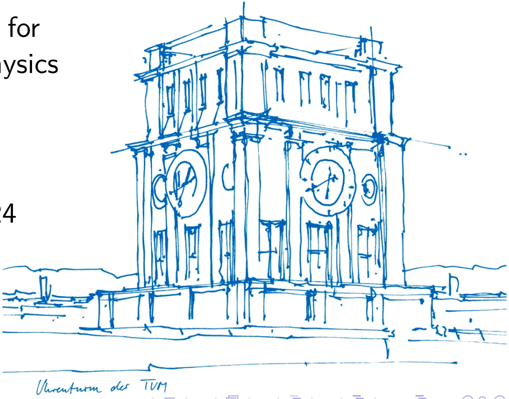
Technical University of Munich
Department of Physics

False vacuum decay of excited states from finite-time instantons

Effective Theories for
Nonperturbative Physics

Nils Wagner

13^{th} August 2024



Supervisor: Prof. Dr. Björn Garbrecht
Research Group: T70 (TUM)

Outline

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Motivation

[1] Sakharov (1967), *Pisma Zh. Eksp. Teor. Fiz.* vol. *5*
[2] Garbrecht (2020), *Prog. Part. Nucl. Phys.* vol. *110*

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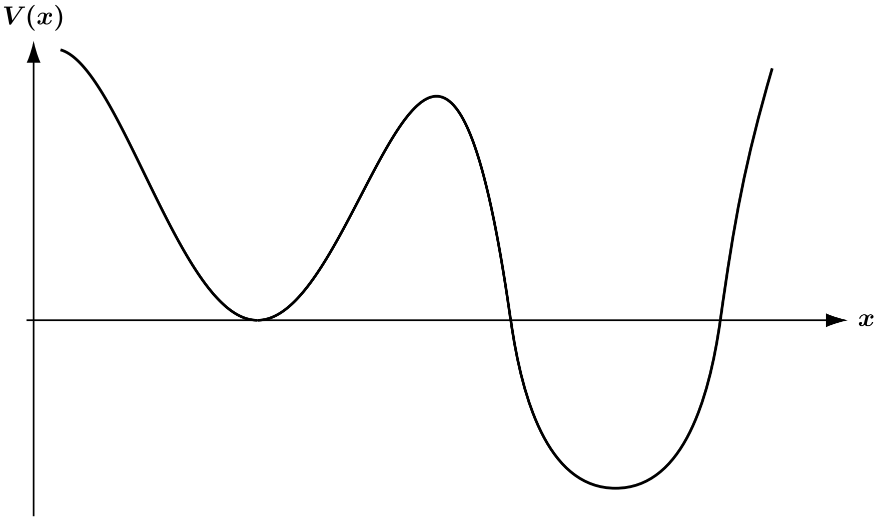
Later goals: Get a full real-time picture of tunneling in quantum mechanics, then incorporate finite-temperature effects.

Generic tunneling potential



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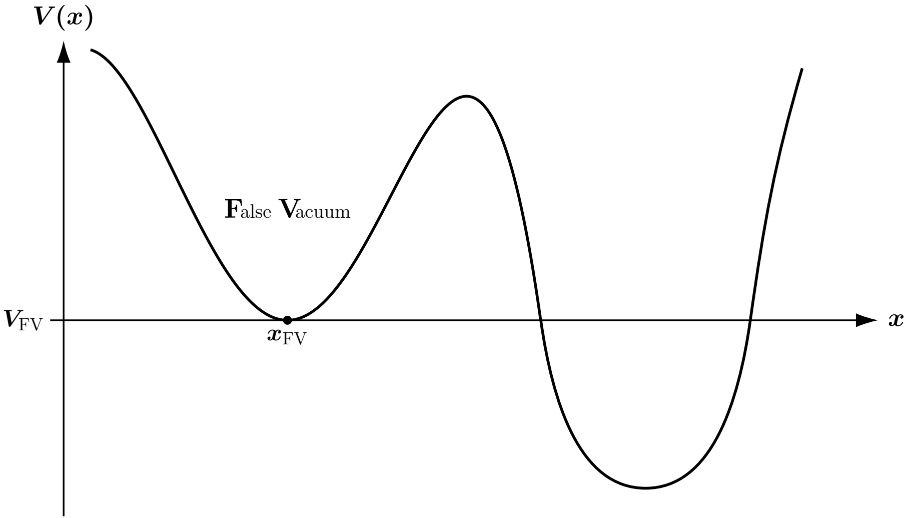

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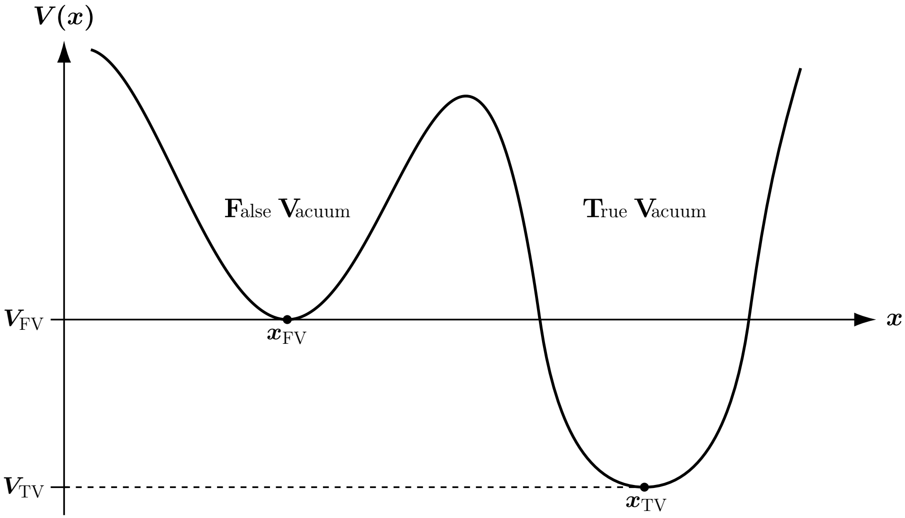

