
Hamiltonian Truncation Effective Theory

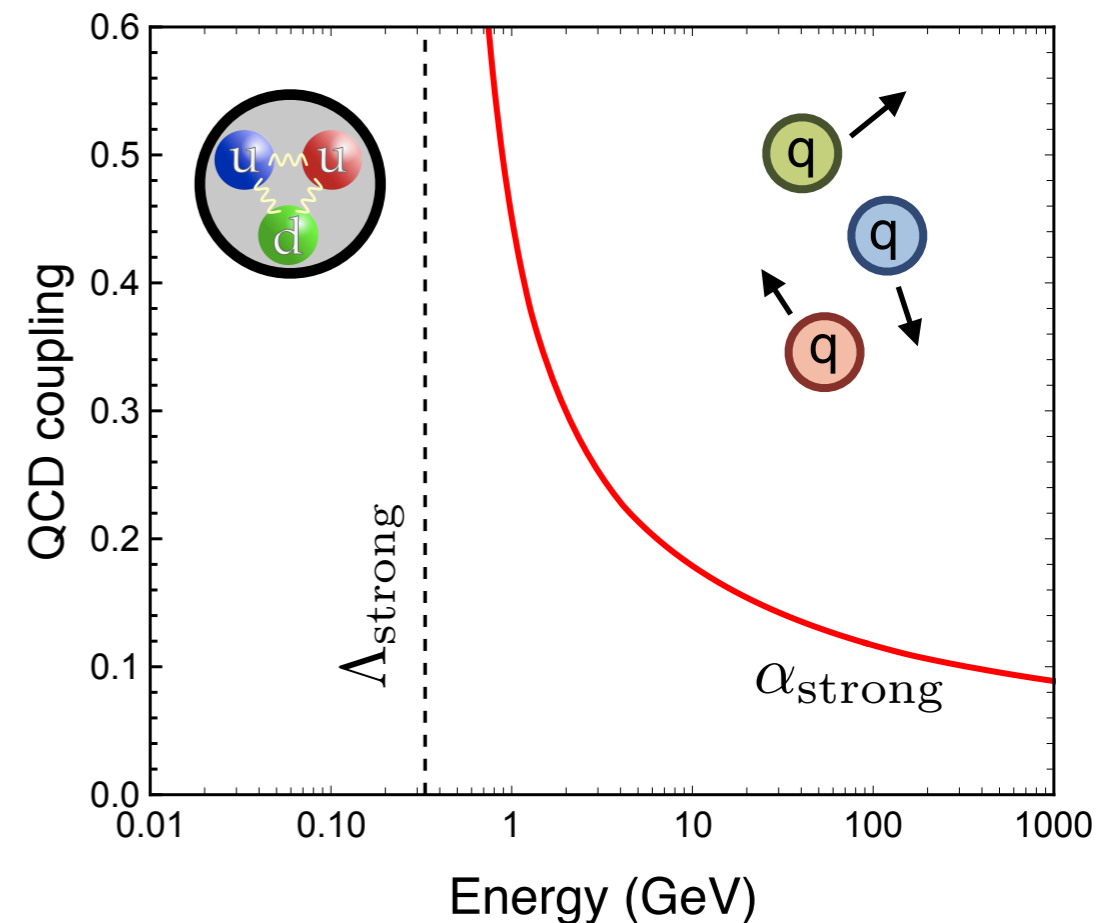


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Effective Theories for
Nonperturbative Physics
MITP
August, 2024

In Collaboration with T. Cohen (CERN & EPFL),
K. Farnsworth (Case Western), M. Luty (UC Davis)

Hamiltonian Truncation Effective Theory (HTET)

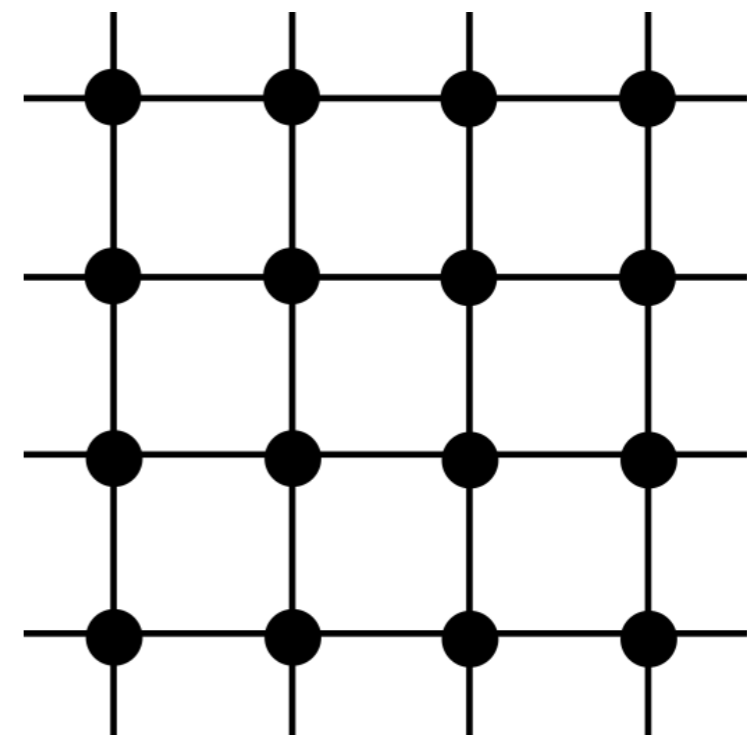
- ❖ We want to be able to understand particle physics at strong coupling
- ❖ Perturbation theory breaks down at confinement
- ❖ HTET is a nonperturbative strategy for probing strong coupling
- ❖ HTET is an attempt to impose some EFT-inspired order-by-order control over nonperturbative methods to probe strongly coupled theories



Alternative to the Lattice?

- ❖ Lattice field theory is the state of the art method to probe strong coupling with errors that are quantifiable and under control
- ❖ Lattice methods have some shortcomings
 - ❖ Tricky to model chiral fermions
Karsten, Smit (1981) Kaplan (1992)
 - ❖ Difficult to obtain dynamical quantities
 - ❖ Explicitly breaks continuous rotational and translational invariance
 - ❖ Requires ever increasing computational resources

Wilson (1974)



Viabale alternatives to the lattice?

Hamiltonian Truncation

❖ Truncated Conformal Space Approach:

Yurov, Zamolodchikov (1990), (1991)

$$H = H_{\text{known}} + \lambda V_{\text{pert}}$$

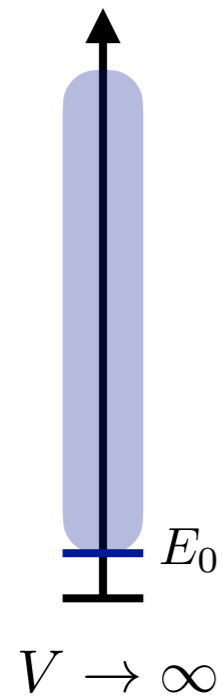
↑
Integrable or conformal
(trivial example: free theory)

Hamiltonian Truncation

- ❖ Truncated Conformal Space Approach: $H = H_{\text{known}} + \lambda V_{\text{pert}}$

Yurov, Zamolodchikov (1990), (1991)

- ❖ H_{known} provides the set of unperturbed energy eigenstates

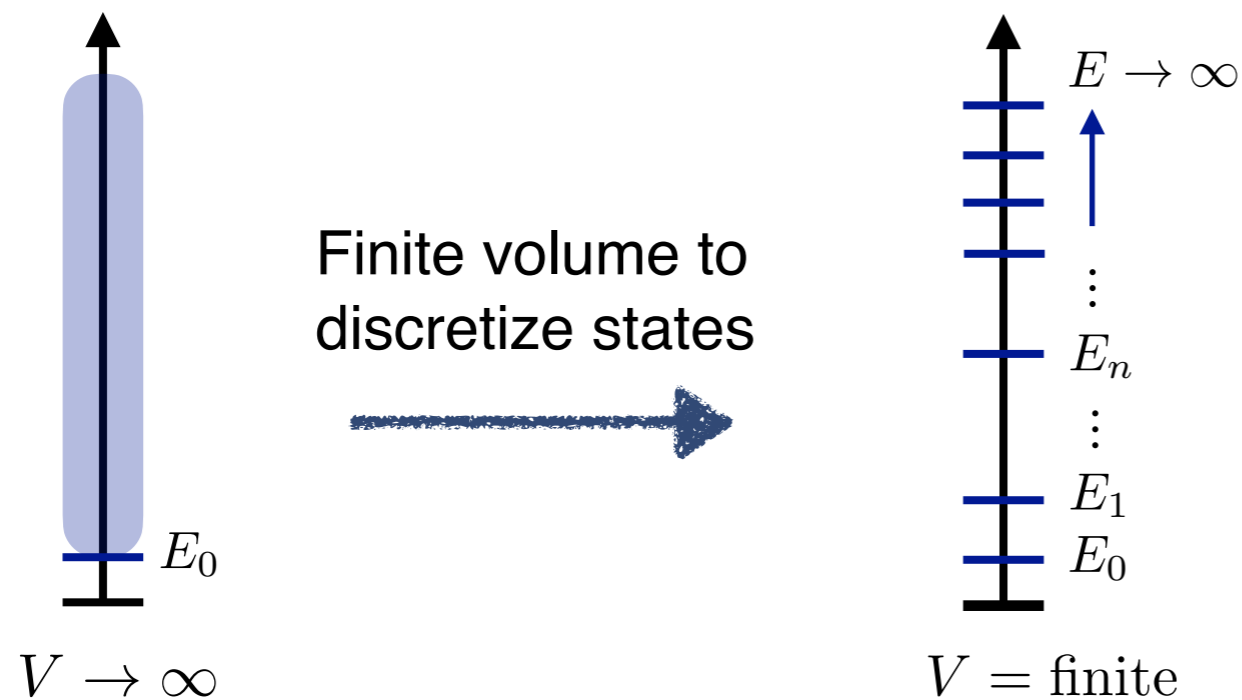


Hamiltonian Truncation

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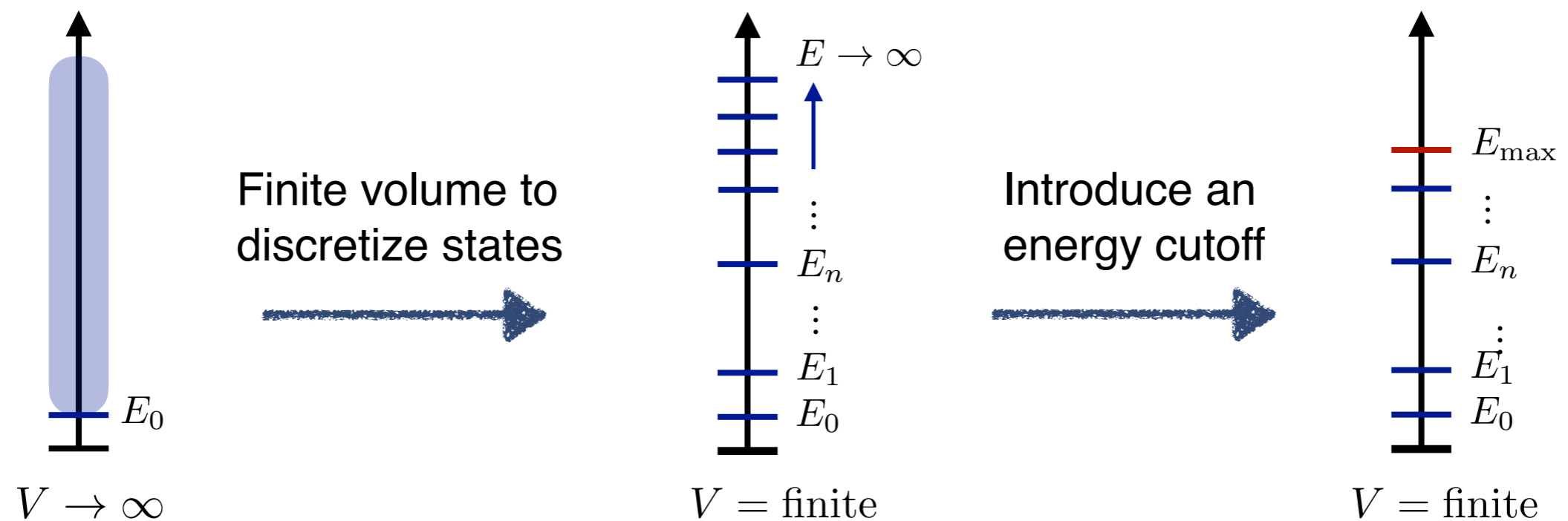


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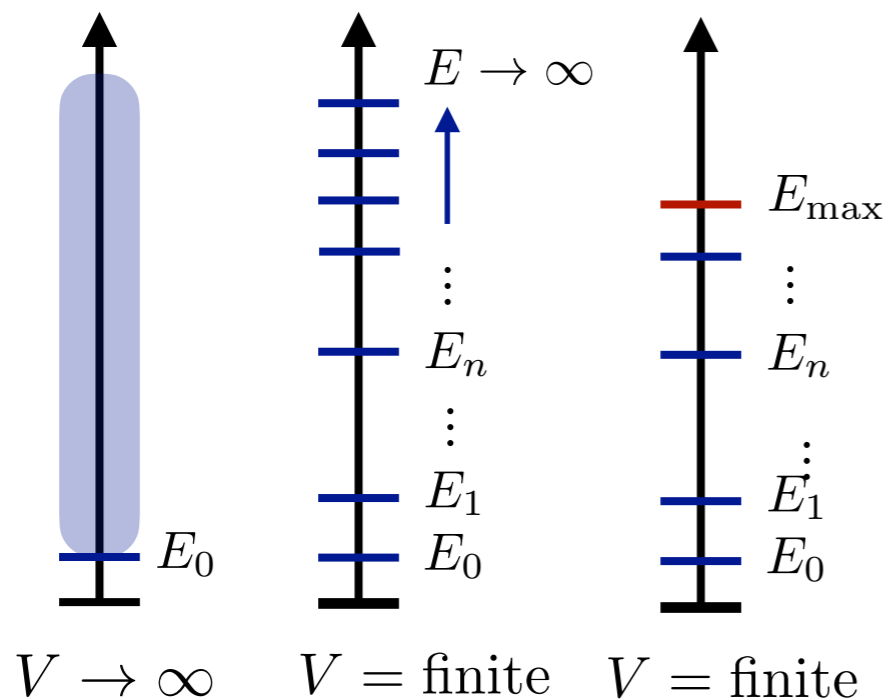
Hamiltonian Truncation

- ❖ Truncated Conformal Space Approach:

$$H = H_{\text{known}} + \lambda V_{\text{pert}}$$

Yurov, Zamolodchikov (1990), (1991)

- ❖ Truncate the basis:



(finite Volume to discretize states)

- ❖ Simply compute

$$H_{ij}^{\text{trunc}} = E_i \delta_{ij} + \lambda \langle E_i | V_{\text{pert}} | E_j \rangle$$

- ❖ Results from diagonalizing H_{trunc} are surprisingly effective due to the *relevancy* of the perturbing operator

Hamiltonian Truncation

A Cheap Alternative to the Lattice?

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❖ It's not cheaper, there is an exponential wall

❖ Intriguingly good results of the TCSA point to the promise of these methods

❖ Truncation effects are small for wisely-chosen theories

Rychkov, Vitale (2015)

❖ Can we control for these effects?

