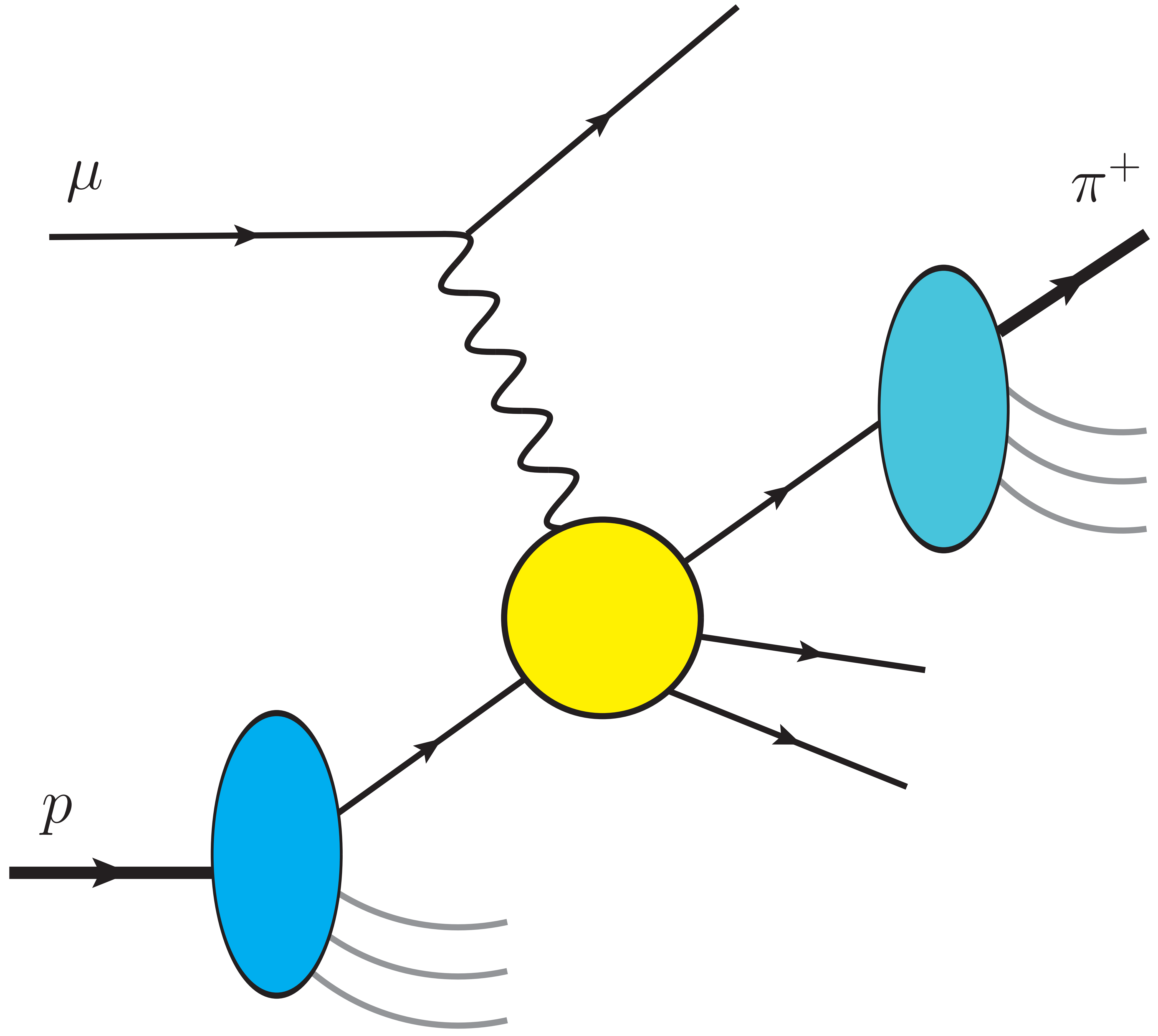
Semi-inclusive DIS at NNLO in QCD



Giovanni Stagnitto

UNIVERSITÀ DEGLI STUDI DI MILANO BICOCCA

INFN

with Leonardo Bonino and Thomas Gehrmann

[2401.16281]

+ Markus Löchner and Kay Schönwald

[2404.08597]

MITP (Mainz), 22.10.2024


Hello, it’s me again!
Bonjour

Disclaimer

I will talk about:

Semi-Inclusive Deep-Inelastic Scattering at Next-to-Next-to-Leading Order in QCD

Leonardo Bonino (Zurich U.), Thomas Gehrmann (Zurich U.), Giovanni Stagnitto (Milan Bicocca U.) (Jan 29, 2024)

Published in: *Phys.Rev.Lett.* 132 (2024) 25, 251901 • e-Print: 2401.16281 [hep-ph]

Polarized semi-inclusive deep-inelastic scattering at NNLO in QCD

Leonardo Bonino (Zurich U.), Thomas Gehrmann (Zurich U.), Markus Löchner (Zurich U.), Kay Schönwald (Zurich U.), Giovanni Stagnitto (Milan Bicocca U.) (Apr 12, 2024)

e-Print: 2404.08597 [hep-ph]

Next week there will be a talk by Vajravelu Ravindran about:

Next-to-Next-to-Leading Order QCD Corrections to Semi-Inclusive Deep-Inelastic Scattering #

Saurav Goyal (IMSc, Chennai and HBNI, Mumbai), Sven-Olaf Moch (Hamburg U., Inst. Theor. Phys. II), Vaibhav Pathak (IMSc, Chennai and HBNI, Mumbai), Narayan Rana (NISER, Jatni), V. Ravindran (IMSc, Chennai and HBNI, Mumbai) (Dec 29, 2023)

Published in: *Phys.Rev.Lett.* 132 (2024) 25, 251902 • e-Print: 2312.17711 [hep-ph]

NNLO QCD corrections to polarized semi-inclusive DIS

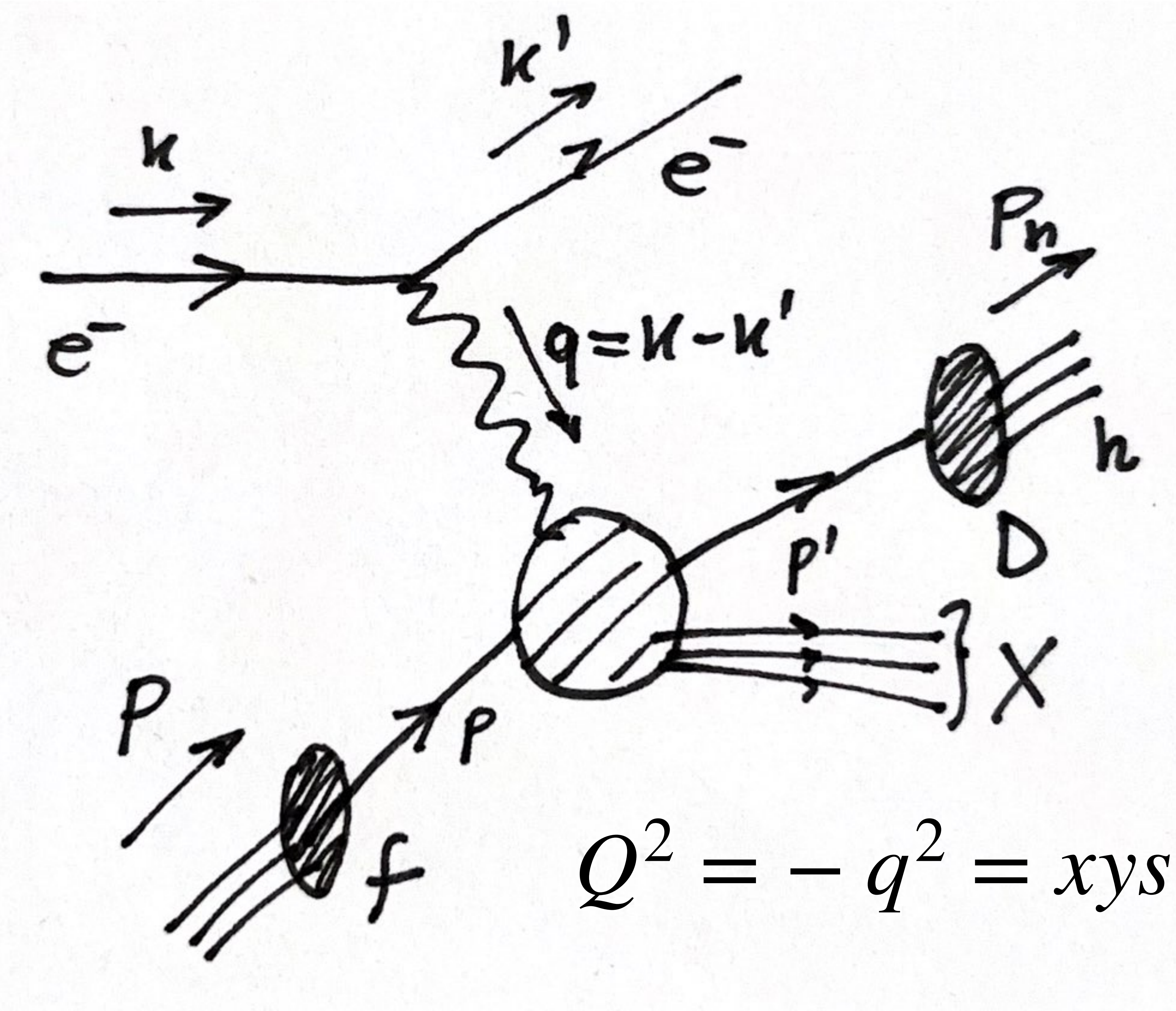
Saurav Goyal, Roman N. Lee, Sven-Olaf Moch, Vaibhav Pathak, Narayan Rana et al. (Apr 15, 2024)

e-Print: 2404.09959 [hep-ph]

Both calculations (unpolarized and polarized) are in agreement

What is semi-inclusive DIS (SIDIS)?

$$\ell(k) + p(P) \to \ell(k') + h(P_h) + X$$



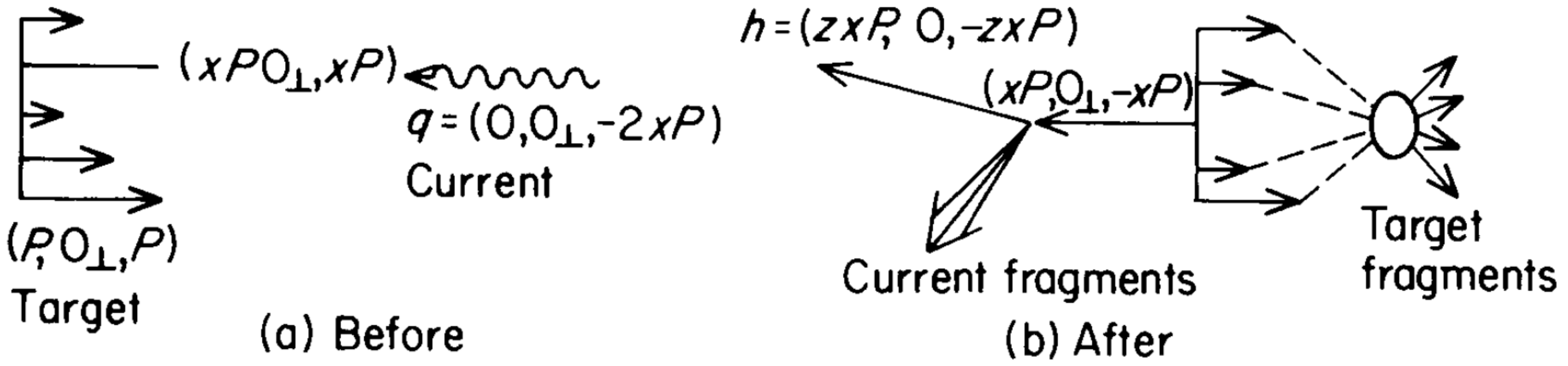
$$Q^2 = -q^2 = xys$$

*we assume only photon exchange ($Q \ll M_Z$)

$$x = \frac{Q^2}{2P \cdot q}$$ *Bjorken variable* (momentum fraction of the parton)

$$y = \frac{P \cdot q}{P \cdot k}$$ *Inelasticity* (energy transfer) (related to polarisation of virtual photon)

$$z = \frac{P \cdot P_h}{P \cdot q}$$ In Breit frame, is the fraction of the parton's longitudinal momentum carried of by the observed hadron



Outlook

Unpolarised

$$\ell(k)\,p(P) \to \ell(k')\,h(P_h)\,X$$

$$\frac{\mathrm{d}^3\sigma^h}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} = \frac{4\pi\alpha^2}{Q^2}\left[\frac{1+(1-y)^2}{2y}\mathscr{F}_T^h(x,z,Q^2) + \frac{1-y}{y}\mathscr{F}_L^h(x,z,Q^2)\right]$$

Longitudinally polarised

$$\overrightarrow{\ell}(k)\,\vec{p}(P) \to \ell(k')\,h(P_h)\,X$$

$$\frac{1}{2}\left(\frac{\mathrm{d}^3\sigma^h(\uparrow\uparrow)}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} - \frac{\mathrm{d}^3\sigma^h(\uparrow\downarrow)}{\mathrm{d}x\mathrm{d}y\mathrm{d}z}\right) = \frac{4\pi\alpha^2}{Q^2}\frac{1-(1-y)^2}{2y}\mathscr{G}_1^h(x,z,Q^2)$$

Outlook

Unpolarised

$$\ell(k)\, p(P) \to \ell(k')\, h(P_h)\, X$$

$$\frac{\mathrm{d}^3\sigma^h}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} = \frac{4\pi\alpha^2}{Q^2}\left[\frac{1+(1-y)^2}{2y}\mathscr{F}_T^h(x,z,Q^2) + \frac{1-y}{y}\mathscr{F}_L^h(x,z,Q^2)\right]$$

Longitudinally polarised

$$\overrightarrow{\ell}(k)\, \vec{p}(P) \to \ell(k')\, h(P_h)\, X$$

$$\frac{1}{2}\left(\frac{\mathrm{d}^3\sigma^h(\uparrow\uparrow)}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} - \frac{\mathrm{d}^3\sigma^h(\uparrow\downarrow)}{\mathrm{d}x\mathrm{d}y\mathrm{d}z}\right) = \frac{4\pi\alpha^2}{Q^2}\frac{1-(1-y)^2}{2y}\mathscr{G}_1^h(x,z,Q^2)$$

Motivation

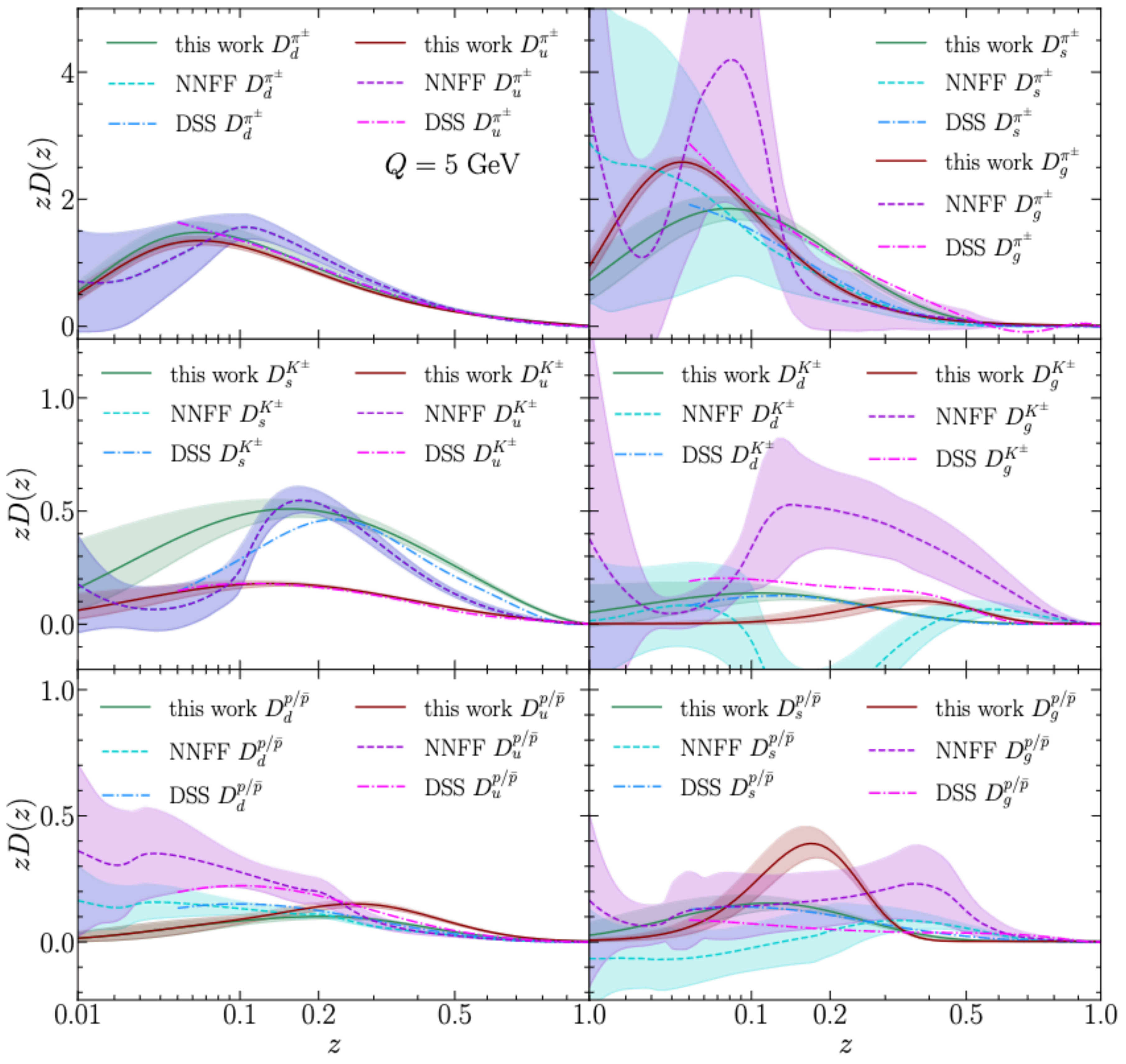
Global fits of fragmentation functions (FFs)

	Experiments	N_{pt}	χ^2	χ^2/N_{pt}
	ATLAS jets †	446	350.8	0.79
	ATLAS Z/γ+jet †	15	31.8	2.12
	CMS Z/γ+jet †	15	17.3	1.15
	LHCb Z+jet	20	30.6	1.53
	ALICE inc. hadron	147	150.6	1.02
	STAR inc. hadron	60	42.2	0.70
pp	pp sum	703	623.3	0.89
	TASSO	8	7.0	0.88
	TPC	12	11.6	0.97
	OPAL	20	16.3	0.81
	OPAL (202 GeV) †	17	24.2	1.42
	ALEPH	42	31.4	0.75
	DELPHI	78	36.4	0.47
	DELPHI (189 GeV)	9	15.3	1.70
	SLD	198	211.6	1.07
e^+e^-	SIA sum	384	353.8	0.92
	H1 †	16	12.5	0.78
	H1 (asy.) †	14	12.2	0.87
	ZEUS †	32	65.5	2.05
	COMPASS (06I)	124	107.3	0.87
	COMPASS (16p)	97	56.8	0.59
ep	SIDIS sum	283	254.4	0.90
	Global total	1370	1231.5	0.90

[2401.02781]

Fits routinely done at NLO by different groups, using data from e^+e^-, ep and pp colliders e.g. very recent global fit by [Gao, Liu, Shen, Xing, Zhao '24].

It exploits a new code FMNLO [Liu, Shen, Zhou, Gao '23], a wrapper around MG5 aMC@NLO, to compute arbitrary processes at the LHC with fragmentation at NLO.

