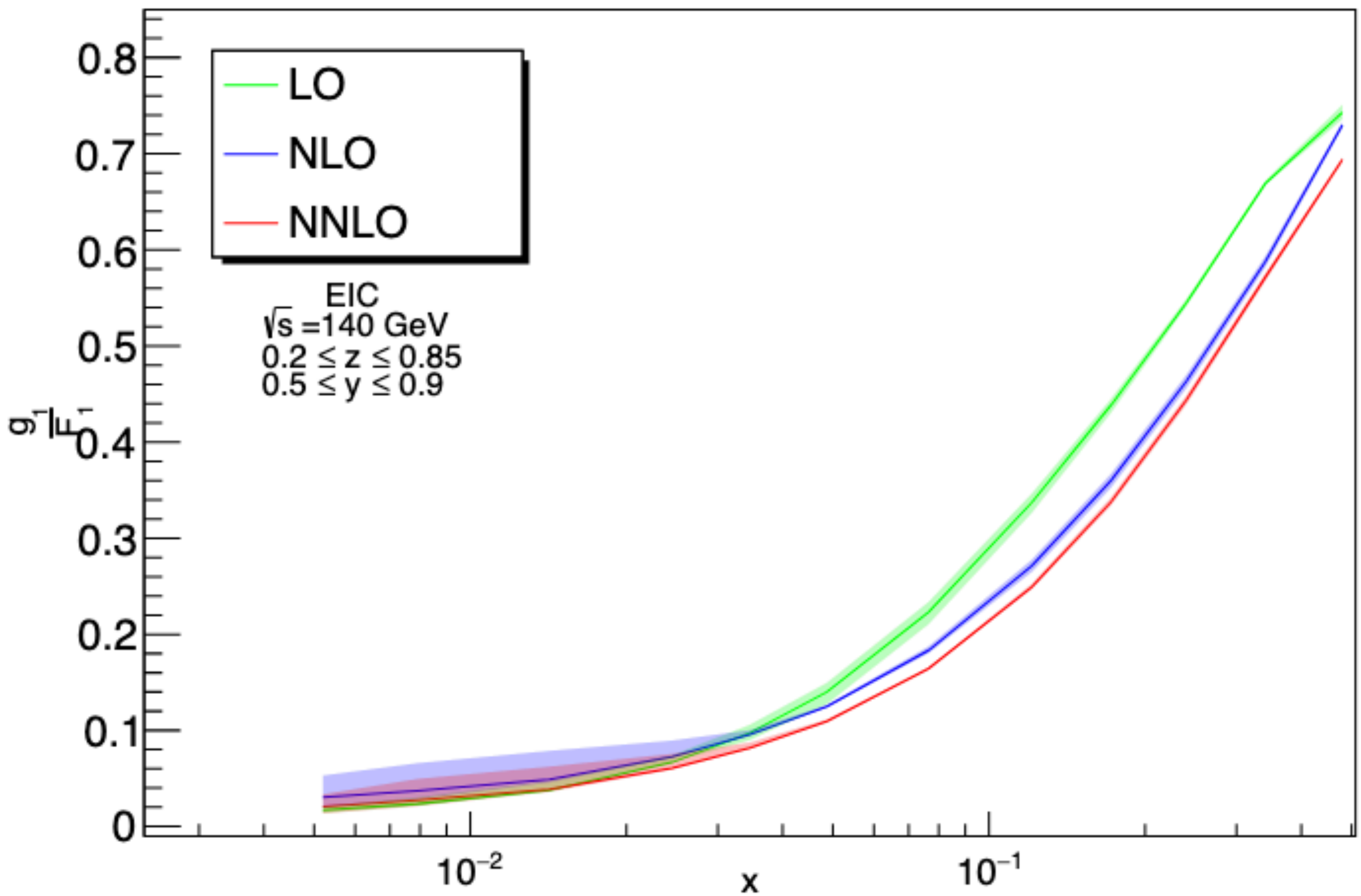


FIG. 19: Asymmetry g_1/F_1 as a function of x at EIC energy $\sqrt{s} = 140$ GeV.