Delouche, Miro, Ingoldby (2023)

Elias-Miro, Rychkov Vitale (2017)

Miro, Ingoldby ('22, '21)

Elias-Miro, Hardy (2020)

❖ We want to develop an improvement scheme to incorporate the effects of H_{lh} , **order by order**

Cohen, Farnsworth, RH, Luty (2021)

The Hamiltonian

- ❖ Split the Hamiltonian up into the free and interacting pieces

$$H = H_{\text{free}} + V$$



- ❖ We want to diagonalize H , but its basis is infinite

Strong coupling in IR

- ❖ Look at a finite corner

$$H = \begin{pmatrix} H_{ll} & H_{lh} \\ H_{hl} & H_{hh} \end{pmatrix}$$

- ❖ The “low” and “high” states in the basis are defined by an energy cutoff E_{max}

$$H_{\text{free}}|\text{state}_i\rangle = E_i|\text{state}_i\rangle \quad E_i \leq E_{\text{max}} \Rightarrow |\text{state}_i\rangle \in \mathcal{H}_l \quad (\text{finite})$$

$$E_i > E_{\text{max}} \Rightarrow |\text{state}_i\rangle \in \mathcal{H}_h \quad (\text{infinite})$$

Truncating the Hamiltonian

- ❖ The easiest thing to do is just truncate the Hamiltonian (basically TCSA)

$$H = \begin{pmatrix} H_{ll} & H_{lh} \\ H_{hl} & H_{hh} \end{pmatrix} \Rightarrow H_{\text{eff}} = (H_{ll})$$

- ❖ The main source of error is mixing coming from H_{lh}
- ❖ We want to develop an improvement scheme to incorporate the effects of H_{lh} ,
order by order
- ❖ Our strategy will be to define **an operator** in the fundamental theory and **match** it to the effective theory to determine corrections to H_{eff}

Hamiltonian Truncation Effective Theory Strategy

❖ Fundamental theory:

$$H = H_{\text{free}} + V$$

Free theory defines basis:

$$H_{\text{free}}|\text{state}_i\rangle = E_i|\text{state}_i\rangle$$

❖ The energy eigenstates are: $H|i\rangle = \mathcal{E}_i|i\rangle$
and can be written as

$$\mathcal{E}_i = E_i + \mathcal{E}_{1i} + \mathcal{E}_{2i} + \dots \quad \mathcal{E}_{ni} = \mathcal{O}(V^n)$$

❖ We want to define a calculable, finite-dimensional H_{eff} such that:

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots \quad H_n = \mathcal{O}(V^n)$$

Matching the Transition Matrix

❖ Time evolution: $U_{\text{IP}}(t_f, t_i) = T \exp \left\{ -i \int_{t_i}^{t_f} dt V_{\text{IP}}(t) \right\} \quad V_{\text{IP}}(t) = e^{iH_0 t} V e^{-\epsilon t} e^{-iH_0 t}$

❖ Define an operator to match: $\langle f | \Sigma | i \rangle \equiv \lim_{t_f \rightarrow \infty} \langle f | U_{\text{IP}}(t_f, 0) | i \rangle$

❖ Implicit relation can be evaluated iteratively order-by-order

$$U_{\text{IP}}(t_f, t_i) = \mathbb{1} - i \int_{t_i}^{t_f} dt U_{\text{IP}}(t_f, t) V_{\text{IP}}(t)$$

❖ This gives an expansion:

$$\langle f | \Sigma | i \rangle = \delta_{fi} + \frac{\langle f | V | i \rangle}{E_f - E_i} + \sum_{\alpha} \frac{\langle f | V | \alpha \rangle \langle \alpha | V | i \rangle}{(E_f - E_i)(E_f - E_{\alpha})} + \mathcal{O}(V^3)$$

Matching the Transition Matrix

❖ More convenient to define: $\langle f|\Sigma|i\rangle = \delta_{fi} + \frac{\langle f|T|i\rangle}{E_f - E_i}$

$$\langle f|T|i\rangle_{\text{fund}} = \langle f|V|i\rangle + \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}} + \sum_{\alpha,\beta} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle f|V|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \mathcal{O}(V^4)$$

$$\langle f|T|i\rangle_{\text{eff}} = \langle f|V_{\text{eff}}|i\rangle + \sum_{\alpha}^{\leq} \frac{\langle f|V_{\text{eff}}|\alpha\rangle\langle\alpha|V_{\text{eff}}|i\rangle}{E_f - E_{\alpha}} + \sum_{\alpha,\beta}^{\leq} \frac{\langle f|V_{\text{eff}}|\alpha\rangle\langle\alpha|V_{\text{eff}}|\beta\rangle\langle\beta|V_{\text{eff}}|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \mathcal{O}(V^4)$$

- ❖ Matching is accomplished by setting the T matrix equal in both theories and evaluating H_i

$$H_{\text{eff}} = H_0 + V_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

Matching the Transition Matrix

$$\langle f|T|i\rangle_{\text{fund}} = \langle f|V|i\rangle + \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}} + \dots$$

$$\langle f|T|i\rangle_{\text{eff}} = \langle f|V_{\text{eff}}|i\rangle + \sum_{\alpha}^{\lessdot} \frac{\langle f|V_{\text{eff}}|\alpha\rangle\langle\alpha|V_{\text{eff}}|i\rangle}{E_f - E_{\alpha}} + \dots$$

- ❖ The matching defines H_{eff} , analogous to normal QM perturbation theory:

$$\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle$$

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^{\gtrdot} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}}$$

$$\langle f|H_3|i\rangle_{\text{eff}} = \sum_{\alpha}^{\gtrdot} \sum_{\beta}^{\gtrdot} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \sum_{\alpha}^{\lessdot} \sum_{\beta}^{\gtrdot} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\beta})(E_{\alpha} - E_{\beta})}$$

...

2D ϕ^4 Theory

- ❖ Test this method out on a specific theory:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad [m] = 1 \quad [\lambda] = 2$$

- ❖ Relevant couplings: weakly coupled in UV, strongly coupled in IR

- ❖ Quantized on a circle of radius R : $H_0 = \sum_k \omega_k a_k^\dagger a_k \quad \omega_k = \sqrt{(k/R)^2 + m_Q^2}$

- ❖ The goal is to calculate H_{eff} , which will be an expansion in V :

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

Power Counting

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots \quad H_n \sim \mathcal{O}(V^n)$$

- ❖ Very roughly from dimensional analysis: $V \sim \lambda \phi^4$ $[\lambda] = 2$

$$H_n \sim \frac{\lambda^n}{E_{\text{max}}^{2n-2}} \quad H_2 \sim \frac{\lambda^2}{E_{\text{max}}^2} \quad H_3 \sim \frac{\lambda^3}{E_{\text{max}}^4}$$

- ❖ Concretely, we calculate H_{eff} as:

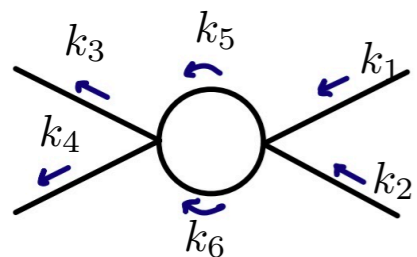
$$\langle f | H_2 | i \rangle_{\text{eff}} = \sum_{\alpha} \frac{\langle f | V | \alpha \rangle \langle \alpha | V | i \rangle}{E_f - E_{\alpha}} \quad \text{uses} \quad V = \int dx \left[\frac{1}{2} m_V^2 : \phi^2 : + \frac{\lambda}{4!} : \phi^4 : \right]$$

- ❖ Develop Feynman rules to more easily evaluate these sums

Diagrammatic Calculation of the Transition Matrix

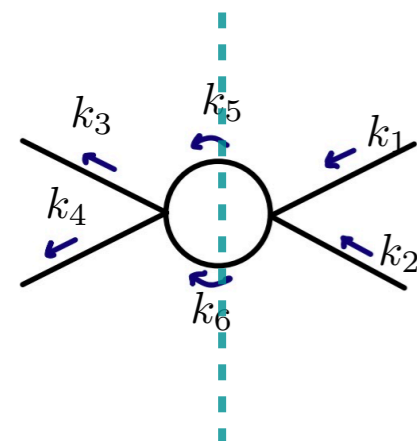
- ❖ For example, part of the transition matrix in the fundamental theory can be found using:

$$\langle f|T|i\rangle_{\text{fund}} \ni \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}}$$



$$= \frac{1}{8} \left(\frac{\lambda}{2\pi R} \right)^2 \sum_{k_1, \dots, k_6} \delta_{k_1+k_2, k_5+k_6} \langle f|\phi_4^- \phi_3^- \phi_2^+ \phi_1^+|i\rangle \frac{1}{2\omega_{k_5}} \frac{1}{2\omega_{k_6}} \frac{1}{\omega_3 + \omega_4 - \omega_5 - \omega_6 + i\epsilon}$$

- ❖ In the effective theory, we instead find:



$$= \frac{1}{8} \left(\frac{\lambda}{2\pi R} \right)^2 \sum_{k_1, \dots, k_6} \delta_{k_1+k_2, k_5+k_6} \langle f|\phi_4^- \phi_3^- \phi_2^+ \phi_1^+|i\rangle \frac{1}{2\omega_{k_5}} \frac{1}{2\omega_{k_6}} \frac{\Theta(E_{\text{max}} - E_f + \omega_3 + \omega_4 - \omega_5 - \omega_6)}{\omega_3 + \omega_4 - \omega_5 - \omega_6 + i\epsilon}$$

The mass parameter

❖ 2D ϕ^4 theory is super-renormalizable $[m] = 1$ $[\lambda] = 2$

❖ The UV divergences can be eliminated by normal ordering



❖ We still implemented a “renormalization” scheme

Delouche, Miro, Ingoldby,
arXiv:2312.09221

❖ This will aid in demonstrating separation of IR and UV scales

❖ We can also use m_Q as a variational parameter to test convergence

❖ Start with a **Wilsonian cutoff** in the **fundamental theory** $\sum_k \rightarrow \sum_{k \leq \Lambda R}$

We’ve compactified onto a circle for finite V , so momenta are now discrete

❖ Arrange $\Lambda \gg E_{\max}$ and eventually take $\Lambda \rightarrow \infty$

The mass parameter

- ❖ Define the coefficient of ϕ^2 :

$$m_V^2 = m_0^2 - m_Q^2 + \frac{\lambda}{8\pi R} \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}$$

The mass parameter

- ❖ Define the coefficient of ϕ^2 :

$$m_V^2 = m_0^2 - m_Q^2 + \underbrace{\frac{\lambda}{8\pi R} \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}}_{\text{Log divergent piece}} \rightarrow \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k} \sim \int^{\Lambda} \frac{dk}{\omega_k} \sim \ln \Lambda$$

The mass parameter

- ❖ Define the coefficient of ϕ^2 :

$$m_V^2 = m_0^2 - \underbrace{m_Q^2}_{\text{Quantization mass:}} + \frac{\lambda}{8\pi R} \underbrace{\sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}}_{\text{Log divergent piece}}$$

Quantization mass:
 $\omega_k = \sqrt{(k/R)^2 + m_Q^2}$

Log divergent piece
 $\sum_{|k| \leq \Lambda R} \frac{1}{\omega_k} \sim \int^{\Lambda} \frac{dk}{\omega_k} \sim \ln \Lambda$

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The mass parameter

- Define the coefficient of ϕ^2 :

$$m_V^2 = \underbrace{m_0^2}_{\substack{\text{Bare mass,} \\ \text{absorbs} \\ \text{divergence}}} - \underbrace{m_Q^2}_{\substack{\text{Quantization mass:} \\ \omega_k = \sqrt{(k/R)^2 + m_Q^2}}} + \frac{\lambda}{8\pi R} \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}$$

Log divergent piece

$$\sum_{|k| \leq \Lambda R} \frac{1}{\omega_k} \sim \int^{\Lambda} \frac{dk}{\omega_k} \sim \ln \Lambda$$

The mass parameter

- Define the coefficient of ϕ^2 :

$$m_V^2 = m_0^2 - m_Q^2 + \frac{\lambda}{8\pi R} \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}$$

k_2 — k_1
 $= m_V^2 \delta_{k_1 k_2}$

m_0^2 : Bare mass, absorbs divergence

m_Q^2 : Quantization mass:
 $\omega_k = \sqrt{(k/R)^2 + m_Q^2}$

Log divergent piece:
 $\sum_{|k| \leq \Lambda R} \frac{1}{\omega_k} \sim \int^{\Lambda} \frac{dk}{\omega_k} \sim \ln \Lambda$

The mass parameter

- Define the coefficient of ϕ^2 :

$$m_V^2 = m_0^2 - m_Q^2 + \frac{\lambda}{8\pi R} \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}$$

$k_2 \text{---} \bullet \text{---} k_1$
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Bare mass, absorbs divergence
 Quantization mass:
 $\omega_k = \sqrt{(k/R)^2 + m_Q^2}$

Log divergent piece
 $\sum_{|k| \leq \Lambda R} \frac{1}{\omega_k} \sim \int^{\Lambda} \frac{dk}{\omega_k} \sim \ln \Lambda$

- To demonstrate separation of scales, we introduce m_R , which depends on a renormalization scale μ

$$m_R^2(\mu) = m_0^2 + \frac{\lambda}{8\pi R} \sum_{\mu R < |k| \leq \Lambda R} \frac{1}{\omega_k}, \quad m_V^2 = m_R^2(\mu) - m_Q^2 + \frac{\lambda}{8\pi R} \sum_{|k| < \mu R} \frac{1}{\omega_k}$$

$$m_{\text{NO}} = m_R(\mu = 0)$$

Operator Matching $\mathcal{O}(V)$

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

- ❖ Matching conditions to define H_{eff}

$$\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle \longleftarrow \text{Start here}$$

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}}$$

$$\langle f|H_3|i\rangle_{\text{eff}} = \sum_{\alpha}^> \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \sum_{\alpha}^< \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\beta})(E_{\alpha} - E_{\beta})}$$

Operator Matching $\mathcal{O}(V)$

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

- ❖ Matching conditions to define H_{eff}

$$\langle f | H_1 | i \rangle_{\text{eff}} = \langle f | V | i \rangle \longleftarrow \text{Start here}$$

Effective theory

$$H_1 = \int dx \left[\frac{1}{2} m_{V_1}^2 : \phi^2 : + \frac{\lambda_1}{4!} : \phi^4 : \right]$$

Fundamental Theory

$$V = \int dx \left[\frac{1}{2} m_V^2 : \phi^2 : + \frac{\lambda}{4!} : \phi^4 : \right]$$

- ❖ Matching yields (trivially at this order):

$$\lambda_1 = \lambda$$

$$m_{V_1}^2 = m_V^2$$

Diagrammatic Matching $\mathcal{O}(V)$

❖ Matching at first order: $\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle$

$$\begin{array}{c} \diagup \cdot \diagdown \\ \diagdown \cdot \diagup \end{array} + \begin{array}{c} \diagup \cdot \\ \diagdown \end{array} + \begin{array}{c} \diagdown \cdot \\ \diagup \end{array} + \begin{array}{c} \diagup \cdot \diagdown \\ \diagdown \cdot \diagup \end{array} + \begin{array}{c} \diagup \cdot \diagdown \\ \diagdown \cdot \diagup \end{array} = \frac{\lambda}{4!} \int dx \langle f|:\phi^4:|i\rangle$$

$$\begin{array}{c} \cdot \\ \text{---} \end{array} + \begin{array}{c} \diagup \cdot \\ \diagdown \end{array} + \begin{array}{c} \diagdown \cdot \\ \diagup \end{array} = \frac{1}{2} m_V^2 \int dx \langle f|:\phi^2:|i\rangle.$$

❖ Same diagrams in both effective theory and fundamental theory at tree level

❖ Trivially gives: $H_{\text{eff}} = H_0 + \int dx \left[\frac{1}{2} m_V^2 : \phi^2 : + \frac{\lambda}{4!} : \phi^4 : \right] + \dots$

Matching at Second Order $\mathcal{O}(V^2)$

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

- ❖ Matching conditions to define H_{eff}

$$\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle \longleftarrow \text{Started here } \checkmark$$

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}}$$

$$\langle f|H_3|i\rangle_{\text{eff}} = \sum_{\alpha}^> \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \sum_{\alpha}^< \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\beta})(E_{\alpha} - E_{\beta})}$$

Matching at Second Order $\mathcal{O}(V^2)$

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

- ❖ Matching conditions to define H_{eff}

$$\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle \longleftarrow \text{Started here } \checkmark$$

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}} \longleftarrow \text{Now we compute this}$$

$$\langle f|H_3|i\rangle_{\text{eff}} = \sum_{\alpha}^> \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \sum_{\alpha}^< \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\beta})(E_{\alpha} - E_{\beta})}$$

Diagrammatic Matching $\mathcal{O}(V^2)$: Four External Legs

❖ Matching at second order:
$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}} = T_2 \Big|_{\text{fund}} - T_2 \Big|_{\text{eff}}$$

❖ This will determine the coupling constants of
$$H_2 = \int dx \left[\frac{1}{2} m_{V_2}^2 : \phi^2 : + \frac{\lambda_2}{4!} : \phi^4 : \right]$$

❖ Vertex correction diagrams

$$\langle f|T_2|i\rangle = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} + \text{[Diagram 6]} + \text{[Diagram 7]}$$

❖ Diagram evaluation example:

$$\text{[Diagram 1]} - \left[\text{[Diagram 1]} \right]_{\text{eff}} = \frac{\lambda^2}{128\pi^2 R^2} \sum_{1,\dots,4} \delta_{12,34} \langle f|\phi_4^{(-)}\phi_3^{(-)}\phi_2^{(+)}\phi_1^{(+)}|i\rangle \sum_{5,6} \delta_{56,34} \frac{\Theta(\omega_5 + \omega_6 - \omega_3 - \omega_4 + E_f - E_{\text{max}})}{\omega_5\omega_6(\omega_3 + \omega_4 - \omega_5 - \omega_6)}$$

The Local Approximation

- ❖ Diagram evaluation example:

$$\begin{array}{c} 3 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \begin{array}{c} 5 \\ \diagup \\ \bullet \\ \diagdown \\ 6 \end{array} \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \end{array} - \left[\begin{array}{c} 3 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \begin{array}{c} 5 \\ \diagup \\ \bullet \\ \diagdown \\ 6 \end{array} \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \end{array} \right]_{\text{eff}} = \frac{\lambda^2}{128\pi^2 R^2} \sum_{1,\dots,4} \delta_{12,34} \langle f | \phi_4^{(-)} \phi_3^{(-)} \phi_2^{(+)} \phi_1^{(+)} | i \rangle \sum_{5,6} \delta_{56,34} \frac{\Theta(\omega_5 + \omega_6 - \omega_3 - \omega_4 + E_f - E_{\max})}{\omega_5 \omega_6 (\omega_3 + \omega_4 - \omega_5 - \omega_6)}$$

- ❖ The full answer will have complicated dependence on IR scales, but this can be expanded out assuming separation of scales:

$$\underbrace{m_Q \lesssim \omega_{1234} \lesssim E_{i,f} \ll E_{\max}}_{\text{IR scales}} \quad \Rightarrow \quad \omega_{5,6} \gtrsim E_{\max}$$

From the step function

- ❖ The **local approximation**: set all external energies = 0

$$\begin{array}{c} 3 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \begin{array}{c} 5 \\ \diagup \\ \bullet \\ \diagdown \\ 6 \end{array} \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \end{array} - \left[\begin{array}{c} 3 \\ \diagdown \\ \bullet \\ \diagup \\ 4 \end{array} \begin{array}{c} 5 \\ \diagup \\ \bullet \\ \diagdown \\ 6 \end{array} \begin{array}{c} 1 \\ \diagup \\ \bullet \\ \diagdown \\ 2 \end{array} \right]_{\text{eff}} = \frac{\lambda^2}{128\pi^2 R^2} \sum_{1,\dots,4} \delta_{12,34} \langle f | \phi_4^{(-)} \phi_3^{(-)} \phi_2^{(+)} \phi_1^{(+)} | i \rangle \sum_k \frac{\Theta(2\omega_k - E_{\max})}{\omega_k^3} [1 + \mathcal{O}(E_{i,f}/E_{\max})]$$

Matching at Second Order $\mathcal{O}(V^2)$

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

- ❖ Matching conditions to define H_{eff}

$$\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle \longleftarrow \text{Started here } \checkmark$$

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}} \longleftarrow \text{We are in the process of computing this}$$

$$\langle f|H_3|i\rangle_{\text{eff}} = \sum_{\alpha}^> \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \sum_{\alpha}^< \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\beta})(E_{\alpha} - E_{\beta})}$$

Matching at Second Order $\mathcal{O}(V^2)$

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots$$

- ❖ Matching conditions to define H_{eff}

$$\langle f|H_1|i\rangle_{\text{eff}} = \langle f|V|i\rangle \longleftarrow \text{Started here } \checkmark$$

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}} \longleftarrow \text{We are in the process of computing this}$$

$$\langle f|H_3|i\rangle_{\text{eff}} = \sum_{\alpha}^> \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\alpha})(E_f - E_{\beta})} + \sum_{\alpha}^< \sum_{\beta}^> \frac{\langle f|V|\alpha\rangle\langle\alpha|V|\beta\rangle\langle\beta|V|i\rangle}{(E_f - E_{\beta})(E_{\alpha} - E_{\beta})}$$

Where do we stop? Go back to power counting.

