Proving chiral symmetry breaking in QCD from 't Hooft anomaly matching

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The Abdus Salam International Centre for Theoretical Physics

Based on hep-th/2212.02930, 2404.02967, 2404.02971,
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Complementary purposes of our papers:

- hep-th/2212.02930
  Revisiting the literature, and clarifying the assumptions that lies in the proofs that have been considered, with simplified examples

- hep-th/2404.02967
  Presenting our new proof in full generality but without examples

- hep-th/2404.02971
  Exemplifying all the arguments and proofs in the companion papers, with many detailed examples

Apologies for missing important references during the talk, please find the references in our papers.

Infrared phases of QCD

- QCD: (3+1) d $SU(N_c)$ Yang-Mills theory coupled to N_f massless quarks in the fundamental representation
- It is well known that the infrared phases depend on the values of N_c and N_f:

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- It is well known that the infrared phases depend on the values of N_c and N_f:

  1) Infrared free quarks and gluons for $N_f \geq 11N_c/2$

  2) Interacting CFT for $N_f^\star \leq N_f < 11N_c/2$

  3) Chiral symmetry breaking for $2 \leq N_f < N_f^\star$

  4) Gapped with unique vacuum for $N_f = 1$

  5) The θ parameter becomes physical for $N_f = 0$:
  gapped with unique vacuum for generic θ; two degenerate vacua at $\theta = \pi$

Infrared phases of QCD

- The picture of infrared phases is mainly based on empirical evidences, lattice results and educated guessworks
- Very little has been rigorously/coherently derived
- Other exotic phases of QCD may be possible

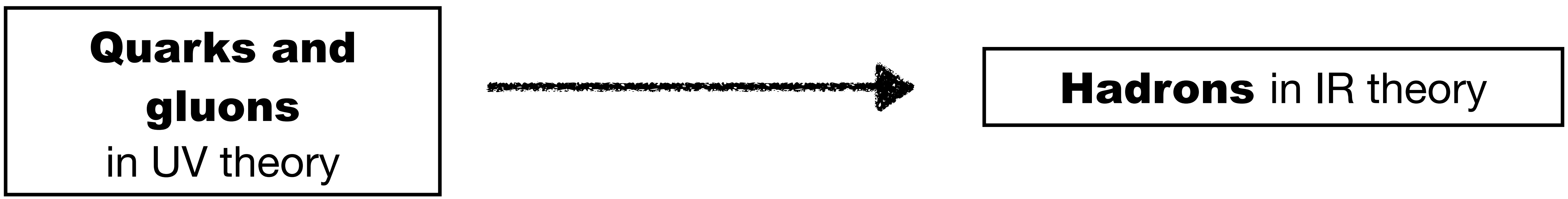
Unsolved

Yang-Mills & The Mass Gap

Experiment and computer simulations suggest the existence of a “mass gap” in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

The chiral symmetry breaking phase

- Characterized by the following RG flow



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Quarks and gluons in UV theory → **Hadrons** in IR theory

- Featured by **color-singlet hadrons** and **the chiral symmetry breaking pattern**

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \to SU(N_f)_V \times U(1)_B$$

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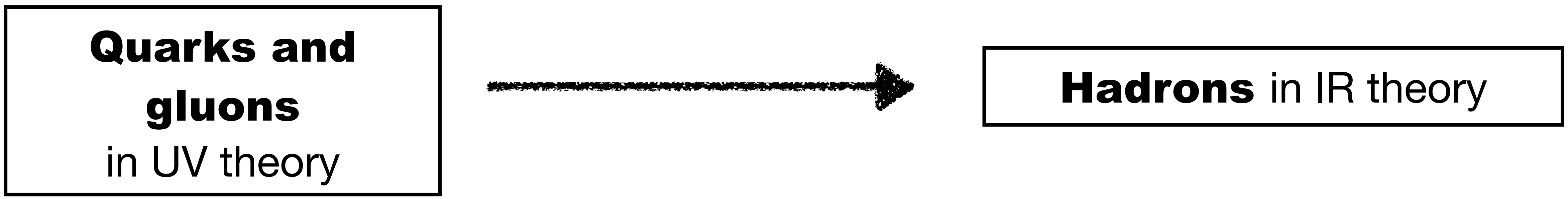

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$$SU(N_f)_L \times SU(N_f)_R \times U(1)_B \to SU(N_f)_V \times U(1)_B$$

- The fact that all the hadrons must be color singlet is conventionally denoted as "**confinement**". More precisely, it is "**color screening**" for dynamical quarks in the fundamental representation, where the Wilson line obeys perimeter law.

