Nontrivial topology and Vacuum Energy in gauge theories

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- This talk is mostly based on several recent papers where we attempt to reveal the nature of vacuum energy in gauge theories in general and strongly coupled QCD in particular

- The key element in our studies is the so-called topological parameter θ

- The natural application of our our findings is cosmology as some observables such as vacuum energy $E(\theta)$ are highly sensitive to θ at <u>arbitrary large distances</u> even in a gapped theories such as QCD.

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Inflation and gauge field holonomy

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Cosmological magnetic field and dark energy as two sides of the same coin

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Few thoughts on θ and the electric dipole moments

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1. preliminary: energy in QCD

- We want to argue that there is a novel type of energy in strongly coupled QCD. This energy has "non-dispersive" nature, and can not be expressed in terms of conventional scattering amplitudes.

- It explicitly contradicts to the "folk theorem" that the S-matrix contains all the information about all physical observables.

- All these novel effects are due to the nontrivial topological sectors in the gauge systems and tunnelling transitions between them.

- The effect is non-local in nature, and can not be expressed in terms of local curvature in gradient expansion. It is expressed in terms of a non-local characteristics of the system -**the holonomy**.

- We apply these ideas to cosmological vacuum energy, de-Sitter space (Dark Energy, inflation)
- We want to test these ideas with explicit computations in Hyperbolic space and the so-called "deformed QCD" model.
- We want to argue that this portion of the QCD energy is amazingly close to the observed Dark Energy (DE) today.
- We also want to argue that the observed magnetic field (with huge correlation length on the level of 1 Gps) is the direct manifestation of this DE.

2. Topological susceptibility

- A convenient way to explain the nature of new type of vacuum energy is to study the topological susceptibility (*it is the key element in the resolution of the so-called U(1) problem in QCD, Witten, Veneziano, 1979*).

$$\chi_{YM} = \int d^4x \, \langle q(x), q(0) \rangle \neq 0 \qquad \frac{\partial^2 E_{\rm vac}(\theta)}{\partial \theta^2} = \chi_{YM}$$

To avoid confusion: This is the Wick's T-product, not Dyson's

- χ_{YM} does not vanish, though $q(x) \sim \partial_\mu K^\mu(x)$. It has ``wrong sign", see below. It can not be related to any physical propagating degrees of freedom. Furthermore, it has a pole in momentum space

$$\lim_{k\to 0} \int d^4x e^{ikx} \langle K_\mu(x), K_\nu(0) \rangle \sim \frac{k_\mu k_\nu}{k^4}$$

- There is a massless pole (Veneziano ghost), but there are no any physical massless states in the system.

$$\chi_{dispersive} \sim \lim_{k\to 0} \sum_n \frac{\langle 0|q|n\rangle\langle n|q|0\rangle}{-k^2 - m_n^2} < 0,$$

- Conventional physical degrees of freedom always contribute with sign (-) while one needs sign (+) to satisfy WI and resolve the U(1) problem

$$\chi_{non-dispersive} = \int d^4x \, \langle q(x), q(0)\rangle = \frac{1}{N^2}|E_{vac}| > 0$$

- Conventional terms (related to propagating degrees of freedom) always produce $\exp\left(-\Lambda_{QCD} L\right)$ behaviour at large distances.

- Witten simply postulated this term, while Veneziano assumed the unphysical field, the so-called the "Veneziano ghost" to saturate "wrong" sign in χ.

- In some models this contact non-dispersive term with "wrong" sign (+) can be explicitly computed. It is originated from the tunnelling effects between the <u>degenerate topological sectors of the theory</u>.

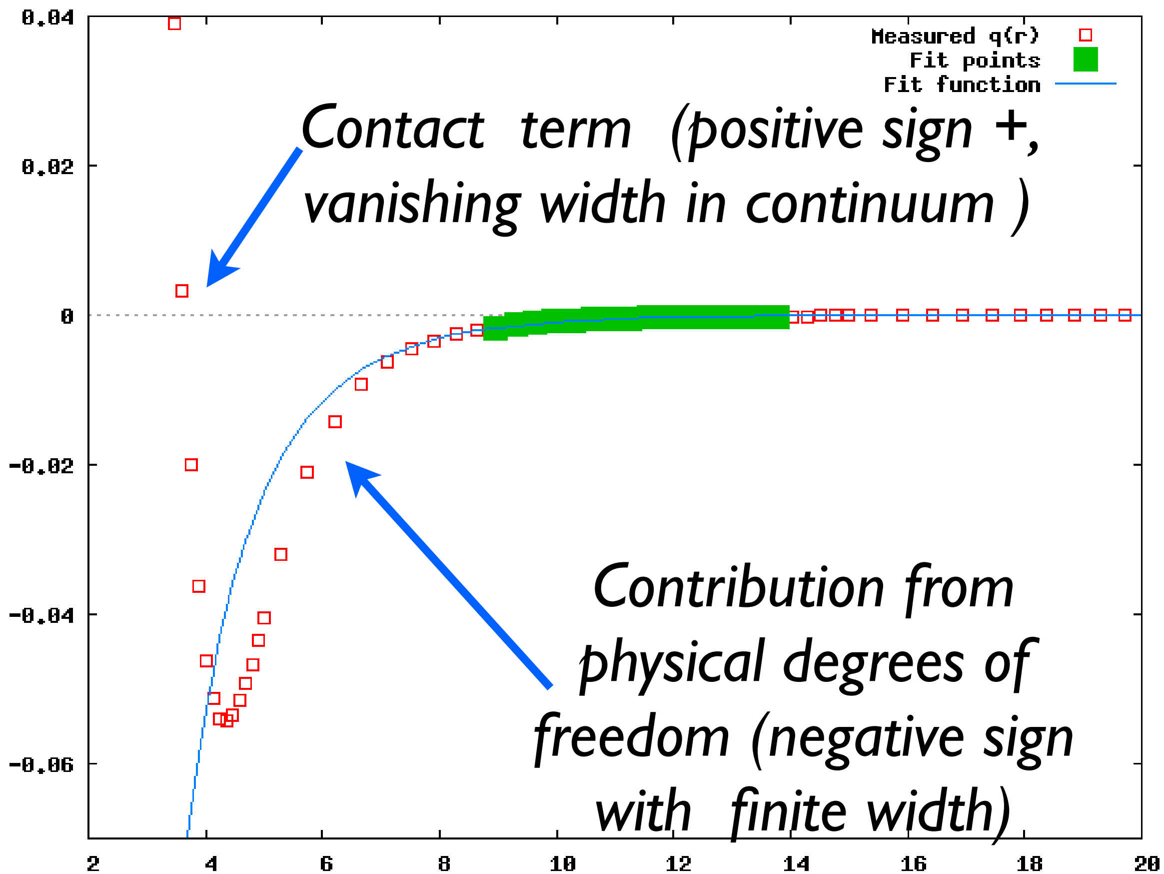
- These contributions can not be described in terms of conventional degrees of freedom (wrong sign);