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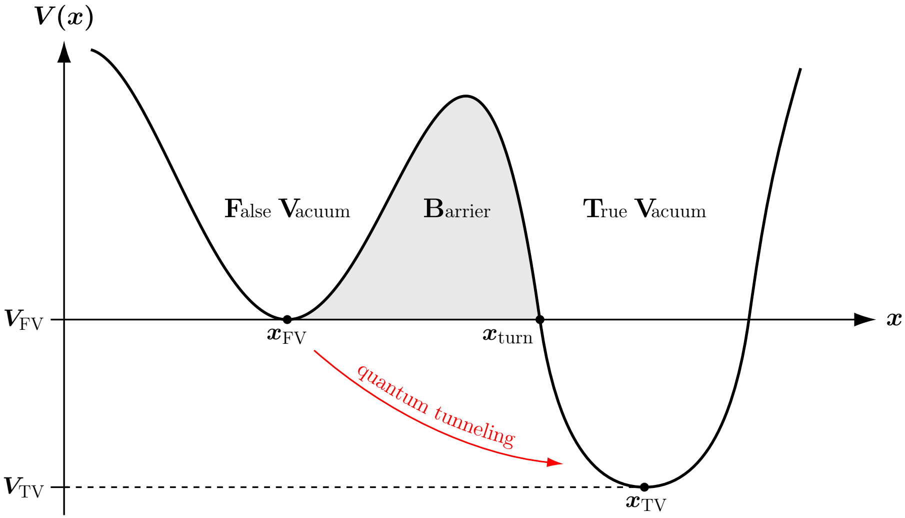

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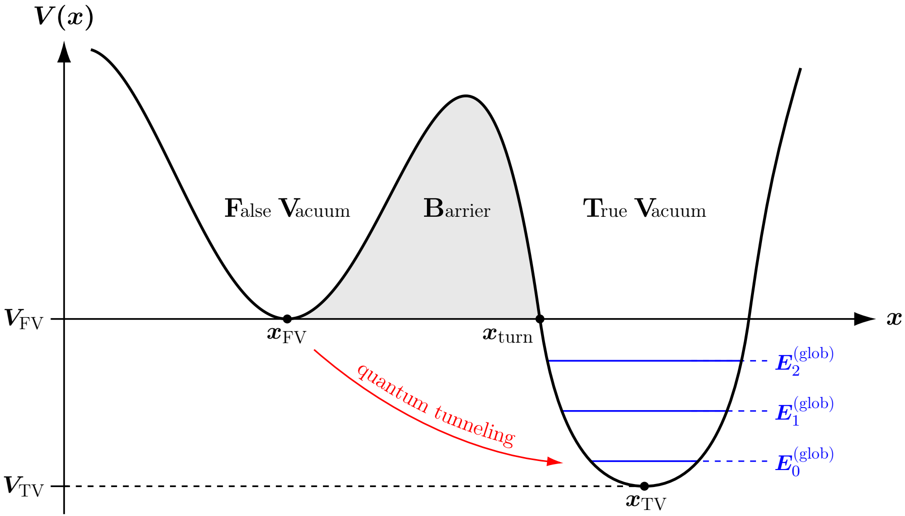

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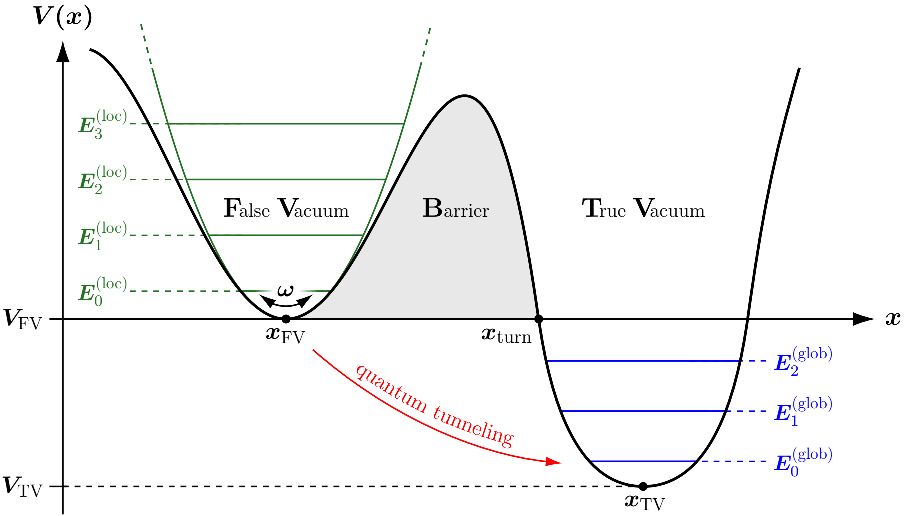

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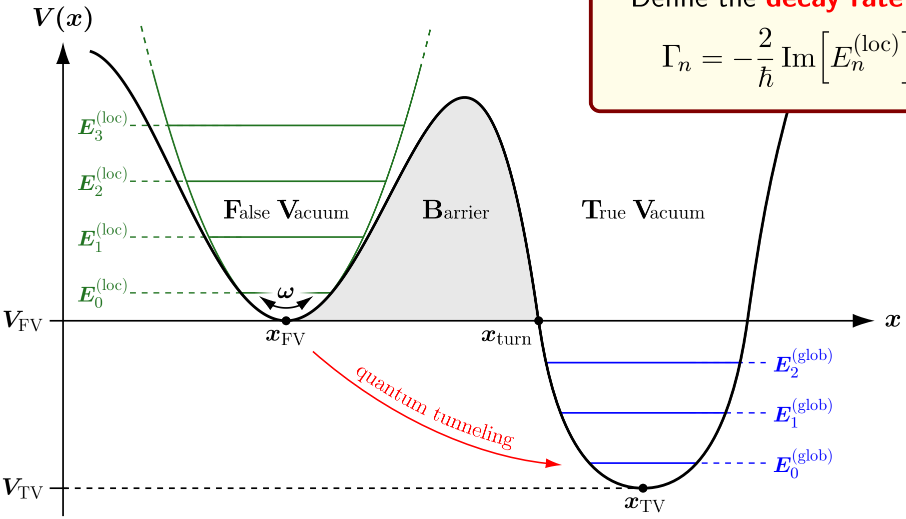

Define the **decay rate** as:

$$\Gamma_n = -\frac{2}{\hbar}\,\mathrm{Im}\left[E_n^{(\mathrm{loc})}\right]$$

Extracting decay rates

There exist numerous methods of attributing a meaningful imaginary part to the local energies $E_n^{(\mathrm{loc})}$, fitting into roughly two categories:

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- Functional techniques based on the (Euclidean) propagator, employing path integrals [5, 6, etc.]

directly extendable to field theory

Outline

Late-time behavior of the Euclidean propagator

Employing the spectral representation, we can project out the (global) ground state energy from the late-time behavior of the Euclidean propagator

$$K_{\mathrm{E}}\left(x_0, x_T\,; T\right) = \sum_{n=0}^{\infty} \overline{\psi_n^{(\mathrm{glob})}(x_0)}\, \psi_n^{(\mathrm{glob})}(x_T)\, \exp\left[-\frac{E_n^{(\mathrm{glob})} T}{\hbar}\right] ,$$