Power Counting

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots \quad H_n \sim \mathcal{O}(V^n)$$

❖ Very roughly from dimensional analysis: $V \sim \lambda \phi^4 \quad [\lambda] = 2$

$$H_n \sim \frac{\lambda^n}{E_{\text{max}}^{2n-2}} \quad H_2 \sim \frac{\lambda^2}{E_{\text{max}}^2} \quad H_3 \sim \frac{\lambda^3}{E_{\text{max}}^4}$$

Power Counting

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots \quad H_n \sim \mathcal{O}(V^n) \quad V \sim \lambda \phi^4 \quad [\lambda] = 2$$

- ❖ What is the dependence on E_{max} for each of these progressive corrections?

$$H_n \sim \frac{\lambda^n}{E_{\text{max}}^{2n-2}} \sum_a \int C_{na} \left(\frac{H_0}{E_{\text{max}}}, \frac{R^{-1}}{E_{\text{max}}}, \frac{m_Q}{E_{\text{max}}} \right) \mathcal{O}_{na} \left(\frac{\partial x}{E_{\text{max}}}, \frac{\partial t}{E_{\text{max}}}, \phi \right)$$

$$H_2 \sim \frac{\lambda^2}{E_{\text{max}}^2} \sum_a \int \left[\phi^2 + \phi^4 + \frac{R^{-1} + H_0}{E_{\text{max}}} (1 + \phi^2 + \phi^4) + \mathcal{O}(1/E_{\text{max}}^2) \right]$$

$$\uparrow \\ \mathcal{O}(1/E_{\text{max}}^3)$$

$$H_3 \sim \frac{\lambda^3}{E_{\text{max}}^4}$$

Local Approximation

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots \quad H_n \sim \mathcal{O}(V^n) \quad V \sim \lambda\phi^4 \quad [\lambda] = 2$$

- ❖ What is the dependence on E_{max} for each of these progressive corrections?

$$H_n \sim \frac{\lambda^n}{E_{\text{max}}^{2n-2}} \sum_a \int C_{na} \left(\frac{H_0}{E_{\text{max}}}, \frac{R^{-1}}{E_{\text{max}}}, \frac{m_Q}{E_{\text{max}}} \right) \mathcal{O}_{na} \left(\frac{\partial x}{E_{\text{max}}}, \frac{\partial t}{E_{\text{max}}}, \phi \right)$$

$$H_2 \sim \frac{\lambda^2}{E_{\text{max}}^2} \sum_a \int \left[\phi^2 + \phi^4 + \frac{R^{-1} + H_0}{E_{\text{max}}} (1 + \phi^2 + \phi^4) + \mathcal{O}(1/E_{\text{max}}^2) \right]$$

$$H_3 \sim \frac{\lambda^3}{E_{\text{max}}^4}$$

- ❖ **Local approximation:** only keep terms up to $\mathcal{O}(1/E_{\text{max}}^2)$

➔ We can stop at H_2

Diagrammatic Matching $\mathcal{O}(V^2)$: Four External Legs

- Adding together the other diagrams builds the full operator, giving:

$$\langle f|T_2|i\rangle = \text{[Diagram 1]} + \text{[Diagram 2]} + \text{[Diagram 3]} + \text{[Diagram 4]} + \text{[Diagram 5]} + \text{[Diagram 6]} + \text{[Diagram 7]}$$

$$T_2\Big|_{\text{fund}} - T_2\Big|_{\text{eff}} \simeq -\frac{3\lambda^2}{4!(16\pi R)} \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^3} \int dx : \phi^4 :$$

- Matching determines:

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha} \frac{\langle f|V|\alpha\rangle \langle \alpha|V|i\rangle}{E_f - E_{\alpha}}$$

$$H_2 \simeq \frac{\lambda_2}{4!} \int dx : \phi^4 : = T_2\Big|_{\text{fund}} - T_2\Big|_{\text{eff}} \Rightarrow \lambda_2 = -\frac{3\lambda^2}{16\pi R} \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^3}$$

Diagrammatic Matching $\mathcal{O}(V^2)$: Two External Legs

- ❖ Mass correction:

$$\langle f | T_2 | i \rangle = \text{[Diagrammatic expansion of mass correction terms]}$$

The diagrammatic expansion shows the following terms:

- 1. A circle with two external legs (left and right).
- 2. A circle with two external legs and a self-energy loop on the top leg.
- 3. A circle with two external legs and a self-energy loop on the bottom leg.
- 4. A circle with two external legs and a self-energy loop on both legs.
- 5. A circle with two external legs and a tadpole loop on the top leg.
- 6. A circle with two external legs and a tadpole loop on the bottom leg.
- 7. A circle with two external legs and a tadpole loop on both legs.

- ❖ Evaluate diagrams

$$2 \text{---} \text{[Diagram with legs 2, 1, 3, 4, 5]} \text{---} 1 - \left[2 \text{---} \text{[Diagram with legs 2, 1, 3, 4, 5]} \text{---} 1 \right]_{\text{eff}} = \frac{1}{6} \left(\frac{\lambda}{2\pi R} \right)^2 \sum_{1, \dots, 5} \delta_{1,2} \delta_{1,345} \langle f | \phi_2^{(-)} \phi_1^{(+)} | i \rangle \frac{1}{2\omega_3} \frac{1}{2\omega_4} \frac{1}{2\omega_5} \frac{\Theta(\omega_3 + \omega_4 + \omega_5 - E_{\max})}{\omega_2 - \omega_3 - \omega_4 - \omega_5}$$

- ❖ UV/IR mixing: some intermediate momenta can be large, while others small
- ❖ Separation of scales is not manifest diagram-by-diagram, but only when all diagrams are added together

Diagrammatic Matching $\mathcal{O}(V^2)$: Two External Legs

- ❖ Matching and evaluating all diagrams with two external legs gives:

$$\langle f|H_2|i\rangle_{\text{eff}} = \sum_{\alpha} \frac{\langle f|V|\alpha\rangle\langle\alpha|V|i\rangle}{E_f - E_{\alpha}}$$

$$H_2 \simeq \frac{1}{2}m_{V_2}^2 \int dx : \phi^2 : = T_2 \Big|_{\text{fund}} - T_2 \Big|_{\text{eff}} = \begin{array}{c} \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \\ + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---} \end{array}$$

- ❖ And we find the coupling in H_2

$$m_{V_2}^2 = \frac{\lambda}{16\pi R} \left[\frac{\lambda}{6\pi R} \sum_{345} \delta_{k_3+k_4+k_5,0} \frac{\Theta(\omega_3 + \omega_4 + \omega_5 - E_{\text{max}})}{\omega_3\omega_4\omega_5(\omega_3 + \omega_4 + \omega_5)} - m_V^2 \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^3} \right]$$

Separation of Scales

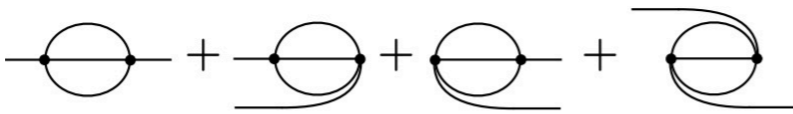
- ❖ The coupling constants in the non-normal-ordered H_{eff} should not have any UV/IR mixing, using $\mu \sim E_{\text{max}}$

$$H_2 = \int dx \left[\frac{1}{2} m_2^2 \phi^2 + \frac{\lambda_2}{4!} \phi^4 \right]$$

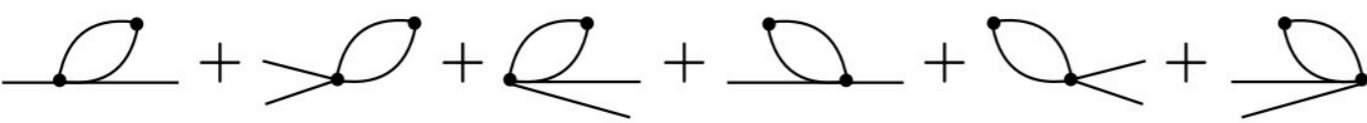
The NO operator coefficients can have UV/IR mixing, as they are essentially renormalized at $\mu = 0$

$$m_2^2 + m_Q^2 = m_R^2(\mu = E_{\text{max}})$$

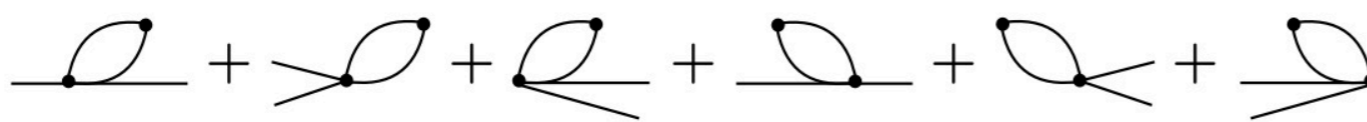
- ❖ The UV/IR contributions are:

①  $\Rightarrow \frac{\lambda}{16\pi R} \frac{\lambda}{6\pi R} \sum_{345} \delta_{345,0} \frac{\Theta(\omega_3 + \omega_4 + \omega_5 - E_{\text{max}})}{\omega_3 \omega_4 \omega_5 (-\omega_3 - \omega_4 - \omega_5)}$

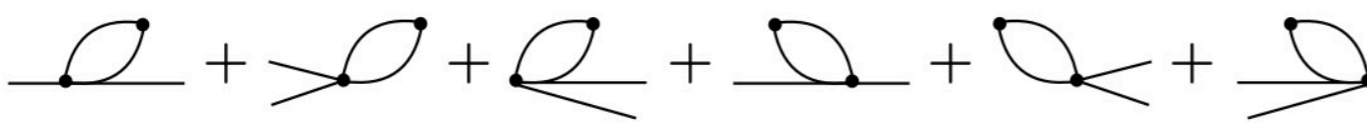
$$\simeq \frac{\lambda}{16\pi R} \frac{\lambda}{6\pi R} \times 3 \sum_{|k'| \ll k_{\text{max}}} \frac{1}{\omega_{k'}} \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^2 (-2\omega_k)}$$

②  $\Rightarrow -\frac{\lambda}{16\pi R} \underbrace{m_V^2}_{\text{IR sensitive}} \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^3}$

Separation of Scales

②  $\Rightarrow -\frac{\lambda}{16\pi R} \underbrace{m_V^2}_{\text{IR sensitive}} \sum_k \frac{\Theta(2\omega_k - E_{\max})}{\omega_k^3}$

$$m_V^2 = m_0^2 - m_Q^2 + \frac{\lambda}{8\pi R} \sum_{|k| \leq \Lambda R} \frac{1}{\omega_k}$$

 $\ni -\frac{\lambda}{16\pi R} \frac{\lambda}{8\pi R} \sum_{|k'| \ll k_{\max}} \frac{1}{\omega_{k'}} \sum_k \frac{\Theta(2\omega_k - E_{\max})}{\omega_k^3}$

③ “Un-normal ordering” the ϕ^4 interaction $\phi^4 =: \phi^4 : + 6 \frac{1}{4\pi R} \sum_k \frac{1}{\omega_k} : \phi^2 : + \text{const.}$

Contributes to ϕ^2 : $\Rightarrow -\frac{1}{2} Z \lambda_2 = -\frac{3\lambda^2}{16\pi R} \left(-\frac{1}{2} \frac{1}{4\pi R} \sum_{|k'| \ll k_{\max}} \frac{1}{\omega_{k'}} \right) \sum_k \frac{\Theta(2\omega_k - E_{\max})}{\omega_k^3}$

Separation of Scales

- ❖ Summing all of these mixed IR/UV contributions together yields:

$$\text{---} \bigcirc \text{---} + \text{---} \text{---} \text{---} \text{---} : \phi^4 :$$

$$\textcircled{1} + \textcircled{2} + \textcircled{3} = \frac{\lambda^2}{16\pi^2 R^2} \underbrace{\left[-\frac{3}{12} - \frac{1}{8} + \frac{3}{8} \right]}_{=0} \sum_{|k'| \ll k_{\max}} \frac{1}{\omega_{k'}} \sum_k \frac{\Theta(2\omega_k - E_{\max})}{\omega_k^3}$$

- ❖ The UV/IR mixing vanishes! Separation of scales is manifest.
- ❖ This justifies the local approximation, can expand in powers of $E_{i,f} / E_{\max}$

Matching Results and Implementation

- ❖ Final answer at this order:

$$H_{\text{eff}} = H_0 + \int dx \left[\frac{1}{2} (m_V^2 + m_{V_2}^2) : \phi^2 : + \frac{\lambda^2 + \lambda_2}{4!} : \phi^4 : \right]$$