- They are inherently non-local in nature as they are related to the tunnelling processes which are formulated in terms of the non-local large gauge transformation operator and holonomy;

- These terms may exhibit the long range features even through QCD has a gap (similar to the CM topological insulators);

- The effects have been explained in terms χ_{YM}. However, the θ -dependent portion of energy $E_{\rm vac}(\theta)$ (relevant for the cosmological applications) has all these unusual features due to the relation

$$\chi_{YM} = \frac{\partial^2 E_{\rm vac}(\theta)}{\partial^2 \theta}\Big|_{\theta=0}$$



The topological susceptibility $\chi(r)$ as a function of r. Wrong sign for χ is well established phenomenon; it has been tested on the lattice (plot above is from C. Bernard et al, LATTICE 2007). This $\chi(r=0)$ contribution is not related to any physical degrees of freedom, and can be interpreted as a contact term.

3. Warm up example: Maxwell system in 2D

- 2D Maxwell theory is exactly solvable model. It is an empty theory as it does not support any propagating DoF. Still, it has non-trivial dynamics.

- The partition function for θ vacua is known (in Hamiltonian approach):

$$E_n(\theta) = \frac{1}{2}\left(n + \frac{\theta}{2\pi}\right)^2 e^2 L, \qquad \mathcal{Z}(V,\theta) = \sum_{n\in\mathbb{Z}} e^{-\frac{e^2 V}{2}\left(n+\frac{\theta}{2\pi}\right)^2}$$

- We want to reproduce $\mathcal{Z}(V,\theta)$ using path integral computations as it can be generalized to 4d system. Instanton configurations, topological charge density Q, classical action are:

- $$\int \mathrm{d}^2x\, Q(x) = k, \quad eE^{(k)} = \frac{2\pi k}{V}, \qquad Q = \frac{e}{2\pi}E \qquad \frac{1}{2}\int d^2x E^2 = \frac{2\pi^2 k^2}{e^2 V}.$$

$$\mathcal{Z}(\theta) = \sum_{k\in\mathbb{Z}} \int \mathcal{D}A^{(k)} e^{-\frac{1}{2}\int d^2x E^2 + i\frac{e\theta}{2\pi}\int d^2x\ E(x)}$$

- Partition function is quadratic and can be easily evaluated. The Euclidean computations (with boundary conditions up to large gauge transformations) produce identically the same results as the Hamiltonian approach (with conventional periodic boundary conditions)

$$\mathcal{Z}(\theta) = \sum_{n\in\mathbb{Z}} e^{-\frac{e^2V}{2}\left(n+\frac{\theta}{2\pi}\right)^2} = \sqrt{\frac{2\pi}{e^2V}} \sum_{k\in\mathbb{Z}} e^{-\frac{2\pi^2k^2}{e^2V} + ik\theta}.$$

- Topological susceptibility is <u>finite</u> (in infinite volume limit) and saturated by very large topological winding numbers, $k \sim \sqrt{e^2V} \gg 1$. Infrared regularization (boundary conditions) are essential at every step (I.Sachs , A. Wipf)

$$\chi \equiv \lim_{k\to 0} \int \mathrm{d}^2x\ e^{ikx} \langle TQ(x)Q(0)\rangle = -\frac{1}{V}\cdot \frac{\partial^2 \ln \mathcal{Z}(\theta)}{\partial\theta^2}\Big|_{\theta=0} = \frac{e^2}{4\pi^2}.$$

- The integrand $(\delta^2(x) - \text{function})$ for the topological susceptibility is saturated by uniform fluxes filling the entire space-time volume (IR not UV physics). This "non-dispersive" contact term is not related to any propagating degrees of freedom

$$\langle Q(x)Q(0)\rangle = \frac{e^2}{4\pi^2}\delta^2(x). \qquad \int d^2x \langle Q(x), Q(0)\rangle \sim \int d^2x\delta^2(x) \sim \int d^2x\ \partial_\mu \left(\frac{x_\mu}{x^2}\right)$$

- Is this contact non-dispersive contribution to the vacuum energy physically observable?