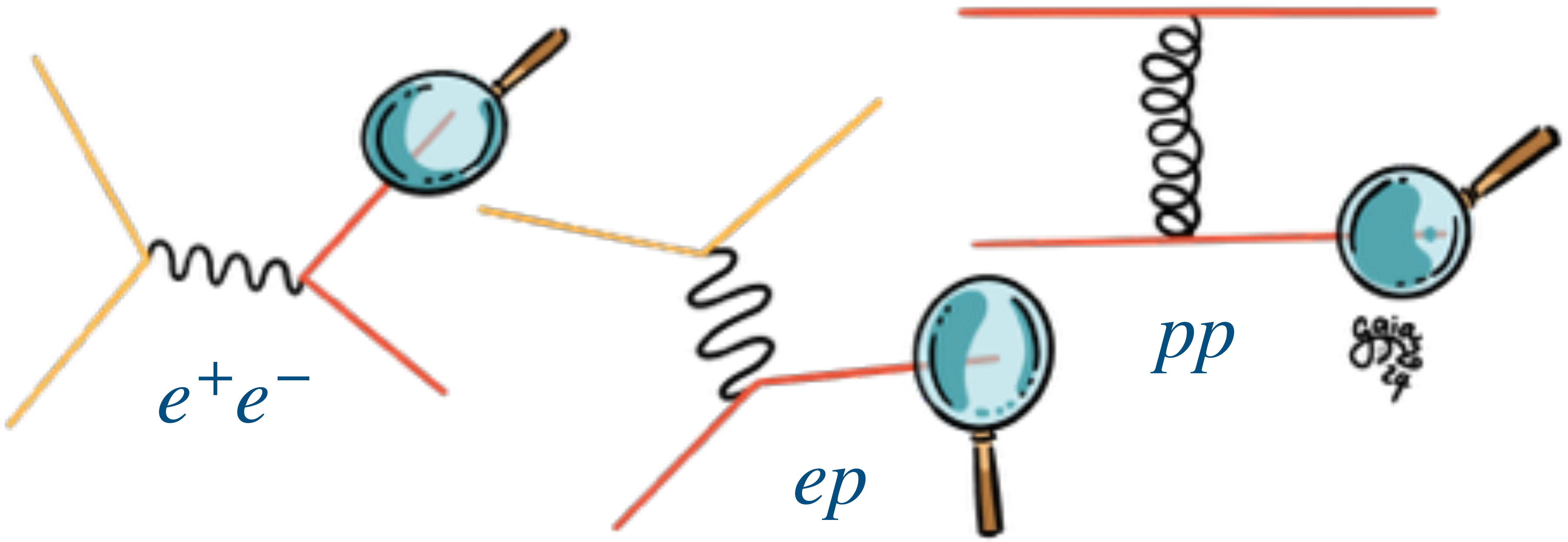


Motivation

Global fits of fragmentation functions (FFs)

The short-distance cross sections for one-particle inclusive processes are known up to:

- NNLO in e^+e^- (SIA) [Rijken, Van Neerven '96,'97] [Mitov, Moch, Vogt '06]
- NLO in ep (SIDIS) [Altarelli, Ellis, Martinelli, Pi '79] [Baier, Fey '79] (NNLO is this work)
- NLO in pp [Aversa, Chiappetta, Greco, Guillet '89]



Therefore, fits at NNLO limited to e^+e^- data
[Bertone, Carrazza, Hartland, Nocera, Rojo '17] [Anderle, Ringer, Stratmann '15] [xFitter '21]

Motivation

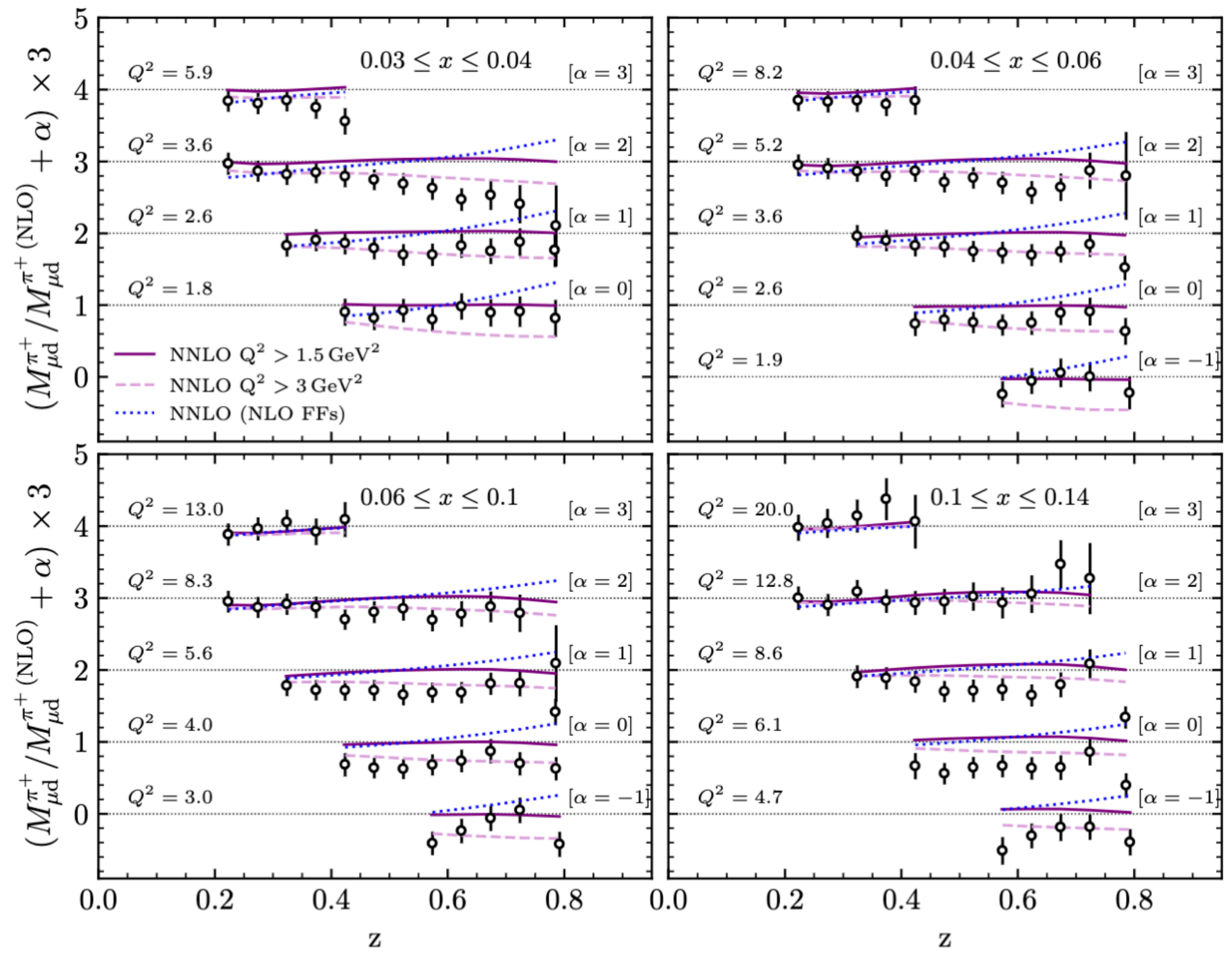
Global fits of fragmentation functions (FFs)

Recent fits at aNNLO with e^+e^- and ep data [Borsa, Sassot, de Florian, Vogelsang '22] [Abdul Khalek, Bertone, Khoudli, Nocera '22], exploiting approximate NNLO results for SIDIS obtained from threshold resummation [Abele, De Florian, Vogelsang '21,'22]

	Experiment	$Q^2 \geq 1.5\,\mathrm{GeV}^2$			$Q^2 \geq 2.0\,\mathrm{GeV}^2$			$Q^2 \geq 2.3\,\mathrm{GeV}^2$			$Q^2 \geq 3.0\,\mathrm{GeV}^2$		
		#data	NLO	NNLO	#data	NLO	NNLO	#data	NLO	NNLO	#data	NLO	NNLO
e^+e^-	SIA	288	1.05	0.96	288	0.91	0.87	288	0.90	0.91	288	0.93	0.86
ep	COMPASS	510	0.98	1.14	456	0.91	1.04	446	0.91	0.92	376	0.94	0.93
ep	HERMES	224	2.24	2.27	160	2.40	2.08	128	2.71	2.35	96	2.75	2.26
	TOTAL	1022	1.27	1.33	904	1.17	1.17	862	1.17	1.13	760	1.16	1.07

It is very encouraging that our NNLO analysis based on the approximate NNLO corrections for SIDIS shows an overall improvement in χ^2 relative to NLO once we go beyond $Q^2 = 2\,\mathrm{GeV}^2$. It is an interesting question, however, why the situation is opposite when the lower cut $Q^2 \geq 1.5\,\mathrm{GeV}^2$ is used. We first note that the lack

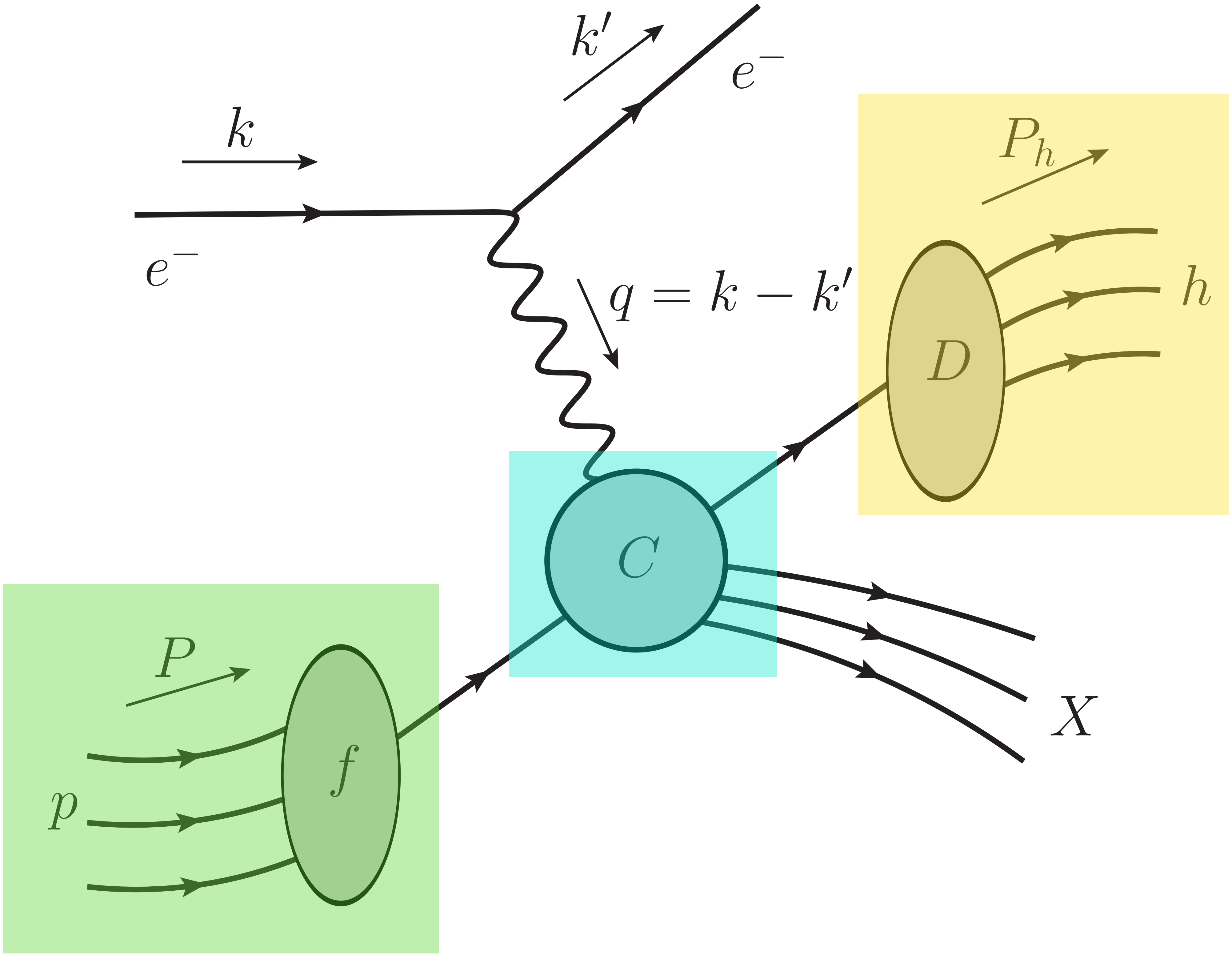
[2202.05060]



Our work will enable a consistent NNLO fit with SIDIS data.

Unpolarised SIDIS structure functions

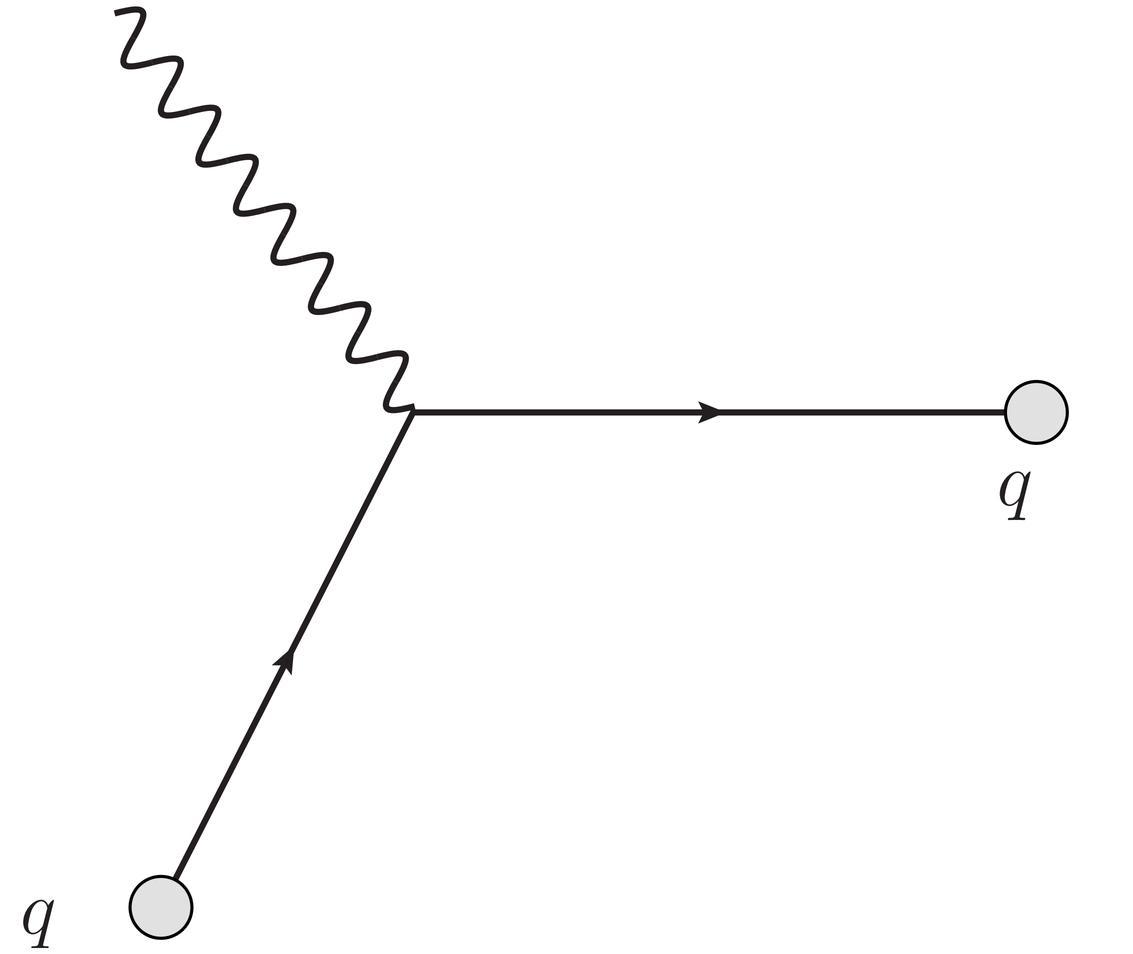
$$\mathcal{F}_i^h(x,z,Q^2) = \sum_{p,p'} \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} \int_z^1 \frac{\mathrm{d}\hat{z}}{\hat{z}} f_p\left(\frac{x}{\hat{x}},\mu_F^2\right) D_{p'}^h\left(\frac{z}{\hat{z}},\mu_A^2\right) \mathcal{C}^i_{p'p}\left(\hat{x},\hat{z},Q^2,\mu_R^2,\mu_F^2,\mu_A^2\right), \quad i = T, L$$



$$\mathcal{C}^i_{p'p} = C^{i,(0)}_{p'p} + \frac{\alpha_s(\mu_R^2)}{2\pi} C^{i,(1)}_{p'p} + \left(\frac{\alpha_s(\mu_R^2)}{2\pi}\right)^2 C^{i,(2)}_{p'p} + \mathcal{O}(\alpha_s^3)$$

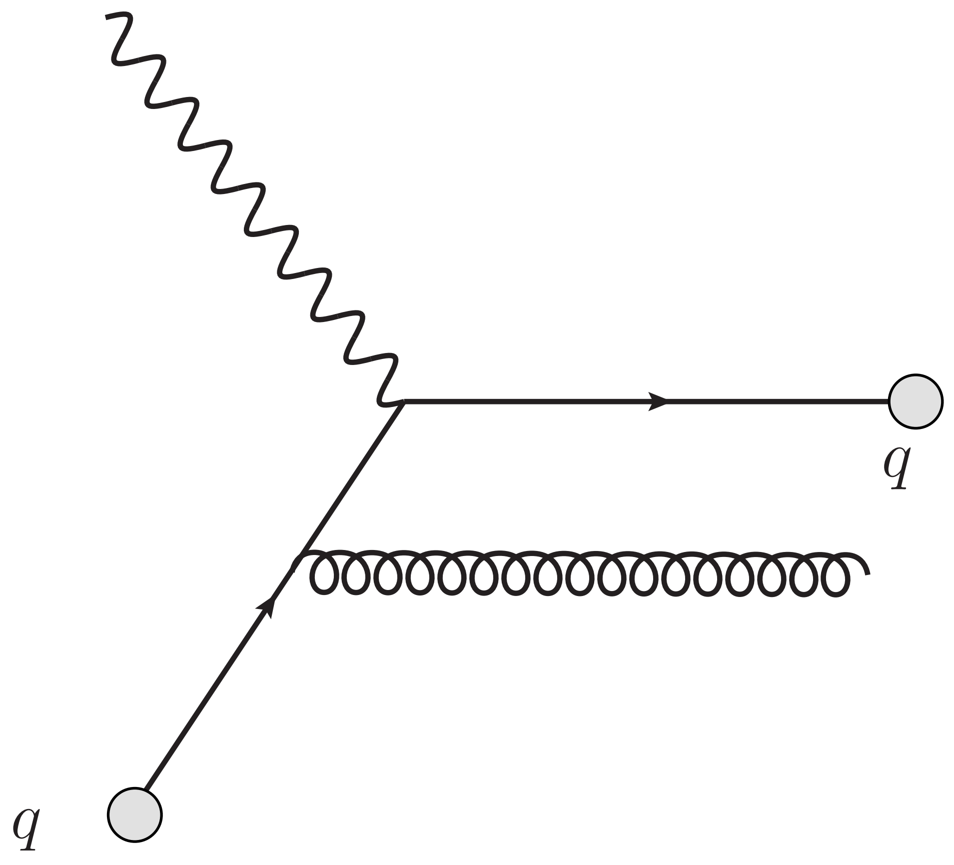
$$C^{T,(0)}_{qq} = e_q^2 \delta(1-\hat{x})\delta(1-\hat{z})$$

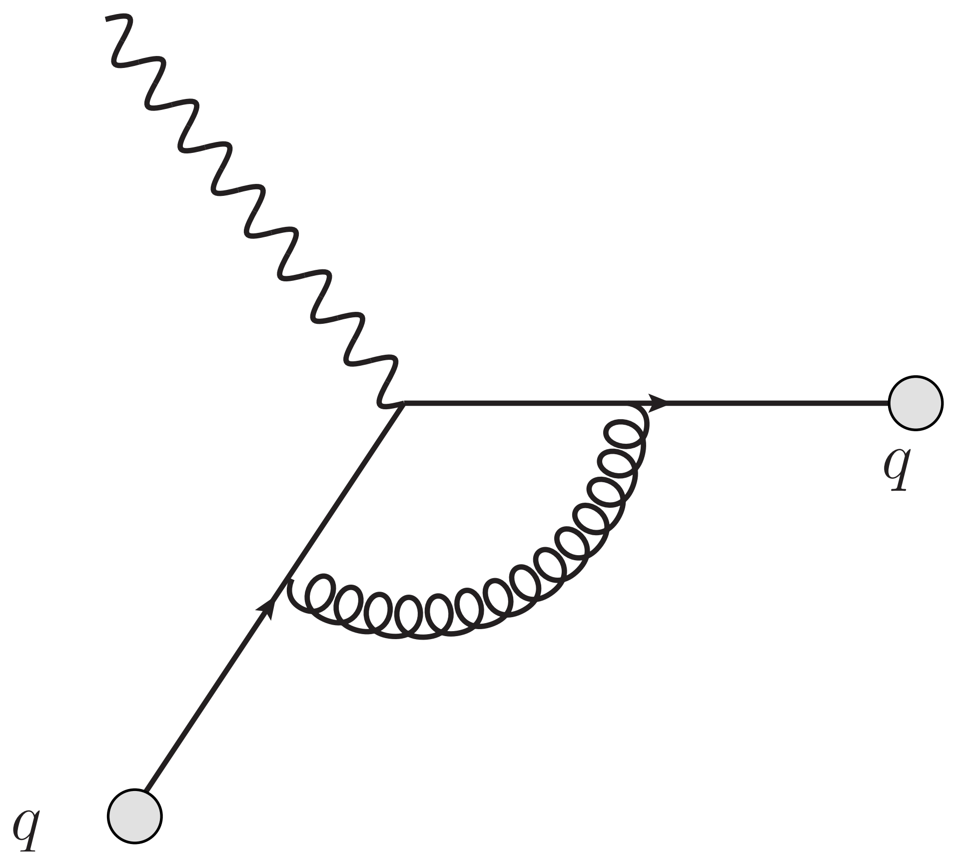
$$C^{L,(0)}_{qq} = 0$$



SIDIS @ NLO

[Altarelli, Ellis, Martinelli, Pi ’79][Baier, Fey ’79]





$$
\begin{aligned}
C_{qq}^{T,(1)}(\hat{x},\hat{z}) \;&=\; e_q^2 C_F \Bigg[-8\delta(1-\hat{x})\delta(1-\hat{z}) \\
&+\delta(1-\hat{x})\left[\tilde{P}_{qq}(\hat{z})\ln\frac{Q^2}{\mu_F^2} + L_1(\hat{z}) + L_2(\hat{z}) + (1-\hat{z})\right] \\
&+\delta(1-\hat{z})\left[\tilde{P}_{qq}(\hat{x})\ln\frac{Q^2}{\mu_F^2} + L_1(\hat{x}) - L_2(\hat{x}) + (1-\hat{x})\right] \\
&+\frac{2}{(1-\hat{x})_+(1-\hat{z})_+} - \frac{1+\hat{z}}{(1-\hat{x})_+} - \frac{1+\hat{x}}{(1-\hat{z})_+} \\
&+2(1+\hat{x}\hat{z})\Bigg],
\end{aligned}
\qquad (49)
$$