In this talk

- We are going to derive **chiral symmetry breaking** as a consequence of **confinement** in QCD, following the seminal work of 't Hooft in 1979

NATURALNESS, CHIRAL SYMMETRY, AND SPONTANEOUS

CHIRAL SYMMETRY BREAKING

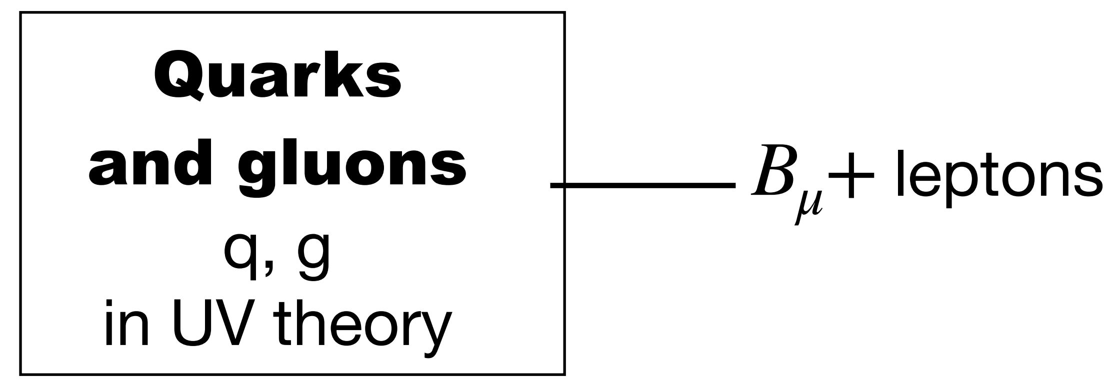
G. 't Hooft

- The above statement will soon be turned into a precise algebraic problem, and we will solve it.

’t Hooft anomaly matching conditions

- Consider a QCD-like theory with $G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$

- ’t Hooft: weakly gauging $G[N_f]$ and adding spectator fermions (leptons), which are charged only under $G[N_f]$ but not under color, to cancel the anomalies of quarks