Computationally costly to build matrices

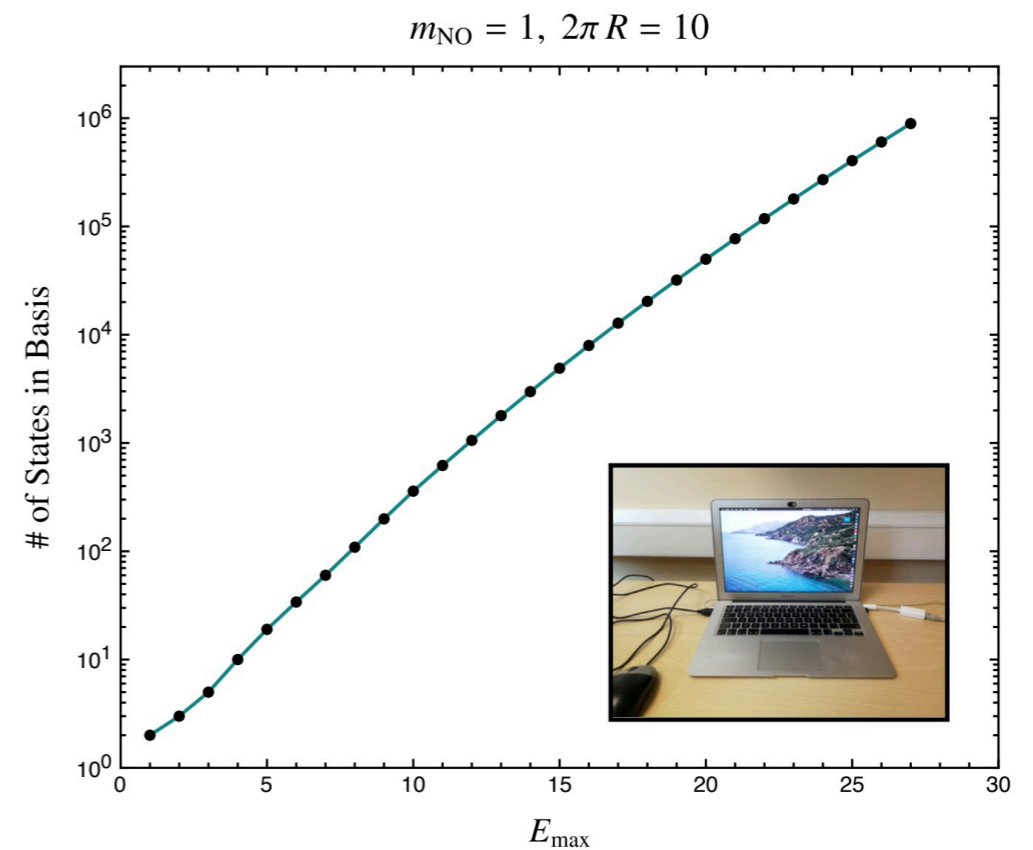
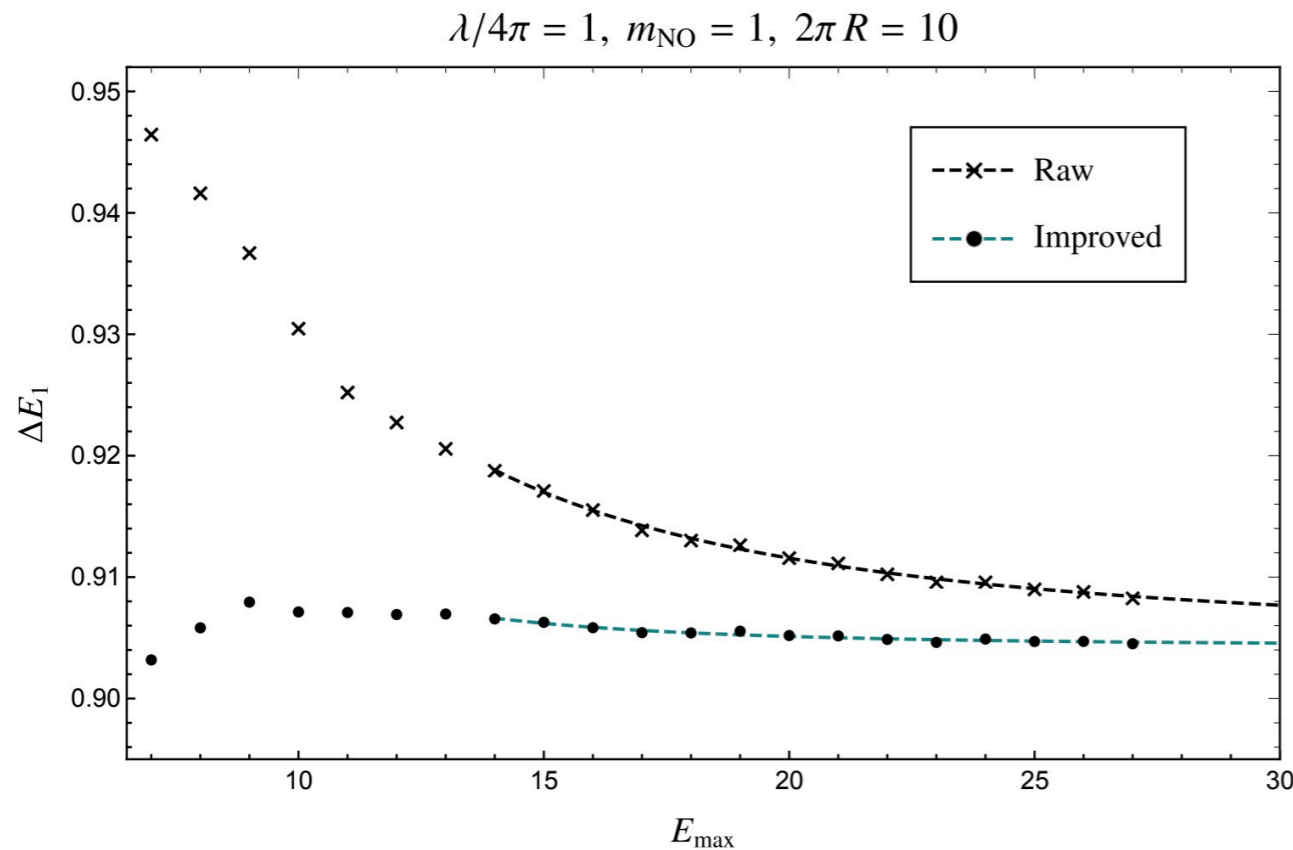
- ❖ Tuning the couplings, adding matrices together, and diagonalizing is cheap
- ❖ Corrected couplings*

$$m_{V_2}^2 = \frac{\lambda}{16\pi R} \left[\frac{\lambda}{6\pi R} \sum_{345} \delta_{k_3+k_4+k_5,0} \frac{\Theta(\omega_3 + \omega_4 + \omega_5 - E_{\text{max}})}{\omega_3\omega_4\omega_5(\omega_3 + \omega_4 + \omega_5)} - m_V^2 \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^3} \right]$$

$$\lambda_2 = -\frac{3\lambda^2}{16\pi R} \sum_k \frac{\Theta(2\omega_k - E_{\text{max}})}{\omega_k^3}$$

*local approximation, neglecting external momenta $E_{i,f} \ll E_{\text{max}}$

Improved Probe of ΔE_1



$$\Delta E_1 \equiv E_1 - E_0$$

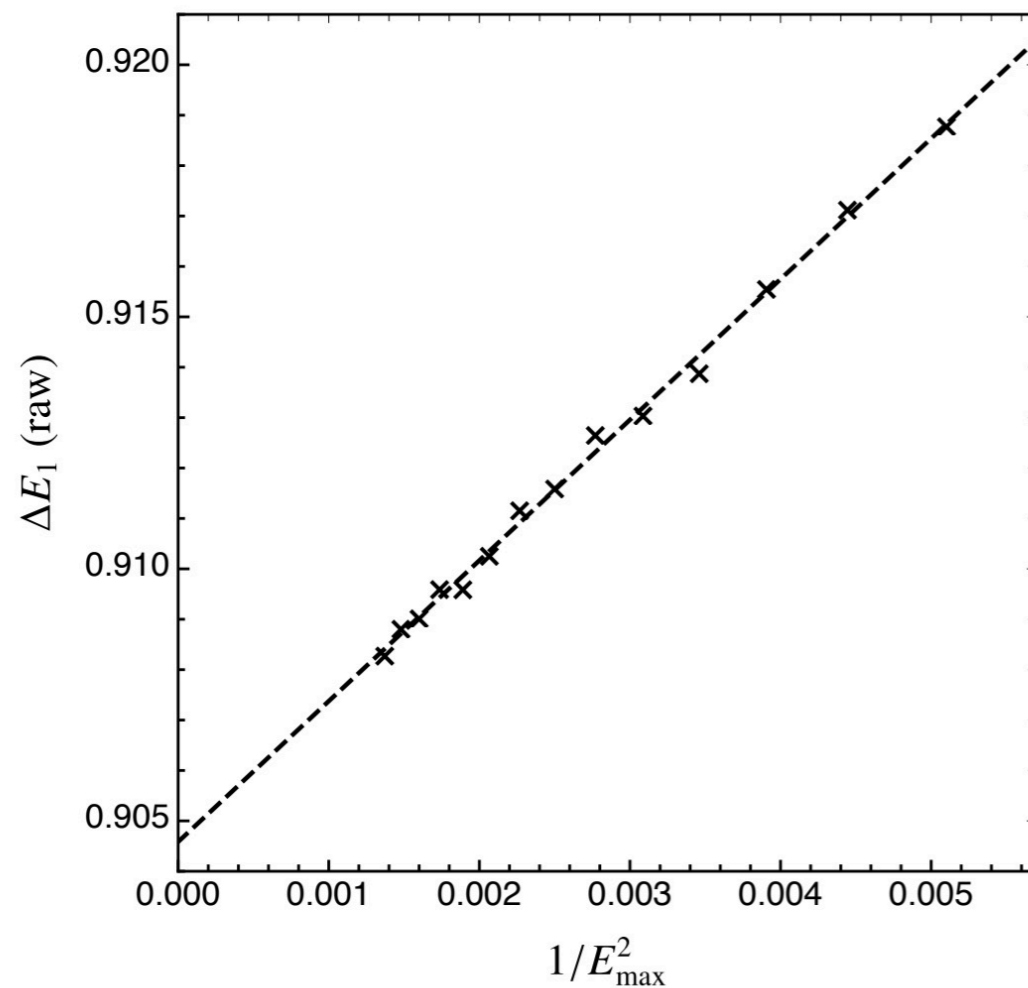
- ❖ Very quick approach to the correct first excited energy level
- ❖ Results obtained on a laptop comparable to others requiring a grid

Elias-Miro, Rychkov Vitale (2017)

Error Scaling with E_{\max}

Raw Truncation

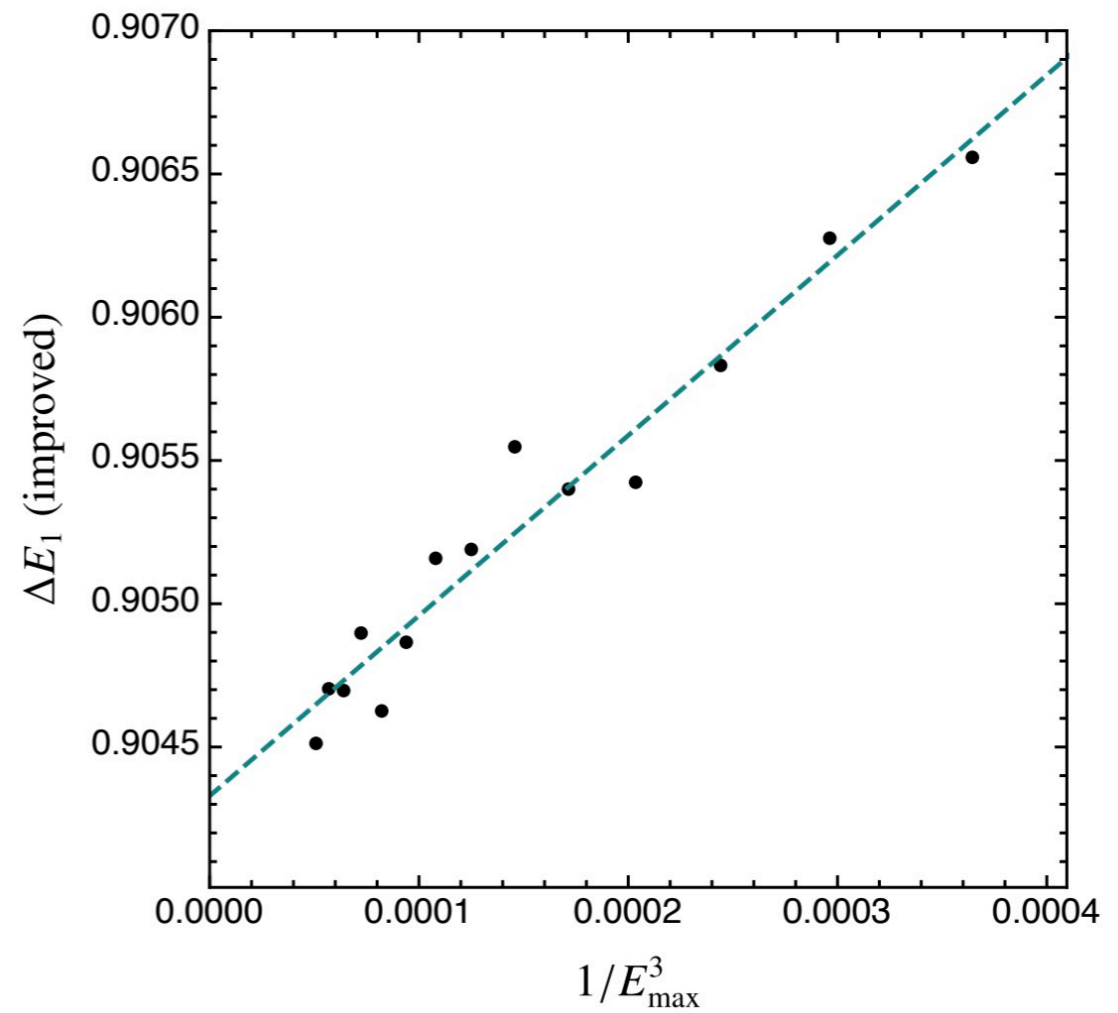
$\lambda/4\pi = 1, m_{\text{NO}} = 1, 2\pi R = 10$



❖ $1/E_{\max}^2$ scaling

Truncation + Improvement

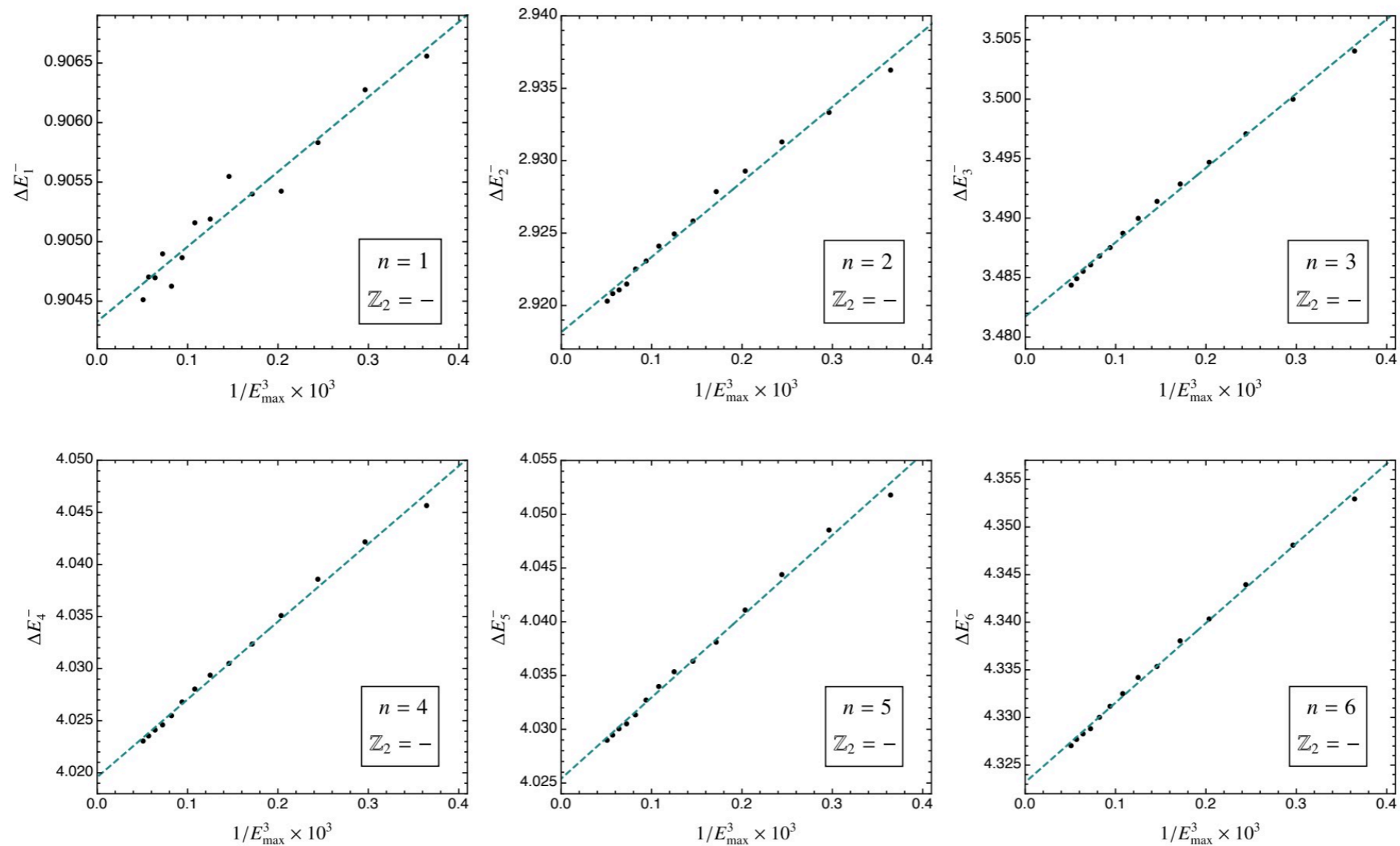
$\lambda/4\pi = 1, m_{\text{NO}} = 1, 2\pi R = 10$



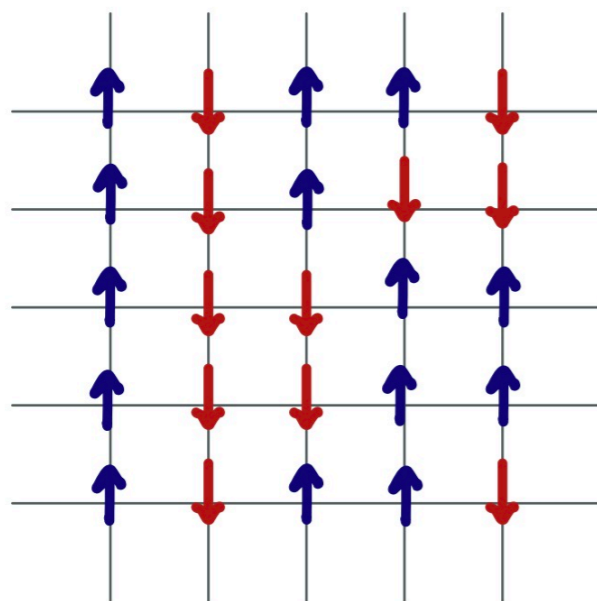
❖ $1/E_{\max}^3$ scaling

Excited States

❖ $1/E_{\max}^3$ scaling persists:



2D Ising Model Checks



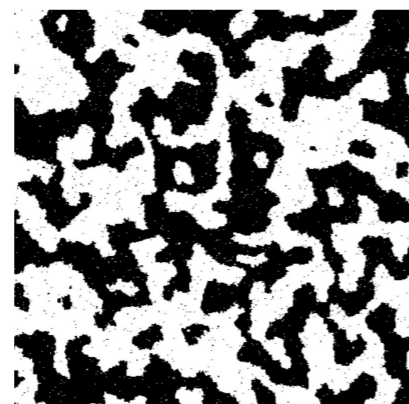
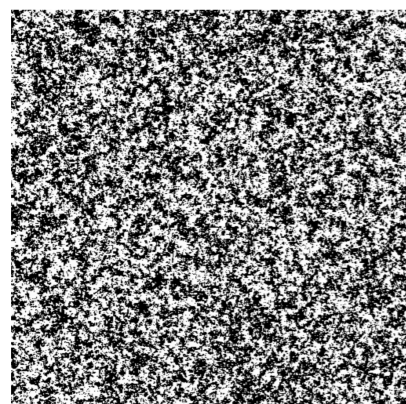
- 2D ϕ^4 is in the same universality class as the 2D Ising model

Onsager, Phys. Rev. 05, 117 (1944)

Anand, Genest, Katz, Khandker, Walters, arXiv:1704.04500

- Expect energy spectrum of 2D ϕ^4 to map onto operator dimensions of 2D Ising model
- 2D Ising Model phase transition:

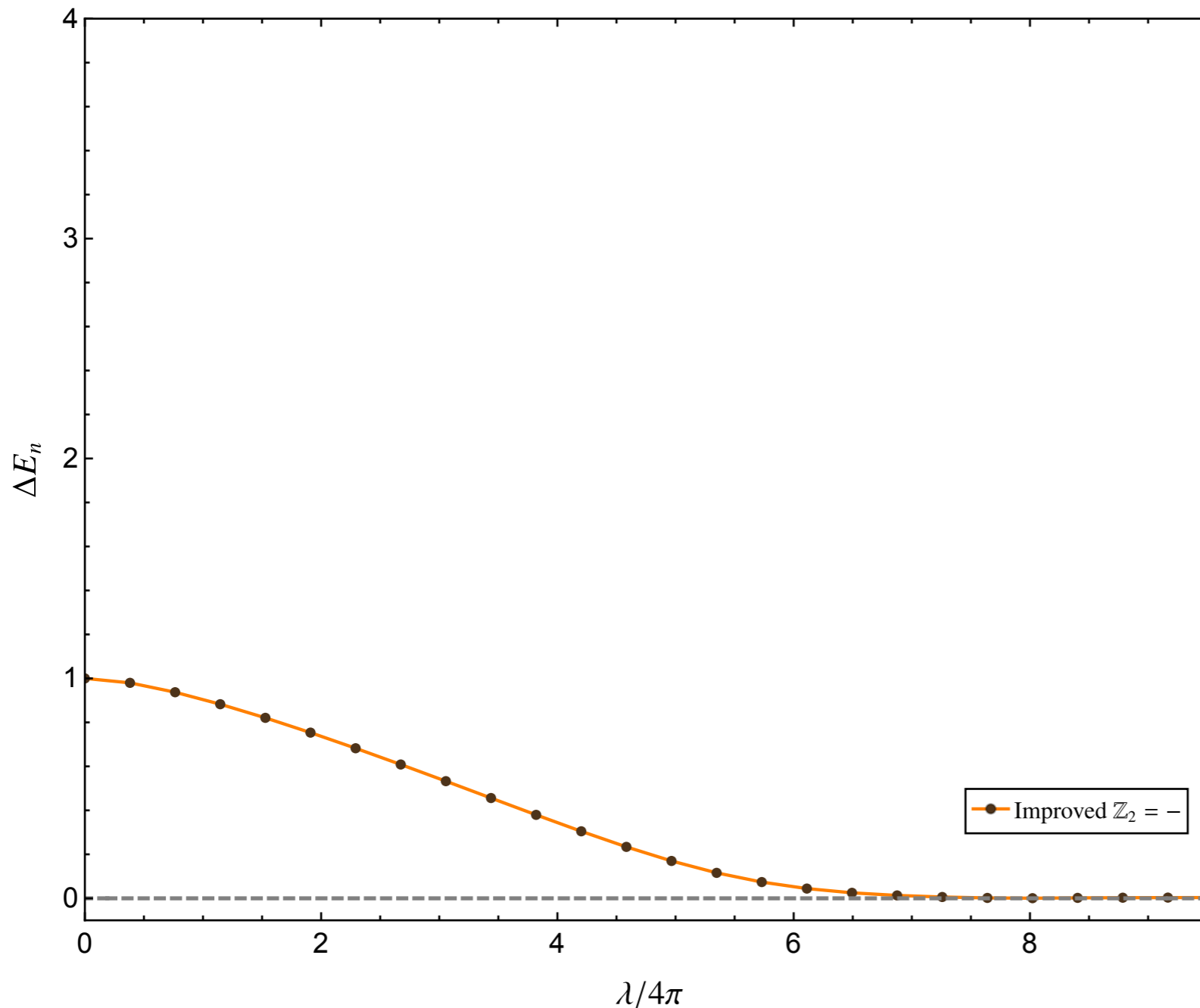
Decreasing T →



Demidov, www.ibiblio.org/e-notes/Perc/ising.htm

Probing the Critical Coupling

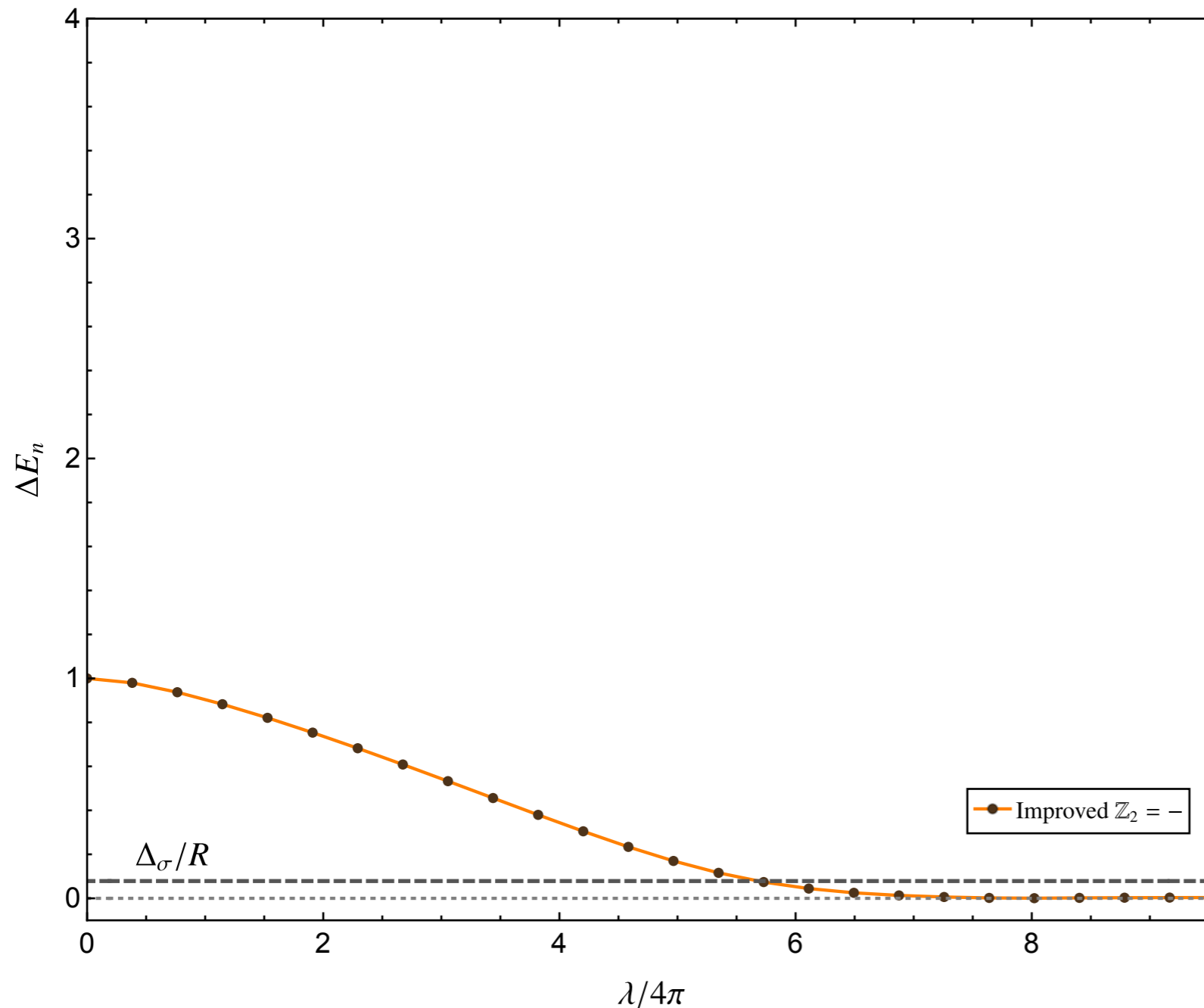
$$E_{\max} = 27, m_{\text{NO}} = 1, 2\pi R = 10$$



- What happens to the spectrum at larger coupling?
- Degenerate ground state emergence for strong coupling
- Indicates Z_2 symmetry is spontaneously broken

Probing the Critical Coupling

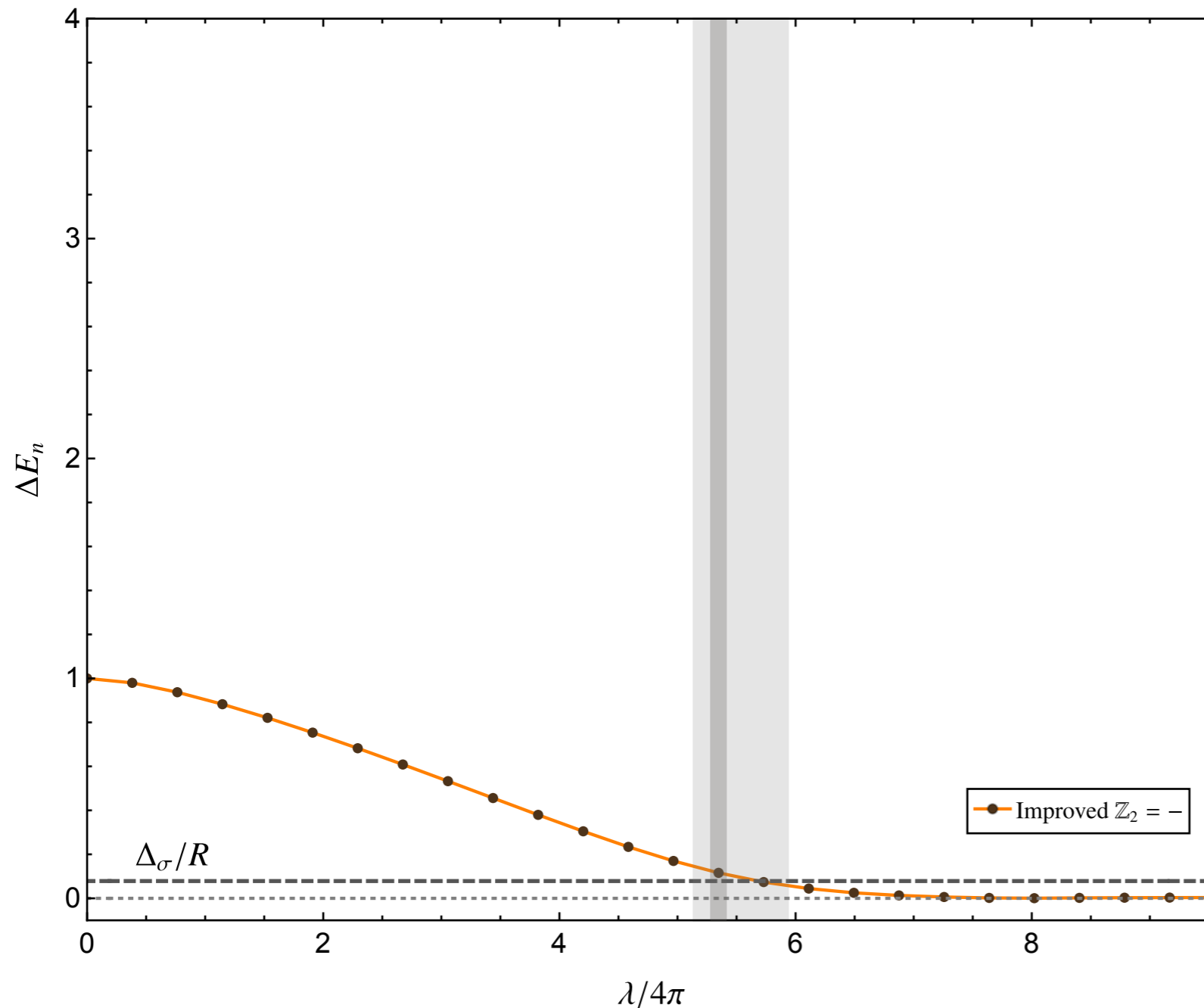
$$E_{\max} = 27, m_{\text{NO}} = 1, 2\pi R = 10$$



- What happens to the spectrum at larger coupling?
- 2D ϕ^4 is in same universality class as the 2D Ising model
- Should reproduce its spectrum near the critical coupling

Probing the Critical Coupling

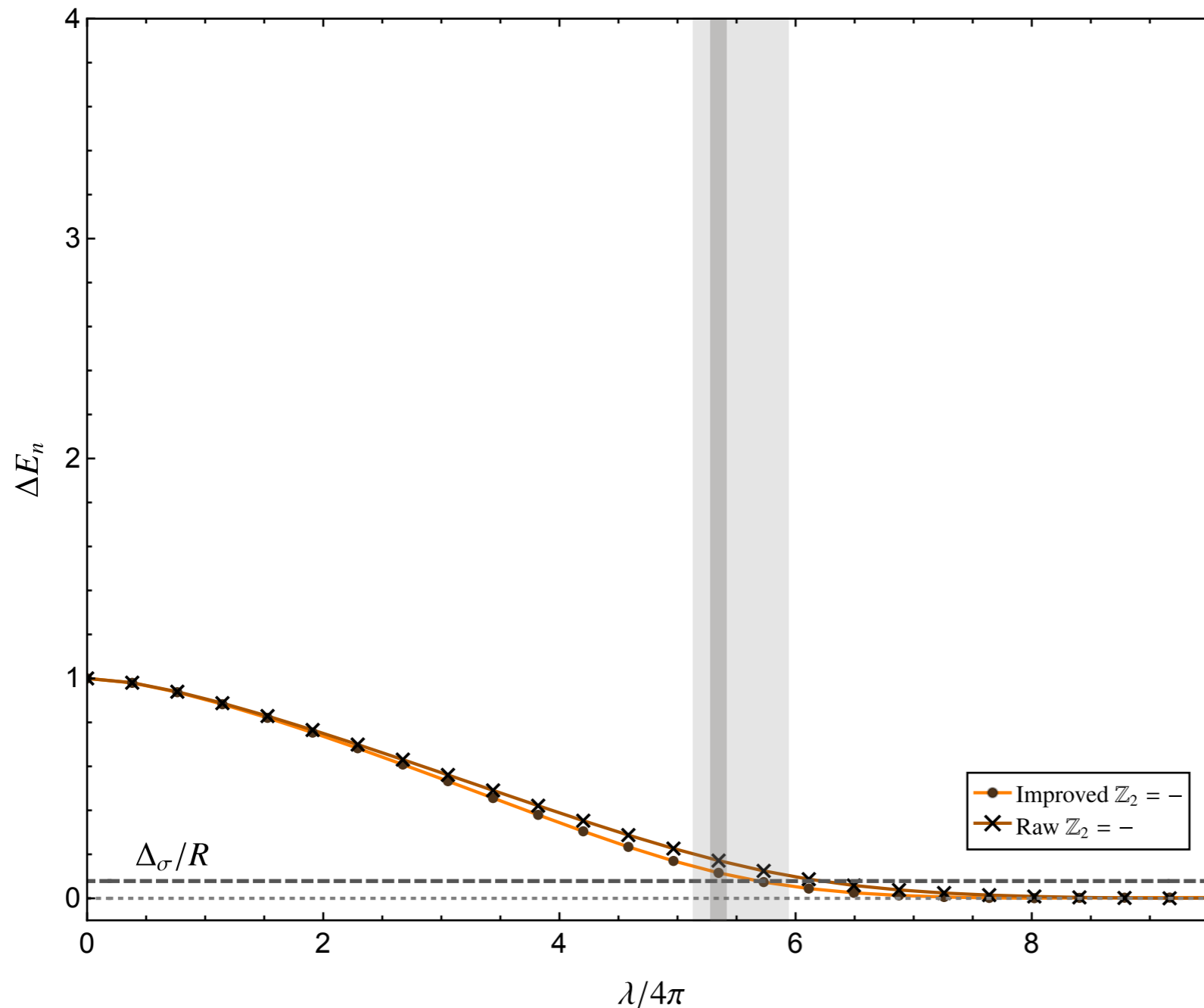
$$E_{\max} = 27, m_{\text{NO}} = 1, 2\pi R = 10$$