Going Beyond NNLO -- Threshold corrections:

Threshold Limit zi -> 1

$$\left(\frac{\ln(1-z_i)}{(1-z_i)}\right)_+ \qquad \delta(1-z_i) \qquad \text{SV}$$

$$\log^k(1-z_i), \quad k=0,\cdots\infty \qquad \text{next to SV}$$

drop $\quad (1-z_i)^k, \quad k=1,\cdots\infty$

Going Beyond NNLO -- Threshold corrections:

Mass Factorisation:

$$(\Delta)\hat{\mathcal{C}}_{i,ab}(x',z',\varepsilon) = (\Delta)\Gamma_{c\leftarrow a}(x',\mu_F^2,\varepsilon) \otimes (\Delta)\mathcal{C}_{i,cd}(x',z',\mu_F^2,\varepsilon)\tilde{\otimes}\tilde{\Gamma}_{b\leftarrow d}(z',\mu_F^2,\varepsilon)$$

Explicitly

$$\begin{aligned}(\Delta)\hat{\mathcal{C}}_{J,qq}(x',z',\varepsilon) = \; & (\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,qq}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow q} + (\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,q\bar{q}}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow \bar{q}} \\ & + (\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,qg}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow g} + (\Delta)\Gamma_{\bar{q}\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,\bar{q}q}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow q} \\ & + (\Delta)\Gamma_{\bar{q}\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,\overline{qq}}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow \bar{q}} + (\Delta)\Gamma_{\bar{q}\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,\bar{q}g}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow g} \\ & + (\Delta)\Gamma_{g\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,gq}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow q} + (\Delta)\Gamma_{g\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,g\bar{q}}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow \bar{q}} \\ & + (\Delta)\Gamma_{g\leftarrow q} \otimes (\Delta)\mathcal{C}_{J,gg}\tilde{\otimes}\tilde{\Gamma}_{q\leftarrow g}\,.\end{aligned}$$

Keeping only SV and NSV terms:

$$(\Delta)\hat{\mathcal{C}}_{J,qq}(x',z',\varepsilon)=(\Delta)\Gamma_{q\leftarrow q} \otimes (\Delta)\mathcal{C}^{\mathrm{sv+nsv}}_{J,qq} \otimes \tilde{\Gamma}_{q\leftarrow q}$$

Remarkably Simple

Threshold Limits:

Mass Factorisation of

Off-diagonal channel:

$$
\begin{aligned}
(\Delta)\hat{\mathcal{C}}_{i,qg}(x',z',\varepsilon) = \; & (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,qq}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow q} + (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,q\bar{q}}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow\bar{q}} \\
& + (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,qg}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow g} + (\Delta)\Gamma_{\bar{q}\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,\bar{q}q}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow q} \\
& + (\Delta)\Gamma_{\bar{q}\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,\overline{qq}}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow\bar{q}} + (\Delta)\Gamma_{\bar{q}\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,\bar{q}g}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow g} \\
& + (\Delta)\Gamma_{g\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,gq}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow q} + (\Delta)\Gamma_{g\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,g\bar{q}}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow\bar{q}} \\
& + (\Delta)\Gamma_{g\leftarrow q}\otimes(\Delta)\mathcal{C}_{i,gg}\tilde{\otimes}\tilde{\Gamma}_{g\leftarrow g}\,.
\end{aligned}
$$

Similarly, dropping beyond NSV accuracy terms,

$$
\begin{aligned}
(\Delta)\hat{\mathcal{C}}_{i,qg}(x',z',\varepsilon) = \; & (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}^{\mathrm{sv+nsv}}_{J,qq}\otimes\tilde{\Gamma}_{\bar{q}\leftarrow g} \\
& + (\Delta)\Gamma_{q\leftarrow q}\otimes(\Delta)\mathcal{C}^{\mathrm{sv+nsv}}_{J,qg}\otimes\tilde{\Gamma}_{g\leftarrow g}
\end{aligned}
$$

Not So Simple

Diagonal channel:

Factor out the Form Factor $\hat{F}_{J,q}(Q^2,\varepsilon)$ Contribution

For diagonal channel in SV+NSV limit we can write,

$$(\Delta)\mathcal{C}^{\mathrm{sv+nsv}}_{J,qq}(Q^2,\mu_F^2,x',z',\epsilon) = \sigma_0^{-1}\left|\hat{F}_{J,q}(Q^2,\varepsilon)\right|^2 \delta(1-x')\delta(1-z')\otimes$$
$$\left(\left((\Delta)\Gamma_{q\leftarrow q}\right)^{-1}(x',\mu_F^2,\varepsilon)\otimes(\Delta)\mathcal{S}_{J,qq}(Q^2,x',z',\varepsilon)\otimes\left(\tilde{\Gamma}_{q\leftarrow q}\right)^{-1}(z',\mu_F^2,\varepsilon)\right)$$

Where the soft functions is defined by:

$$(\Delta)\mathcal{S}_{J,qq}(Q^2,x',z',\varepsilon) = \frac{(\Delta)\hat{\mathcal{C}}_{J,qq}(Q^2,x',z',\varepsilon)}{\left|\hat{F}_{J,q}(Q^2,\varepsilon)\right|^2}$$

No radiation gives simple result to all orders in QCD coupling

$$(\Delta)\mathcal{S}_{J,qq}\Big|_{\text{no rad.}} = \delta(1-x')\delta(1-z').$$

Form Factor

- The infrared and UV structures of the FFs through Sudakov K+G equation provide deep insight about the underlying quantum dynamics.
- In QCD, the UV divergences present in the FF can be removed by renormalisation. One finds that IR singularities factorises the way UV singularities do. That is, we can write

$$\hat{F}_q(Q^2, \varepsilon) = Z_{\hat{F}_q}(Q^2, \mu_s^2, \varepsilon) F_{q,fin}(Q^2, \mu_s^2, \varepsilon)$$

where $Z_{\hat{F}_q}$ is IR singular and $F_{q,fin}$ is IR finite, μ_s is IR factorisation scale. Renormalisation group invariance implies,

$$\mu_s^2 \frac{d}{d\mu_s^2} \ln Z_{\hat{F}_q} = \gamma_{\hat{F}_q},$$

where, $\gamma_{\hat{F}_q}$ has a remarkable structure because of double-poles

$$\gamma_{\hat{F}_q} = \frac{1}{2}\left(A_q(\mu_s^2) \log\left(\frac{Q^2}{\mu_s^2}\right) + 2B_q(\mu_s^2) + f_q(\mu_s^2) \right)$$

Form Factor

To understand the structure of finite part F_q, we set up the differential equation with respect to Q^2 called Sudakov K+G differential equation,

$$Q^2 \frac{d}{dQ^2} \ln \hat{F}_q(Q^2, \varepsilon) = \Gamma_{\hat{F}_q}(Q^2, \varepsilon) = \frac{1}{2}\Big(K_q(\mu_s^2, \varepsilon) + G_q(Q^2, \mu_s^2, \varepsilon)\Big)$$

where, kernel K_q contains IR singularities while G_q is IR finite,

$$K_q = 2\frac{d \ln Z_{\hat{F}_q}}{d \ln(Q^2)}, \qquad G_q = 2\frac{d \ln F_{q,fin}}{d \ln(Q^2)}$$

Also,

$$\mu_s^2 \frac{d}{d\mu_s^2} K_q(\mu_s^2, \varepsilon) = -A_q(\mu_s^2) = -\mu_s^2 \frac{d}{d\mu_s^2} G_q(Q^2, \mu_s^2, \varepsilon)$$

Solution to K+G equation takes the form:

$$\hat{F}_q(Q^2, \varepsilon) = \exp\left(\int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{\hat{F}_q}(\lambda^2, \varepsilon)\right)$$

with $\hat{F}_q(Q^2 = 0, \epsilon) = 1$. The FF is known to fourth order in QCD.

Altarelli-Parisi Evolution Equation

AP kernels $\Gamma_{q\leftarrow q}(x', \mu_F^2, \varepsilon)$ and $\tilde{\Gamma} q \leftarrow q(z', \mu_F^2, \varepsilon)$ removes collinear singularities arising from incoming partons and outgoing partons respectively at the factorisation scale μ_F.