- The ultimate answer is "yes" as the anomalous Ward Identities $\chi = 0$ (when physical massless fermions are introduced into the system) can be only satisfied if the contact term is not zero.

$$\chi = \frac{e^2}{4\pi^2}\int d^2x \left[\delta^2(x) - \frac{e^2}{2\pi^2}K_0(\mu|x|)\right] = \frac{e^2}{4\pi^2}\left[1 - \frac{e^2}{\pi}\frac{1}{\mu^2}\right] = \frac{e^2}{4\pi^2}[1-1] = 0.$$

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3. Definition of the Gravitating Energy

- We assume that the relevant (gravitating) energy which enters the Friedman's equation is the difference $\Delta E = (E_{\mathrm{FLRW}}(H) - E_{\mathrm{Mink}})$ similar to computations of the Casimir energy, when the difference ΔE is observed. This assumption was, in fact, originally formulated by Zeldovich in 1967.

- We can not (by technical reasons) to perform the computations in FLRW background. However, we can proceed with computations in a toy model formulated on hyperbolic space $\mathbb{H}^3_\kappa \times \mathbb{S}^1_{\kappa^{-1}}$ when role of $H \sim 10^{-33}\ eV$ plays IR cutoff parameter $\kappa \to 0$

- We want to argue that a nontrivial holonomy generates a linear correction $\Delta E(\kappa) \sim \kappa$ in contrast with conventional expectation $\Delta E \sim R\,[\mathbb{H}^3_\kappa] \sim \kappa^2$

- Technically, we want to see a linear (rather than very small quadratic κ^2) correction in the ratio

$$\frac{E_{\rm vac}[\mathbb{H}^3_\kappa \times \mathbb{S}^1_{\kappa^{-1}}]}{E_{\rm vac}[\mathbb{R}^3 \times \mathbb{S}^1]} \simeq 1 + \mathcal{O}\left(\frac{\kappa}{\Lambda_{QCD}}\right).$$

- If the same pattern persists in real FLRW Universe one could estimate (see slides below)

$$\Delta E_{\rm vac} \sim \kappa \sim L^{-1} \sim \Lambda^4_{QCD}\left(\frac{1}{\Lambda_{QCD} L}\right) \sim (10^{-3}{\rm eV})^4$$

- In other words, we interpret the observed Dark Energy as a modification of the QCD vacuum $\sim \kappa$ energy due to a nontrivial topology (not expressible in terms of local curvature $R\,[\mathbb{H}^3_\kappa] \sim \kappa^2$)

- It has the same non-dispersive nature (can not be expressed in terms of propagating DOF), it is non-local in nature (not expressible in terms of the local curvature), and it has a positive sign.

- Historical comments: many people from different fields had advocated (after *Zeldovich, 1967*) a similar idea on the RHS for the Friedman's equation

$$\Delta E(L) = [E(L) - E_{\mathrm{Mink}}]$$

- $$E(L) \equiv -(\beta V)^{-1} \ln \mathcal{Z}$$

- *James Bjorken (particle physics), 2001,*

  Ralf Schuetzhold (GR), PRL, 2002;

  Grisha Volovik (CM physics), 2008 +many more

- I personally adopted this idea in 2009, mostly due to the intense (and never ending) discussions with Grisha Volovik in the relation with his COSLAB (Cosmology in a Laboratory) activities.

4. Holonomy and the linear correction $\kappa \sim 1/\mathcal{T}$ in hyperbolic space $\mathbb{H}^3_\kappa \times \mathbb{S}^1_{\kappa^{-1}}$

- Normally it is expected that all corrections due to the time-dependent (curved) background are proportional to the local curvature $R\left[\mathbb{H}^3_\kappa\right] \sim \kappa^2$

- We want to test these ideas in gauge theories with nontrivial holonomy. In this case corrections are not reduced to the local observables. The IR regularization plays key role in all computations.

- Specifically, we compute the ratio which explicitly shows the linear correction $\sim \kappa$ (IR cutoff)

$$\frac{E_{\rm vac}[\mathbb{H}^3_\kappa \times \mathbb{S}^1_{\kappa^{-1}}]}{E_{\rm vac}[\mathbb{R}^3 \times \mathbb{S}^1]} \simeq \left(1 - \frac{\nu\bar{\nu}}{2} \cdot \frac{\kappa}{\Lambda_{QCD}}\right). \quad E_{\rm vac} \equiv -\frac{1}{\beta V} \ln \mathcal{Z}$$

- The computations are based on KvBLL calorons with nontrivial holonomy (Kraan-van Baal-Lee-Lu)

$$\frac{1}{2}TrL = \frac{1}{2}Tr\mathcal{P}\exp\left(i\int_0^\beta dx_4 A_4(x_4, |\mathbf{x}| \to \infty)\right) = \cos(\pi\nu)$$

- Normally, nontrivial holonomy ($\nu \neq 0, 1$) generates zero contribution to the partition function in thermodynamical limit. However, the KvBLL configurations are known to generate IR -finite contribution to the free energy (in huge contrast with conventional instantons).

- The KvBLL configurations can be thought as a superposition of "N" different monopoles which carry the fractional topological charge $Q = \pm 1/N$

- Confinement can be understood as **percolation** of these fractionally charged monopoles which enter the partition function in sets of "N" .

- The crucial role in generating this result is zero-mode determinant. These modes are drastically different in hyperbolic and in Euclidean spaces.