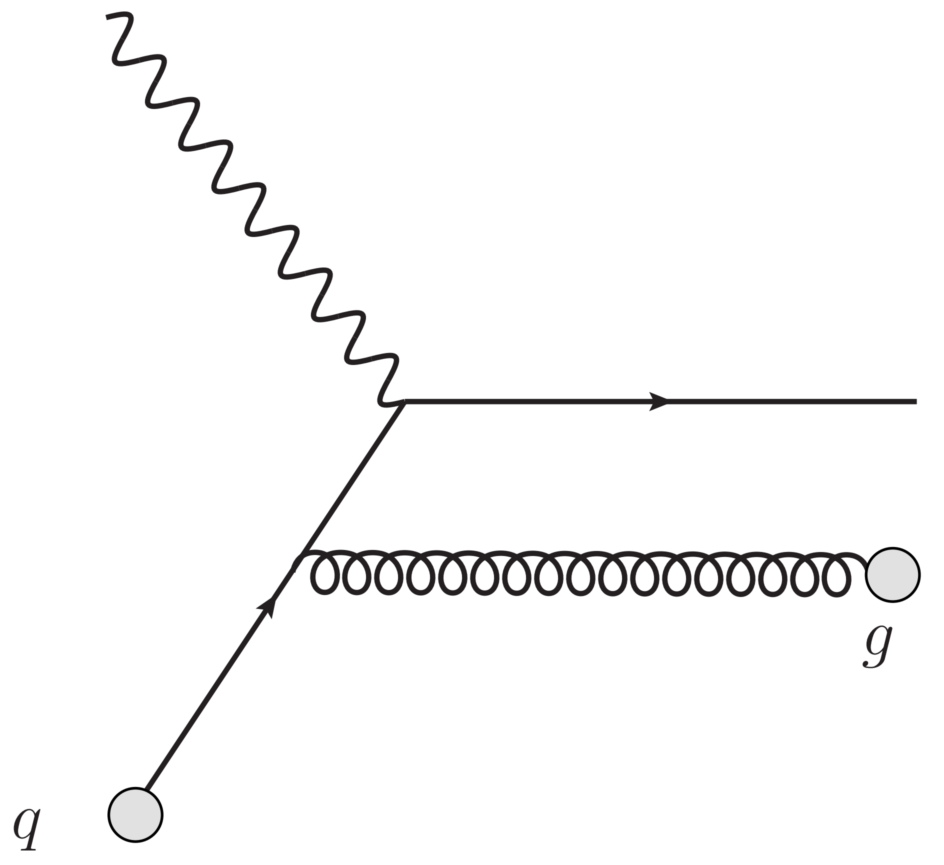
$$
C_{qq}^{L,(1)}(\hat{x},\hat{z}) \;=\; 4e_q^2 C_F \hat{x}\hat{z},
$$

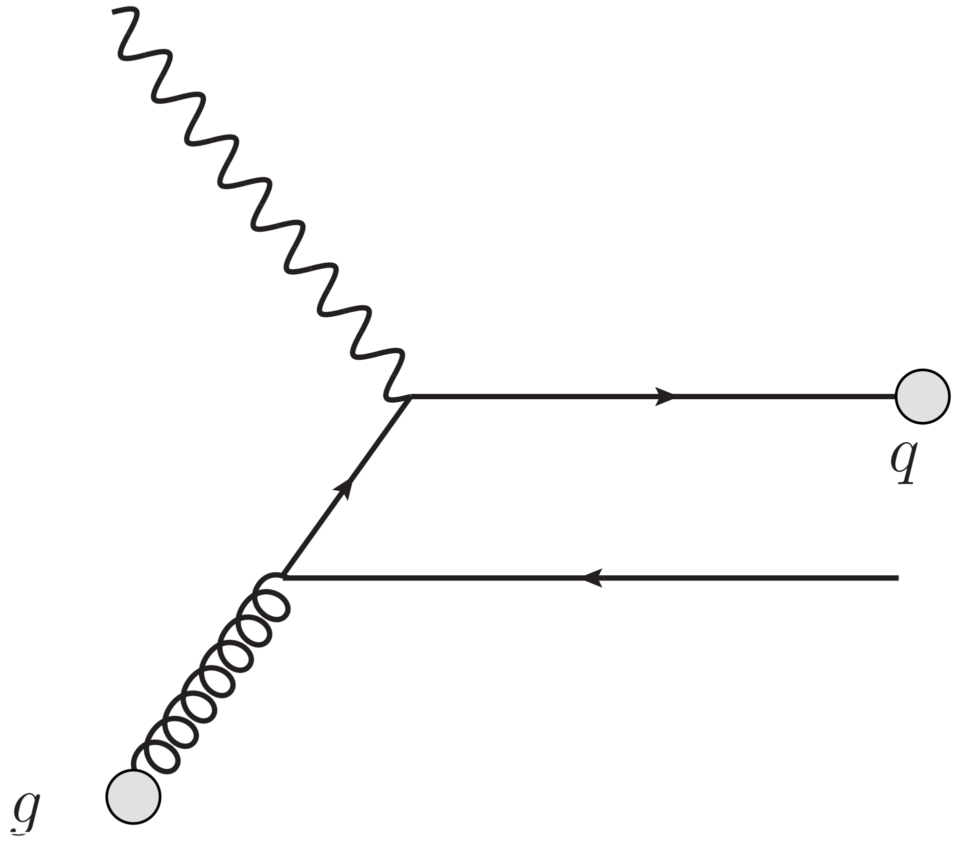
$$
\begin{aligned}
\tilde{P}_{qq}(\xi) &= \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi), \\
\tilde{P}_{gq}(\xi) &= \frac{1+(1-\xi)^2}{\xi}, \\
\tilde{P}_{qg}(\xi) &= \xi^2 + (1-\xi)^2, \\
L_1(\xi) &= (1+\xi^2)\left(\frac{\ln(1-\xi)}{1-\xi}\right)_+, \\
L_2(\xi) &= \frac{1+\xi^2}{1-\xi}\ln\xi,
\end{aligned}
\qquad (52)
$$

Screenshots from
[Anderle, Ringer, Vogelsang ’12]

SIDIS @ NLO

[Altarelli, Ellis, Martinelli, Pi ’79][Baier, Fey ’79]





$$C_{gq}^{T,(1)}(\hat{x},\hat{z}) = e_q^2 C_F \Big[\tilde{P}_{gq}(\hat{z}) \left(\delta(1-\hat{x}) \ln\left(\frac{Q^2}{\mu_F^2} \hat{z}(1-\hat{z}) \right) + \frac{1}{(1-\hat{x})_+} \right) + \hat{z}\delta(1-\hat{x}) + 2(1+\hat{x}-\hat{x}\hat{z}) - \frac{1+\hat{x}}{\hat{z}} \Big], \tag{50}$$

$$C_{qg}^{T,(1)}(\hat{x},\hat{z}) = e_q^2 T_R \Big[\delta(1-\hat{z}) \left[\tilde{P}_{qg}(\hat{x}) \ln\left(\frac{Q^2}{\mu_F^2} \frac{1-\hat{x}}{\hat{x}} \right) + 2\hat{x}(1-\hat{x}) \right] + \tilde{P}_{qg}(\hat{x}) \left\{ \frac{1}{(1-\hat{z})_+} + \frac{1}{\hat{z}} - 2 \right\} \Big], \tag{51}$$

$$C_{gq}^{L,(1)}(\hat{x},\hat{z}) = 4 e_q^2 C_F \hat{x}(1-\hat{z}),$$

$$C_{qg}^{L,(1)}(\hat{x},\hat{z}) = 8 e_q^2 T_R \hat{x}(1-\hat{x}).$$

$$\begin{aligned} \tilde{P}_{qq}(\xi) &= \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi), \\ \tilde{P}_{gq}(\xi) &= \frac{1+(1-\xi)^2}{\xi}, \\ \tilde{P}_{qg}(\xi) &= \xi^2 + (1-\xi)^2, \\ L_1(\xi) &= (1+\xi^2)\left(\frac{\ln(1-\xi)}{1-\xi}\right)_+, \\ L_2(\xi) &= \frac{1+\xi^2}{1-\xi}\ln\xi, \end{aligned} \tag{52}$$

Screenshots from
[Anderle, Ringer, Vogelsang ’12]

ancillary.inc

```
*************************************************************************************
** SIDIS coefficient functions up to NNLO from:
**
** Semi-inclusive deep-inelastic scattering at NNLO in QCD
** L. Bonino, T. Gehrmann and G. Stagnitto
**
** FORM readable format
**
** Notation, according to eq.(6) of the paper:
**          C[order][component][a2b][label] with
**          - order: 1 = NLO, 2 = NNLO
**          - component: L = Longitudinal, T = transverse
**          - a2b: means a -> b, for a and b partons
**          - label (it can be none, NS, PS, 1, 2, 3)
**
** Symbols:
**         NC = 3: number of colours
**         NF = 5: number of active flavours
**
** Scales:
**        LMUR = ln(muR^2/Q2)
**        LMUF = ln(muF^2/Q2)
**        LMUA = ln(muA^2/Q2)
**        with Q2 = -q2, invariant mass of the photon
**             muR: renormalisation scale
**             muF: initial-state factorisation scale
**             muA: final-state factorisation scale
**
** Functions:
**           Li2(a) = PolyLog(2,a)
**           Li3(a) = PolyLog(3,a)
**           sqrtxz1 = sqrt(1 - 2*z + z*z + 4*x*z)
**           poly2   = 1 + 2*x + x*x - 4*x*z
**           sqrtxz2 = sqrt(poly2)
**           sqrtxz3 = sqrt(x/z)
**           InvTanInt(x) = int_0^x dt arctan(t)/t : Arctangent integral
**           T(region): Heaviside Theta function
**
** Distributions:
**          Dd([1-x]) is the Dirac delta function of argument [1-x]
**          Dn(a,[1-x]) = (ln^a(1-x)/(1-x))_+ (plus-prescription) for a = 1,2,3
**          same for z
**
** Kinematic regions in (x,z)-plane: as defined in 2201.06982
**          ui = Ui for i = 1,2,3,4
**          ri = Ri, ti = Ti for i = 1,2
**          Ri, Ti and Ui defined in eq. (5.9), (5.12) and (5.16)
**
** Constants: pi, zeta3 = Zeta(3) with Zeta Riemann Zeta function
**
*************************************************************************************
```

1.1 MB of size

SIDIS @ NNLO

$$C^{i,(2)}_{qq} = C^{i,(2)}_{\bar{q}\bar{q}} = e_q^2 C^{i,\mathrm{NS}}_{qq} + \left(\sum_j e_{q_j}^2\right) C^{i,\mathrm{PS}}_{qq} ,$$

$$C^{i,(2)}_{\bar{q}q} = C^{i,(2)}_{q\bar{q}} = e_q^2 C^{i}_{\bar{q}q} ,$$

$$C^{i,(2)}_{q'q} = C^{i,(2)}_{\bar{q}'\bar{q}} = e_q^2 C^{i,1}_{q'q} + e_{q'}^2 C^{i,2}_{q'q} + e_q e_{q'} C^{i,3}_{q'q} ,$$

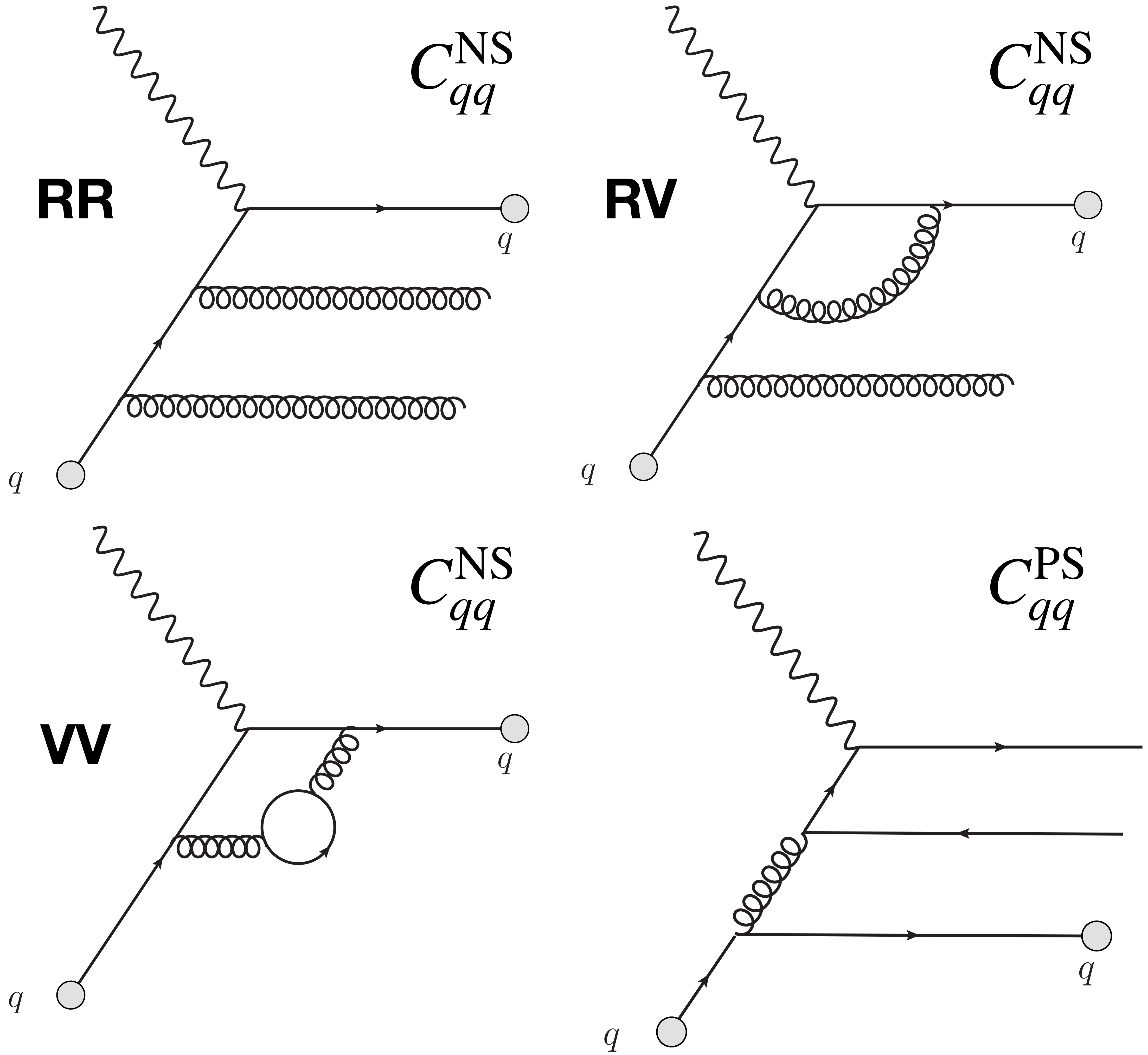
$$C^{i,(2)}_{\bar{q}'q} = C^{i,(2)}_{q'\bar{q}} = e_q^2 C^{i,1}_{q'q} + e_{q'}^2 C^{i,2}_{q'q} - e_q e_{q'} C^{i,3}_{q'q} ,$$

$$C^{i,(2)}_{gq} = C^{i,(2)}_{g\bar{q}} = e_q^2 C^{i}_{gq} ,$$

$$C^{i,(2)}_{qg} = C^{i,(2)}_{\bar{q}g} = e_q^2 C^{i}_{qg} ,$$

$$C^{i,(2)}_{gg} = \left(\sum_j e_{q_j}^2\right) C^{i}_{gg} ,$$

SIDIS @ NNLO



$$C_{qq}^{i,(2)} = C_{\bar{q}\bar{q}}^{i,(2)} = e_q^2 C_{qq}^{i,\mathrm{NS}} + \left(\sum_j e_{q_j}^2\right) C_{qq}^{i,\mathrm{PS}} ,$$

$$C_{\bar{q}q}^{i,(2)} = C_{q\bar{q}}^{i,(2)} = e_q^2 C_{\bar{q}q}^{i} ,$$

$$C_{q'q}^{i,(2)} = C_{\bar{q}'\bar{q}}^{i,(2)} = e_q^2 C_{q'q}^{i,1} + e_{q'}^2 C_{q'q}^{i,2} + e_q e_{q'} C_{q'q}^{i,3} ,$$

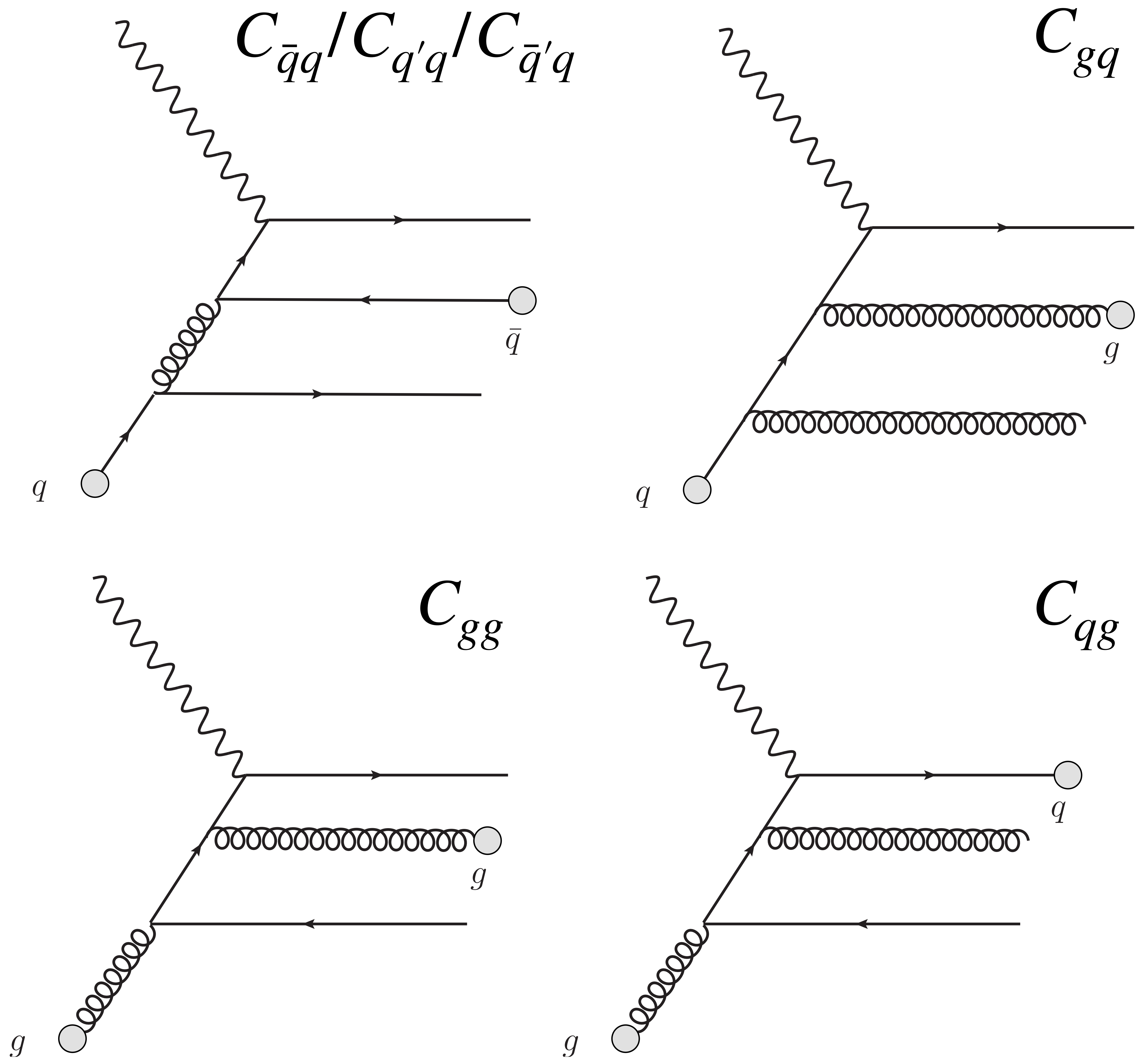
$$C_{\bar{q}'q}^{i,(2)} = C_{q'\bar{q}}^{i,(2)} = e_q^2 C_{q'q}^{i,1} + e_{q'}^2 C_{q'q}^{i,2} - e_q e_{q'} C_{q'q}^{i,3} ,$$

$$C_{gq}^{i,(2)} = C_{g\bar{q}}^{i,(2)} = e_q^2 C_{gq}^{i} ,$$

$$C_{qg}^{i,(2)} = C_{\bar{q}g}^{i,(2)} = e_q^2 C_{qg}^{i} ,$$

$$C_{gg}^{i,(2)} = \left(\sum_j e_{q_j}^2\right) C_{gg}^{i} ,$$

SIDIS @ NNLO



$$C_{qq}^{i,(2)} = C_{\bar{q}\bar{q}}^{i,(2)} = e_q^2 C_{qq}^{i,\mathrm{NS}} + \left(\sum_j e_{q_j}^2\right) C_{qq}^{i,\mathrm{PS}}\,,$$

$$C_{\bar{q}q}^{i,(2)} = C_{q\bar{q}}^{i,(2)} = e_q^2 C_{\bar{q}q}^{i}\,,$$

$$C_{q'q}^{i,(2)} = C_{\bar{q}'\bar{q}}^{i,(2)} = e_q^2 C_{q'q}^{i,1} + e_{q'}^2 C_{q'q}^{i,2} + e_q e_{q'} C_{q'q}^{i,3}\,,$$

$$C_{\bar{q}'q}^{i,(2)} = C_{q'\bar{q}}^{i,(2)} = e_q^2 C_{q'q}^{i,1} + e_{q'}^2 C_{q'q}^{i,2} - e_q e_{q'} C_{q'q}^{i,3}\,,$$

$$C_{gq}^{i,(2)} = C_{g\bar{q}}^{i,(2)} = e_q^2 C_{gq}^{i}\,,$$

$$C_{qg}^{i,(2)} = C_{\bar{q}g}^{i,(2)} = e_q^2 C_{qg}^{i}\,,$$

$$C_{gg}^{i,(2)} = \left(\sum_j e_{q_j}^2\right) C_{gg}^{i}\,,$$

Details of the calculation

VV: well-known two-loop quark form factor in space-like kinematics

RV: one-loop squared matrix elements in terms of one-loop bubble and box integrals, which are known in exact form in ε. For fixed $\hat{x}$ and $\hat{z}$, the phase space integral is fully constrained:

$$C^{\mathrm{RV}}_{j\leftarrow i} \propto \int \mathrm{d}\Phi_2(k_j, k_k; k_i, q)\, \delta\left(z - x\frac{(k_i + k_j)^2}{Q^2}\right) \left|\mathcal{M}^{\mathrm{RV}}\right|^2 \propto \mathcal{J}(x,z) \left|\mathcal{M}^{\mathrm{RV}}\right|^2(x,z)$$

Only expansions in the end-point distributions $\hat{x} = 1$ and $\hat{z} = 1$ are required.