$$\mathscr{A}(q) + \mathscr{A}(l) = 0$$

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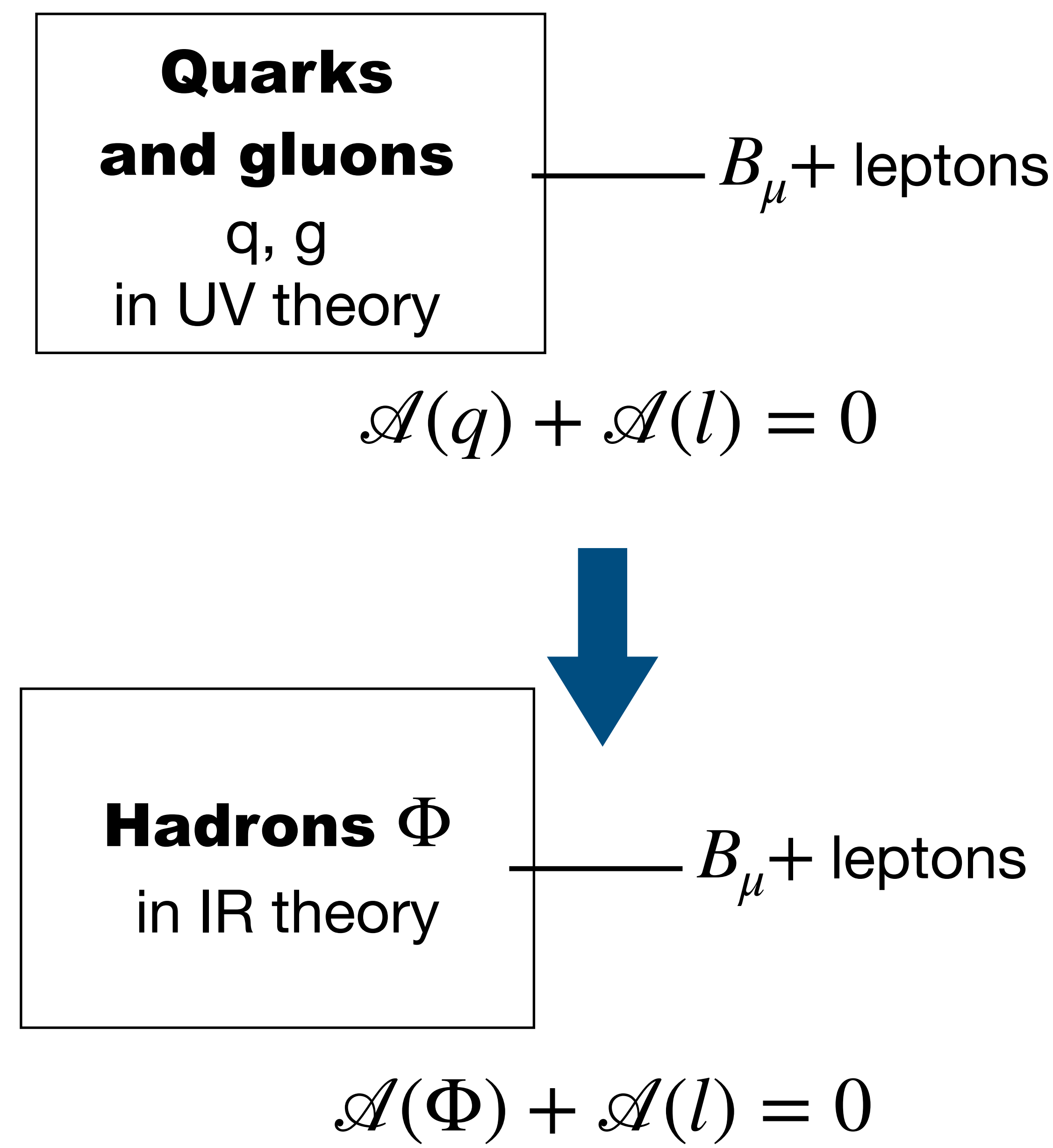
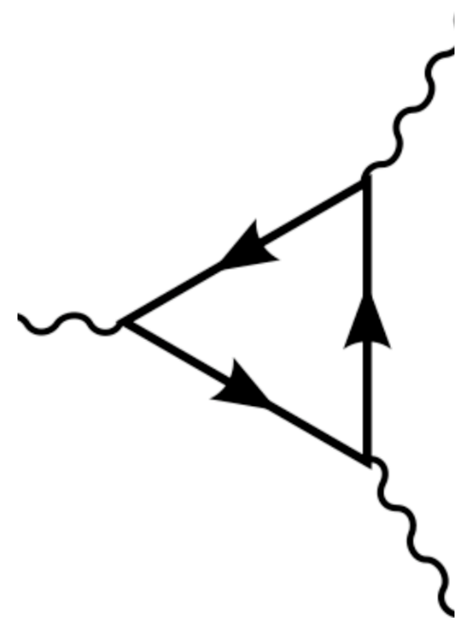
- Anomalies match in the UV and IR
  $$\mathscr{A}(q) = \mathscr{A}(\Phi)$$



’t Hooft anomalies in QCD

- For our purpose, let us consider the perturbative $[SU(N_f)_{L,R}]^3$ and $[SU(N_f)_{L,R}]^2U(1)_B$ ’t Hooft anomalies



- It is possible to have ’t Hooft anomalies involving one-form symmetries by identifying the discrete quotient correctly. This offers finer probes to the strong dynamics, but we will not consider them.

- Anomalies can be matched at infrared by
  1) Pions from chiral symmetry breaking
  2) Massless composite spin-1/2 fermions

- Weinberg-Witten theorem states that no massless particles with spin > 1/2 can exist which are charged under $G[N_f]$

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$$n_q - n_{\bar{q}} = bN_c$$

- Massless particles are classified by irreps of $G[N_f]$

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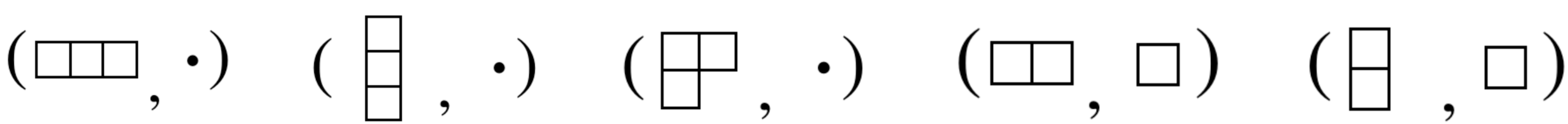
- The assumption of confinement (i.e. color screening) implies that the numbers of constituent quarks and antiquarks have to satisfy the constraint

$$n_q - n_{\bar{q}} = bN_c$$

- For example, for $N_c = 3$ and $2 < N_f < N_f^\star$ one can consider the following spectrum of baryons (e.g. $n_{\bar{q}} = 0$) with $b = 1$