- This difference in these two cases is determined by asymptotic behaviour (different cutoff: $v \longleftrightarrow \rho$)

$$A_4^M(r) = \left(v \coth(vr) - \frac{1}{r} \right) \frac{\tau^3}{2} \qquad \text{on} \qquad \mathbb{R}^3$$

$$A_\chi^M(\rho) = \Big((\nu+1) \coth\left[(\nu+1)\kappa\rho\right] - \coth \kappa\rho \Big) \frac{\kappa\tau^3}{2} \quad \text{on} \quad \mathbb{H}^3_\kappa$$

- Eventually, this difference translates into the difference in fugacities (and <u>vacuum energies</u>) as claimed above

$$f^2 = \left[\frac{4\pi\beta\Lambda^4_{QCD}}{g^4} \right]^2 \cdot \langle \frac{[1 + 2\pi\nu\bar{\nu}\frac{r_{12}}{\beta}]}{(\Lambda_{QCD}\ r_{12})^{2/3}} [1 + 2\pi\nu\frac{r_{12}}{\beta}]^{\frac{8}{3}\nu - 1} [1 + 2\pi\bar{\nu}\frac{r_{12}}{\beta}]^{\frac{8}{3}\bar{\nu} - 1} \rangle$$

- Though linear terms $\sim r_{12}/\beta$ can be modified as a result of interactions, they cannot be exactly cancelled. The effect is proportional to holonomy $\nu(1-\nu)$, and vanishes for conventional instantons with $\nu = 0, 1$

- With these assumptions the non-dispersive correction to the energy (at very small $\kappa \to 0$) is

$$\frac{E_{\rm vac}[\mathbb{H}^3_\kappa \times \mathbb{S}^1_{\kappa^{-1}}]}{E_{\rm vac}[\mathbb{R}^3 \times \mathbb{S}^1]} \simeq \left(1 - \frac{\nu\bar{\nu}}{2} \cdot \frac{\kappa}{\Lambda_{QCD}}\right).$$

- For the same effect to persist in our FLRW Universe one should have a nontrivial part $\mathbb{S}^1_{\kappa^{-1}}$ (spatial or temporal) along which the holonomy to be computed. The $\mathbb{S}^1_{\kappa^{-1}}$ has Universe size $L = 2\pi\kappa^{-1}$

- No contradictions with observations as long as size is sufficient large $L \geq H^{-1}$, in which case the DE is

$$\Delta E_{\rm vac} \sim \kappa \sim L^{-1} \sim \Lambda^4_{QCD}\left(\frac{1}{\Lambda_{QCD} L}\right) \sim (10^{-3}{\rm eV})^4$$

- Q: *How a system with a gap could be ever sensitive to arbitrary large distances?*

- A1: The long range order in gapped QCD is similar to Aharonov -Casher effect. If one inserts an external charge into superconductor when electric field is screened $\exp(-r/\lambda)$ a neutral magnetic fluxon will be still sensitive to external charge at arbitrary large distances.

- A2: <u>Long range order</u> in the system emerges because the large gauge transformation operator and holonomy are non-local operators sensitive to far IR-physics, similar to "modular operator" in Aharonov -Casher effect.

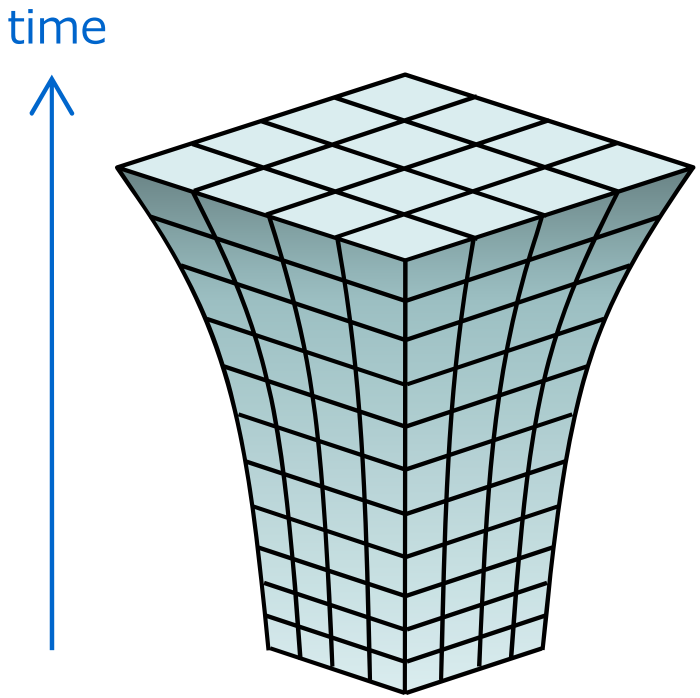
- Are there other hints on a linear dependence on cosmological scale $L \sim H^{-1}$ in a gapped system? (locality suggests quadratic behaviour as $R \sim H^2$)