- What happens to the spectrum at larger coupling?
- 2D ϕ^4 is in same universality class as the 2D Ising model
- Shaded: Critical coupling region

Probing the Critical Coupling

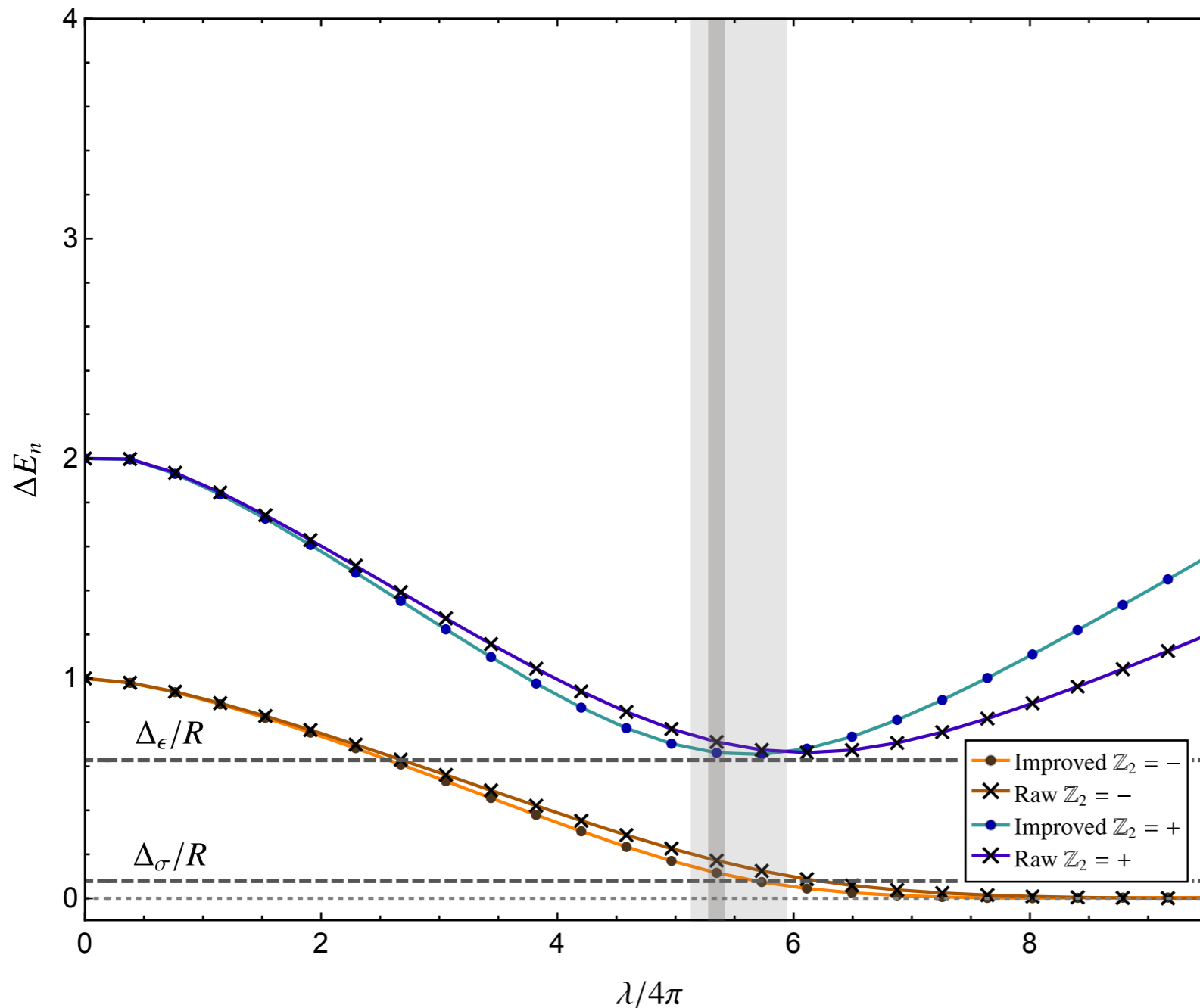
$$E_{\max} = 27, m_{\text{NO}} = 1, 2\pi R = 10$$



- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- The **improved theory** shows better agreement when compared to the **raw truncation**

Probing the Critical Coupling

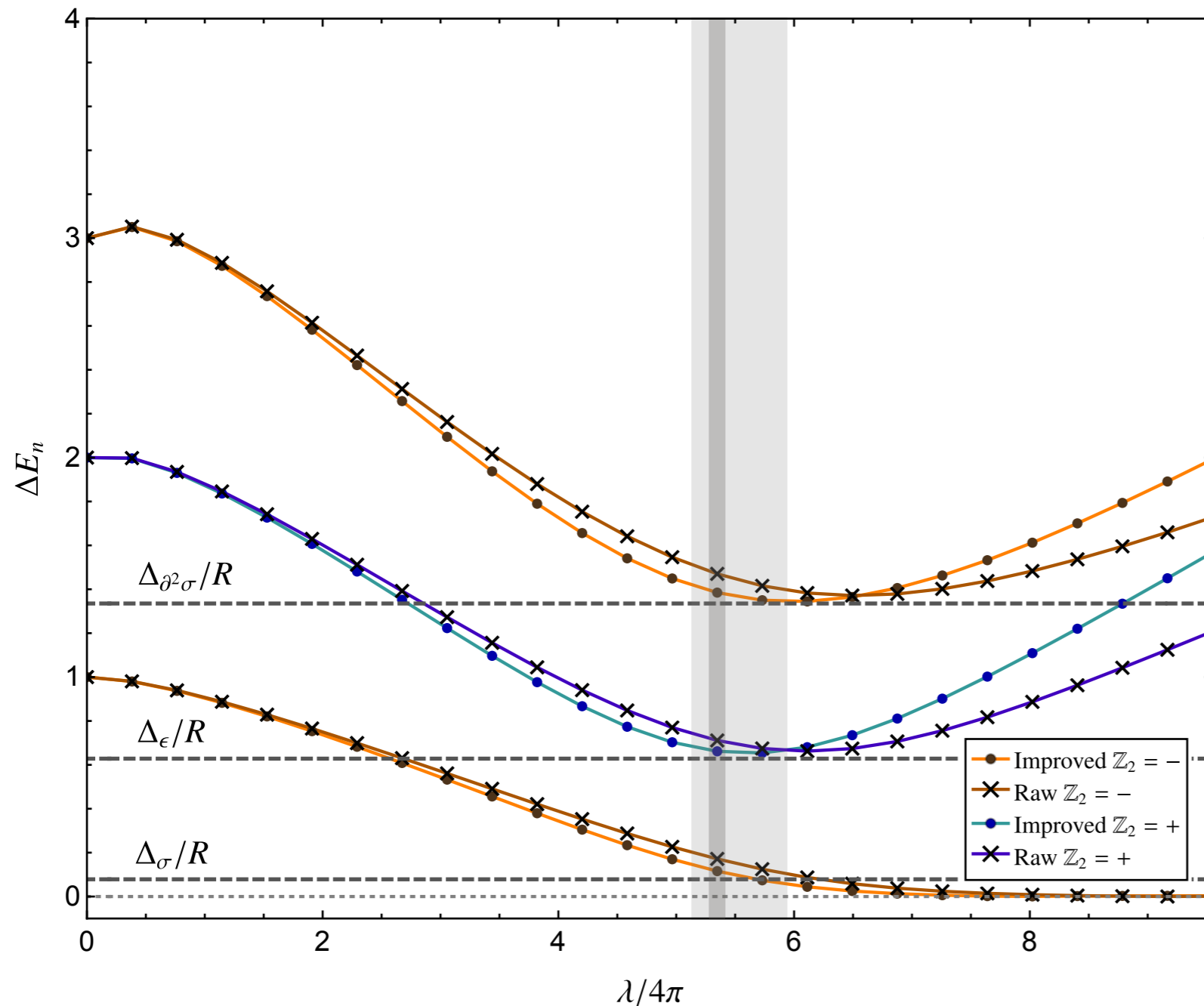
$$E_{\max} = 27, m_{\text{NO}} = 1, 2\pi R = 10$$



- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- The second excited state also probes the 2D Ising model prediction

Probing the Critical Coupling

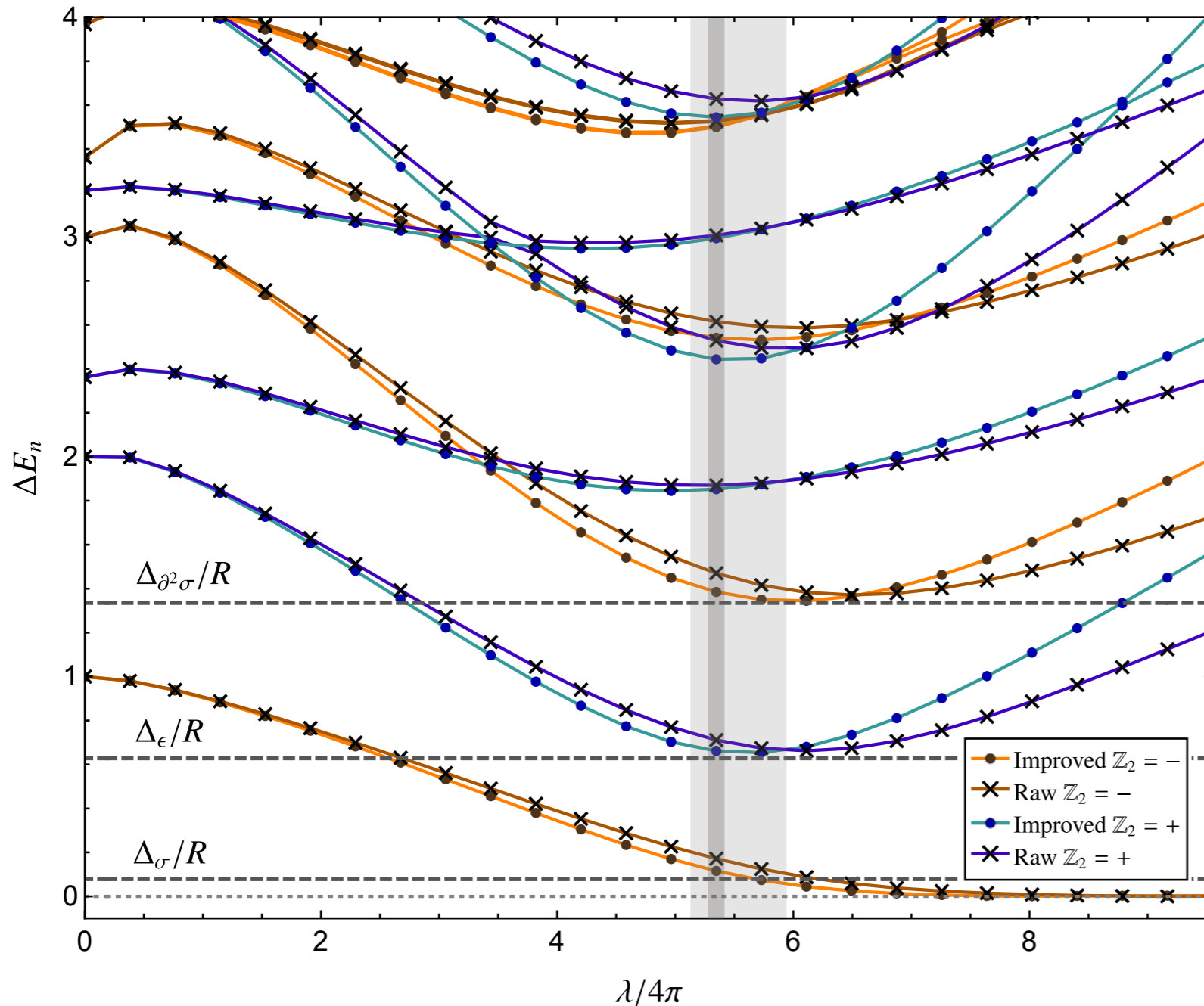
$$E_{\max} = 27, m_{\text{NO}} = 1, 2\pi R = 10$$



- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- The third excited state also probes the 2D Ising model prediction
- No \mathbb{Z}_2 degeneracy for higher excited states, probably an artifact of finite R

Probing the Critical Coupling

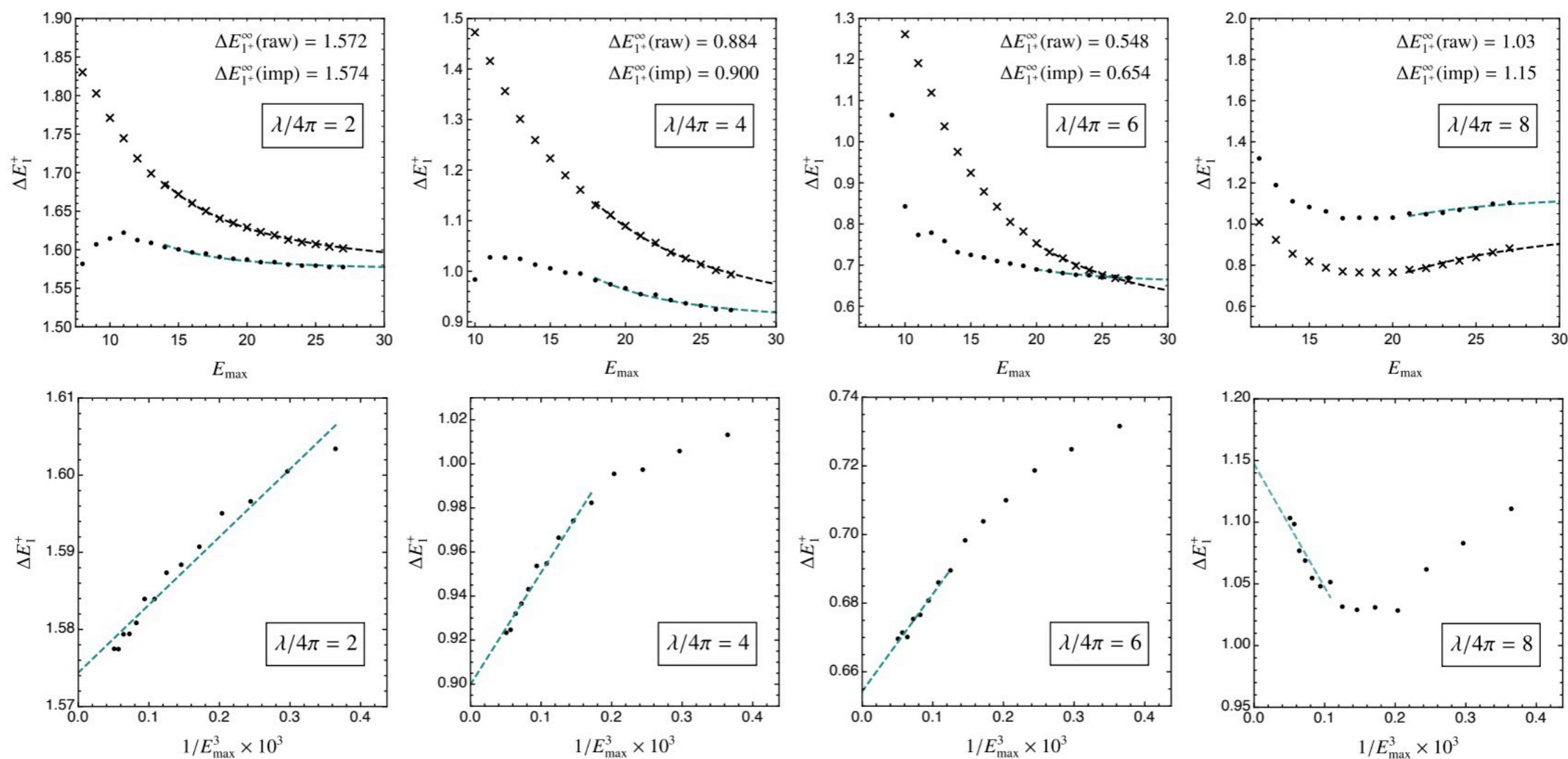
$$E_{\max} = 27, m_{\text{NO}} = 1, 2\pi R = 10$$



- Dark colors: **raw**
- Light colors: **improved**
- Shaded: Critical coupling region
- Odd/Even states
- The full spectrum
- No Z_2 degeneracy for higher excited states, probably an artifact of finite R