- More examples, see 2404.02971

- If chiral symmetry is not broken, then the spectrum of massless fermions must satisfy anomaly matching conditions (AMC):

$$\sum_{r \in \mathcal{R}[N_f]} \ell(r)\, A_i(r) = N_c\, A_i(r_{q_L})$$

anomaly coefficient
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anomaly coefficient
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$$\sum_{r\in\mathcal{R}[N_f]} \ell(r)\, A_i(r) = N_c\, A_i(r_{q_L})$$

Index $\ell(r) \equiv$ # times r appears in the spectrum with helicity $+1/2$ $-$ # times r appears in the spectrum with helicity $-1/2$

Clearly 1) all indices must be integers for a physical spectrum

2) the index vanishes for vectorlike matter.

3) Nontrivial indices (i.e. $\ell(r) > 1$) imply enhanced symmetry in the infrared.

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- The statement can be proven to be true when
  — N_c is even such that the infrared spectrum is bosonic
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To have some intuition, see the following example...

- Consider $N_c = 3$ and $N_f > 2$, and the following spectrum of massless composite fermions (also with their parity-conjugated partners) with the corresponding indices:

Tensor	$(\square\square\square\,,\ \cdot)$	$\left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\,,\ \cdot\right)$	$\left(\begin{smallmatrix}\square\square\\\square\end{smallmatrix}\,,\ \cdot\right)$
Index	ℓ_a	ℓ_b	ℓ_c
Tensor	$(\square\square\,,\ \square)$	$\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}\,,\ \square\right)$	
Index	ℓ_d	ℓ_e	

$[SU(N_f)_L]^3$

$$\frac{(N_f+3)(N_f+6)}{2}\ell_a + \frac{(N_f-3)(N_f-6)}{2}\ell_b + (N_f^2-9)\ell_c + \frac{N_f(N_f+7)}{2}\ell_d + \frac{N_f(N_f-7)}{2}\ell_e = 3$$

$U(1)_B\ [SU(N_f)_L]^2$

$$\frac{(N_f+2)(N_f+3)}{2}\ell_a + \frac{(N_f-2)(N_f-3)}{2}\ell_b + (N_f^2-3)\ell_c + \frac{N_f(N_f+3)}{2}\ell_d + \frac{N_f(N_f-3)}{2}\ell_e = 1$$

No integral solution exists when $N_f = 0 \bmod 3$

Prime factor —

In $QCD[N_c, mp]$*, where* p *is a prime factor of* N_c *and* m *a positive integer, there exist no integral solutions of the* $[SU(mp)_{L,R}]^2 U(1)_V$ *AMC. Therefore,* χSB *must occur in* $QCD[N_c, mp]$ *if the theory confines.*

- We proved this statement in full generality, see 2404.02967

Additional constraints needed

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- Question: can we find additional constraints that can be used together with AMC?
- Answer: Yes, the so-called Persistent Mass Condition (PMC)

— The intuition is to deform the massless theory with small quark masses and keep track of the symmetries. This is another probe which is allowed only in vectorlike theories.

Persistent Mass Conditions

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- Proven by Vafa and Witten with mild assumptions
- As a remark, PMC implies that the vectorlike part of $G[N_f]$ cannot be spontaneously broken (i.e. the Vafa-Witten theorem)

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2) The bound on the quark propagator in the background of gauge fields (with some technicalities on smearing):

$$|S_A(x,y)| \leq e^{-m|x-y|}$$

- Let $B(x)$ be an operator with nonzero charge under flavor symmetry group. If all quarks have bare mass m, it follows that

$$|\langle B^\dagger(x) B(y)\rangle| \leq e^{-m\cdot n|x-y|}$$

n = number of quark propagators

- Now, let m be the bare mass of one flavor, and ϵ that of the others, with $\epsilon \to 0$. Let $B(x)$ be an operator, it follows that

$$|\langle B(x)^\dagger B(y)\rangle| \leq e^{-(n_H \cdot m + n_L \cdot \epsilon)|x-y|}$$