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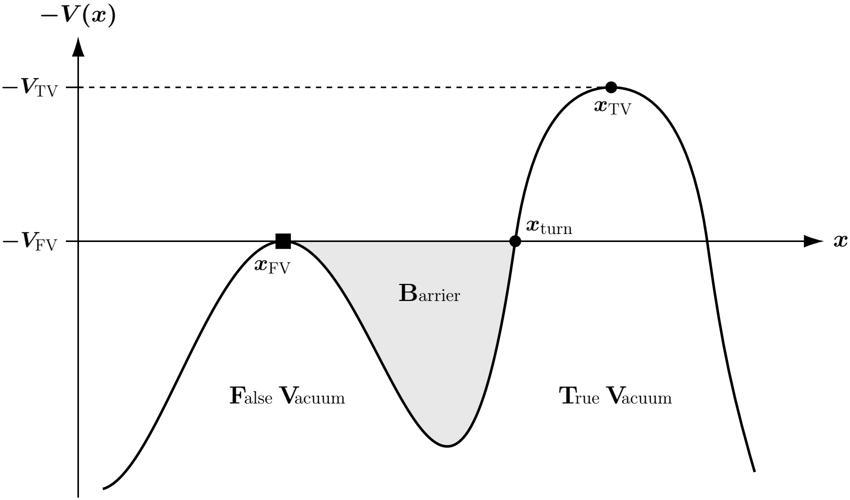
One hereby chooses $x_0 = x_T = x_{\mathrm{FV}}$ for convenience [5,6].

Critical trajectories for large T



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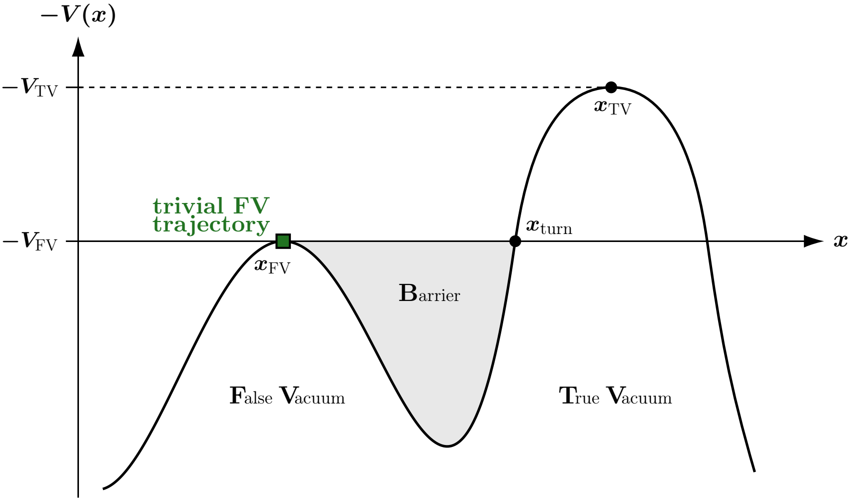

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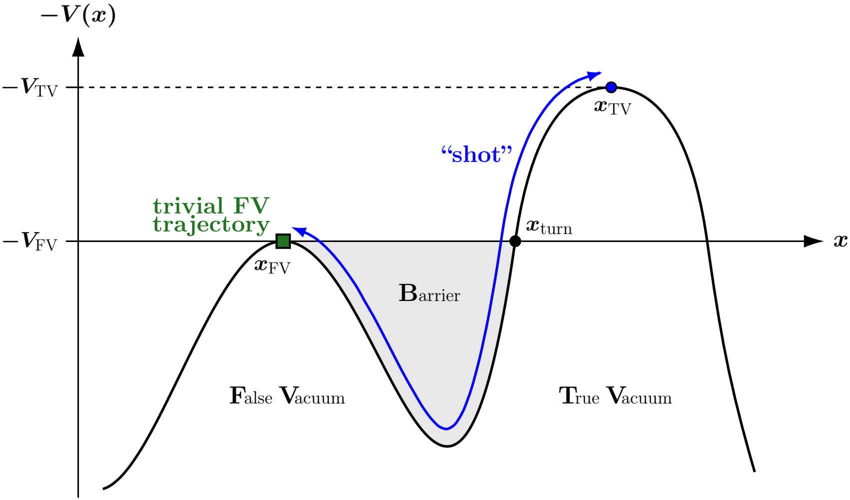

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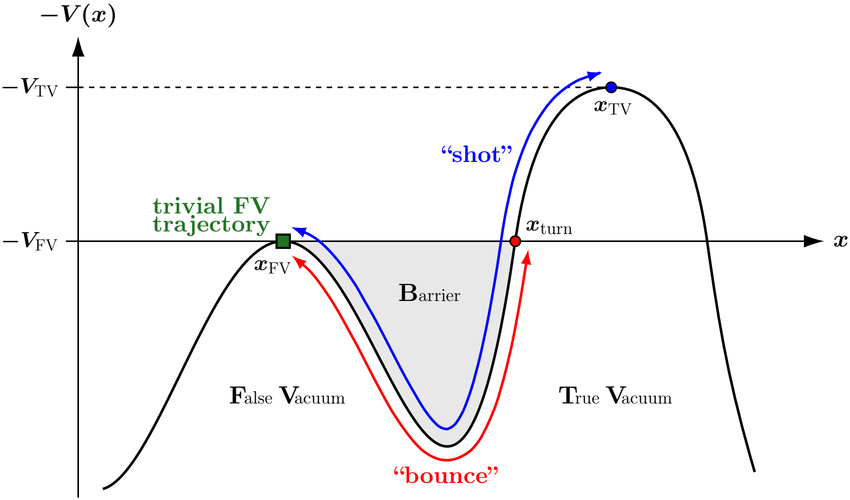

Coleman's conjecture

We require information about $E_0^{(\mathrm{loc})}$, so how is the relation

$$E_0^{(\mathrm{glob})} = -\hbar \lim_{T\to\infty} \left\{ T^{-1} \log \left[\underbrace{K_{\mathrm{E}}^{(\mathrm{shot})}(T)}_{\text{dominant}} + \underbrace{K_{\mathrm{E}}^{(\mathrm{FV})}(T) + \cancel{K_{\mathrm{E}}^{(\mathrm{bounce})}(T)}}_{\text{exponentially suppressed}} \right] \right\}$$