In threshold limit, the AP kernels $(\mathbb{\Gamma}_{q\rightarrow q})$ satisfy evolution equation and are controlled by AP splitting functions $\mathbb{P}_{qq}$.

$$\mu_F^2 \frac{d}{d\mu_F^2} \mathbb{\Gamma}_{q\leftarrow q}(\mu_F^2, \xi, \varepsilon) = \frac{1}{2} \mathbb{P}_{qq}(\mu_F^2, \xi) \otimes \mathbb{\Gamma}_{q\leftarrow q}(\mu_F^2, \xi, \varepsilon),$$

Where the AP slitting function takes all order simple form

$$\mathbb{P}_{qq}(\mu_F^2, \xi) = 2\left(\frac{A_q(\mu_s^2)}{1-\xi} + B_q(\mu_s^2)\delta(1-\xi) + \mathbb{C}_q(\mu_s^2) \ln(1-\xi) + \mathbb{D}_q(\mu_s^2) \right)$$

Solution:

$$\mathbb{\Gamma}_{q\leftarrow q}(\mu_F^2, \xi, \varepsilon) = \mathbf{C} \exp\left(\frac{1}{2} \int_0^{\mu_F^2} \frac{d\lambda^2}{\lambda^2} \mathbb{P}_{qq}(\lambda^2, \xi, \varepsilon) \right)$$

Soft Function:

- Since $\mathcal{S}_{J,qq}(Q^2, x', z', \varepsilon)$ is RG invariant w.r.t μ_R and μ_F implies that the derivative w.r.t Q^2 of $\mathcal{C}_{J,qq}$ has to be a function of only Q^2 and x', z'.
- Also, the first term is finite and the second term contains both singular and finite parts via K+G equation of $\Gamma_{\hat{F},q}$, which implies we can write,

We can write a similar K-G eq. for soft-collinear function,

$$Q^2 \frac{d}{dQ^2} \mathcal{S}_{J,qq}(Q^2, x', z', \varepsilon) = \Gamma_{\mathcal{S},q}(Q^2, x', z', \varepsilon) \otimes \mathcal{S}_{J,qq}(Q^2, x', z', \varepsilon)$$

where,

$$\Gamma_{\mathcal{S},q} = Q^2 \frac{d}{dQ^2} \left(\mathbf{C} \ln \mathcal{C}_{J,qq}(Q^2, \mu_R^2, \mu_F^2, x', z')) - \ln \left| \hat{F}_q(Q^2) \right|^2 \delta(1 - x') \delta(1 - z') \right)$$

First term is finite and second contains both singular and finite

$$\Gamma_{\mathcal{S},q}(Q^2, x', z', \varepsilon) = \frac{1}{2} \left(\overline{K}_q(\mu_s^2, x', z', \varepsilon) + \overline{G}_q(Q^2, \mu_s^2, x', z', \varepsilon) \right)$$

K+G equation

Soft function:

K+G equation for Soft function implies factorisation:

$$\mathcal{S}_q(Q^2, x', z', \varepsilon) = Z_{\mathcal{S},q}(Q^2, \mu_s^2, x', z', \varepsilon) \otimes \mathcal{S}_{q,\mathrm{fin}}(Q^2, \mu_s^2, x', z', \varepsilon)$$

Renormalisation group equation:

$$\mu_s^2 \frac{d}{d\mu_s^2} Z_{\mathcal{S},q}(\mu_s^2, Q^2, x', z', \varepsilon) = \gamma_{\mathcal{S},q}(\mu_s^2, Q^2, x', z') \otimes Z_{\mathcal{S},q}(\mu_s^2, Q^2, x', z', \varepsilon)$$

Remarkable all order structure:

$$\gamma_{\mathcal{S},c} = \xi_1(\mu_s^2, x', z') \ln(Q^2/\mu_s^2) + \xi_2(\mu_s^2, x', z')$$

Solution:

$$\boxed{\mathcal{S}_q(Q^2, x', z', \varepsilon) = \mathbf{C} \exp\left(\int_0^{Q^2} \frac{d\lambda^2}{\lambda^2} \Gamma_{\mathcal{S}_q}(\lambda^2, x', z', \varepsilon) \right)} = \mathbf{C} \exp\left(2\Phi_q(Q^2, x', z', \varepsilon)\right)$$

Soft function:

Finiteness of Coefficient functions

Pole structure of Form factor and AP kernels

Renormalisation group invariance

Transcentendality structure of perturbative results upto Two loops

Ansatz for the solutions (SV + NSV)

$$\Phi_q^{SV} = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2(1-x')(1-z')}{\mu^2} \right)^{i\frac{\varepsilon}{2}} S_\varepsilon^i \left[\frac{i\varepsilon}{4(1-z')(1-z')} \varphi_q^{(i)}(\varepsilon) \right]$$

$$\Phi_q^{NSV} = \sum_{i=1}^{\infty} \hat{a}_s^i \left(\frac{Q^2(1-x')(1-z')}{\mu^2} \right)^{i\frac{\varepsilon}{2}} S_\varepsilon^i \frac{i\varepsilon}{4} \left[\frac{\varphi^{NSV,(i)}(z',\varepsilon)}{1-x'} + \frac{\varphi_q^{NSV,(i)}(x',\varepsilon)}{1-z'} \right]$$

Resumed Coefficient Function

Diagonal CF in the threshold:

$$(\Delta)\mathcal{C}^{\mathrm{sv+nsv}}_{J,qq}(Q^2,\mu_F^2,x',z',\epsilon) = \sigma_0^{-1}\left|\hat{F}_{J,q}(Q^2,\varepsilon)\right|^2 \delta(1-x')\delta(1-z')\otimes$$
$$\left(\left((\Delta)\Gamma_{q\leftarrow q}\right)^{-1}(x',\mu_F^2,\varepsilon)\otimes(\Delta)\mathcal{S}_{J,qq}(Q^2,x',z',\varepsilon)\otimes\left(\tilde{\Gamma}_{q\leftarrow q}\right)^{-1}(z',\mu_F^2,\varepsilon)\right)$$

Resumed CF:

$$\mathcal{C}^{\mathrm{sv+nsv}}_{qq}(Q^2,\mu_F^2,x',z',\varepsilon) = \mathbf{C}\exp\left(\Psi_{d,q}(Q^2,\mu_R^2,\mu_F^2,x',z',\varepsilon)\right)\Big|_{\varepsilon=0}$$

$$\Psi^q_d = \frac{\delta(\overline{x}')}{2}\left(\left\{\int_{\mu_F^2}^{Q^2\overline{z}'}\frac{d\lambda^2}{\lambda^2}\mathcal{P}^q\left(a_s(\lambda^2),\overline{z}'\right)+\mathcal{Q}^q_d\left(a_s(q_2^2),\overline{z}'\right)\right)_+\right.$$
$$+\frac{1}{4}\left(\frac{1}{\overline{x}'}\left\{\mathcal{P}^q\left(a_s(q_{12}^2),\overline{z}'\right)+2L^q\left(a_s(q_{12}^2),\overline{z}'\right)\right.\right.$$
$$\left.\left.+Q^2\frac{d}{dQ^2}\left(\mathcal{Q}^q_d\left(a_s(q_2^2),\overline{z}'\right)+2\varphi^f_{d,q}\left(a_s(q_2^2),\overline{z}'\right)\right)\right\}\right)_+$$
$$+\frac{1}{2}\delta(\overline{x}')\,\delta(\overline{z}')\ln\left(g^q_{d,0}\left(a_s(\mu_F^2)\right)\right)+\left(\overline{x}'\leftrightarrow\overline{z}'\right)$$

$\overline{x}' = (1-x')$ and $\overline{z}' = (1-z')$, $q_{12} = Q^2\overline{x}'\overline{z}'$ and $q_1 = Q^2\overline{x}'$.

Resumed Exponents:

$$P^q(a_s,\overline{x}') = 2\left(\frac{A^q(a_s)}{(\overline{x}')_+} + B^q(a_s)\delta(\overline{x}') + L^q(a_s,\overline{x}')\right),$$

$$\tilde{P}_q(a_s,\overline{z}') = 2\left(\frac{A^q(a_s)}{(\overline{z}')_+} + B^q(a_s)\delta(\overline{z}') + \tilde{L}^q(a_s,\overline{z}')\right),$$

$$\mathcal{P}^q(a_s,\overline{x}') = P^q(a_s,\overline{x}') - 2B^q(a_s)\delta(\overline{x}')$$

$$\mathcal{P}^q(a_s,\overline{z}') = \tilde{P}^q(a_s,\overline{z}') - 2B^q(a_s)\delta(\overline{z}')$$