RR: integrations over three-particle phase space with multi-loop techniques:

$$C^{\mathrm{RR}}_{j\leftarrow i} \propto \int \mathrm{d}\Phi_3(k_j, k_k, k_l; k_i, q)\, \delta\left(z - x\frac{(k_i + k_j)^2}{Q^2}\right) \left|\mathcal{M}^{\mathrm{RR}}\right|^2$$

Reduction to master integrals using IBP identities, 13 integral families, 21 master integrals. Solved using differential equations, boundary terms obtained by integrating over $\hat{z}$ and comparing to master integrals relevant to inclusive version.

Details of the calculation

VV: well-known two-loop quark form factor in space-like kinematics

RV: one-loop squared matrix elements in terms of one-loop bubble and box integrals, which are known in exact form in ε. For fixed $\hat{x}$ and $\hat{z}$, the phase space integral is fully constrained:

$$C^{\mathrm{RV}}_{j\leftarrow i} \propto \int \mathrm{d}\Phi_2(k_j, k_k; k_i, q)\, \delta\left(z - x\frac{(k_i+k_j)^2}{Q^2}\right) \left|\mathcal{M}^{\mathrm{RV}}\right|^2 \propto \mathcal{J}(x,z)\ \left|\mathcal{M}^{\mathrm{RV}}\right|^2(x,z)$$

Only expansions in the end-point distributions $\hat{x}=1$ and $\hat{z}=1$ are required.

RR: integrations over three-particle phase space with multi-loop techniques:

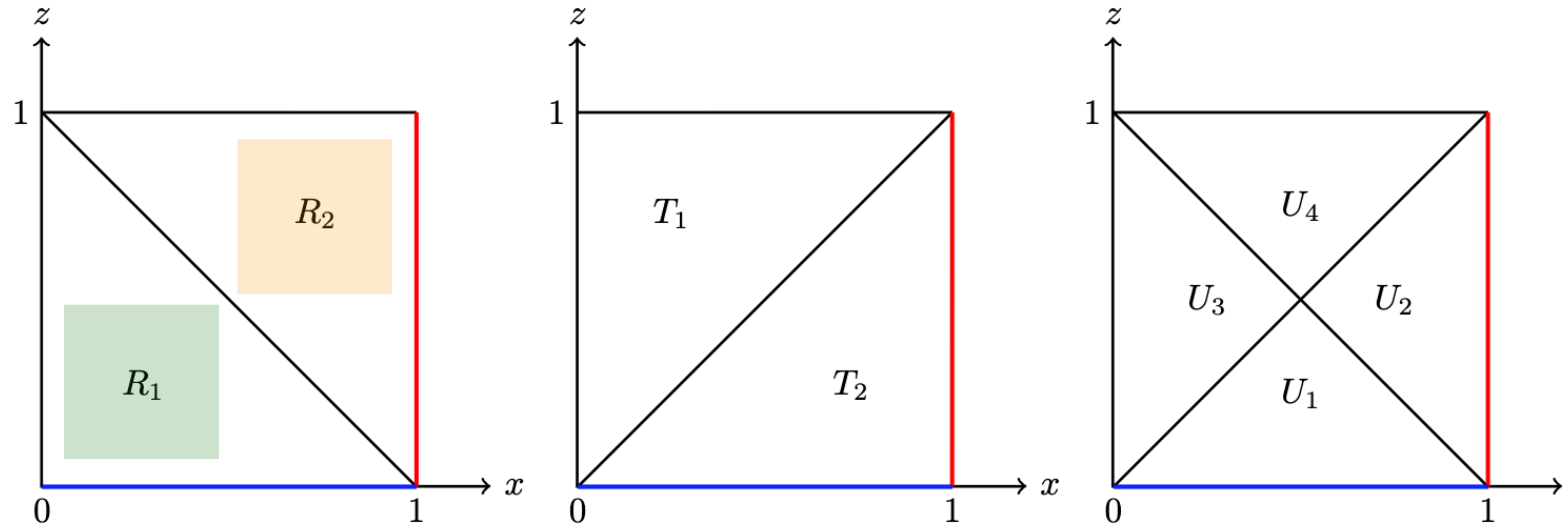
$$C^{\mathrm{RR}}_{j\leftarrow i} \propto \int \mathrm{d}\Phi_3(k_j, k_k, k_l; k_i, q)\, \delta\left(z - x\frac{(k_i+k_j)^2}{Q^2}\right) \left|\mathcal{M}^{\mathrm{RR}}\right|^2$$

Reduction to master integrals using IBP identities, 13 integral families, 21 master integrals.
Solved using differential equations, boundary terms obtained by integrating over $\hat{z}$ and comparing to master integrals relevant to inclusive version.

Analytic continuation in the real-virtual

[Gehrmann, Schürmann '22]

To avoid ambiguities associated with the analytic continuation of boxes, we segment the (x, z) plane into four sectors, where manifestly real-valued expressions are obtained.



$$
\begin{aligned}
&\mathrm{Box}(s_{ij}, s_{ik}) \\
&= \frac{2(1-2\epsilon)}{\epsilon} A_{2,LO} \frac{1}{s_{ij}s_{ik}} \\
&\times \Bigg[\left(\frac{s_{ij}s_{ik}}{s_{ij}-s_{ijk}}\right)^{-\epsilon} {}_2F_1\left(-\epsilon, -\epsilon; 1-\epsilon; \frac{s_{ijk}-s_{ij}-s_{ik}}{s_{ijk}-s_{ij}}\right) \\
&+ \left(\frac{s_{ij}s_{ik}}{s_{ik}-s_{ijk}}\right)^{-\epsilon} {}_2F_1\left(-\epsilon, -\epsilon; 1-\epsilon; \frac{s_{ijk}-s_{ij}-s_{ik}}{s_{ijk}-s_{ik}}\right) \\
&- \left(\frac{-s_{ijk}s_{ij}s_{ik}}{(s_{ij}-s_{ijk})(s_{ik}-s_{ijk})}\right)^{-\epsilon} {}_2F_1\left(-\epsilon, -\epsilon; 1-\epsilon; \frac{s_{ijk}(s_{ijk}-s_{ij}-s_{ik})}{(s_{ijk}-s_{ij})(s_{ijk}-s_{ik})}\right) \Bigg]
\end{aligned}
$$

Example:

in $\mathrm{Box}(s_{12}, s_{23})$ we use

$$
\begin{aligned}
a_1(s_{12}, s_{23}) &= \frac{s_{123}-s_{12}-s_{23}}{s_{123}-s_{12}} = -\frac{z}{1-x-z}\,, \\
a_2(s_{12}, s_{23}) &= \frac{s_{123}-s_{12}-s_{23}}{s_{123}-s_{23}} = z\,, \qquad R_1 \\
a_3(s_{12}, s_{23}) &= \frac{s_{123}\,s_{13}}{(s_{13}+s_{23})(s_{12}+s_{13})} = -\frac{x\,z}{1-x-z}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a}_1(s_{12}, s_{23}) &= 1 - \frac{1}{a_1(s_{12}, s_{23})} = \frac{1-x}{z}\,, \qquad R_2 \\
\tilde{a}_3(s_{12}, s_{23}) &= 1 - \frac{1}{a_3(s_{12}, s_{23})} = \frac{(1-x)(1-z)}{xz}
\end{aligned}
$$

Details of the calculation

VV: well-known two-loop quark form factor in space-like kinematics

RV: one-loop squared matrix elements in terms of one-loop bubble and box integrals, which are known in exact form in ε. For fixed $\hat{x}$ and $\hat{z}$, the phase space integral is fully constrained:

$$C^{\mathrm{RV}}_{j\leftarrow i} \propto \int \mathrm{d}\Phi_2(k_j, k_k; k_i, q)\,\delta\left(z - x\frac{(k_i+k_j)^2}{Q^2}\right)\left|\mathscr{M}^{\mathrm{RV}}\right|^2 \propto \mathscr{J}(x,z)\ \left|\mathscr{M}^{\mathrm{RV}}\right|^2(x,z)$$

Only expansions in the end-point distributions $\hat{x}=1$ and $\hat{z}=1$ are required.

RR: integrations over three-particle phase space with multi-loop techniques:

$$C^{\mathrm{RR}}_{j\leftarrow i} \propto \int \mathrm{d}\Phi_3(k_j, k_k, k_l; k_i, q)\,\delta\left(z - x\frac{(k_i+k_j)^2}{Q^2}\right)\left|\mathscr{M}^{\mathrm{RR}}\right|^2$$

Reduction to master integrals using IBP identities, 13 integral families, 21 master integrals.
Solved using differential equations, boundary terms obtained by integrating over $\hat{z}$ and comparing to master integrals relevant to inclusive version.

Real-real master integrals

[Bonino, Gehrmann, Marcoli, Schürmann, GS ’24]

$$I[-3,7] \propto \int d^dk_j d^dk_l \delta(D_9)\delta(D_{10})\delta(D_{11})\delta(D_{12})\frac{D_3}{D_7}$$

In order to compute the integral over z, to match the boundary condition, we need to retain the end-point singularities in $z = 0$.
In this limit, the identified particle momentum become soft, and propagators $D_5 = (k_j + k_k)^2$ and $D_8 = (k_j + k_l)^2$ become singular.
Master integrals $I[3,5,8]$ and $I[4,5,8]$ require special care.

$$q + p \to k_k + k_l + k_j$$

$$\begin{aligned}
D_1 &= (q-k_j)^2\,,\\
D_2 &= (p+q-k_j)^2\,,\\
D_3 &= (p-k_l)^2\,,\\
D_4 &= (q-k_l)^2\,,\\
D_5 &= (p+q-k_l)^2\,,\\
D_6 &= (q-k_j-k_l)^2\,,\\
D_7 &= (p-k_j-k_l)^2\,,\\
D_8 &= (k_j+k_l)^2\,,\\
D_9 &= k_j^2\,,\\
D_{10} &= k_l^2\,,\\
D_{11} &= (q+p-k_j-k_l)^2\,,\\
D_{12} &= (p-k_j)^2 + Q^2\frac{z}{x}\,,
\end{aligned}$$

cut propagators

set of denominator factors

family	master	deepest pole	at $x=1$	at $z=1$
	$I[0]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
A	$I[5]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[2,3,5]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
B	$I[7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[-2,7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[-3,7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[2,3,7]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
C	$I[5,7]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[3,5,7]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
D	$I[1]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[1,4]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[1,3,4]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
E	$I[1,3,5]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
G	$I[1,3,8]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
H	$I[1,4,5]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
I	$I[2,4,5]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
J	$I[4,7]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[3,4,7]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
K	$I[3,5,8]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
L	$I[4,5,7]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
M	$I[4,5,8]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$

Table 1. Summary of the double real radiation master integrals.

Example of master integral: $I[3,5,8]$

$$\frac{\partial I[358](Q^2,x,z)}{\partial Q^2} = -\frac{2(1+\epsilon)}{Q^2} I[358](Q^2,x,z)\,,$$

$$\frac{\partial I[358](Q^2,x,z)}{\partial x} = \left(\frac{1+2\epsilon}{1-x} + \frac{2+2\epsilon}{x}\right) I[358](Q^2,x,z)\,,$$

$$\begin{aligned}\frac{\partial I[358](Q^2,x,z)}{\partial z} &= -\frac{1+2\epsilon}{z} I[358](Q^2,x,z) - \frac{2x^3(1-2\epsilon)^2(1+z)}{(Q^2)^3(1-x)^2\epsilon z^2(1-z)^2} I[0](Q^2,x,z) \\ &+ \frac{2x^2\epsilon}{(Q^2)^2(1-x)z^2} I[5](Q^2,x,z)\,. \end{aligned} \tag{3.14}$$

$$I[358](Q^2,x,z) = N_\Gamma \left(\frac{1-2\epsilon}{\epsilon}\right)^2 (Q^2)^{-2-2\epsilon}(1-x)^{-1-2\epsilon}x^{2+2\epsilon}z^{-1-2\epsilon}I'[358](z)\,, \tag{3.15}$$

$$\begin{aligned}\frac{\partial I'[358](z)}{\partial z} &= -\frac{4(1-2\epsilon)^2}{\epsilon} z^{\epsilon}(1-z)^{-1-2\epsilon} - \frac{2(1-2\epsilon)^2}{\epsilon} z^{-1+\epsilon}(1-z)^{-2\epsilon} \\ &+ \frac{2(1-2\epsilon)^2}{\epsilon} z^{-1+\epsilon}\,{}_2F_1(\epsilon, 2\epsilon; 1+\epsilon; z) \\ &- \frac{2(1-2\epsilon)^2}{\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} z^{-1}\,. \end{aligned} \tag{3.16}$$

Example of master integral: $I[3,5,8]$

Solution, with unknown boundary constant C':

$$
\begin{aligned}
I'[358](z) = &-\frac{4(1-2\epsilon)^2}{\epsilon(1+\epsilon)} z^{1+\epsilon}\, {}_2F_1(1+\epsilon, 1+2\epsilon; 2+\epsilon; z) \\
&-\frac{2(1-2\epsilon)^2}{\epsilon^2} z^{\epsilon}\, {}_2F_1(\epsilon, 2\epsilon; 1+\epsilon; z) \\
&+\frac{2(1-2\epsilon)^2}{\epsilon^2} z^{\epsilon}\, {}_3F_2(\epsilon, \epsilon, 2\epsilon; 1+\epsilon, 1+\epsilon; z) \\
&-\frac{2(1-2\epsilon)^2}{\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} \ln(z) + C'
\end{aligned}
$$

Singular behaviour at $z \to 0$ cannot be expressed as ansatz containing a finite number of term of the form $z^{-1-n\epsilon}$.

Naive expansion yields $1/\epsilon$ as most singular piece. But the inclusive result has $1/\epsilon^3$

$$\int_0^1 \mathrm{d}z\, z^{-1-2\epsilon} = -\frac{1}{2\epsilon}\,, \qquad \int_0^1 \mathrm{d}z\, \ln(z) z^{-1-2\epsilon} = -\frac{1}{4\epsilon^2}$$

$$I_{\mathrm{inc}}[358](Q^2,x) = \frac{3(1-2\epsilon)(4-6\epsilon)(2-6\epsilon)}{\epsilon^3} \frac{x^3}{(Q^2)^3(1-x)^2} I[0](Q^2,x)\,,$$

$$I_{\mathrm{inc}}[0](Q^2,x) = N_\Gamma (Q^2)^{1-2\epsilon}(1-x)^{1-2\epsilon} x^{-1+2\epsilon} \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-3\epsilon)}\,.$$

The $1/\epsilon^3$ term is originating by the integral of $\ln(z)$!

Assembling and checking the result

After inserting the master integrals and expanding in distributions in $x = 1$ and $z = 1$,
the sum VV+VR+RR still contain UV and IR pole terms. They are removed by:

- renormalising the strong coupling (in $\overline{\mathrm{MS}}$ ren. scheme)
- adding the mass factorisation counterterms, both initial- and final-state (in $\overline{\mathrm{MS}}$ fac. scheme)

Checks:

- Scale dependent terms are found to be as predicted by RGE
- We used the underlying RR, RV and VV subprocess matrix elements to re-derive the inclusive NNLO coefficient functions.
- We integrated specific subprocess contributions over the final-state momentum $\hat{z}$ and we recovered the respective contributions to the inclusive coefficient function.
- Comparison to approximate results
- Comparison to partial results

Comparison to approximate results

[Abele, De Florian, Vogelsang ’21,’22]

By expanding the NNLL threshold resummation (i.e. resummation of dominant terms in the $\hat{x} \to 1$ and/or $\hat{z} \to 1$ limit) , approximate corrections have been derived at NNLO and at N3LO

We then have for the leading-power part:

$$\begin{aligned}
\Delta^{(2),C_F}_{qq,\mathrm{LP}} &= \frac{1}{2}\left(\delta_x\,\mathcal{D}^3_z+\delta_z\,\mathcal{D}^3_x\right)+\frac{3}{2}\left(\mathcal{D}^0_x\mathcal{D}^2_z+\mathcal{D}^0_z\mathcal{D}^2_x+2\,\mathcal{D}^1_x\mathcal{D}^1_z\right)\\
&- \left(4+\frac{\pi^2}{3}\right)\left(\mathcal{D}^0_x\mathcal{D}^0_z+\delta_x\,\mathcal{D}^1_z+\delta_z\mathcal{D}^1_x\right)+2\zeta(3)\left(\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x\right)\\
&+ \delta_x\,\delta_z\left(\frac{511}{64}-\frac{15\zeta(3)}{4}+\frac{29\pi^2}{48}-\frac{7\pi^4}{360}\right)\\
&+ \left[\delta_x\,\mathcal{D}^1_z+\delta_z\,\mathcal{D}^1_x+\mathcal{D}^0_x\,\mathcal{D}^0_z+\frac{3}{2}\left(\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x\right)+\delta_x\,\delta_z\left(\frac{9}{8}-\frac{\pi^2}{6}\right)\right]\ln^2\frac{\mu_F^2}{Q^2}\\
&+ \left[-\frac{3}{2}\left(\delta_x\,\mathcal{D}^2_z+\delta_z\,\mathcal{D}^2_x+2\,\mathcal{D}^0_x\,\mathcal{D}^1_z+2\,\mathcal{D}^0_z\,\mathcal{D}^1_x+\mathcal{D}^0_x\,\mathcal{D}^0_z+\delta_x\,\mathcal{D}^1_z+\delta_z\,\mathcal{D}^1_x\right)\right.\\
&\left.+\left(4+\frac{\pi^2}{3}\right)\left(\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x\right)+\delta_x\,\delta_z\left(-5\zeta(3)+\frac{\pi^2}{4}+\frac{93}{16}\right)\right]\ln\frac{\mu_F^2}{Q^2}\,,
\end{aligned}$$

while the dominant NLP terms are given by

$$\Delta^{(2),C_F}_{qq,\mathrm{NLP}} = -\frac{3}{2}\left(\mathcal{D}^2_x+\mathcal{D}^2_z+2\,\mathcal{D}^1_x\,\ell^1_z+2\,\mathcal{D}^1_z\,\ell^1_x+\mathcal{D}^0_x\,\ell^2_z+\mathcal{D}^0_z\,\ell^2_x\right)-\frac{1}{2}\left(\delta_x\,\ell^3_z+\delta_z\,\ell^3_x\right)\,.$$

$$\begin{aligned}
\Delta^{(2),C_A}_{qq} &= -\frac{11}{24}\left(\delta_x\,\mathcal{D}^2_z+\delta_z\,\mathcal{D}^2_x+2\,\mathcal{D}^0_x\,\mathcal{D}^1_z+2\,\mathcal{D}^0_z\,\mathcal{D}^1_x\right)+\left(\frac{67}{36}-\frac{\pi^2}{12}\right)\left(\mathcal{D}^0_x\,\mathcal{D}^0_z+\delta_x\,\mathcal{D}^1_z+\delta_z\,\mathcal{D}^1_x\right)\\
&+ \left(\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x\right)\left(\frac{7\zeta(3)}{4}+\frac{11\pi^2}{72}-\frac{101}{54}\right)+\delta_x\,\delta_z\left(\frac{43\zeta(3)}{12}+\frac{17\pi^4}{720}-\frac{1535}{192}-\frac{269\pi^2}{432}\right)\\
&+ \frac{11}{24}\left[\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x+\frac{3}{2}\,\delta_x\,\delta_z\right]\ln^2\frac{\mu_F^2}{Q^2}\\
&+ \left[-\left(\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x\right)\left(\frac{67}{36}-\frac{\pi^2}{12}\right)+\delta_x\,\delta_z\left(\frac{3\zeta(3)}{2}-\frac{11\pi^2}{36}-\frac{17}{48}\right)\right]\ln\frac{\mu_F^2}{Q^2}\,.
\end{aligned}\tag{64}$$

$$\begin{aligned}
\Delta^{(2),N_f}_{qq} &= \frac{1}{12}\left(\delta_x\,\mathcal{D}^2_z+\delta_z\,\mathcal{D}^2_x+2\,\mathcal{D}^0_x\,\mathcal{D}^1_z+2\,\mathcal{D}^0_z\,\mathcal{D}^1_x\right)-\frac{5}{18}\left(\mathcal{D}^0_x\,\mathcal{D}^0_z+\delta_x\,\mathcal{D}^1_z+\delta_z\,\mathcal{D}^1_x\right)\\
&+ \left(\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x\right)\left(\frac{7}{27}-\frac{\pi^2}{36}\right)+\delta_x\,\delta_z\left(\frac{\zeta(3)}{6}+\frac{19\pi^2}{216}+\frac{127}{96}\right)\\
&- \frac{1}{12}\left[\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x+\frac{3}{2}\,\delta_x\,\delta_z\right]\ln^2\frac{\mu_F^2}{Q^2}\\
&+ \left[\frac{5}{18}\left(\delta_x\,\mathcal{D}^0_z+\delta_z\,\mathcal{D}^0_x\right)+\delta_x\,\delta_z\left(\frac{1}{24}+\frac{\pi^2}{18}\right)\right]\ln\frac{\mu_F^2}{Q^2}\,.
\end{aligned}\tag{65}$$