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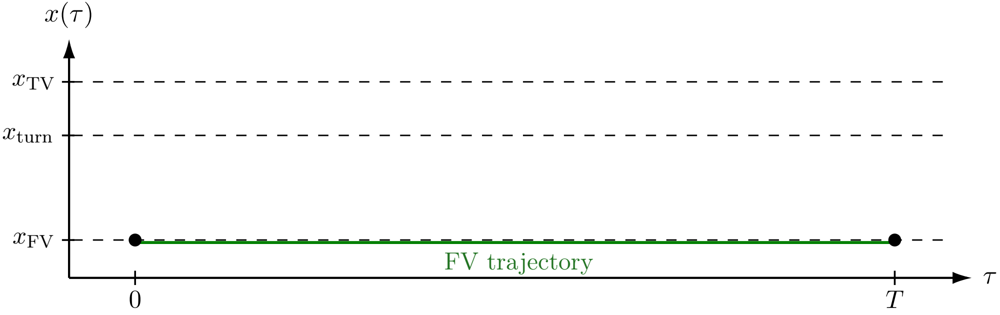

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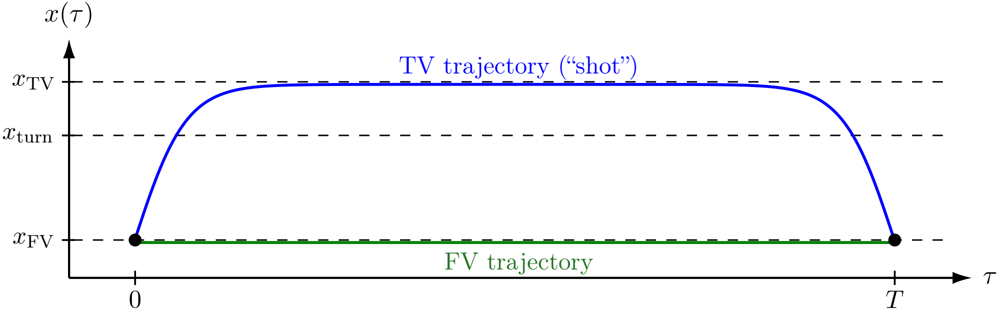

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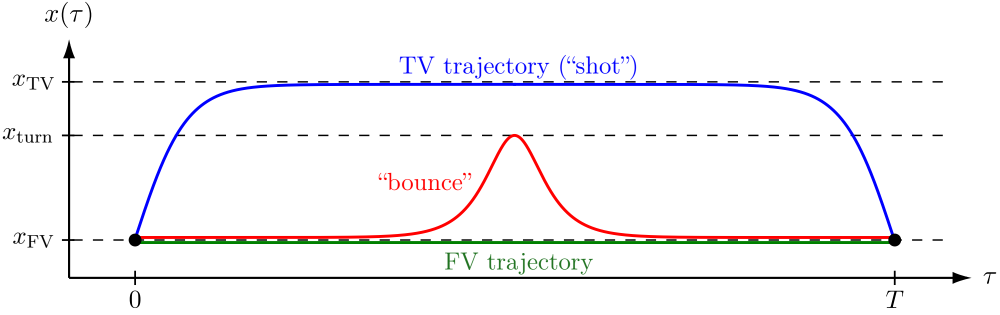

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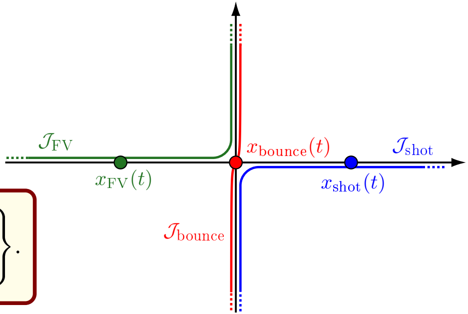

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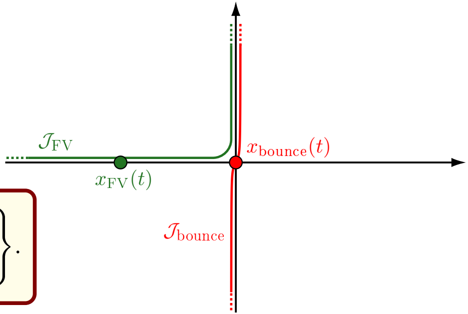

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Important: By virtue of analogy, this formula can be transferred to field theory!

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While incapacitating a straightforward transfer to QFT, we employ this ansatz to gain insights into the instanton method.

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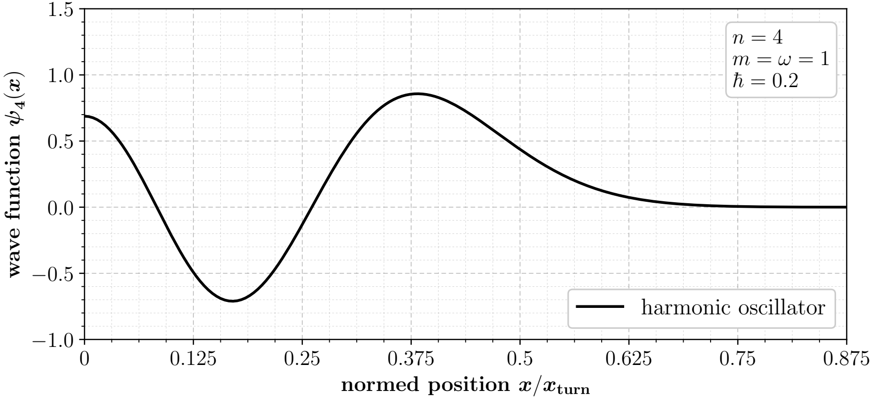

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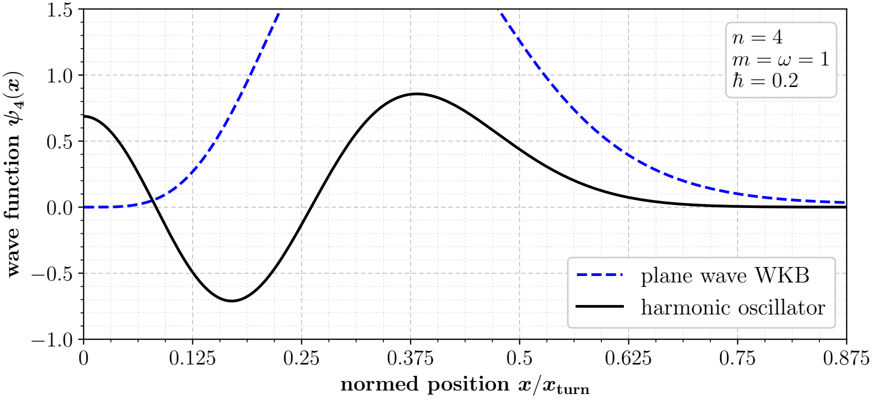