$$\mathcal{Q}^q_d(a_s,\overline{z}') = \frac{2}{\overline{z}'} D^q_d(a_s) + 2\,\varphi^f_{d,q}(a_s,\overline{z}')$$

$$L^q(a_s,\overline{x}') = C^q(a_s)\ln(\overline{x}') + D^q(a_s)$$

Resummation in 2-d N space

The double Mellin transform of $\mathcal{C}_{qq}^{\rm sv+nsv}$ in $\vec{N}$ space as

$$\mathcal{C}_{qq,\vec{N}}^{\rm sv+nsv} = \int_0^1 dx' x'^{N_1-1} \int_0^1 dz' z'^{N_2-1} \mathcal{C}_{qq}^{\rm sv+nsv}(x',z')(Q^2,\mu_R^2,\mu_F^2)$$
$$= g_{d,0}^q(q^2,\mu_R^2,\mu_F^2) \exp\left(\Psi_{d,\vec{N}}^q(Q^2,\mu_F^2)\right) .$$

Where the exponent takes a simple structure

$$\Psi_{d,\vec{N}}^q = g_{d,1}^q(\omega) \ln N_1 + \sum_{i=0}^{\infty} a_s^i \left(\frac{1}{2} g_{d,i+2}^q(\omega) + \frac{1}{N_1} \overline{g}_{d,i}^q(\omega)\right)$$
$$+ \frac{1}{N_1}\left(h_{d,0}^q(\omega,N_1) + \sum_{i=1}^{\infty} a_s^i h_{d,i}^q(\omega,\omega_1,N_1)\right) + (N_1 \leftrightarrow N_2, \omega_1 \leftrightarrow \omega_2)$$

where, $\omega = a_s\beta_0 \ln N_1 N_2$ and $\omega_l = a_s \beta_0 \ln N_l$ for $l = 1, 2$.

Results are various Log accuracies:

Leading log:

$$\mathcal{C}^{\overline{\mathrm{LL}}}_{qq,\vec{N}} = \left(g^q_{d,0,0}\right) \exp\left[g^q_{d,1}(\omega)\ln N_1 + \frac{1}{N_1}\left(\overline{g}^q_{d,0}(\omega) + h^q_{d,0}(\omega, N_1)\right)\right] + (N_1 \leftrightarrow N_2)\,.$$

Next to Leading log:

$$\mathcal{C}^{\overline{\mathrm{NLL}}}_{qq,\vec{N}} = \left(g^q_{d,0,0} + a_s\; g^q_{d,0,1}\right) \exp\left[g^q_{d,1}(\omega)\ln N_1 + \frac{1}{2} g^q_{d,2}(\omega) + \frac{1}{N_1}\left(\overline{g}^q_{d,0}(\omega) + a_s\; \overline{g}^q_{d,1}(\omega) + h^q_{d,0}(\omega, N_1) + a_s\; h^q_{d,1}(\omega, \omega_1, N_1)\right)\right] + (N_1 \leftrightarrow N_2, \omega_1 \leftrightarrow \omega_2)\,.$$

Next to next to Leading log:

$$\mathcal{C}^{\overline{\mathrm{NNLL}}}_{qq,\vec{N}} = \left(g^q_{d,0,0} + a_s\; g^q_{d,0,1} + a_s^2\; g^q_{d,0,2}\right) \exp\left[g^q_{d,1}(\omega)\ln N_1 + \frac{1}{2}\left(g^q_{d,2}(\omega) + a_s\; g^q_{d,3}(\omega)\right) + \frac{1}{N_1}\left(\overline{g}^q_{d,0}(\omega) + a_s\; \overline{g}^q_{d,1}(\omega) + a_s^2\; \overline{g}^q_{d,2}(\omega) + h^q_{d,0}(\omega, N_1) + a_s\; h^q_{d,1}(\omega, \omega_1, N_1) + a_s^2\; h^q_{d,2}(\omega, \omega_1, N_1)\right)\right] + (N_1 \leftrightarrow N_2, \omega_1 \leftrightarrow \omega_2)\,.$$

Conclusions:

- QCD improved Parton Model is useful for SIDIS
- NNLO QCD effects are available
- Resuming soft gluons is possible