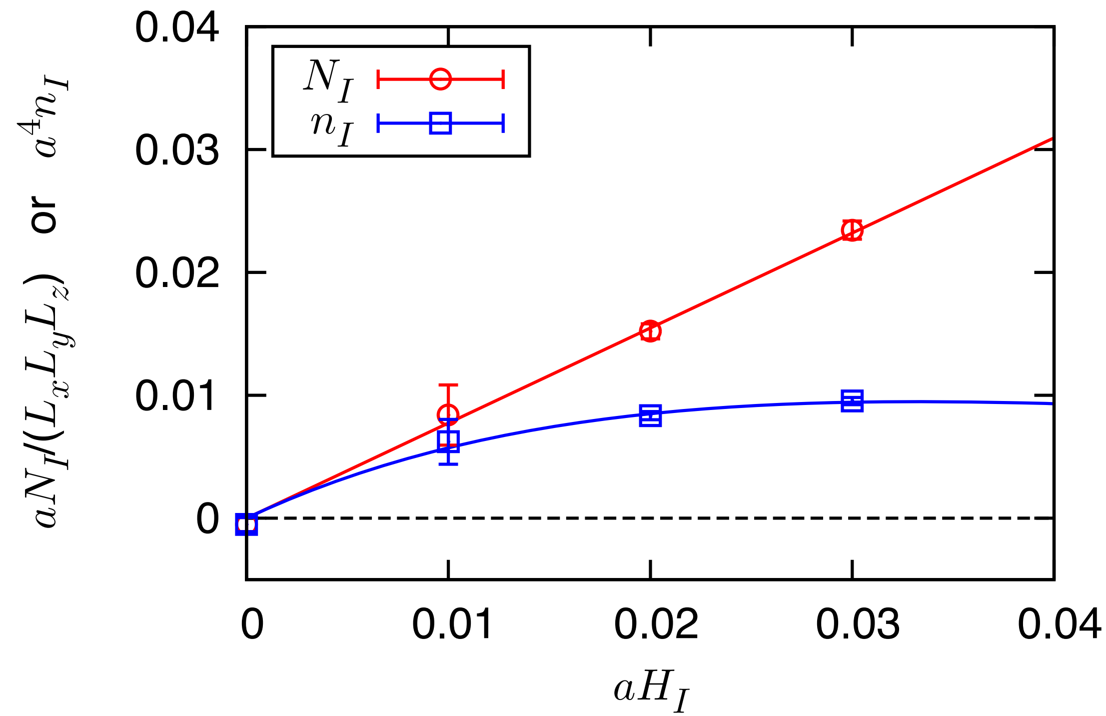
- 1. A number of analytical computations in some simplified models (e.g. deformed QCD).

- 2a. Lattice numerical simulations. In this case the computations of a real part of the energy -momentum tensor $Re\langle T_{\mu\nu}\rangle$ is a hard problem.

- 2b. However, the imaginary (absorptive) portion of the energy-momentum tensor $Im\langle T_{\mu\nu}\rangle$ due to particle production, can be computed, see plot below.

- 2c. Analyticity suggests that the dependence on H must be the same in $Re\langle T_{\mu\nu}\rangle$ and $Im\langle T_{\mu\nu}\rangle$





The plots from A. Yamamoto, arxiv 1405.6665.

1. The expansion in Euclidean space-time was parametrized by the "imaginary" Hubble constant when the lattice action is positively defined;
2. Red curve describes the particle production rate per unit volume per unit time in the background H_I ;
3. The linear dependence on H_I has been computed, $Im[\langle T_{\mu\nu}\rangle] \sim H_I$. It strongly supports our arguments.

5. Applications to the Dark Energy (vacuum energy today)

- Lesson 1: There is a fundamentally new type of the vacuum energy which can not be expressed in terms of any local field.
- Lesson 2: It emerges as a result of tunnelling processes between degenerate topological sectors, and formulated in terms of the "non-dispersive" contact terms and nonlocal holonomy.
- Lesson 3: This term cannot be expressed in terms of a gradient expansion in effective QFT as it is sensitive to arbitrary large parameters $\kappa \sim 1/\mathcal{T}$
- Lesson 4: We identify this new type of energy with cosmological vacuum dark energy (DE) today.

- The obtained relevant parameters are amazingly close to the observed DE values:

$$\mathcal{T}^{-1} \sim H \sim \frac{\Lambda^3_{QCD}}{M^2_{PL}} \sim 10^{-33} eV, \quad \rho_{\rm DE} \sim H\Lambda^3_{QCD} \sim (10^{-3} eV)^4, \quad \mathcal{T} \sim H^{-1} \sim \frac{M^2_{PL}}{\Lambda^3_{QCD}} \sim 10\,{\rm Gyr},$$

- Parameter $\mathcal{T}$ should be thought as IR cutoff in computations in strongly coupled QCD. Contribution to DE is expressed in terms of $\Lambda_{\rm QCD}$

- This energy will be eventually transferred to the Maxwell EM fields (so-called helical instability) on the time scale of $\alpha^{-2} H^{-1} \sim \alpha^{-2} 10^{10}$ years

- As a direct consequence of this energy transfer the large scale magnetic field (with correlation length of entire visible Universe) will be generated. Apparently such enormous correlation scale has been observed (see slides below)