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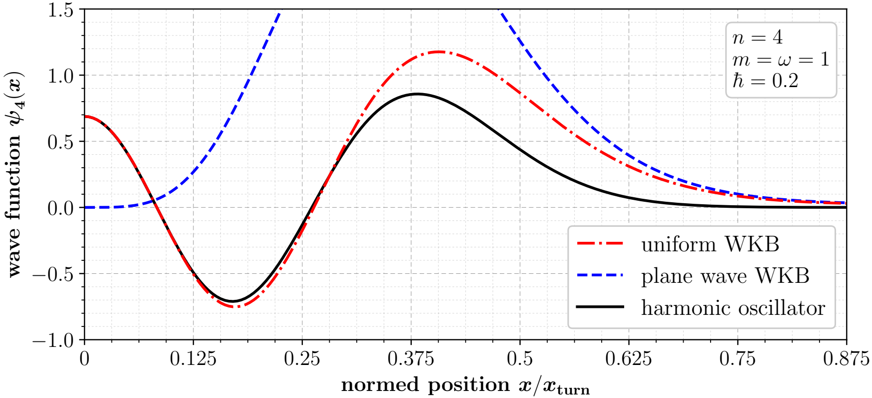

Local wave function

Upshot: Want to affirm the assertion

$$E_n^{(\mathrm{loc})} = -T^{-1}\hbar \, \log\left\{ \int_{-\infty}^{\infty} \mathrm{d}x_0 \int_{-\infty}^{\infty} \mathrm{d}x_T \; \overline{\psi_n^{(\mathrm{loc})}(x_T)} \, \psi_n^{(\mathrm{loc})}(x_0) \, K_{\mathrm{E}}\big(x_0, x_T\,; T\big) \right\}$$

for <u>arbitrary T</u>

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$$\psi_n^{(\mathrm{loc})}(x) = \text{complicated prefactor} \times \underbrace{\exp\left[-\frac{1}{\hbar}\int_0^x\sqrt{2mV(\xi)}\,\mathrm{d}\xi\right]}_{\text{usual WKB suppression factor}}\left[1+\mathcal{O}\big(\sqrt{\hbar}\big)\right].$$

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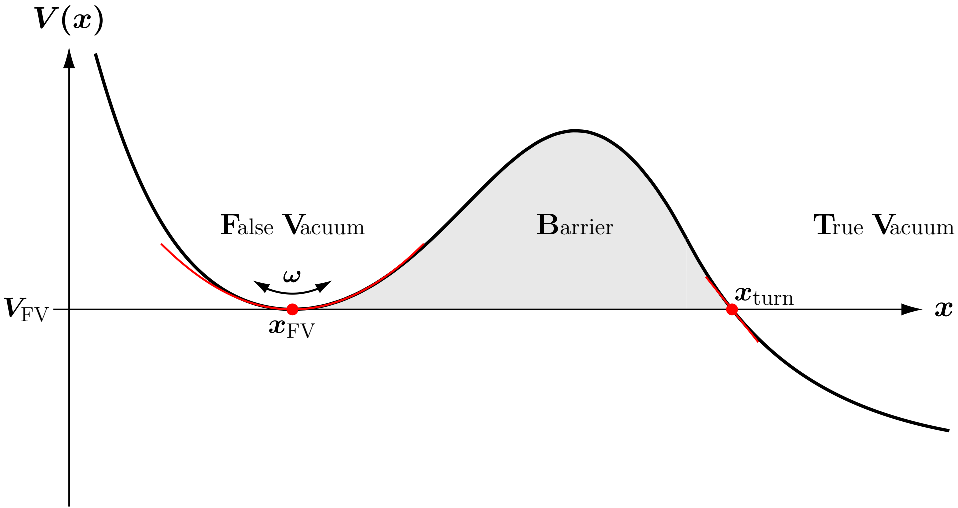
Note that the only n-dependence is inside the (non-exponential) prefactor.

Comparison to traditional WKB procedures



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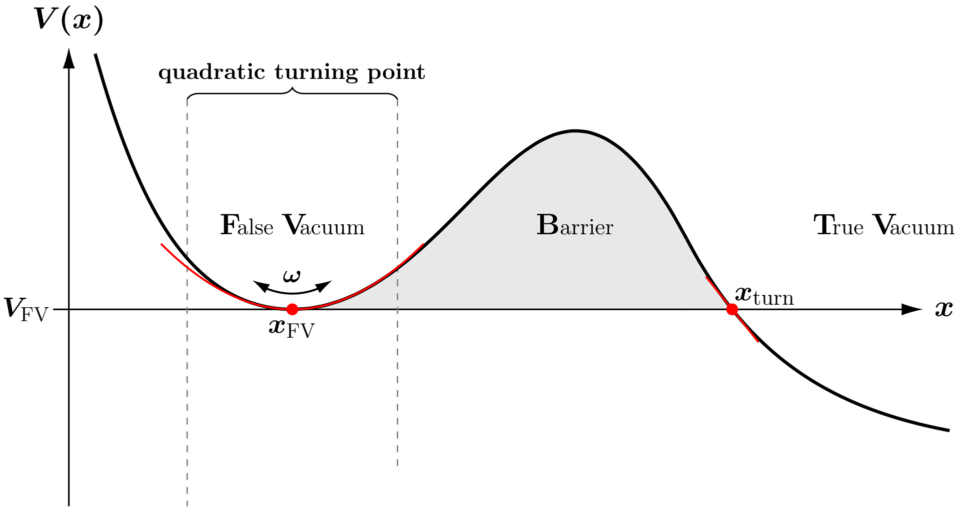

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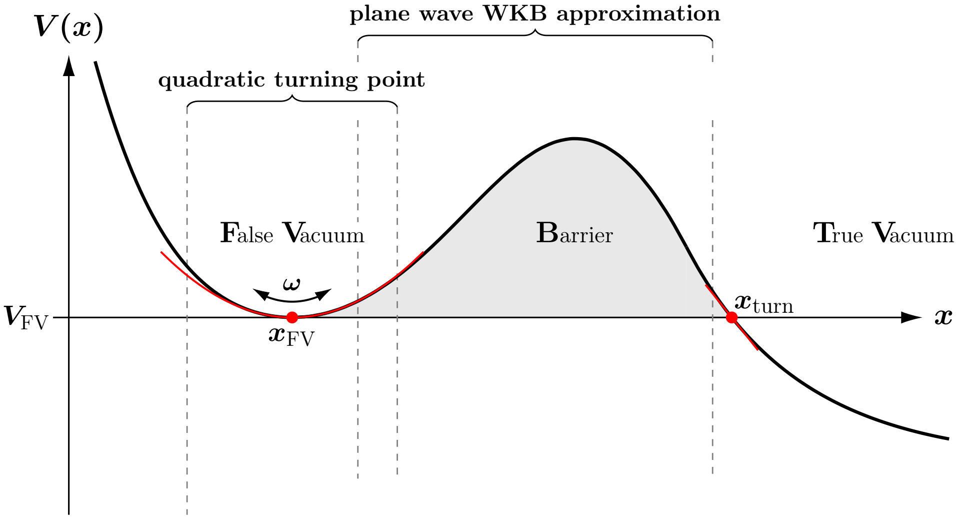