Full agreement with our result

Comparison to partial results

The leading colour contribution to the $q \to q$ non-singlet channel was computed in [Goyal, Moch, Pathak, Rana, Ravindran ’23]

e.g. piece contained in the transverse coefficient function with single distributions in x or z $\longrightarrow$

We found analytical agreement for all terms involving distributions, and numerical agreement for the regular parts

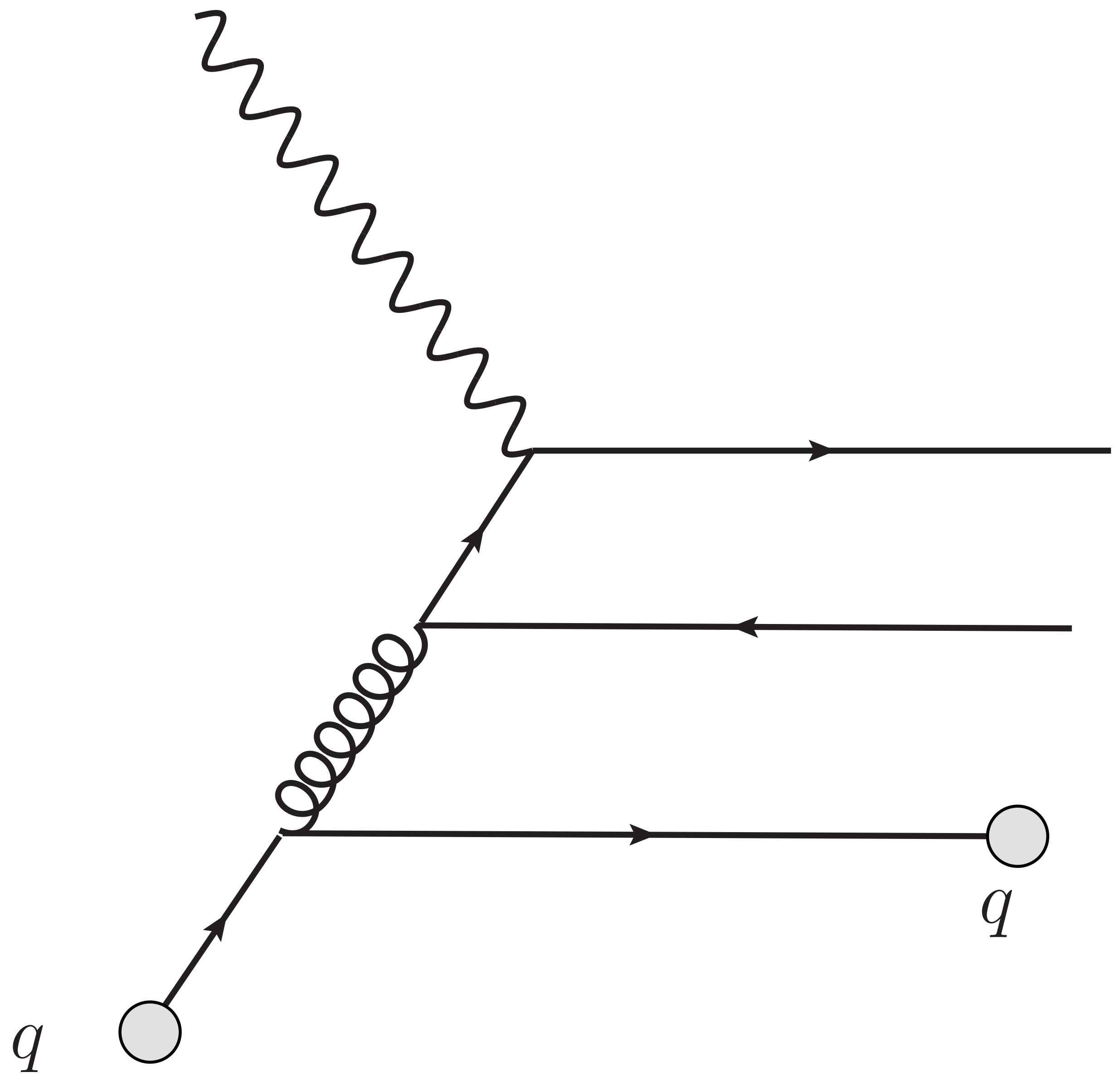
$$
\begin{aligned}
\mathcal{F}^{(2)}_{1,1} = C_F^2 \Big[&\delta_{\bar x}\Big\{2l_z^2(1-4\bar z)+4(1-8\bar z)-8\,\mathrm{Li}_3(\bar z)\tilde z+\frac{25}{3}l_z^3\tilde z-4l_zl_{\bar z}^2\tilde z-4l_{\bar z}^3\tilde z+52\,\mathrm{S}_{12}(\bar z)\tilde z+\mathrm{Li}_2(\bar z)(4(1-6\bar z) \\
&+40l_z\tilde z)+\frac{1}{\bar z}\big(8\,\mathrm{Li}_3(\bar z)-64\,\mathrm{Li}_2(\bar z)l_z-\frac{40}{3}l_z^3+12l_zl_{\bar z}^2-88\,\mathrm{S}_{12}(\bar z)+l_{\bar z}\big(-8\,\mathrm{Li}_2(\bar z)-12l_z^2\big)+l_z\big(-64+24\zeta_2\big)\big) \\
&+l_{\bar z}\big(14+24\tilde z+4l_z(1-2\bar z)+8\,\mathrm{Li}_2(\bar z)\tilde z+10l_z^2\tilde z+16\tilde z\zeta_2\big)+l_z\big(-2+38\tilde z-16\tilde z\zeta_2\big)+8\bar z\zeta_2-16\tilde z\zeta_3\Big\} \\
&+\mathcal{D}_{x,0}\{12+24\tilde z+4l_z(1-3\bar z)+12\,\mathrm{Li}_2(\bar z)\tilde z+16l_z^2\tilde z-4l_zl_{\bar z}\tilde z-12l_{\bar z}^2\tilde z-\frac{1}{\bar z}(16\,\mathrm{Li}_2(\bar z)+24l_z^2-16l_zl_{\bar z})+16\tilde z\zeta_2\} \\
&+\mathcal{D}_{x,1}\{(4l_z\tilde z-24l_{\bar z}\tilde z)\}-12\tilde z\,\mathcal{D}_{x,2}+\delta_{\bar z}\Big\{-4-48\bar x-2l_x^2+\frac{11}{3}l_x^3\tilde x+16l_xl_{\bar x}^2\tilde x-4l_{\bar x}^3\tilde x-24\,\mathrm{S}_{12}(\bar x)\tilde x \\
&+\mathrm{Li}_2(\bar x)(4+8\bar x-12l_x\tilde x)+\frac{1}{\bar x}\big(-8\,\mathrm{Li}_3(\bar x)+16\,\mathrm{Li}_2(\bar x)l_x-4l_x^3-28l_xl_{\bar x}^2+48\,\mathrm{S}_{12}(\bar x)+l_{\bar x}\big(8\,\mathrm{Li}_2(\bar x)+32l_x^2\big) \\
&+l_x\big(64+32\zeta_2\big)\big)+l_{\bar x}\big(14+26\tilde x+4l_x-20l_x^2\tilde x+16\tilde x\zeta_2\big)+l_x\big(-8-34\tilde x-20\tilde x\zeta_2\big)+8\bar x\zeta_2-16\tilde x\zeta_3\Big\} \\
&+\mathcal{D}_{z,0}\{12+28\tilde x+4l_x(1+\bar x)-4\,\mathrm{Li}_2(\bar x)\tilde x-12l_x^2\tilde x+28l_xl_{\bar x}\tilde x-12l_{\bar x}^2\tilde x+\frac{1}{\bar x}(16\,\mathrm{Li}_2(\bar x)+16l_x^2-48l_xl_{\bar x})+16\tilde x\zeta_2\} \\
&+\mathcal{D}_{z,1}\Big\{-\frac{32}{\bar x}l_x+20l_x\tilde x-24l_{\bar x}\tilde x\Big\}-12\tilde x\mathcal{D}_{z,2}\Big]+C_AC_F\Big[\delta_{\bar x}\Big\{4\,\mathrm{Li}_2(\bar z)(1-2\tilde z)+\frac{2}{3}l_z^2(3-11\tilde z)+\frac{1}{27}(396+179\tilde z) \\
&+\frac{1}{9}l_z(1+70\bar z)-4\,\mathrm{Li}_3(\bar z)\tilde z-\frac{5}{3}l_z^3\tilde z+\frac{11}{3}l_{\bar z}^2\tilde z+6\,\mathrm{S}_{12}(\bar z)\tilde z+\frac{1}{\bar z}\big(6\,\mathrm{Li}_2(\bar z)+8\,\mathrm{Li}_3(\bar z)+\frac{62}{9}l_z+\frac{49}{6}l_z^2+\frac{10}{3}l_z^3+6l_zl_{\bar z} \\
&-12\,\mathrm{S}_{12}(\bar z)\big)+l_{\bar z}\Big(\frac{4}{9}(15-41\tilde z)-6l_z\tilde z+4\tilde z\zeta_2\Big)-\frac{4}{3}(3+4\tilde z)\zeta_2-14\tilde z\zeta_3\Big\}+\mathcal{D}_{x,0}\Big\{\frac{2}{9}(39-82\tilde z)-4\,\mathrm{Li}_2(\bar z)\tilde z-6l_z\tilde z \\
&-2l_z^2\tilde z+\frac{22}{3}l_{\bar z}\tilde z+\frac{1}{\bar z}\big(8\,\mathrm{Li}_2(\bar z)+6l_z+4l_z^2\big)+4\tilde z\zeta_2\Big\}+\frac{22}{3}\tilde z\mathcal{D}_{x,1}+\delta_{\bar z}\Big\{\frac{46}{3}+\frac{197}{27}\tilde x+8\,\mathrm{Li}_3(\bar x)\tilde x+\frac{55}{6}l_x^2\tilde x+\frac{11}{3}l_{\bar x}^2\tilde x \\
&+2\,\mathrm{S}_{12}(\bar x)\tilde x+\mathrm{Li}_2(\bar x)\Big(-4l_x\tilde x-\frac{4}{3}(3+4\tilde x)\Big)+l_x\Big(\frac{1}{3}(-13+77\tilde x)-4\tilde x\zeta_2\Big)+\frac{1}{\bar x}\Big(-16\,\mathrm{Li}_3(\bar x)-\frac{83}{6}l_x^2 \\
&+\frac{2}{3}\mathrm{Li}_2(\bar x)(13+12l_x)+\frac{70}{3}l_xl_{\bar x}-4\,\mathrm{S}_{12}(\bar x)+l_x\Big(-\frac{116}{3}+8\zeta_2\Big)\Big)+l_{\bar x}\Big(-\frac{44}{3}l_x\tilde x-\frac{4}{9}(-15+41\tilde x)+4\tilde x\zeta_2\Big) \\
&-\frac{4}{3}(3+4\tilde x)\zeta_2-14\tilde x\zeta_3\Big\}+\mathcal{D}_{z,0}\Big\{4\,\mathrm{Li}_2(\bar x)\tilde x-\frac{44}{3}l_x\tilde x+\frac{22}{3}l_{\bar x}\tilde x+\frac{26}{9}(3-7\tilde x)+\frac{1}{\bar x}\big(-8\,\mathrm{Li}_2(\bar x)+\frac{70}{3}l_x\big)+4\tilde x\zeta_2\Big\} \\
&+\frac{22}{3}\tilde x\mathcal{D}_{z,1}\Big]+\frac{1}{3}C_Fn_F\Big[\delta_{\bar x}\Big\{4-\frac{2}{3}\frac{1}{\bar z}l_z(10+3l_z)-\frac{74}{9}\tilde z+l_z^2\tilde z-2l_{\bar z}^2\tilde z+\frac{8}{3}l_{\bar z}(-3+4\tilde z)+\frac{2}{3}l_z(-12+11\tilde z)+4\tilde z\zeta_2\Big\} \\
&+\mathcal{D}_{x,0}\Big\{\frac{32}{3}\tilde z-8-4l_{\bar z}\tilde z\Big\}-4\tilde z\mathcal{D}_{x,1}+\delta_{\bar z}\Big\{4\,\mathrm{Li}_2(\bar x)\tilde x-4-\frac{38}{9}\tilde x-5l_x^2\tilde x-2l_{\bar x}^2\tilde x+2l_x(2-7\tilde x)+\frac{1}{\bar x}\big(20l_x+10l_x^2 \\
&-8\,\mathrm{Li}_2(\bar x)-16l_xl_{\bar x}\big)+l_{\bar x}\Big(\frac{32}{3}\tilde x-8+8l_x\tilde x\Big)+4\tilde x\zeta_2\Big\}+\mathcal{D}_{z,0}\Big\{\frac{32}{3}\tilde x-8-\frac{16}{\bar x}l_x+8l_x\tilde x-4l_{\bar x}\tilde x\Big\}-4\tilde x\mathcal{D}_{z,1}\Big]\,. \qquad (10)
\end{aligned}
$$

Example of result: C_{qq}^{PS}

$$s_{x/z} = \sqrt{\frac{x}{z}}$$

$$\mathrm{Ti}_2(y) = \int_0^y \frac{\arctan x}{x} \mathrm{d}x$$

$$\begin{aligned}\mathscr{C}_{qq}^{T,\mathrm{PS}} = C_F \Bigg\{ & s_{x/z} P_2(x,z) \Bigg(-\frac{1}{8}\ln(s_{x/z})\arctan(s_{x/z}) + \frac{1}{8}\ln(z s_{x/z})\arctan(z s_{x/z}) - \frac{1}{8}\mathrm{Ti}_2(z s_{x/z}) \\ & + \frac{1}{16}\mathrm{Ti}_2(s_{x/z}) - \frac{1}{16}\mathrm{Ti}_2(-s_{x/z}) \Bigg) - 2\frac{xz+x+z+1}{z}\ln(x)\ln(z) - \frac{1}{16}\ln(x) P_3(x,z) \\ & + \frac{1}{16}\ln(z) P_4(x,z) - \frac{5}{8} P_5(x,z) \Bigg\}, \end{aligned} \tag{14}$$



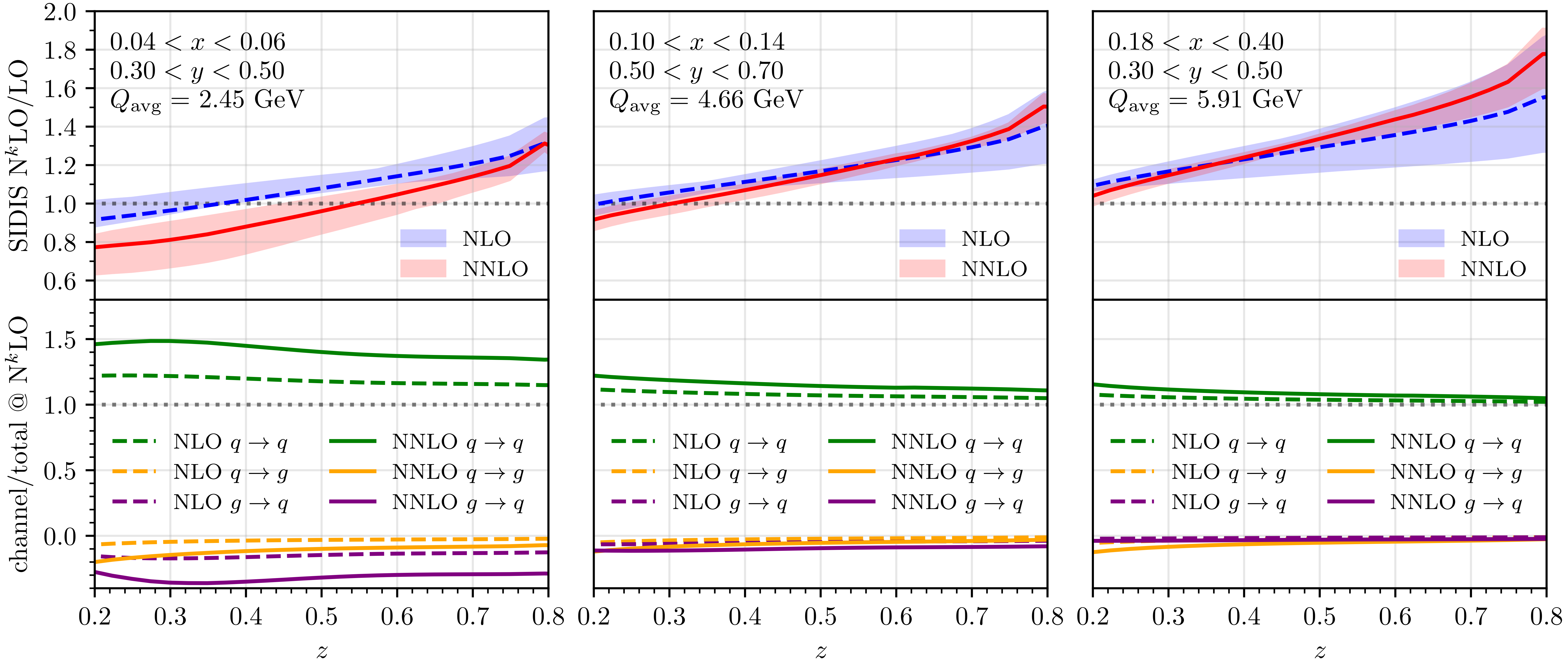
$$\begin{aligned} P_1(x,z) &= \frac{x^2 z + x z^2 + x + z}{xz}, \\ P_2(x,z) &= \frac{5x^4z^2 + 18x^3z^3 + 18x^3z + 5x^2z^4 + 52x^2z^2 + 5x^2 + 18xz^3 + 18xz + 5z^2}{x^2z^2}, \\ P_3(x,z) &= \frac{5x^3z^2 - 5x^3z - 5x^2z^3 - 34x^2z^2 + 34x^2z + 5x^2 - 5xz^3 - 34xz^2 + 34xz + 5x + 5z^2 - 5z}{xz^2}, \\ P_4(x,z) &= \frac{5x^3z^2 + 5x^3z - 5x^2z^3 + 34x^2z^2 + 34x^2z - 5x^2 + 5xz^3 - 34xz^2 - 34xz + 5x - 5z^2 - 5z}{xz^2}, \\ P_5(x,z) &= \frac{x^3z^2 - x^3z + x^2z^3 + 6x^2z^2 - 6x^2z - x^2 - xz^3 - 6xz^2 + 6xz + x - z^2 + z}{xz^2}. \end{aligned} \tag{16}$$

Impact of NNLO corrections

Focus on COMPASS 2016 data for SIDIS charged pion production

(fixed-target experiment, muon beam scattering off an isoscalar target at $\sqrt{s} \simeq 17.35$ GeV)

COMPASS cuts: $Q^2 = xys > 1$ GeV^2, $\sqrt{(P+q)^2}$ (invariant mass of the hadronic system) 5 GeV



Note: FF adopted are the ones of [Borsa, Sassot, De Florian, Stratmann, Vogelsang '22]

Fit on e^+e^- and SIDIS data (including this dataset) at NNLO, using the approximate NNLO for SIDIS