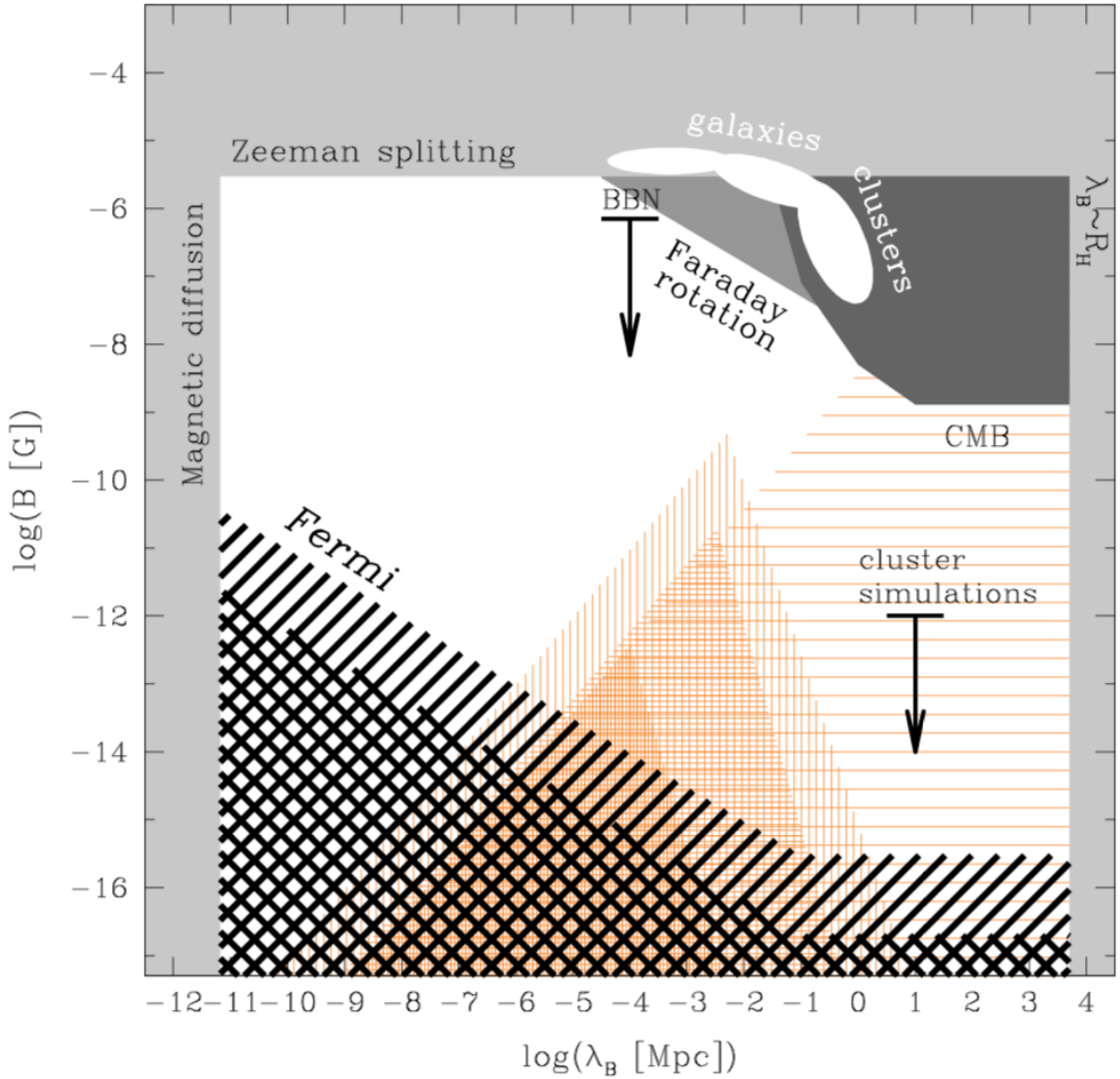
6. Large Scale Magnetic field

- We want to argue that similar to reheating epoch (as discussed above) which happened after inflation with $\bar{\Lambda}_{\rm QCD} \gg M_{\rm EW}$ is happening today when DE transfers its energy to magnetic field energy

- It is known-the B field correlated on enormous (Gpc) scale must exist, see slide below

- We advocate unorthodox mechanism which is dramatically different from all previous approaches: the B- field is generated with enormous coherence scale from DE now (not at earlier times, phase transitions, inflation etc)

- No need for any amplification mechanisms as it is characterized by the largest possible scale at the moment of formation, i.e. today



Constraints on the B field [From Neronov & Vovk]. The B-field correlated on Gpc scales must exist: $10^{-15} G \lesssim B \lesssim 10^{-9} G$

- The starting point is the conventional Effective Lagrangian in terms of the auxiliary field $b(x, H)$

$$\mathcal{L}_{b\gamma\gamma}(x) = \frac{\alpha}{4\pi} N \frac{\sum_i Q_i^2}{N_f} \left[\theta + b(x, H)\right] \cdot F_{\mu\nu} \tilde{F}^{\mu\nu}(x)$$

- It generates well known extra term with $\mu_5 \sim H_0$

$$\vec{\nabla} \times \vec{B} = \sigma \vec{E} + \frac{\alpha}{2\pi} N \frac{\sum_i Q_i^2}{N_f} \cdot \left(\mu_5 \vec{B}\right), \quad \mu_5 \equiv \langle \dot{b}(x, H) \rangle$$

- Similar equations have been studied before (e.g. dynamical axion field). The difference here is that the μ_5 does not satisfy any classical equation of motion as there is no canonical kinetic term for auxiliary field $b(x, H)$. The μ_5 is background field

- The $b(x, H)$ field was introduced as the Lagrange multiplier to account for tunnelling events

- It is known that the presence of the μ_5 term leads to the helical instability. In the present context it implies the generation of the magnetic field on the huge scales where μ_5 is correlated.

- The instability develops for large wavelengths:

$$B(t) = B_0 \exp(\gamma t), \quad k < \frac{\alpha}{\pi}\mu_5, \quad \mu_5 \sim H$$

- This effect leads to the generation of the <u>magnetic</u> <u>field</u> correlated on the enormous scales.