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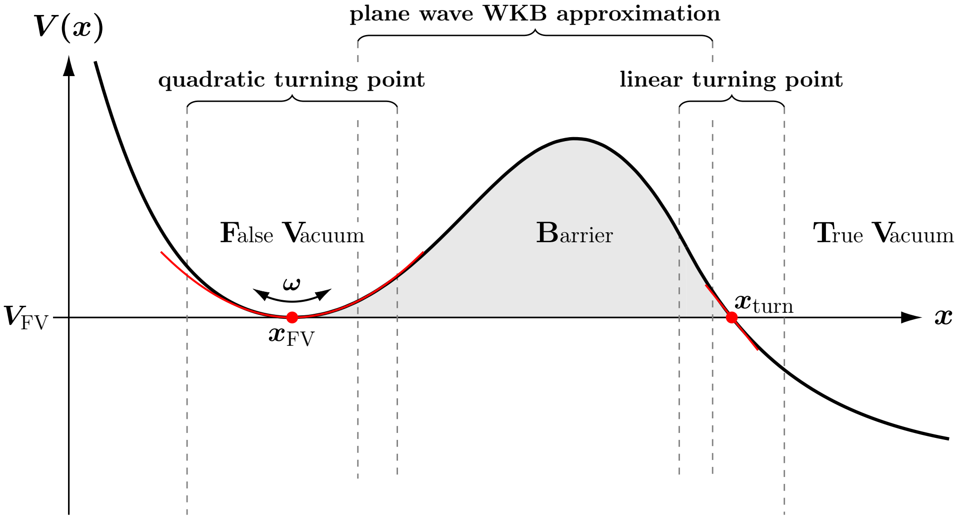

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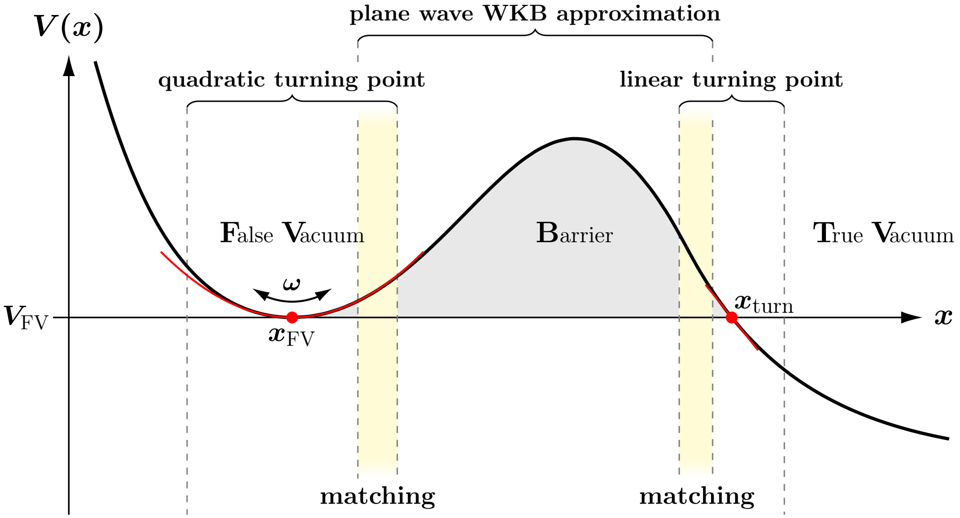
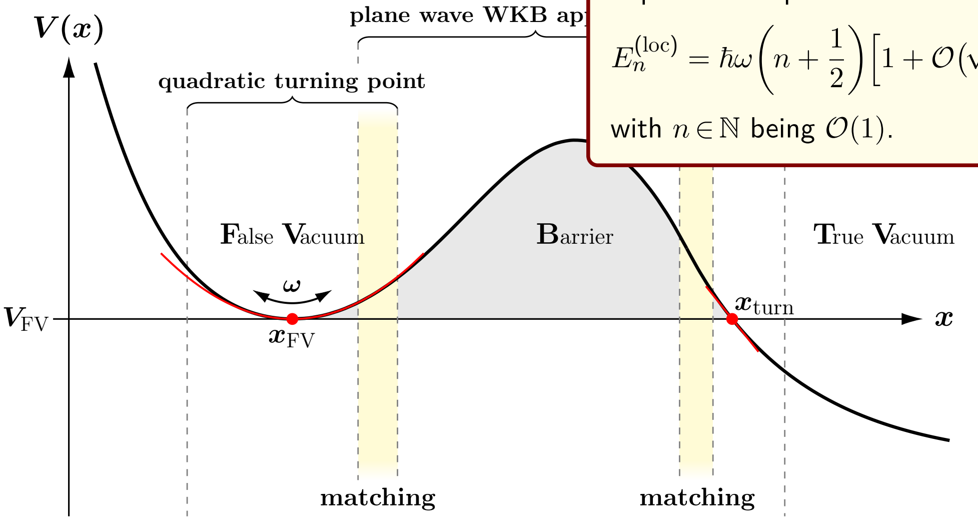


Comparison to traditional WKB procedures

Implicit assumption:

$$E_n^{(\text{loc})} = \hbar\omega\left(n+\frac{1}{2}\right)\left[1+\mathcal{O}(\sqrt{\hbar})\right]$$

with $n \in \mathbb{N}$ being $\mathcal{O}(1)$.