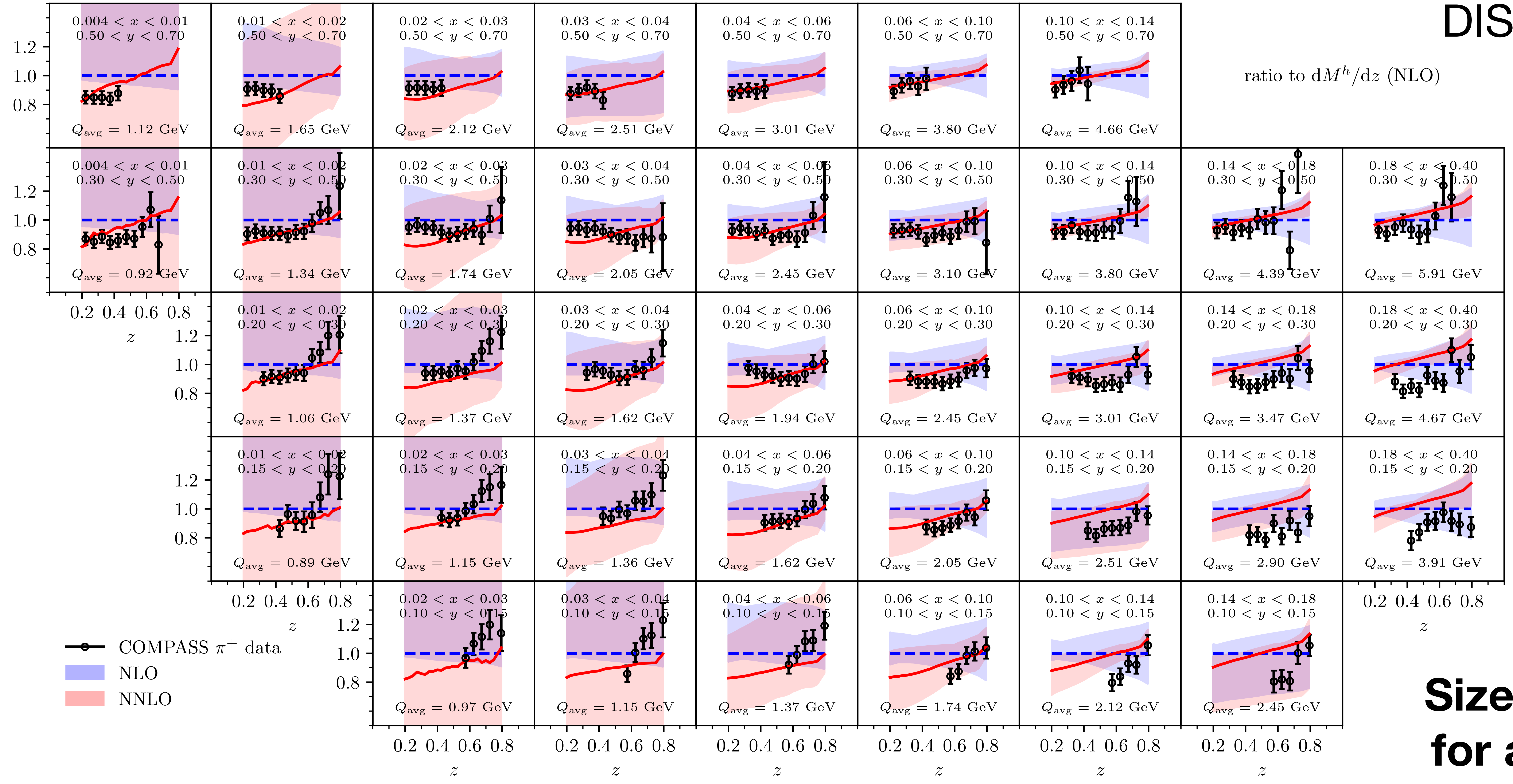
Hadron multiplicities

"ratio of SIDIS over DIS"

$$\frac{\mathrm{d}M^h}{\mathrm{d}z} = \frac{\mathrm{d}^3\sigma^h/\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y}$$

integrated over bins in x and y

DIS from APFEL [Bertone '17]



NNLO improves data description in some bins, but makes it worse in others

Size of NNLO corrections call for a new global fit to assess the impact of SIDIS data

Outlook

Unpolarised

$$\ell(k)\, p(P) \to \ell(k')\, h(P_h)\, X$$

$$\frac{\mathrm{d}^3\sigma^h}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} = \frac{4\pi\alpha^2}{Q^2}\left[\frac{1+(1-y)^2}{2y}\mathscr{F}_T^h(x,z,Q^2) + \frac{1-y}{y}\mathscr{F}_L^h(x,z,Q^2)\right]$$

Longitudinally polarised

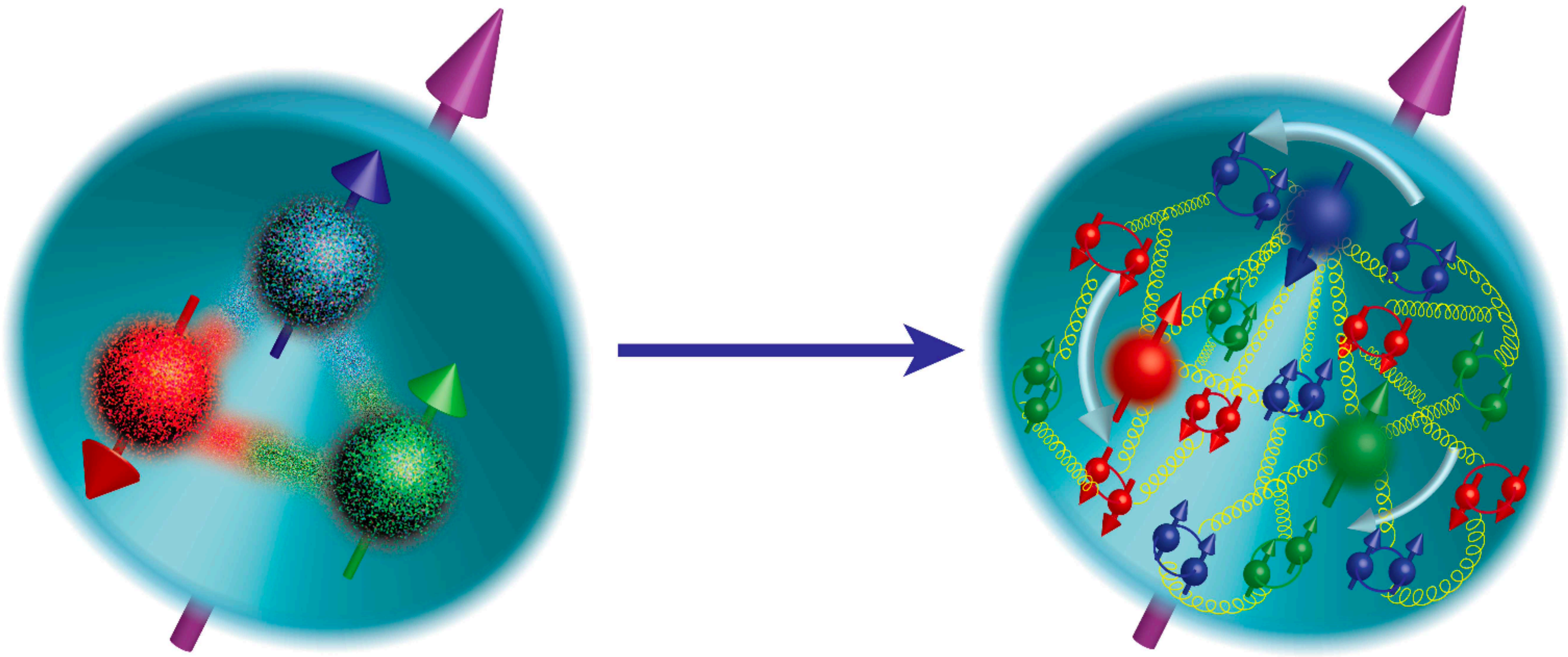
$$\overrightarrow{\ell}(k)\, \vec{p}(P) \to \ell(k')\, h(P_h)\, X$$

$$\frac{1}{2}\left(\frac{\mathrm{d}^3\sigma^h(\uparrow\uparrow)}{\mathrm{d}x\mathrm{d}y\mathrm{d}z} - \frac{\mathrm{d}^3\sigma^h(\uparrow\downarrow)}{\mathrm{d}x\mathrm{d}y\mathrm{d}z}\right) = \frac{4\pi\alpha^2}{Q^2}\frac{1-(1-y)^2}{2y}\mathscr{G}_1^h(x,z,Q^2)$$

Motivation

See talk by Maria Zurek yesterday!

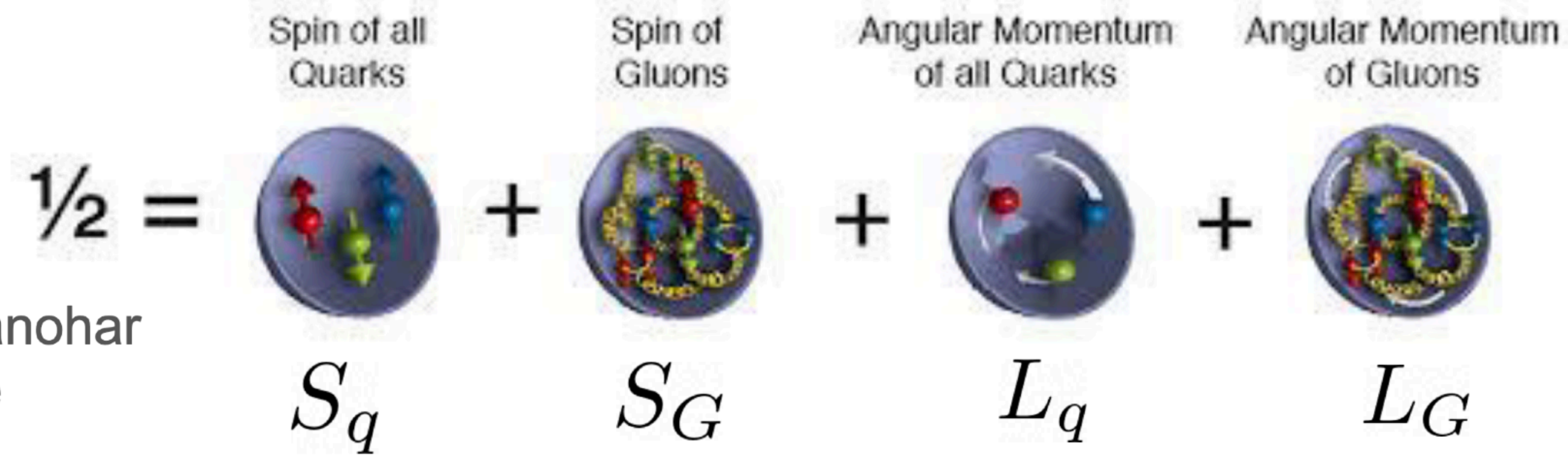
The proton spin puzzle



EMC ‘spin crisis’ (1987):
contribution of quark and anti-quark spins constitute only a small fraction of the proton spin (~ 10%)

Where is the rest?

Jaffe-Manohar sum rule



$$S_q(Q^2) = \frac{1}{2}\int_0^1 \Delta\Sigma(x,Q^2)dx \equiv \frac{1}{2}\int_0^1 \left(\Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}\right)(x,Q^2)dx$$

$$S_g(Q^2) = \int_0^1 \Delta g(x,Q^2)dx \;,$$

polarised PDFs

$$\Delta f(x,Q^2) \equiv f^+(x,Q^2) \; - \; f^-(x,Q^2)$$

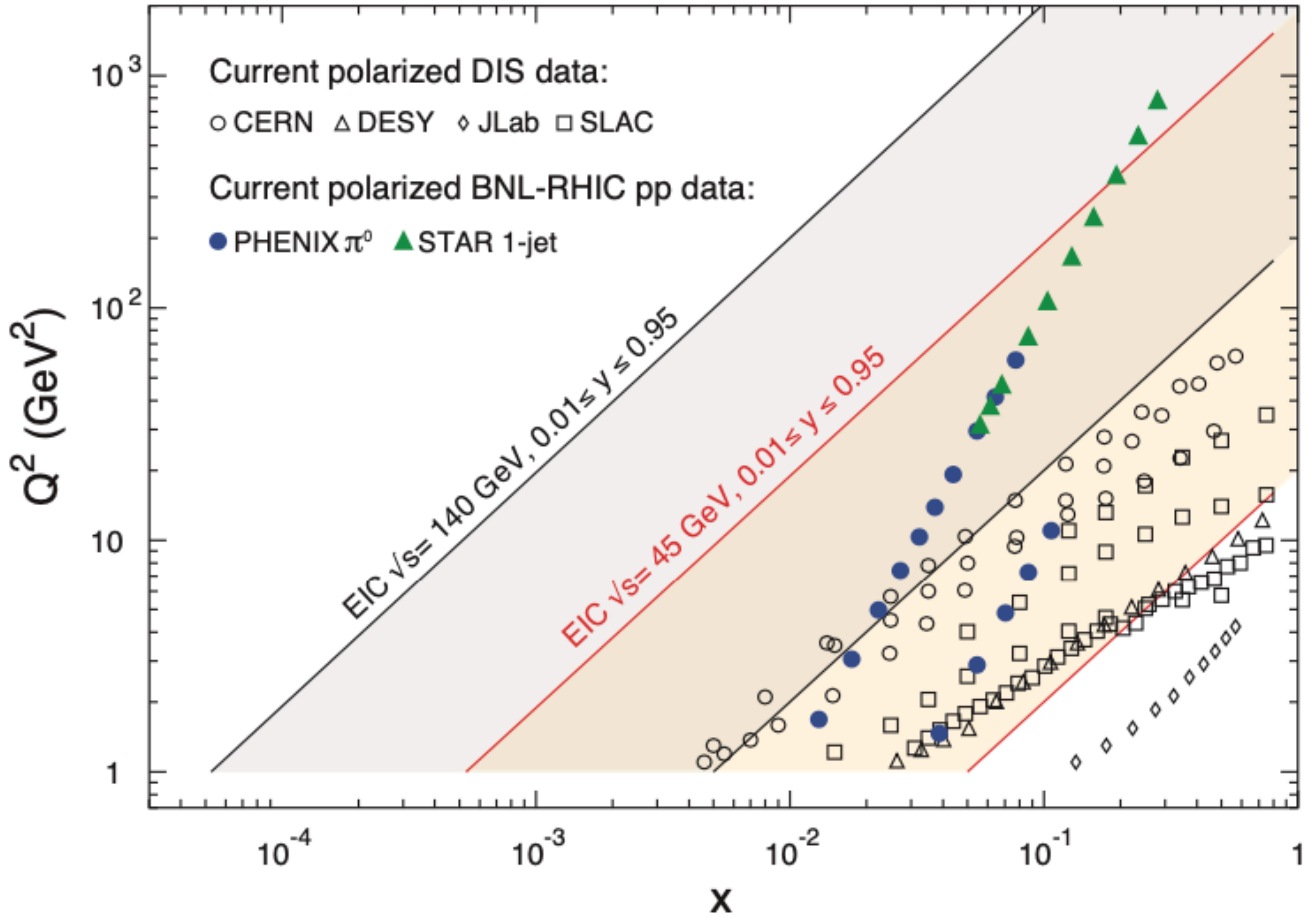
Determined routinely at NLO through global fits
e.g. [NNPDFpol1.0 ’14] [DSSV ’14]

Motivation

See talk by Maria Zurek yesterday!

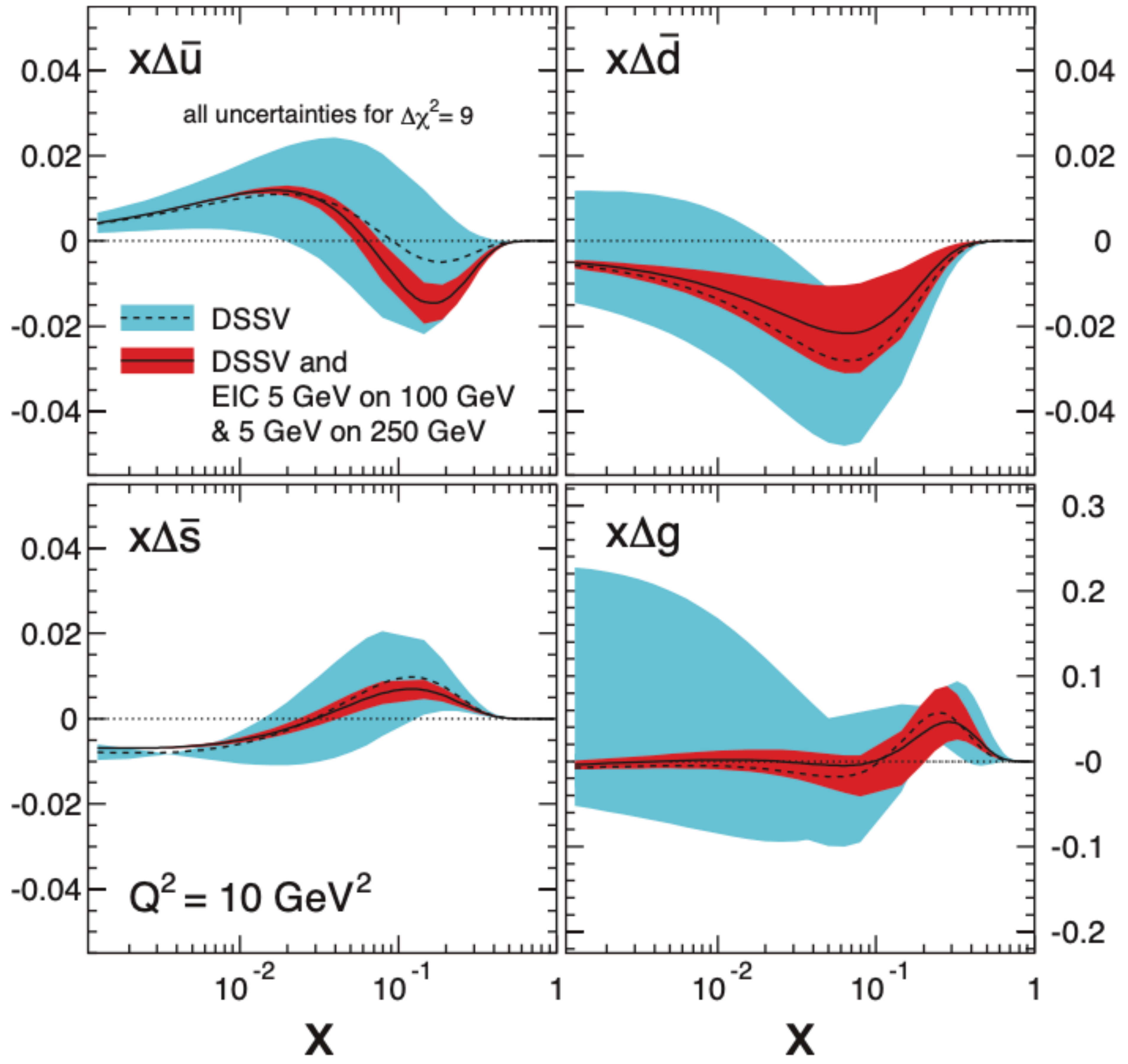
The proton spin puzzle

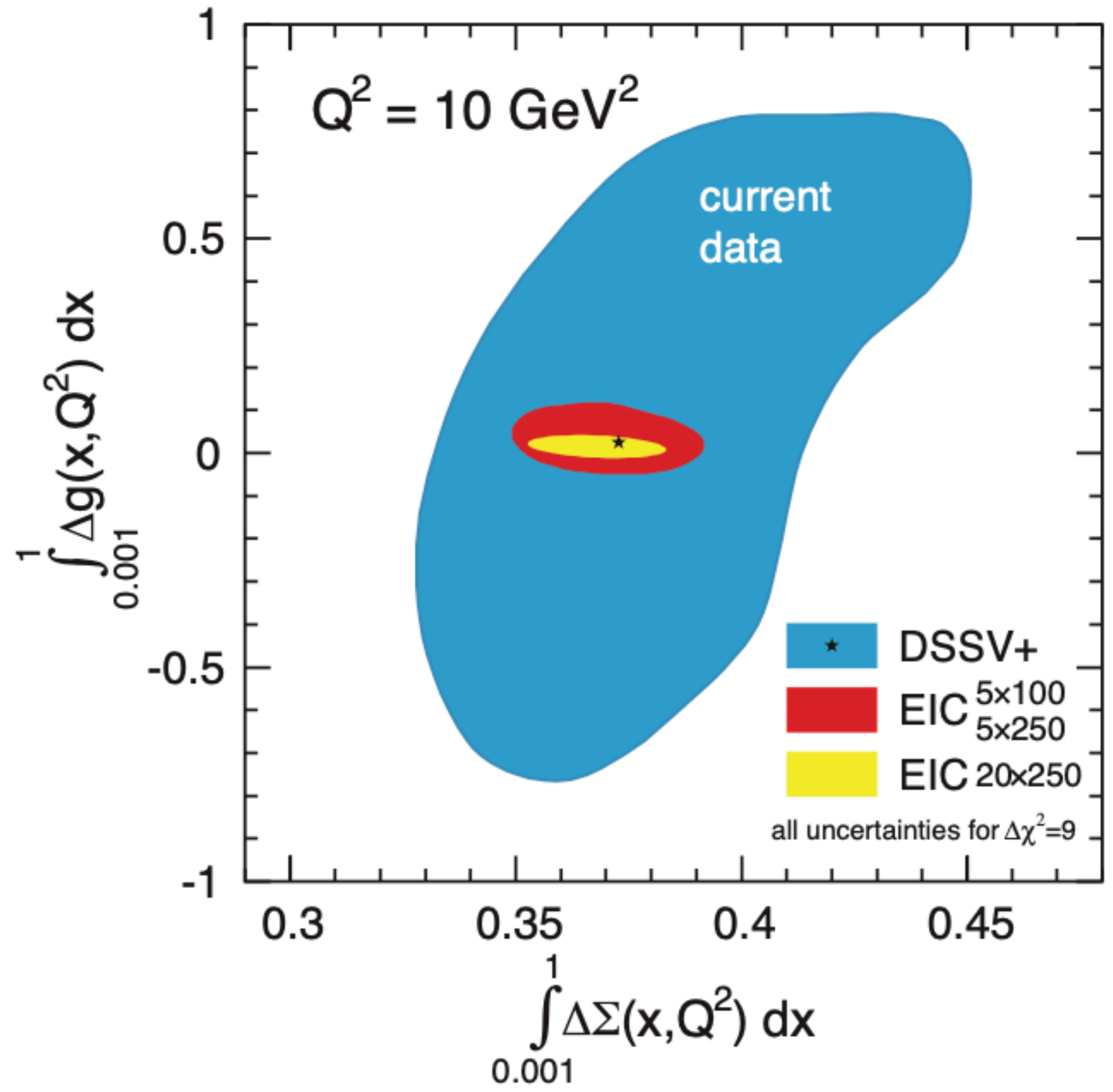
[EIC White Paper, 1212.1701]



EIC will significantly extend the kinematical region covered by previous spin experiments

[Aschenauer, Stratmann, Sassot '12]





EIC has potential to greatly constrain helicity distributions and first moments!