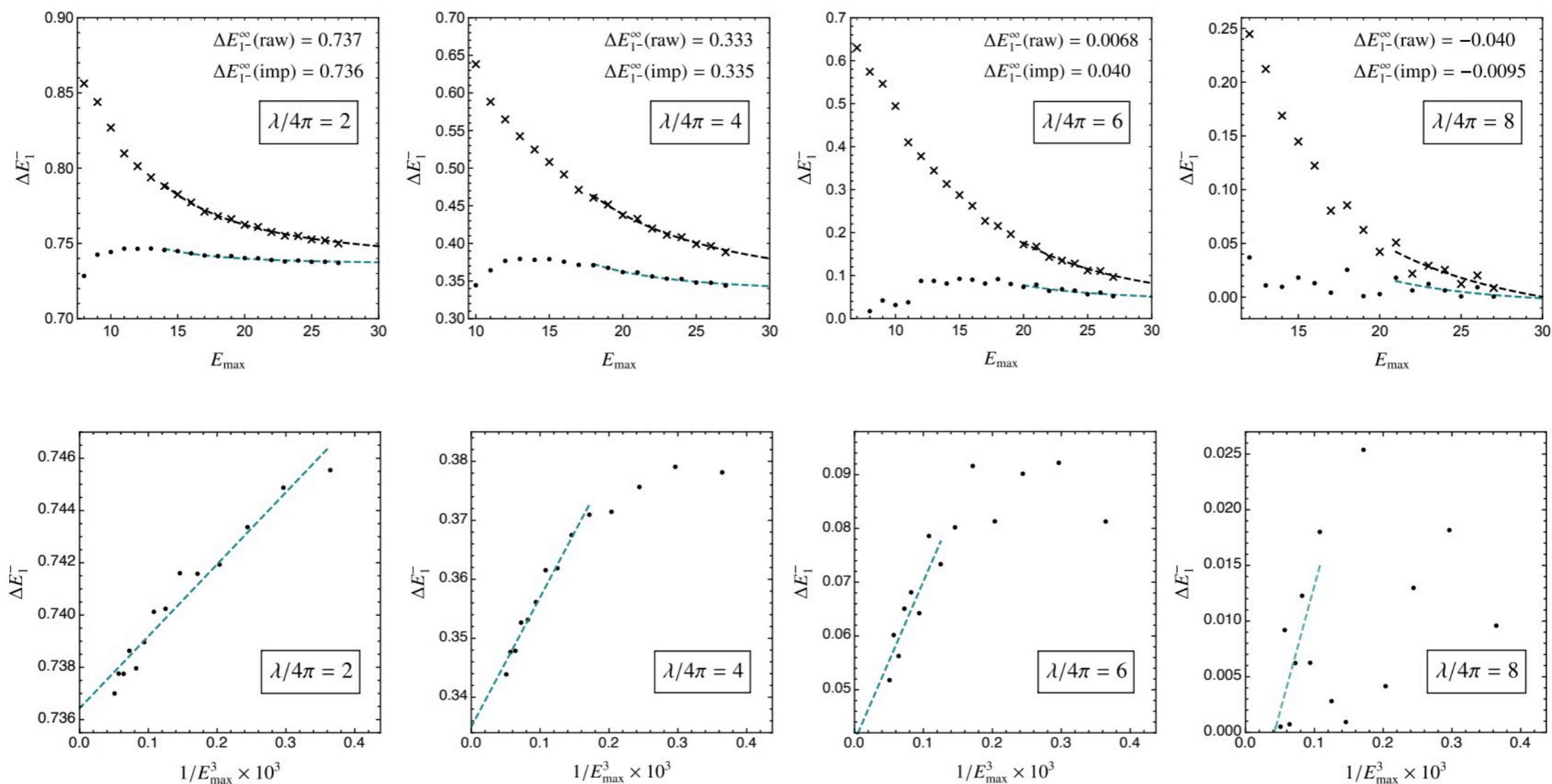
Stronger Couplings (Z_2 even)

- ❖ Need to increase truncation scale to see power-law scaling



Stronger Couplings (Z_2 odd)

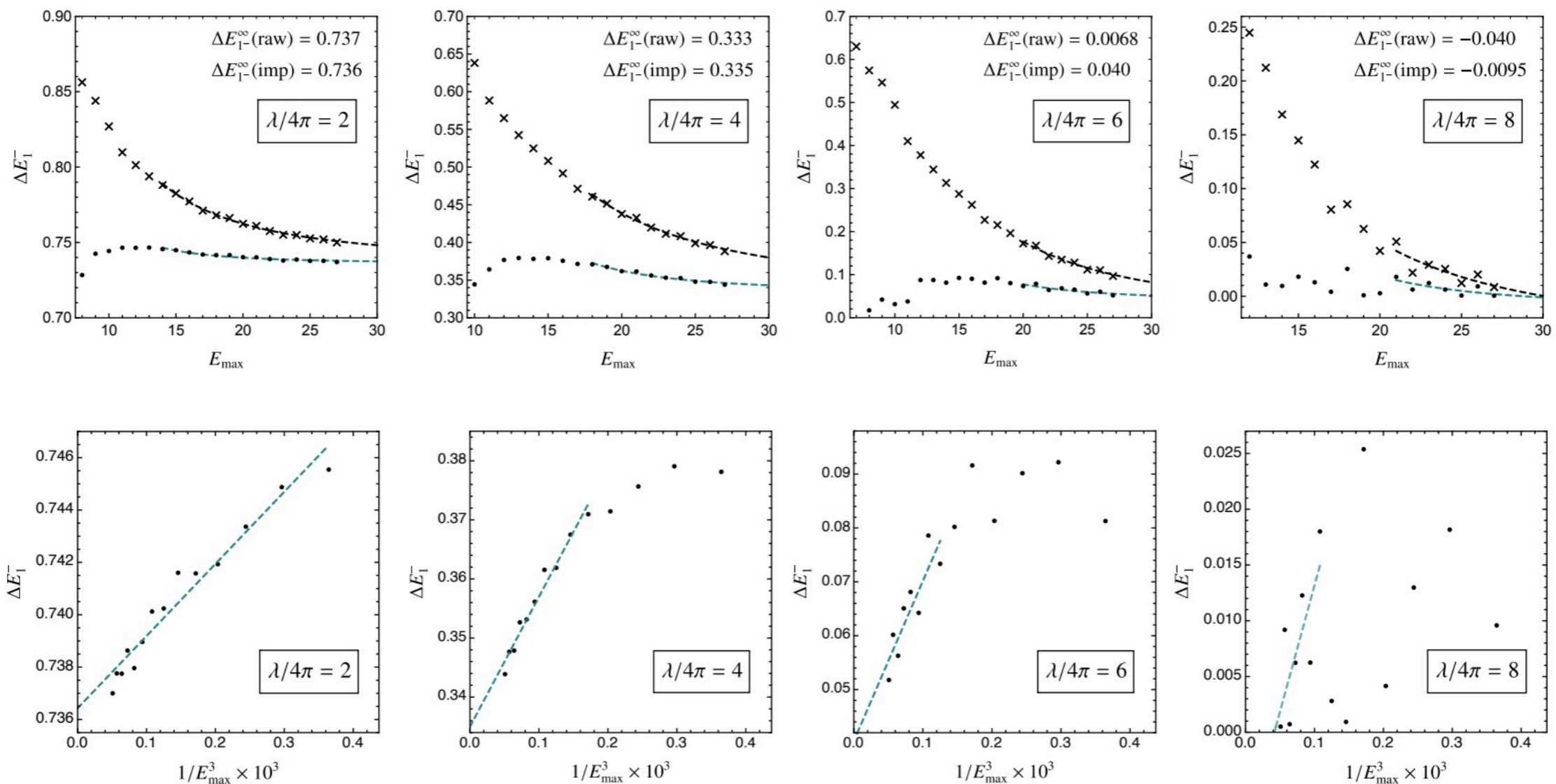
- Need to increase truncation scale to see power-law scaling



Stronger Couplings (Z_2 odd)

- Need to increase truncation scale to see power-law scaling

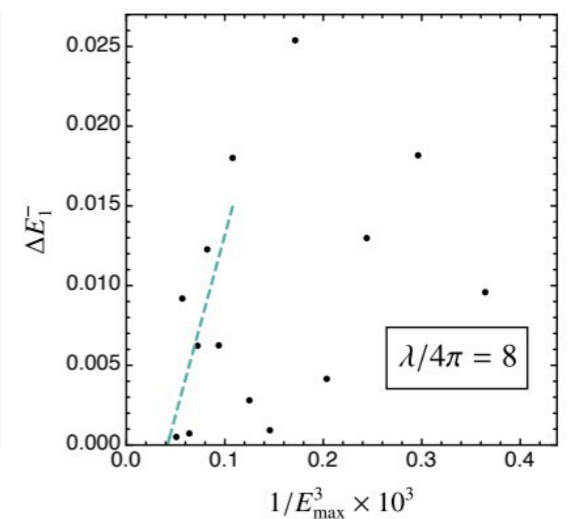
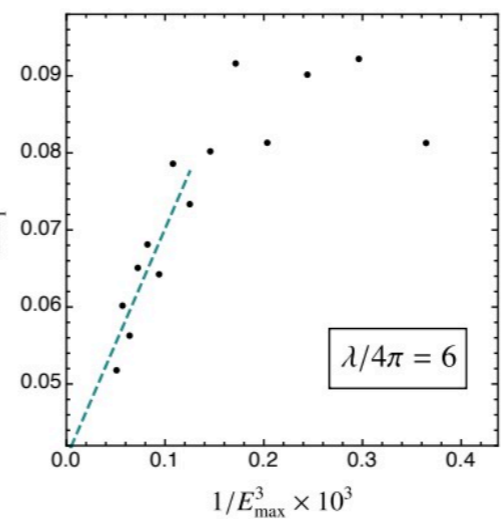
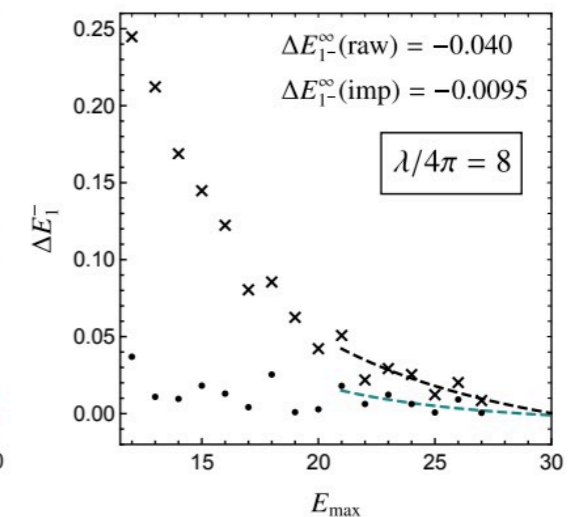
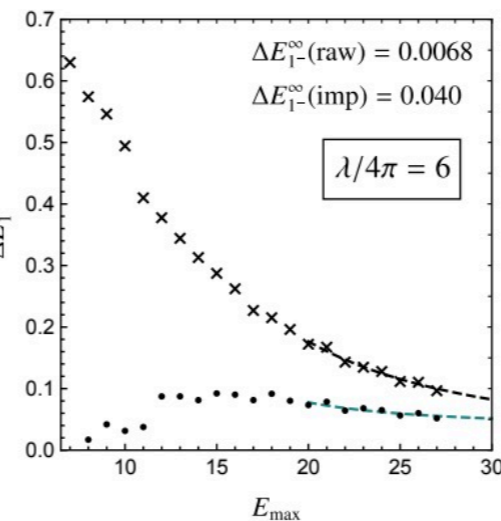
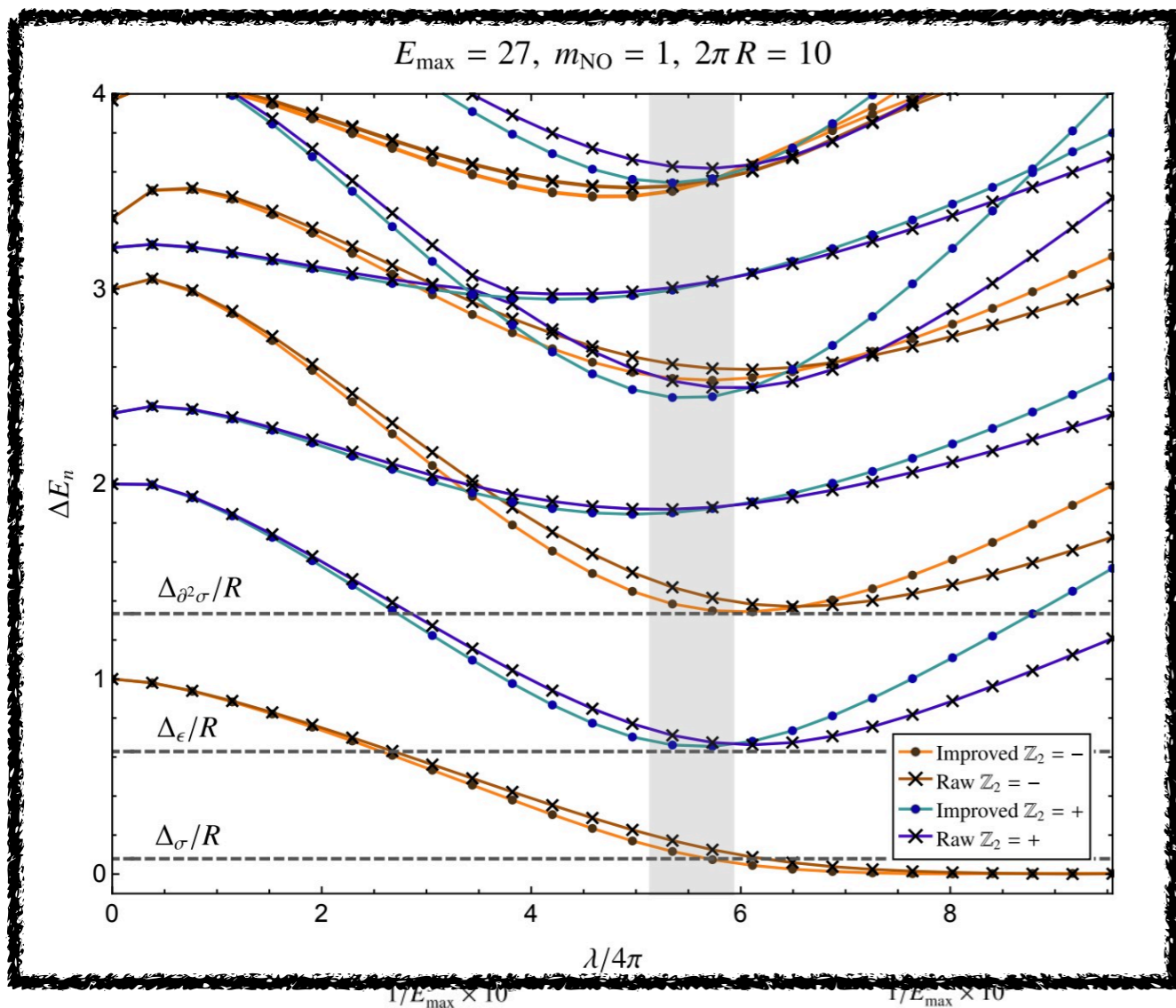
Numerical Noise from degenerate eigenvalues



Stronger Couplings (Z_2 odd)

- Need to increase truncation scale to see power-law scaling

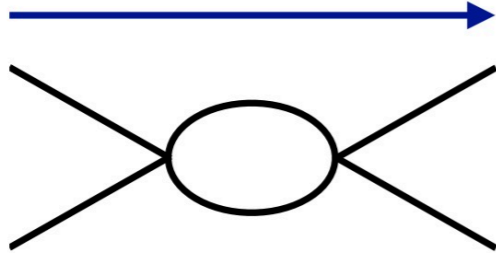
Numerical Noise from degenerate eigenvalues



Demonstrating at NLO

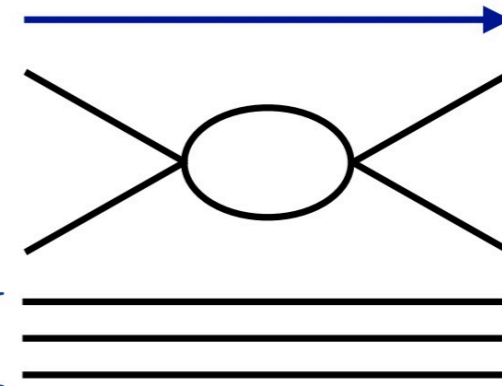
$$H_2 \sim \frac{\lambda^2}{E_{\max}^2} \sum_a \int \left[\phi^2 + \phi^4 + \frac{R^{-1} + H_0}{E_{\max}} (1 + \phi^2 + \phi^4) + \mathcal{O}(1/E_{\max}^2) \right]$$

Initial state to final state



VS.