Methods of evaluation

There are two ways of computing the expression

$$\int_{-\infty}^{\infty} \mathrm{d}x_0 \int_{-\infty}^{\infty} \mathrm{d}x_T \; \overline{\psi_n^{(\mathrm{loc})}(x_T)} \, \psi_n^{(\mathrm{loc})}(x_0) \int_{x(0)=x_0}^{x(T)=x_T} \mathcal{D}_{\mathrm{E}}[\![x]\!] \, \exp\left(-\frac{S_{\mathrm{E}}[\![x]\!]}{\hbar}\right) .$$

Methods of evaluation

[8] Liang & Müller-Kirsten (1994), *PRD* vol. *50*(10)
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Sequential semi-classical evaluation of all integrals involved [8, 9]

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↘ Rewriting the expression into a single composite path integral

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Rewriting the expression into a single composite path integral

Method of choice, manifests symmetries

Evaluating an endpoint-weighted path integral

Let us see what changes when evaluating a composite path integral of the form

$$\int_{\mathcal{C}([0,T])} \mathcal{D}_{\mathrm{E}}[\![x]\!] \; \overline{\psi_n^{(\mathrm{loc})}[x(T)]} \; \psi_n^{(\mathrm{loc})}[x(0)] \; \exp\left(-\frac{S_{\mathrm{E}}[\![x]\!]}{\hbar}\right) .$$

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- Find the critical paths of the <u>full</u> exponent.
- Choose an appropriate function basis around the critical paths, expand the exponent to quadratic order and perform the Gaussian integration (beware of flat directions).
- Add all contributions with their appropriate weight.

Outline

Evaluating an endpoint-weighted path integral: Exponent

Notice: The exponent gets contributions from the local wave functions, seen best when splitting them as

$$\psi_n^{(\mathrm{loc})}(x) = \psi_{n,\text{non-exp}}^{(\mathrm{loc})}(x) \exp\left[-\frac{\psi_{\exp}^{(\mathrm{loc})}(x)}{\hbar}\right].$$

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Full exponent:
$$f_{\exp}[\![x]\!] = \underbrace{\int_0^T \frac{m\dot{x}(t)^2}{2} + V\big[x(t)\big]\,\mathrm{d}t}_{\text{Euclidean action } S_{\mathrm{E}}[\![x]\!]} + \underbrace{\psi_{\exp}^{(\mathrm{loc})}\big[x(T)\big] + \psi_{\exp}^{(\mathrm{loc})}\big[x(0)\big]}_{\text{supplementary boundary contributions from } \psi_n^{(\mathrm{loc})}}.$$

Evaluating an endpoint-weighted path integral: Exponent

Expanding around a critical trajectory $x_{\text{crit}}(t)$ yields the expansion

$$f_{\text{exp}}\left[\!\left[x_{\text{crit}} + \Delta x\right]\!\right] = f_{\text{exp}}[\![x_{\text{crit}}]\!]$$

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$$f_{\exp}\left[\!\left[x_{\text{crit}} + \Delta x\right]\!\right] = f_{\exp}[\![x_{\text{crit}}]\!] - \int_0^T \Big\{\underbrace{m\ddot{x}_{\text{crit}}(t) - V'\big[x_{\text{crit}}(t)\big]}_{\text{Euler-Lagrange equation}}\Big\}\,\Delta x(t)\,\mathrm{d}t$$

$$+ \Big\{\underbrace{m\dot{x}_{\text{crit}}(T) + \psi_{\exp}^{(\text{loc})\prime}\big[x_{\text{crit}}(T)\big]}_{\text{right transversality condition}}\Big\}\,\Delta x(T)$$

$$- \Big\{\underbrace{m\dot{x}_{\text{crit}}(0) - \psi_{\exp}^{(\text{loc})\prime}\big[x_{\text{crit}}(0)\big]}_{\text{left transversality condition}}\Big\}\,\Delta x(0)$$

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$$

Because the variations $\Delta x(t)$ are unconstrained at both temporal boundaries, we encounter additional transversality conditions.

Restrictions on critical paths

Inserting the ususal WKB supression factor for $\psi_{\text{exp}}^{(\text{loc})}$ yields

$$\dot{x}_{\text{crit}}(t)^2 = \frac{2}{m}\left\{V\left[x_{\text{crit}}(t)\right] + E_{\text{crit}}\right\},$$

(integrated) Euler-Lagrange equation

$$\dot{x}_{\text{crit}}(0) = \sqrt{\frac{2}{m}V\left[x_{\text{crit}}(0)\right]},$$

transversality condition at the left boundary

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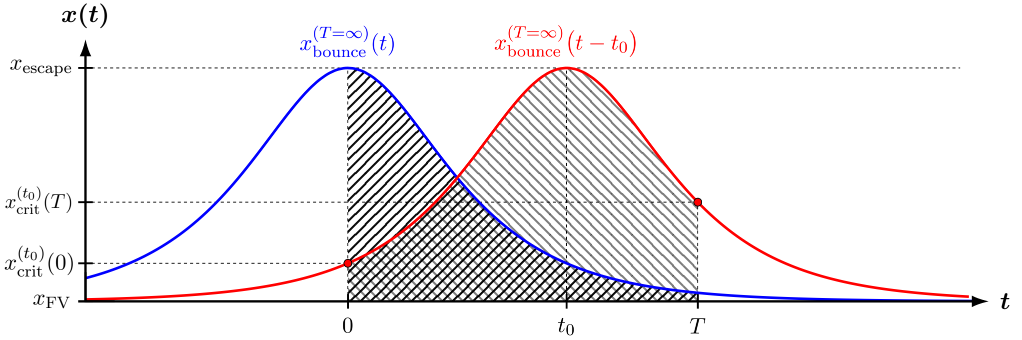
There are only two admissible solutions, namely

1. the trivial FV solution, and
2. (time-translated) bounce motions $\longrightarrow$ one-parameter family of critical paths.

Critical bounce trajectories

Admissible finite-time bounce motions are cutouts of the (time-translated) infinite-time bounce, defined as the representative that turns at exactly $t = 0$.



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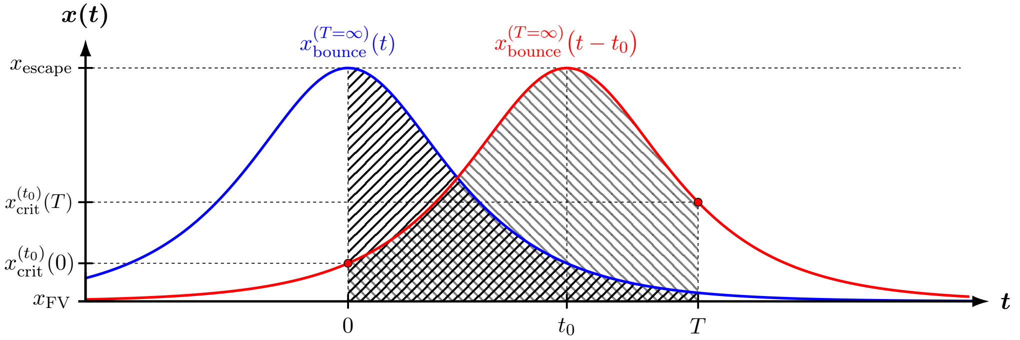