Motivation

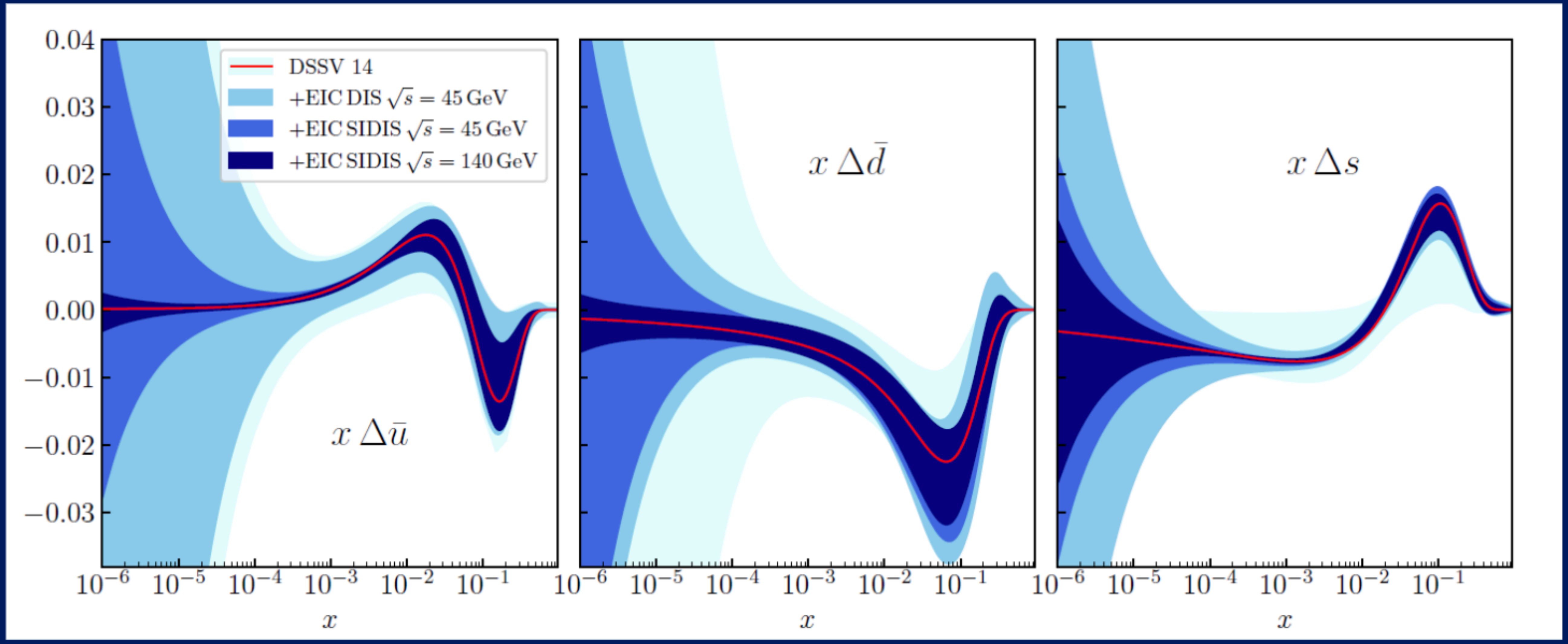
See talk by Maria Zurek yesterday!

The proton spin puzzle

EIC: Improving the flavor-separated helicity distributions of the proton sea through SIDIS

PRD102, 094018 (2020)
DSSV14: PRL113, 012001 (2014)

Christine Aidala @ DIS2024



Access flavor through SIDIS measurements of identified charged pions and kaons.

Current treatment of strangeness assumes $\Delta s = \Delta\bar{s}$ and incorporates constraints from hyperon β decay. In the future could use positive and negative kaons to separate Δs and $\Delta\bar{s}$.

Recent NNLO fits of helicity PDFs

[(BDSSV) Borsa, Stratmann, Vogelsang, De Florian, Sassot ’24]

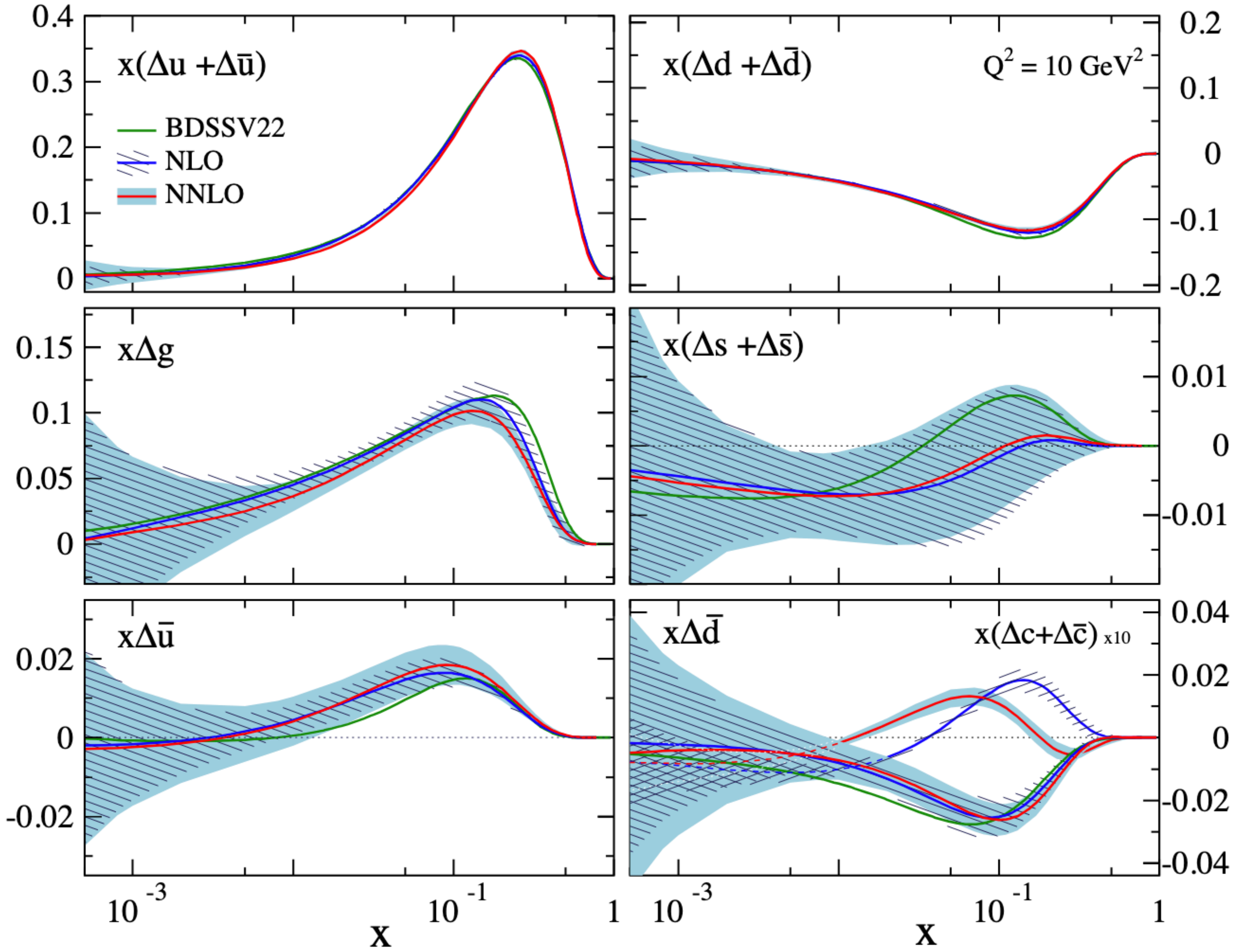
Global fit at aNNLO (DIS + SIDIS + pp), with approximate SIDIS and pp coefficient functions from threshold resummation

TABLE I. Partial and total χ^2 obtained in the fits

	Total	DIS	SIDIS	pp-jets	pp-pions	pp-W
NLO	627.2	302.7	127.6	111.1	63.5	22.3
NNLO	607.5	294.3	122.9	104.0	66.0	20.3
Data points	673	368	114	91	78	22

Remarkable perturbative stability

Total up and down well determined.
Gluon clearly positive.
Positive $\Delta\bar{u}$ and negative $\Delta\bar{d}$.



Recent NNLO fits of helicity PDFs

[(BDSSV) Borsa, Stratmann, Vogelsang, De Florian, Sassot ’24]

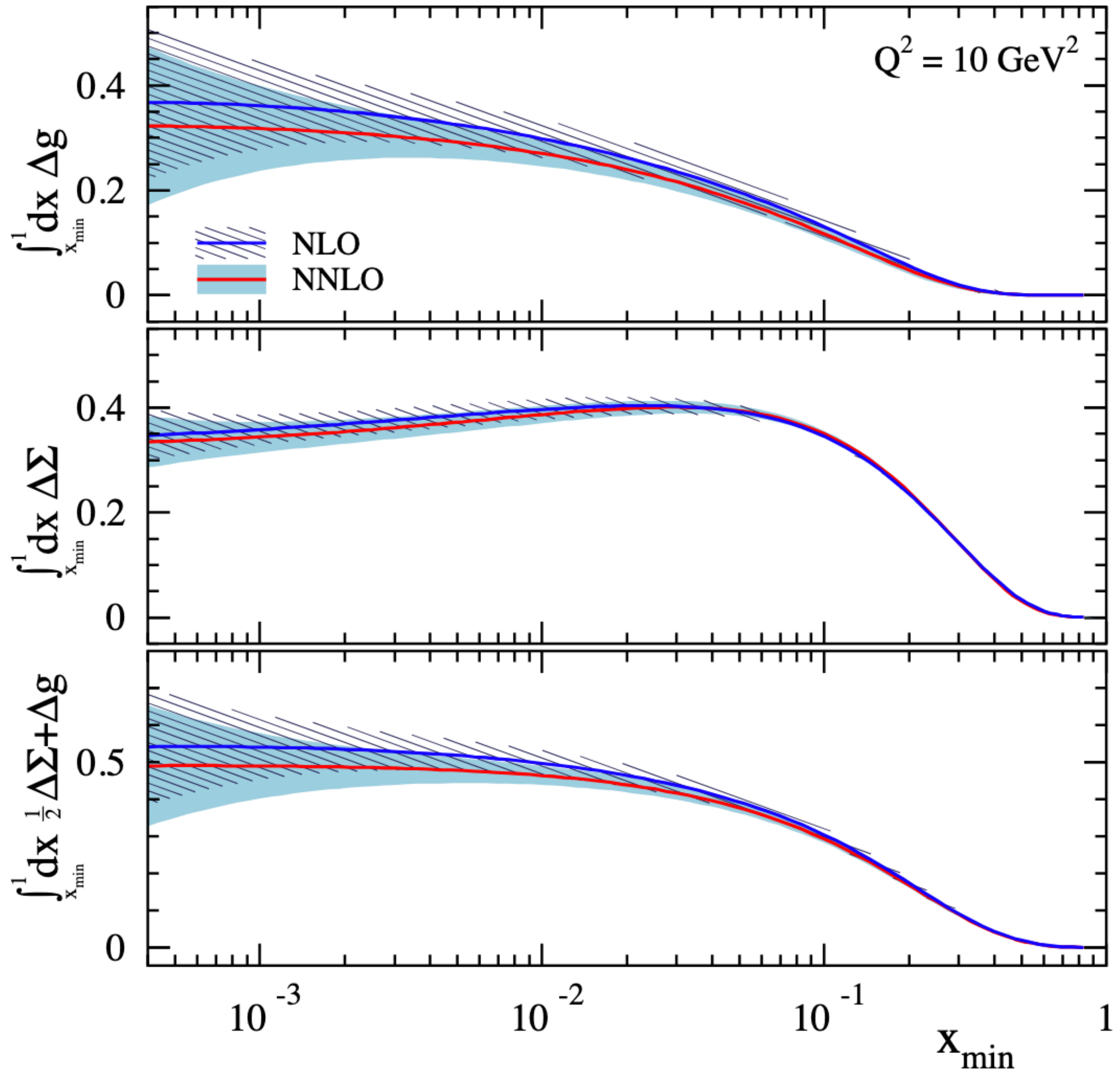
Global fit at aNNLO (DIS + SIDIS + pp), with approximate SIDIS and pp coefficient functions from threshold resummation

TABLE I. Partial and total χ^2 obtained in the fits

	Total	DIS	SIDIS	pp-jets	pp-pions	pp-W
NLO	627.2	302.7	127.6	111.1	63.5	22.3
NNLO	607.5	294.3	122.9	104.0	66.0	20.3
Data points	673	368	114	91	78	22

Remarkable perturbative stability

Combined contribution of quarks and gluon to proton spin approaching 1/2: small contribution by orbital angular momenta?

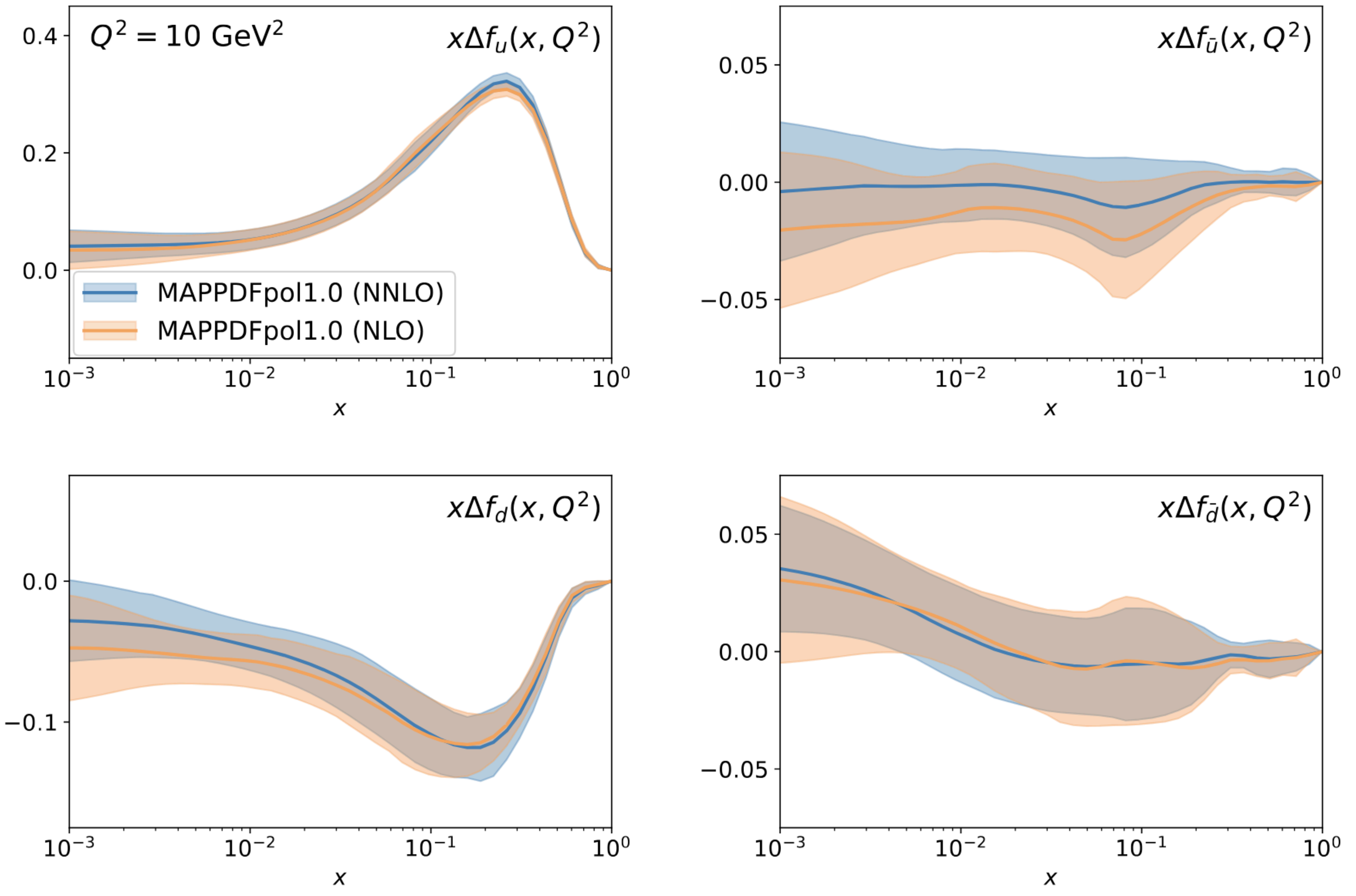


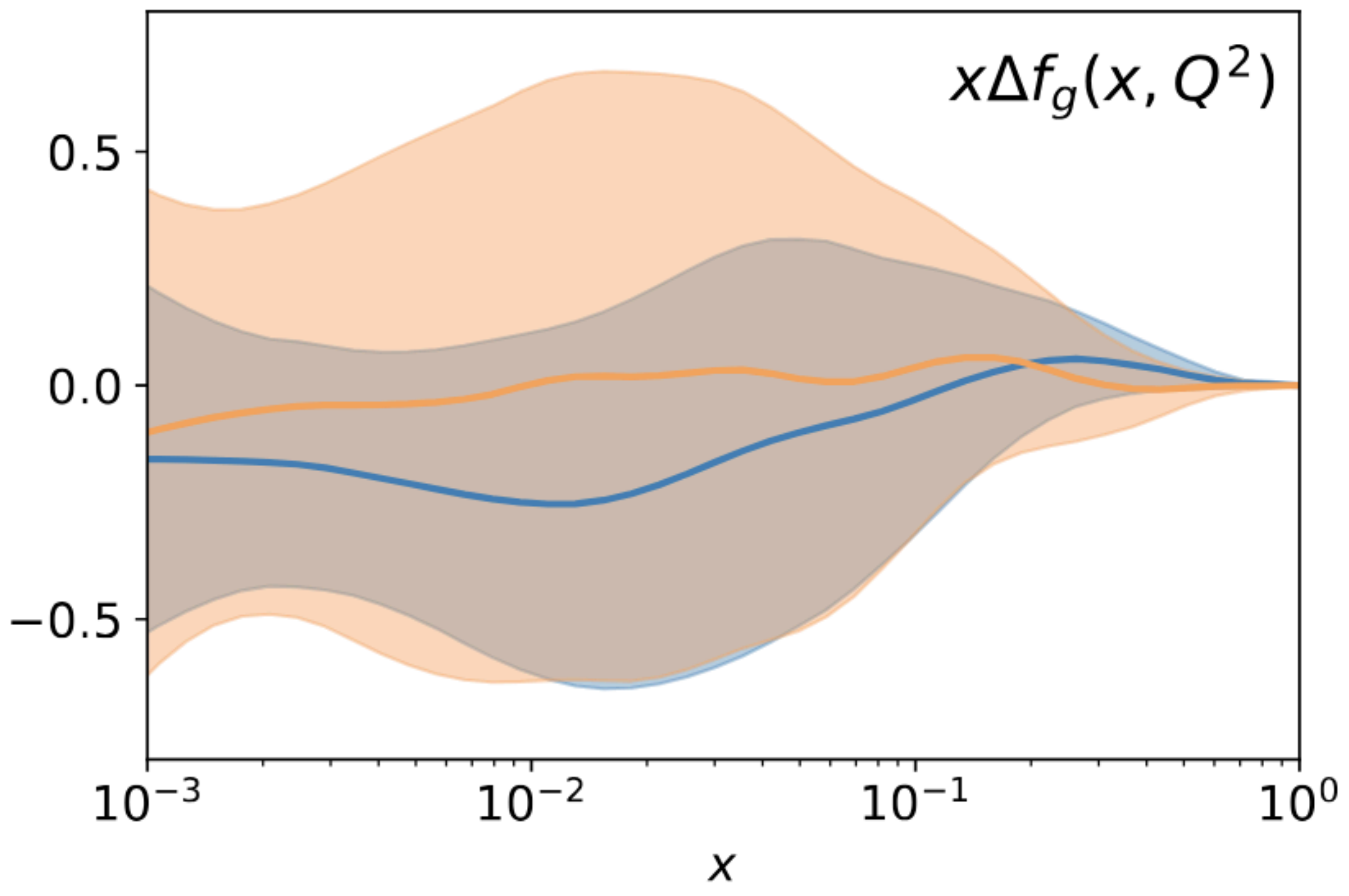
Recent NNLO fits of helicity PDFs

[(MAP) Bertone, Chief, Nocera ’24]

Different methodology: neural-network parametrisation of PDFs with a Monte Carlo representation of their uncertainties.

DIS + SIDIS data only (with aNNLO for SIDIS). Striking difference is the gluon, largely undetermined w/o pp data.





Polarised SIDIS structure function

$$\mathcal{G}_1^h(x,z,Q^2) = \sum_{p,p'} \int_x^1 \frac{\mathrm{d}\hat{x}}{\hat{x}} \int_z^1 \frac{\mathrm{d}\hat{z}}{\hat{z}} \Delta f_p\left(\frac{x}{\hat{x}}, \mu_F^2\right) D_{p'}^h\left(\frac{z}{\hat{z}}, \mu_A^2\right) \Delta\mathscr{C}_{p'p}\left(\hat{x}, \hat{z}, Q^2, \mu_R^2, \mu_F^2, \mu_A^2\right)$$

$\Delta\mathscr{C}_{p'p}$ known up to NLO (see e.g. [De Florian, Stratmann, Vogelsang '97])

What the experiments measure is the longitudinal double-spin asymmetry $A_{\|}$

$$A_{\|} = \frac{\mathrm{d}\sigma^{\uparrow\uparrow} - \mathrm{d}\sigma^{\uparrow\downarrow}}{\mathrm{d}\sigma^{\uparrow\uparrow} + \mathrm{d}\sigma^{\uparrow\downarrow}} \simeq D\,A_1$$ with a (known) kinematical factor D.