- The order of magnitude estimates suggest (present time)

$$B \sim 10^{-10} G$$

Concluding comments on Dark Energy & B field

- QCD vacuum energy is different for different background (Minkowski vs deSitter). This difference generates correct order of magnitude for the **observed** (gravitating) DE today

$$\rho_{\rm DE} \equiv \Delta E \equiv \left(E_{\rm deSitter} - E_{\rm Mink}\right)$$

$$H \sim \frac{\Lambda^3_{\rm QCD}}{M^2_{\rm PL}} \sim 10^{-33}{\rm eV}, \qquad \rho_{\rm DE} \sim H\Lambda^3_{\rm QCD} \sim (10^{-3}{\rm eV})^4$$

- The vacuum energy will be eventually transferred to the magnetic energy in $\alpha^{-2}H^{-1}{\rm years}$

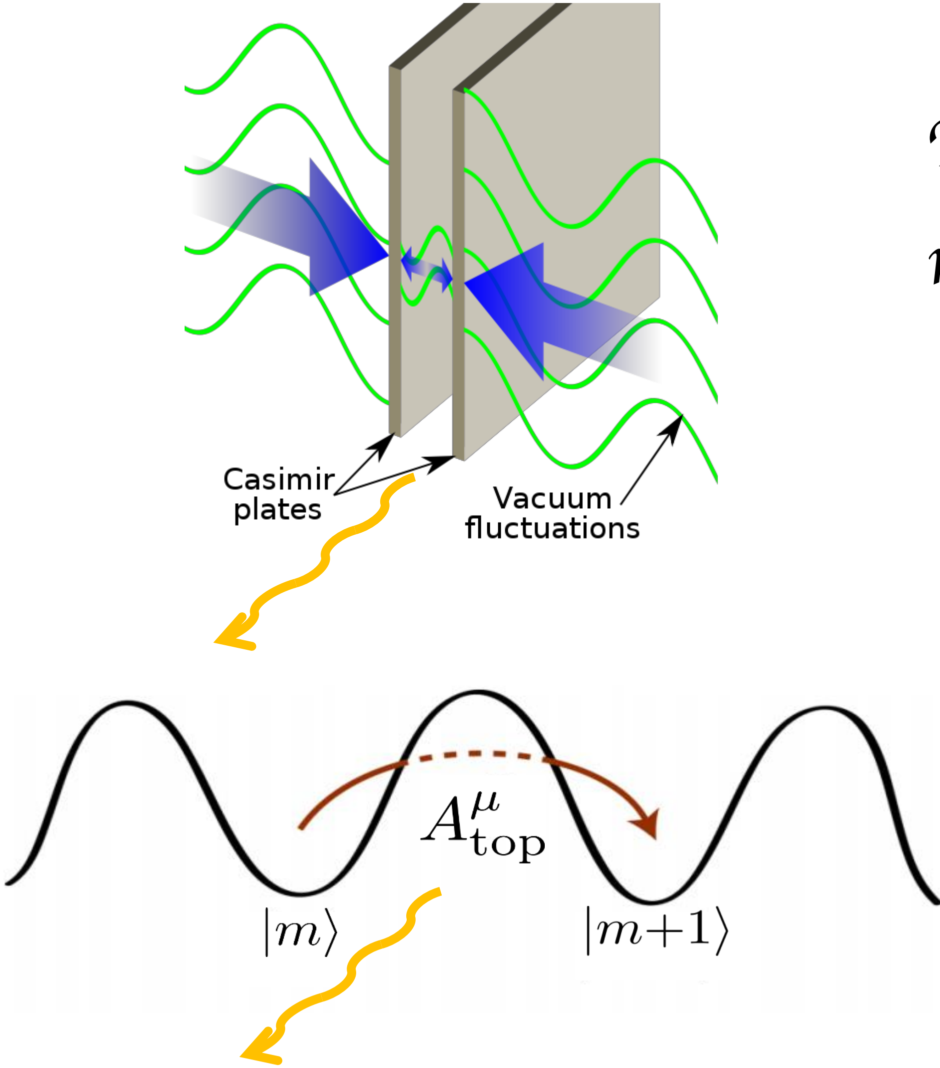
- The magnetic field at present time could be large: $B \sim 10^{-10}G$. It must be correlated on the scale of the entire visible Universe

Proposal: Instead of theoretical speculations I suggest to conduct a real tabletop experiment to study this new type of energy:

- When the Maxwell system is formulated on four-torus there will be an <u>extra contribution</u> to the Casimir pressure, not related to the physical propagating photons with two transverse polarizations (4-torus has nontrivial holonomy).

- This setting based on 4-torus topology should be contrasted with conventional setting when the Casimir energy is generated between two conducting plates (trivial holonomy).

- The Maxwell system on the 4-torus shows all signs (degeneracy, etc) which are normally attributed to the topologically ordered systems.



This picture illustrates a new physics phenomenon

The emission of real physical photons as a result of tunnelling transitions in time dependent background is precisely the effect discussed in this talk: the cosmological magnetic field is generated as a result of variation of the vacuum tunnelling transitions

Extra Slides:

1. on physical meaning of θ_{QED}
2. on physical meaning of $\mathbb{S}^1$

9. On physical meaning of θ_{QED}

- It is known that θ_{QCD} is the physical fundamental parameter of the system on the non-perturbative level

- The θ_{QCD} does not enter the equations of motion as the topological density $\mathrm{Tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$ is the total derivative

- It is normally argued (correctly) that due to the nontrivial mapping $\pi_3[SU(3)] = \mathbb{Z}$ (presence of the instantons) the θ_{QCD} is physical parameter

- It is also commonly argued (incorrectly) that due to triviality of the mapping $\pi_3[U(1)] = 0$ the θ_{QED} is an unobservable parameter of the theory

- While the argument is indeed correct for a topologically trivial vacuum background in 4d Minkowski space-time, it has been known that $\theta_{\rm QED}$ is a physical parameter of the system formulated on a non-simply connected manifold with $\pi_1[U(1)] = \mathbb{Z}$