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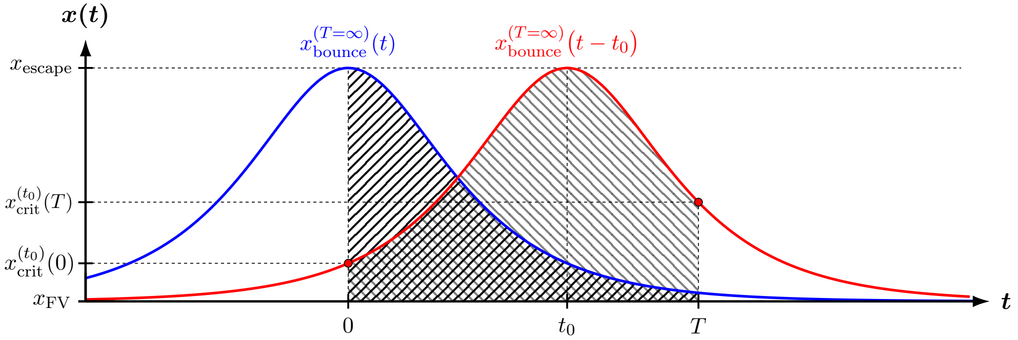

Important: $t_0 \in [0, T]$, otherwise the turning point is not traversed
$\longrightarrow$ volume of the instanton moduli space is exactly T

Exponent for bounce-like trajectories

The important observation is that the full exponent evaluated on the one-parameter family of bounce solutions is given by

$$f_{\exp}\left[\!\left[x_{\text{crit}}^{(t_0)}\right]\!\right] = \psi_{\exp}^{(\text{loc})}\left[x_{\text{crit}}^{(t_0)}(0)\right] + \psi_{\exp}^{(\text{loc})}\left[x_{\text{crit}}^{(t_0)}(T)\right] + S_{\text{E}}\left[\!\left[x_{\text{crit}}^{(t_0)}\right]\!\right]$$

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$$= \int_0^{x_{\text{crit}}^{(t_0)}(0)} \sqrt{2mV(\xi)}\,\mathrm{d}\xi + \int_{x_{\text{crit}}^{(t_0)}(0)}^{x_{\text{turn}}} \sqrt{2mV(\xi)}\,\mathrm{d}\xi + \left[x_{\text{crit}}^{(t_0)}(0) \leftrightarrow x_{\text{crit}}^{(t_0)}(T)\right]$$

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$$= 2\int_0^{x_{\text{turn}}} \sqrt{2mV(\xi)}\,\mathrm{d}\xi = S_{\text{E}}\left[\!\left[x_{\text{bounce}}^{(T=\infty)}\right]\!\right] .$$

This ensures the correct exponential suppression for arbitrary parameter T.

Outline

A glimpse at the fluctuation factor

The quadratic terms in the former expansion read

$$f_{\exp}^{(2)}\left[\!\left[x_{\text{crit}}+\Delta x\right]\!\right]=\frac{m\omega^2}{2}\int_0^T \Delta x(t)\underbrace{\left\{-\frac{\mathrm{d}^2}{\mathrm{d}(\omega t)^2}+\frac{V'\left[x_{\text{crit}}(t)\right]}{m\omega^2}\right\}}_{\text{fluctuation operator } O_{\text{crit}}}\Delta x(t)\,\mathrm{d}t$$

$$+\frac{1}{2}\underbrace{\left\{m\Delta\dot{x}(T)+\psi_{\exp}^{(\text{loc})\prime\prime}\left[x_{\text{crit}}(T)\right]\Delta x(T)\right\}}_{\text{(right) restriction on eigenfunctions}}\Delta x(T)$$

$$-\frac{1}{2}\underbrace{\left\{m\Delta\dot{x}(0)-\psi_{\exp}^{(\text{loc})\prime\prime}\left[x_{\text{crit}}(0)\right]\Delta x(0)\right\}}_{\text{(left) restriction on eigenfunctions}}\Delta x(0)\,.$$

A glimpse at the fluctuation factor

[10] Gel'fand & Yaglom (1960), *J. Math. Phys.* vol. *1(1)*
[11] Kirsten & McKane (2004), *J. Phys. A* vol. *37*

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$$-\frac{1}{2}\Bigg\{\underbrace{m\Delta\dot{x}(0) - \psi^{(\text{loc})\prime\prime}_{\exp}\left[x_{\text{crit}}(0)\right]\Delta x(0)}_{\text{(left) restriction on eigenfunctions}}\Bigg\}\Delta x(0)\,.$$

Sole difference to the usual discussion: Utilize Robin boundary conditions instead of Dirichlet ones for the determinant computation (generalized Gel'fand-Yaglom [10, 11]).

Final result for the decay width

[12] Levit, Negele, & Paltiel (1980), *PRC* vol. *22*(5)
[13] Weiss & Häffner (1983), *PRD* vol. *27*(12)

The aforementioned procedure exactly reproduces the well-known WKB result [12, 13]

$$\Gamma_n = -\frac{2}{\hbar}\,\mathrm{Im}\left[E_n^{(\mathrm{loc})}\right] = \frac{1}{n!}\left(\frac{2m\omega\mathcal{A}^2}{\hbar}\right)^n \underbrace{\sqrt{\frac{m\omega^3\mathcal{A}^2}{\pi\hbar}}\,\exp\left(-\frac{\mathcal{B}}{\hbar}\right)}_{\text{ground state decay width } \Gamma_0}.$$

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Reason: Projection is only valid up to terms of order $\sqrt{\hbar}$.

Concluding remarks

Key conclusion: Solving a composite path integral is in many situations advantageous to sequentially approximating the involved integrals.

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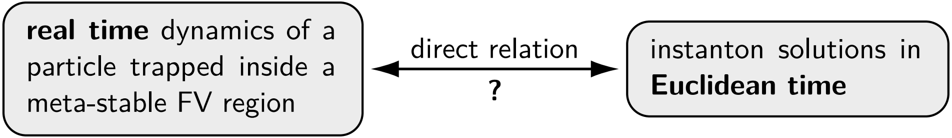

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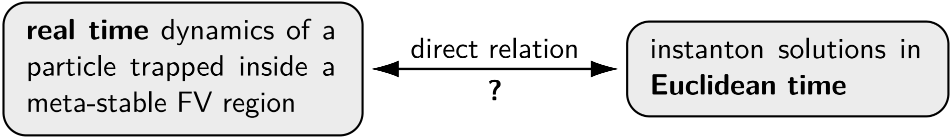

Thanks for your attention!