A_1 is related to the photoabsorption cross sections σ_{J_z},

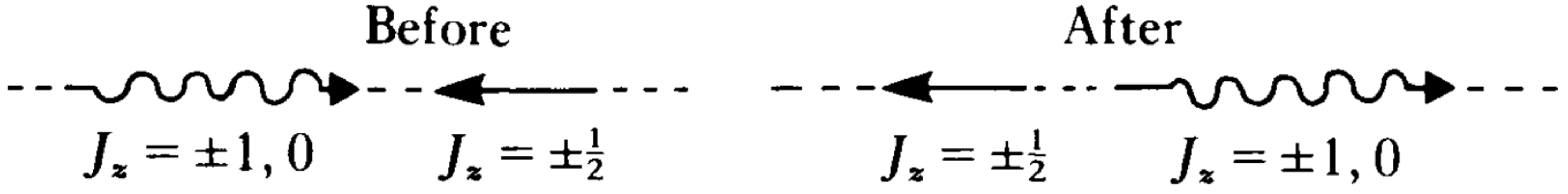
with J_z is the spin of the intermediate photon-nucleon system:

$$A_1 = \frac{\sigma_{1/2} - \sigma_{3/2}}{\sigma_{1/2} + \sigma_{3/2}} = \frac{\mathscr{G}_1}{\mathscr{F}_T}$$

Why $A_1 = \mathcal{G}_1/\mathcal{F}_T$? Physical argument

from [F. Close, *An Introduction to Quarks and Partons*, 1979]

TABLE 13.1
Forward Compton helicity amplitudes



	Initial state γ_V	Initial state P	Intermediate state J_z	Final state γ_V	Final state P
(A)	$+1$	$+\frac{1}{2}$	$+\frac{3}{2}$	$+1$	$+\frac{1}{2}$
(B)	$+1$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+1$	$-\frac{1}{2}$
(C)	$+1$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	$+\frac{1}{2}$
$(\tilde{C})$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	1	$-\frac{1}{2}$
(D)	0	$+\frac{1}{2}$	$+\frac{1}{2}$	0	$+\frac{1}{2}$

$$\gamma(J_z = +1): \quad \gamma^\uparrow + \mathrm{P}^\uparrow \to \sigma_{3/2}$$

$$\gamma(J_z = -1): \quad \gamma_\downarrow + \mathrm{P}^\uparrow \to \sigma_{1/2}$$

Quark moving along the z-axis
(i.e. $k_T = 0$)

$$\gamma^\uparrow + q_\downarrow \to q^\uparrow$$

$$\gamma_\downarrow + q^\uparrow \to q_\downarrow$$

$$q^\uparrow = \sqrt{\left(\frac{E+m}{2E}\right)} \begin{pmatrix} \chi^\uparrow \\ \dfrac{P_z\chi^\uparrow + (P_x + iP_y)\chi_\downarrow}{E+m} \end{pmatrix} \qquad \gamma_\pm = \begin{pmatrix} 0 & \sigma_\pm \\ -\sigma_\pm & 0 \end{pmatrix}$$

$$\sigma_{1/2} \sim \gamma_\downarrow \mathrm{P}^\uparrow \sim \sum_i e_i^2 q^\uparrow \qquad \sigma_{3/2} \sim \gamma^\uparrow \mathrm{P}^\uparrow \sim \sum_i e_i^2 q_\downarrow$$

$$\boxed{A \equiv \frac{\sigma_{1/2} - \sigma_{3/2}}{\sigma_{1/2} + \sigma_{3/2}} = \frac{\sum_i e_i^2 [q_i^\uparrow - q_{i\downarrow}]}{\sum_i e_i^2 [q_i^\uparrow + q_{i\downarrow}]}}$$

Calculation: polarised vs. unpolarised case

Prescription for γ_5

Problem: projector of the hadronic tensor to isolate the $g_1 = \mathscr{G}_1/2$ structure function is:

$$P_{g_1}^{\mu\nu} = \frac{i}{(D-2)(D-3)} \frac{2x}{Q^2} \varepsilon^{\mu\nu\rho\sigma} p_\rho q_\sigma$$

explicit Levi-Civita tensor

Required a consistent treatment in dim. reg.

In addition, we have polarised quark or gluon in the initial state: spin sum with explicit γ_5 or Levi-Civita

We adopt the Larin prescription: setting $\gamma_\mu \gamma_5 = \frac{i}{3!} \varepsilon_{\mu\nu\rho\sigma} \gamma^\nu \gamma^\rho \gamma^\sigma$

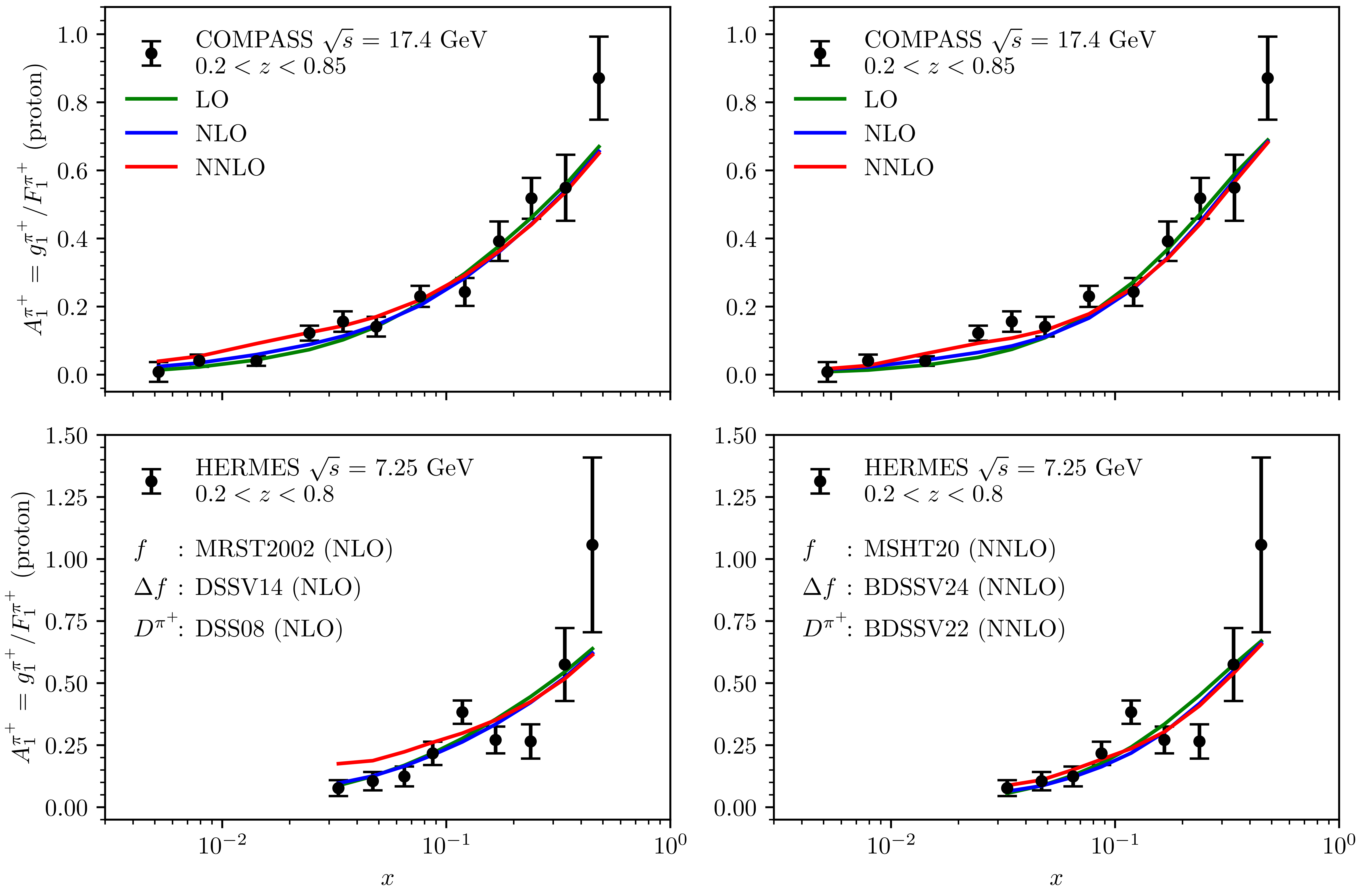
And evaluating traces in D dimensions, and contracting the two Levi-Civita into D-dim metric tensors.

We carry out mass factorization with Larin space-like splitting functions and at the end in order to restore Ward identities we apply the transformation:

$$\begin{aligned} g_1 =& \Delta \mathcal{C}^{\overline{\mathrm{MS}}} \otimes \Delta f^{\overline{\mathrm{MS}}} \\ =& (\Delta \mathcal{C}^{\mathrm{L}} \otimes Z^{-1}) \otimes (Z \otimes \Delta f^{\mathrm{L}}) = \Delta \mathcal{C}^{\mathrm{L}} \otimes \Delta f^{\mathrm{L}} \end{aligned}$$

Numerical results

Comparison to experimental points



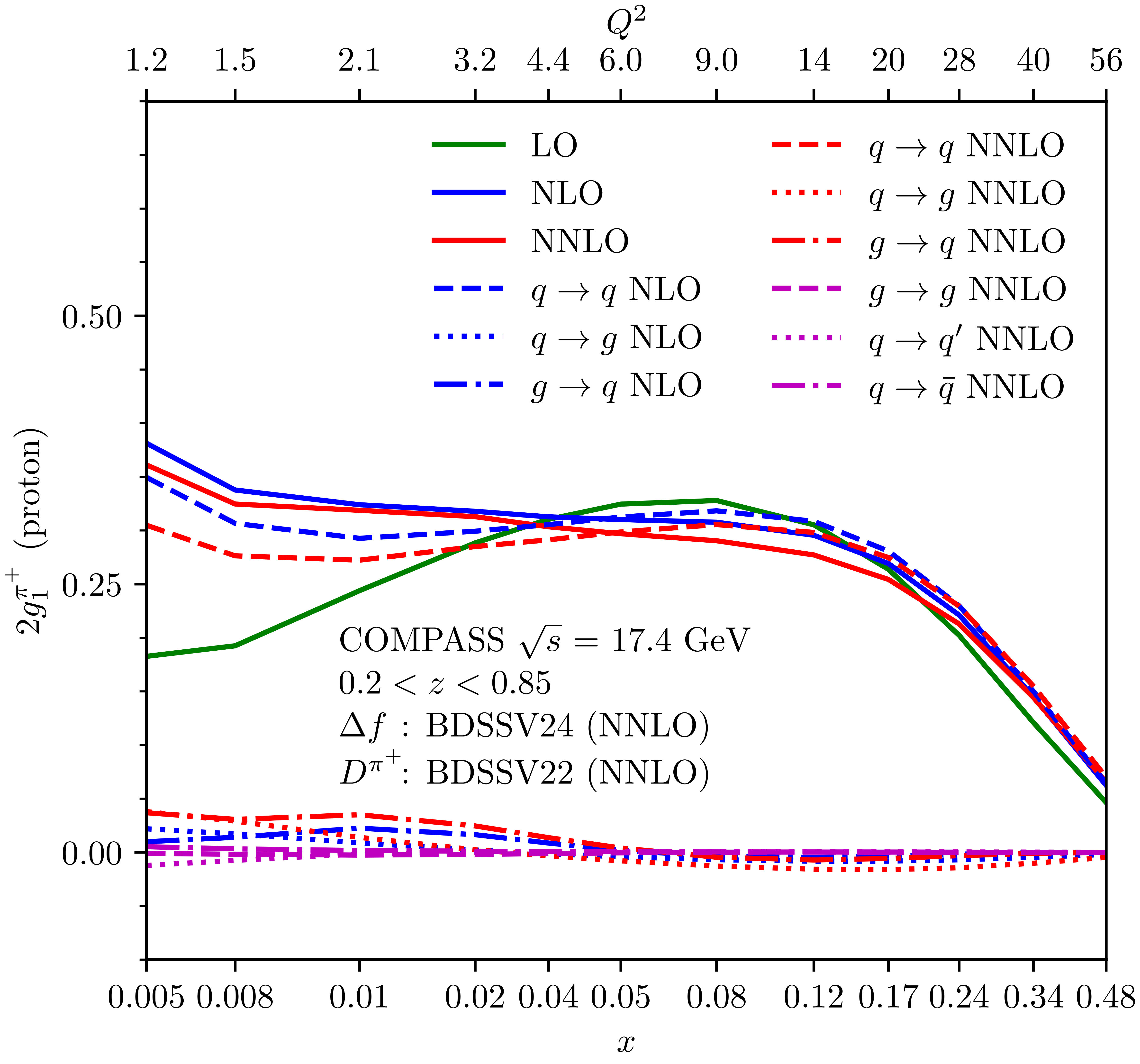
Data points from CERN COMPASS and DESY HERMES for identified π^+ produced over a range of z-values.

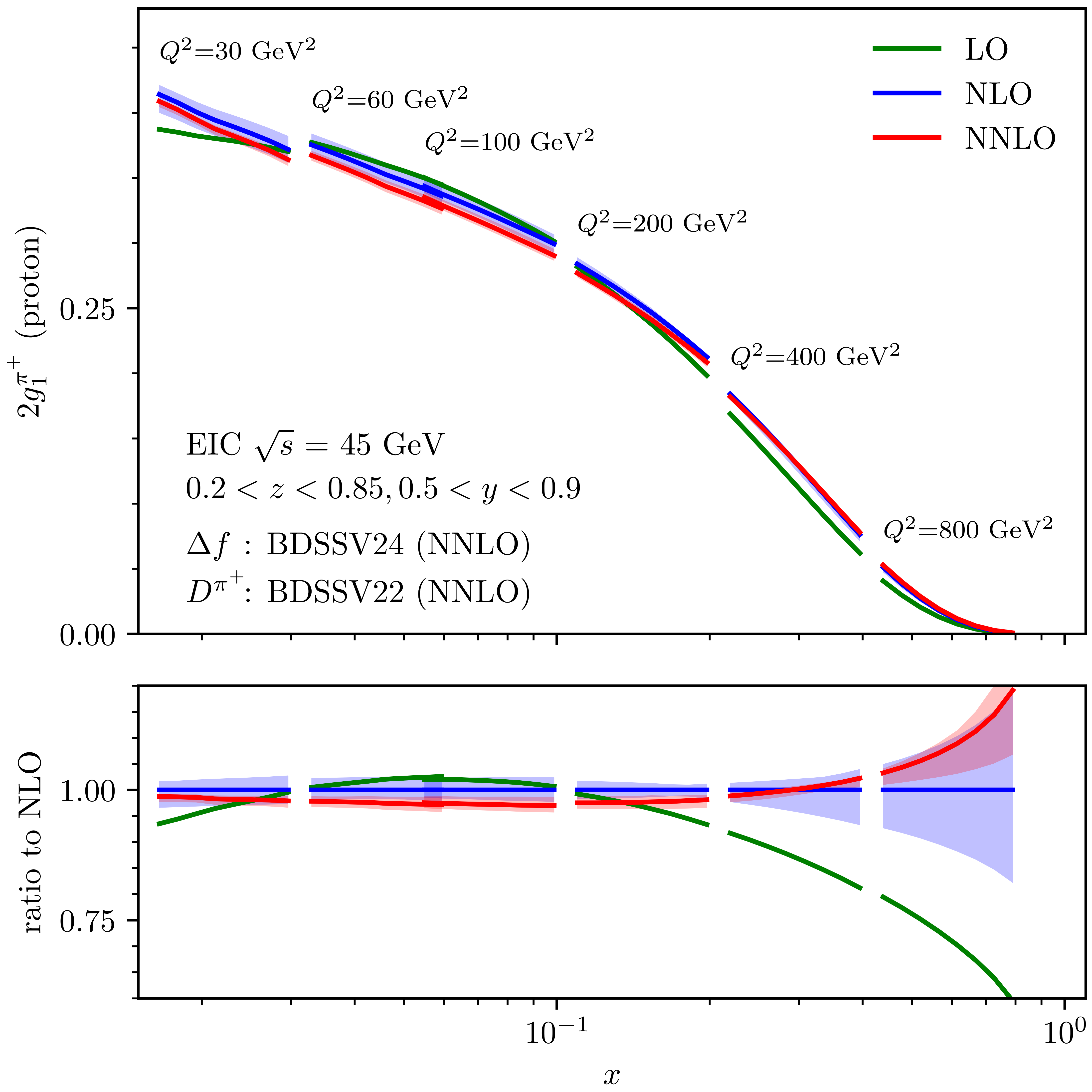
2-dim data points in (x, Q^2): larger (smaller) x implies larger (smaller) Q^2

NNLO corrections can be sizeable, especially at low-x

Numerical results

Channel decomposition and perturbative convergence



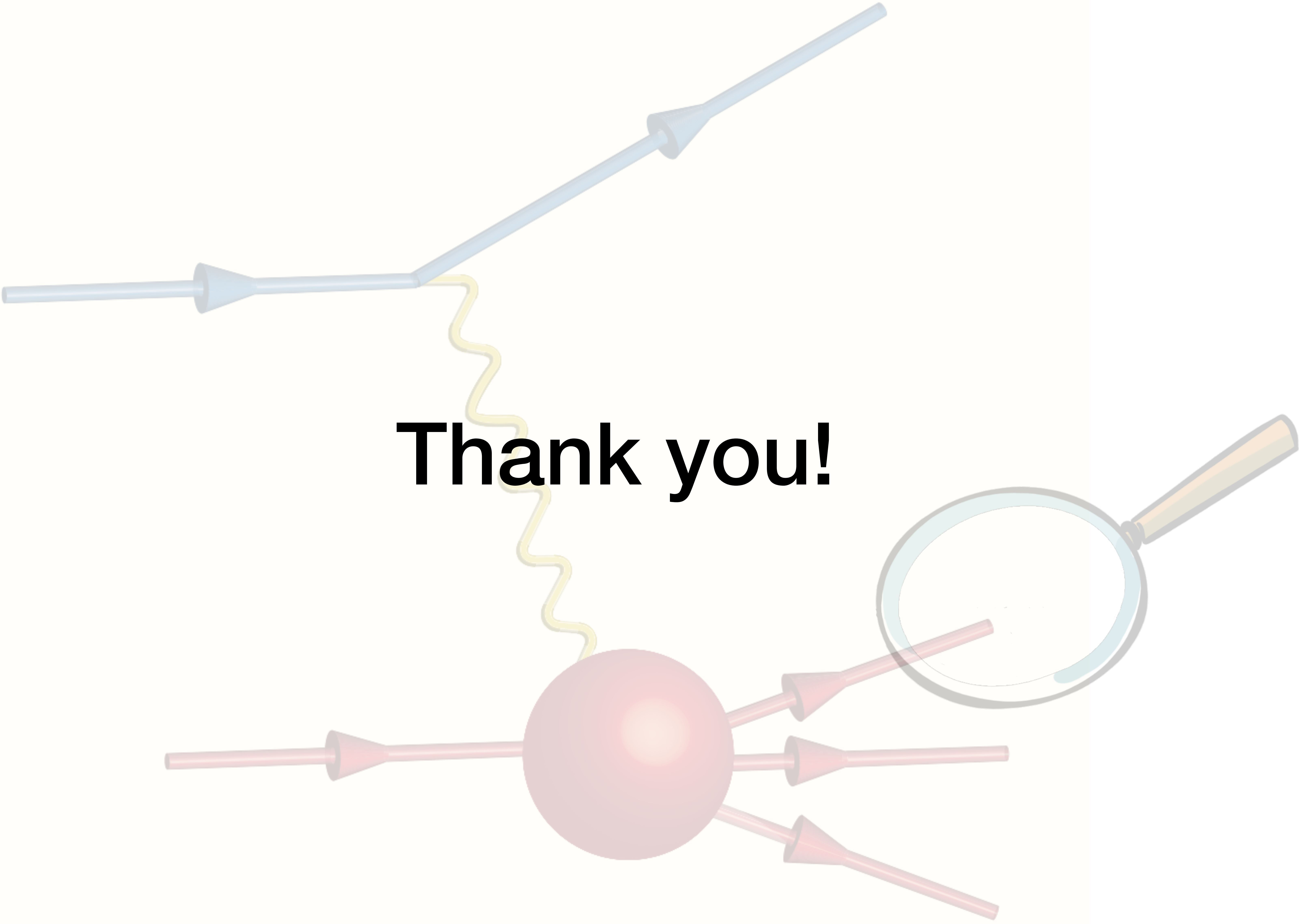


Conclusions

- We computed the NNLO QCD corrections to SIDIS coefficient functions in analytical form, both for unpolarised and longitudinally polarised beam and target. Our results allow for NNLO global fits of fragmentation functions and helicity PDFs.

- Agreement with independent calculations by [Goyal, Lee, Moch, Pathak, Rana, Ravindran 2312.17711, 2404.09959].

- In the antenna subtraction formalism for fully differential NNLO calculations, matrix elements of simpler processes are used as subtraction terms. In case of an identified particle in the final state, **we can recycle the SIDIS coefficient functions as integrated initial-final fragmentation quark-quark antenna functions**! Work in progress towards antenna subtraction for identified particles at hadron colliders



Thank you!