- A simple realization for such a manifold is the presence of a magnetic field in the system

- A well known realization for $\theta_{\rm QED}$ to become a physical parameter is the Witten's effect when electric charge is induced in the background of magnetic monopole

$$e' = -(e\theta_{\rm QED}/2\pi) \quad e \equiv \frac{2\pi}{g}$$

- The effect is proportional to $\theta_{\rm QED}$ itself, and not to the derivative $\dot{\theta}_{\rm QED}$ which enters the Maxwell equations:

$$\vec{j}_a = -\dot{\theta}_{\rm QED} \, \frac{\alpha}{2\pi} \, \vec{B}, \quad \alpha \equiv \frac{e^2}{4\pi}$$

- There are many topological phenomena when θ_{QED} becomes the physically observable parameter. For example, electric field will be induced in the background of magnetic field

$$\langle\vec{E}\rangle_{\mathrm{ind}} = -\frac{\alpha\theta_{\mathrm{QED}}}{\pi}\vec{B}_{\mathrm{ext}}$$

- It can be interpreted as charge separation effect along the magnetic field. Similarly one can argue that electric and magnetic dipole moments are related (proper path-integral computations on $\mathbb{T}^4$):

$$\langle d_{\mathrm{ind}}\rangle = -\frac{\theta_{\mathrm{QED}}\cdot\alpha}{\pi}\langle m_{\mathrm{ind}}\rangle, \quad \alpha \equiv \frac{e^2}{4\pi}$$

- The last equation is exactly reduced to the Witten's formula after identifications

$$\langle m_{\mathrm{ind}}\rangle = gL_3, \quad \langle d_{\mathrm{ind}}\rangle = e'L_3, \quad eg = 2\pi, \quad e' = -(e\theta_{\mathrm{QED}}/2\pi)$$

- Can we apply this relation to the fundamental fields such as electrons, protons, neutrons?

$$\langle d_{\mathrm{ind}}\rangle = -\frac{\theta_{\mathrm{QED}}\cdot\alpha}{\pi}\mu \;\; \text{for} \;\; \mu = \mu_e, \mu_p, \mu_n?$$

- We think the answer is "yes" as the magnetic dipole can be thought as the monopole- anti-monopole pair when the Witten's effect is operational.

- Few historical remarks are in order:

- Similar relation was derived by C.Hill (2015) assuming the time dependent axion background $\theta(t)$. There are many subtle points (gauge fixing, ghost-behaviour) in computations as the perturbation theory obviously produces a trivial vanishing result as $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative

- Flambaum et al (2017) have criticized Hill's computations based on definitions of static limit

- Our formula identically coincides with Hill's expression in the static limit $\theta(t) \to \theta_{\rm QED}$

Concluding comments on abelian θ_{QED}

- 1. Deep mathematical reason for θ_{QED} to become a physically observable parameter is that the gauge cannot be uniquely fixed even in abelian gauge theory on a nontrivial manifold (Gribov's copies emerge in QED, similar to QCD)

- 2. The constraint appears $\theta_{QED} \lesssim 10^{-16}$ if one uses formula (path integral computations), which agrees with Hill's formula in static limit

$$d_e = -\frac{\theta_{\mathrm{QED}} \cdot \alpha}{\pi} \mu_e$$

- 3. It has been advocated for a long time that every gauge field of Nature (including QED) requires a corresponding axion. Strong constraints on θ_{QED} could be manifestation and consequence of such construction

10.Physical meaning of $\mathbb{S}^1$ (the size of $\mathcal{T}$)

(appendix A3 from paper with Barvinsky)

- Originally $\mathcal{T}$ was introduced in Euclidean space for computations in the weak coupling regime with given holonomy.

$$L = \mathcal{P}\exp\left(i\int_0^{\mathcal{T}} dx_4 A_4(x_4, |\mathbf{x}| \to \infty)\right).$$

- It should not be confused with real size in 4d in Minkowski space.

- In weakly coupled gauge theories (such as deformed QCD) all computations can be carried out explicitly with fixed $\mathcal{T}$. One can see explicitly confinement, fractionally 1/N charged monopoles (instanton quarks), generation of the vacuum energy expressed in terms of the auxiliary field $b(x, H)$, etc

- In strongly coupled real QCD (when you start from the very begging from $\mathcal{T} \to \infty$) such computations cannot be done as calorons with nontrivial holonomy cannot be constructed

- How do we know about anything about holonomy defined on $\mathbb{S}^1$ if it was not a part of construction to begin with?

- It turns out that the holonomy can be dynamically generated (emerging) in strongly coupled regime.

- The well known example is $2d\; CP^{N-1}$ model defined on $\mathbb{R}^2$ when the only integer values instantons with trivial holonomy were introduced into the system

- However, when all the instants are taken into account (grand canonical ensemble) the fractional topological charge 1/N dynamically emerges. It looks exactly as we were started with nontrivial holonomy defined on $\mathbb{S}^1$. However, the semiclassical description is not justified.

- Therefore, in this "emergent" case defined on $\mathbb{R}^2$ the effective size $\mathbb{S}^1$ is generated dynamically.

- The main lesson in the present context: the effective size $\mathbb{S}^1$ should be treated as free parameter to be fixed by IR cutoff of system: $\mathcal{T} \sim H^{-1}$

- Sufficiently large size of $\mathbb{S}^1$ is consistent with presently available CMB observations for $\mathcal{T} \gtrsim H^{-1}$