Initial state to final state

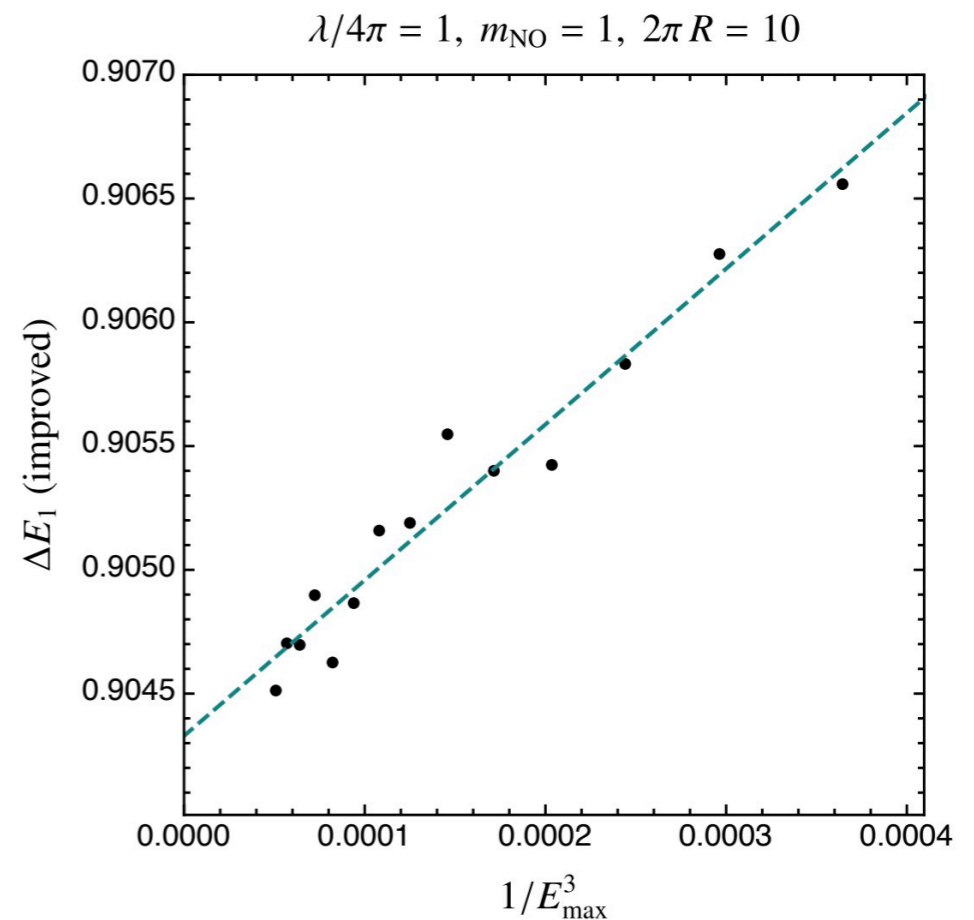
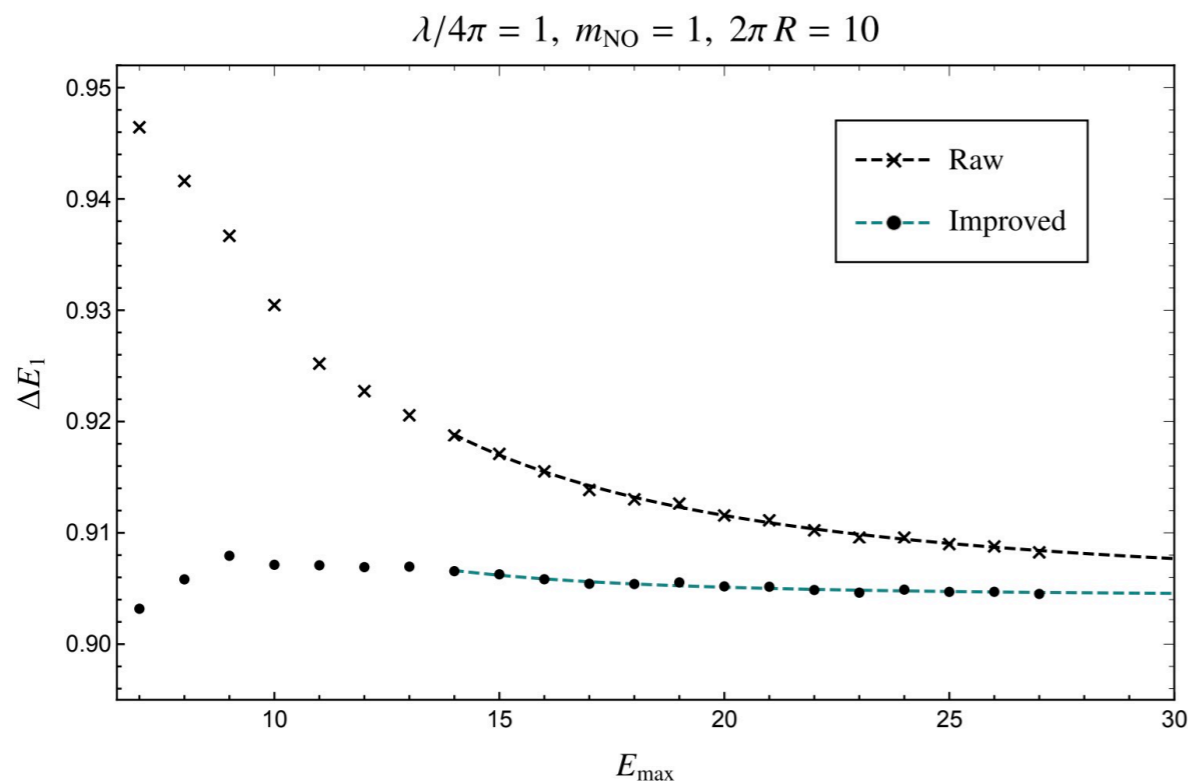


non-local contributions

$$\left[\text{Diagram with external lines 1, 2, 3, 4 and internal lines 5, 6} \right] - \left[\text{Diagram with external lines 1, 2, 3, 4 and internal lines 5, 6} \right]_{\text{eff}} = \frac{\lambda^2}{128\pi^2 R^2} \sum_{1, \dots, 4} \delta_{12,34} \langle f | \phi_4^{(-)} \phi_3^{(-)} \phi_2^{(+)} \phi_1^{(+)} | i \rangle \sum_k \frac{\Theta(2\omega_k - E_{\max})}{\omega_k^3} [1 + \mathcal{O}(E_{i,f}/E_{\max})]$$

$$\left[\text{Diagram with external lines 1, 2, 3, 4 and internal lines 5, 6} \right] - \left[\text{Diagram with external lines 1, 2, 3, 4 and internal lines 5, 6} \right]_{\text{eff}} = \frac{\lambda^2}{128\pi^2 R^2} \sum_{1, \dots, 4} \delta_{12,34} \langle f | \phi_4^{(-)} \phi_3^{(-)} \phi_2^{(+)} \phi_1^{(+)} | i \rangle \sum_{5,6} \delta_{56,34} \frac{\Theta(\omega_5 + \omega_6 - \omega_3 - \omega_4 + E_f - E_{\max})}{\omega_5 \omega_6 (\omega_3 + \omega_4 - \omega_5 - \omega_6)}$$

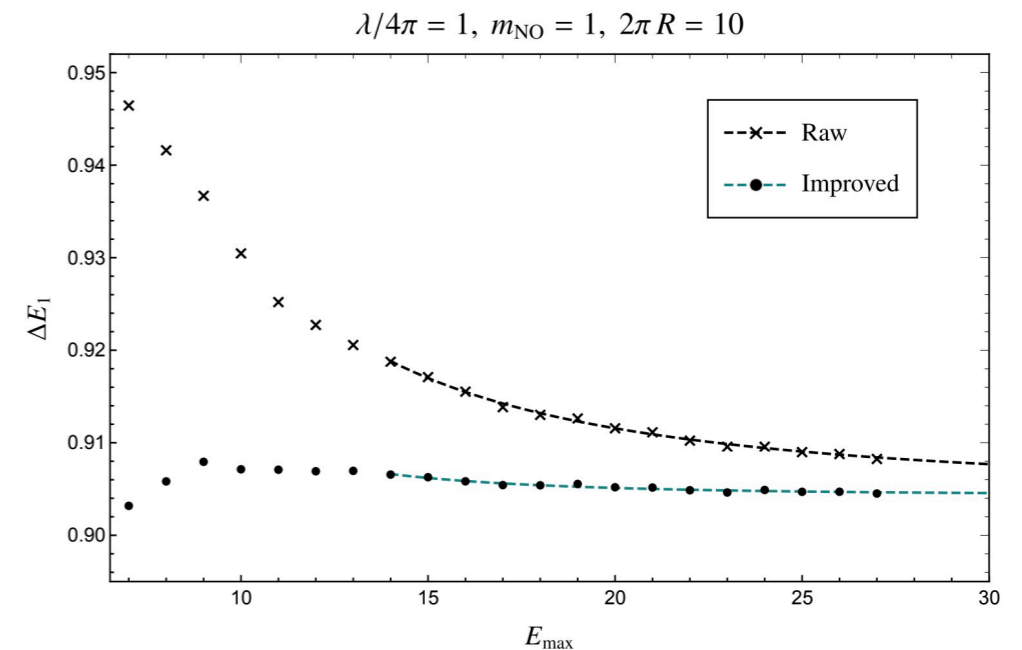
Numerical Noise at NLO



- ❖ By the time the corrections have smoothed to a scaling regime, noise makes it difficult to distinguish $1/E_{\text{max}}^3$ vs. $1/E_{\text{max}}^4$ scaling
- ❖ Strategies: increase R or increase λ

Conclusions

- ❖ Introduced a scheme for systematic, order-by-order improvement for Hamiltonian truncation: HTET
- ❖ Fast results obtained on a laptop
- ❖ Demonstrated improvement from $1/E_{\max}^2$ scaling for raw truncation to $1/E_{\max}^3$ scaling for improved theory
- ❖ This improvement program would be more convincing at NLO
 - ❖ 2D ϕ^4 theory at next order ($1/E_{\max}^4$ scaling)
 - ❖ Next order may require state-dependent counterterms



Thank you!

Back-up Slides

Matching Strategy

- ❖ Start with some initial state at $t_i = 0$, turn off interactions adiabatically for $t > 0$

$$V \rightarrow V e^{-\epsilon t}$$

- ❖ Evolve the state to $t_f \rightarrow +\infty$ and compute its overlap with states E_i

$$\lim_{t_f \rightarrow \infty} \langle f | e^{-iH_{\text{eff}} t_f} | i \rangle = \lim_{\epsilon \rightarrow 0^+} \langle f | T \exp \left\{ -i \int_0^{\infty} dt (H_0 + V e^{-\epsilon t}) \right\} | i \rangle$$

- ❖ Go to the interaction picture:

$$|\psi(t)\rangle_{\text{IP}} = e^{iH_0 t} |\psi(t)\rangle_{\text{SP}} \quad \mathcal{O}_{\text{IP}}(t) = e^{iH_0 t} \mathcal{O}_{\text{SP}} e^{-iH_0 t}$$

$$U_{\text{IP}}(t_f, t_i) = T \exp \left\{ -i \int_{t_i}^{t_f} dt V_{\text{IP}}(t) \right\} \quad V_{\text{IP}}(t) = e^{iH_0 t} V e^{-\epsilon t} e^{-iH_0 t}$$

Matching an Operator

- ❖ We would prefer to match an operator order by order

- ❖ Start by turning off interactions adiabatically $V \rightarrow V e^{-\epsilon t}$

- ❖ Evolving in time, we find an ill-defined phase

$$\lim_{t_f \rightarrow \infty} \langle f | e^{-iH_{\text{eff}} t_f} | i \rangle = \lim_{\epsilon \rightarrow 0^+} \langle f | T \exp \left\{ -i \int_0^\infty dt H_0 + V e^{-it} \right\} | i \rangle$$

- ❖ Can be factored out in the interaction picture

$$|\Psi(t)\rangle_{\text{IP}} = e^{iH_0 t} |\Psi(t)\rangle_{\text{SP}} \quad \mathcal{O}_{\text{IP}} = e^{iH_0 t} \mathcal{O}_{\text{SP}} e^{-iH_0 t}$$

- ❖ Time evolution: $U_{\text{IP}}(t_f, t_i) = T \exp \left\{ -i \int_{t_i}^{t_f} dt V_{\text{IP}}(t) \right\}$

Expansion of the Transition Matrix

❖ Time evolution:

$$U_{\text{IP}}(t_f, t_i) = T \exp \left\{ -i \int_{t_i}^{t_f} dt V_{\text{IP}}(t) \right\} \quad V_{\text{IP}}(t) = e^{iH_0 t} V e^{-\epsilon t} e^{-iH_0 t}$$

$$\lim_{t_f \rightarrow \infty} \langle f | U_{\text{IP}}(t_f, 0) | i \rangle \ni \langle f | i \rangle$$

$$\lim_{t_f \rightarrow \infty} \langle f | U_{\text{IP}}(t_f, 0) | i \rangle \ni -i \int_0^{\infty} dt \langle f | e^{iH_0 t} V e^{-\epsilon t} e^{-iH_0 t} | i \rangle = \langle f | V | i \rangle \frac{1}{E_{fi} + i\epsilon}$$

$$\begin{aligned} \lim_{t_f \rightarrow \infty} \langle f | U_{\text{IP}}(t_f, 0) | i \rangle &\ni - \int_0^{\infty} dt \int_{-\infty}^t dt' \langle f | e^{iH_0 t'} V e^{-\epsilon t'} e^{-iH_0 t'} e^{iH_0 t} V e^{-\epsilon t} e^{-iH_0 t} | i \rangle \\ &= \sum_{\alpha} \int_0^{\infty} dt \langle f | V \frac{1}{iE_{f\alpha} - \epsilon} | \alpha \rangle \langle \alpha | e^{iE_{f\alpha} t + iE_{\alpha i} t - 2\epsilon t} V | i \rangle \\ &= \sum_{\alpha} \frac{\langle f | V | \alpha \rangle \langle \alpha | V | i \rangle}{E_{f\alpha} + i\epsilon E_{fi} + 2i\epsilon} \end{aligned}$$

Power Counting

$$H_{\text{eff}} = H_0 + H_1 + H_2 + \dots \quad H_n \sim \mathcal{O}(V^n) \quad V \sim \lambda \phi^4 \quad [\lambda] = 2$$

- ❖ What is the dependence on E_{max} for each of these progressive corrections?

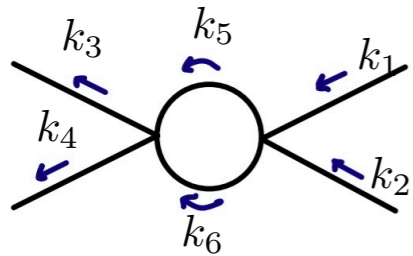
$$H_n \sim \frac{\lambda^n}{E_{\text{max}}^{2n-2}} \sum_a \int C_{na} \left(\frac{H_0}{E_{\text{max}}}, \frac{R^{-1}}{E_{\text{max}}}, \frac{m_Q}{E_{\text{max}}} \right) \mathcal{O}_{na} \left(\frac{\partial x}{E_{\text{max}}}, \frac{\partial t}{E_{\text{max}}}, \phi \right)$$

$$H_2 \sim \frac{\lambda^2}{E_{\text{max}}^2} \sum_a \int \left[\phi^2 + \phi^4 + \frac{R^{-1} + H_0}{E_{\text{max}}} (1 + \phi^2 + \phi^4) + \frac{1}{E_{\text{max}}} (\phi + \phi^3) i \dot{\phi} + \mathcal{O}(1/E_{\text{max}}^2) \right]$$

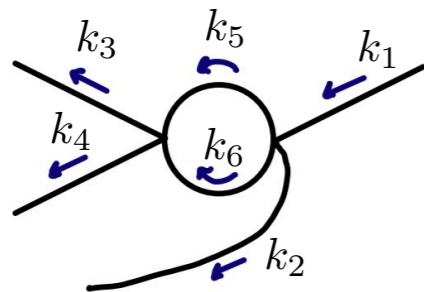
$$H_3 \sim \frac{\lambda^3}{E_{\text{max}}^4}$$

$$\mathcal{O}(1/E_{\text{max}}^3)$$

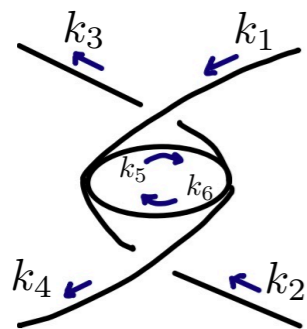
Diagrammatic Representation of Matching



$$\frac{1}{8} \left(\frac{\lambda}{2\pi R} \right)^2 \sum_{k_1, \dots, k_6} \delta_{k_1+k_2, k_5+k_6} \langle f | \phi_4^- \phi_3^- \phi_2^+ \phi_1^+ | i \rangle \frac{1}{2\omega_{k_5}} \frac{1}{2\omega_{k_6}} \frac{1}{\omega_3 + \omega_4 - \omega_5 - \omega_6 + i\epsilon}$$



$$\frac{1}{8} \left(\frac{\lambda}{2\pi R} \right)^2 \sum_{k_1, \dots, k_6} \delta_{k_1, k_2+k_5+k_6} \langle f | \phi_4^- \phi_3^- \phi_2^- \phi_1^+ | i \rangle \frac{1}{2\omega_{k_5}} \frac{1}{2\omega_{k_6}} \frac{1}{\omega_3 + \omega_4 - \omega_5 - \omega_6 + i\epsilon}$$



$$\frac{1}{8} \left(\frac{\lambda}{2\pi R} \right)^2 \sum_{k_1, \dots, k_6} \delta_{k_1+k_2, k_5+k_6} \langle f | \phi_4^- \phi_3^- \phi_2^+ \phi_1^+ | i \rangle \frac{1}{2\omega_{k_5}} \frac{1}{2\omega_{k_6}} \frac{1}{-\omega_1 - \omega_2 - \omega_5 - \omega_6 + i\epsilon}$$

Varying m_Q