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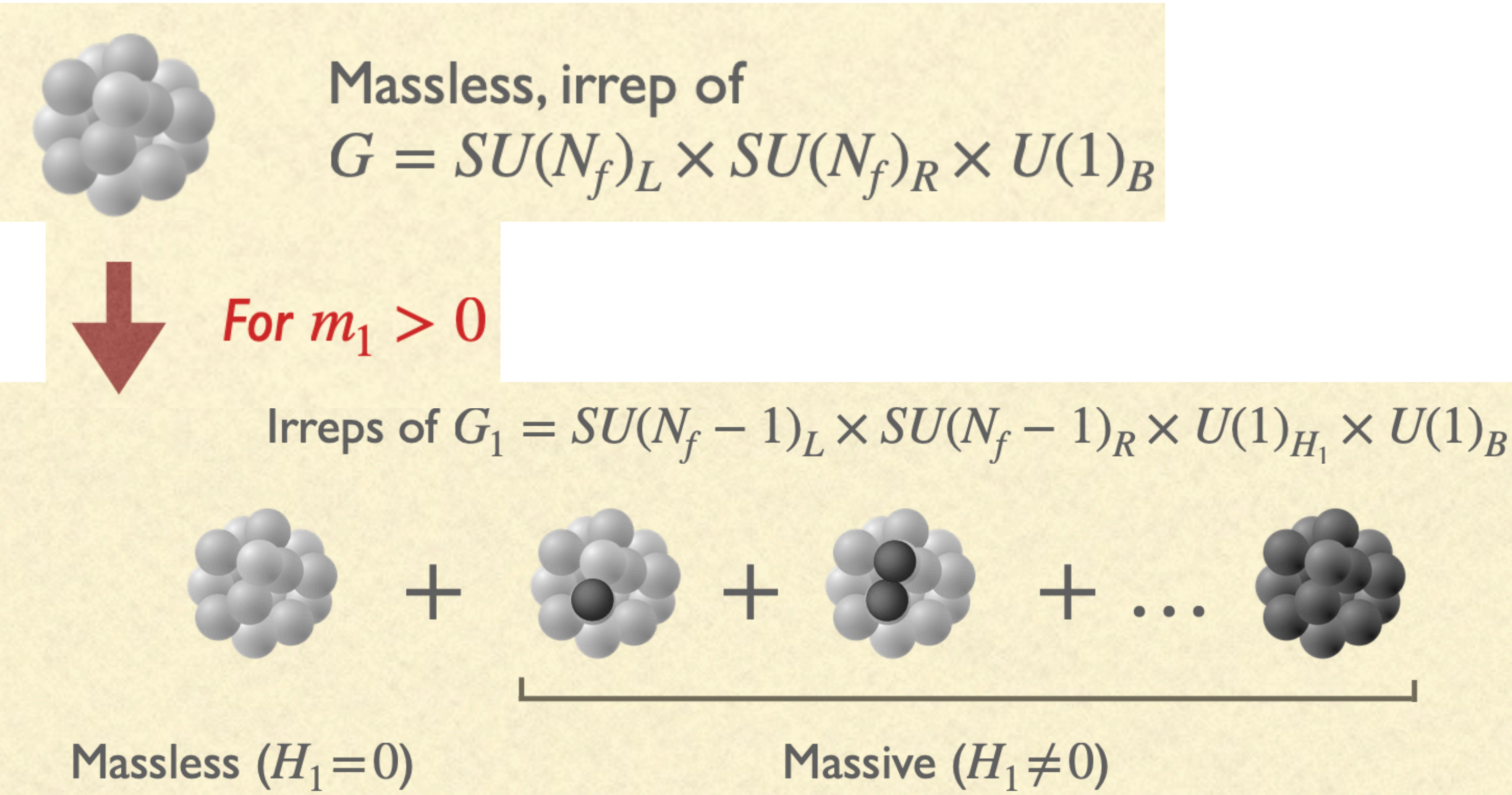
- In the limit $\epsilon \to 0$ the global symmetry $G[N_f]$ reduces to $G[N_f, 1] = SU(N_f - 1)_L \times SU(N_f - 1)_R \times U(1)_B \times U(1)_{H_1}$. The massless particles charged under $U(1)_{H_1}$ (hence with $n_H > 0$) must be massive.

Chiral symmetry and PMC equations

Massless, irrep of
$G = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$

Chiral symmetry and PMC equations



Chiral symmetry and PMC equations

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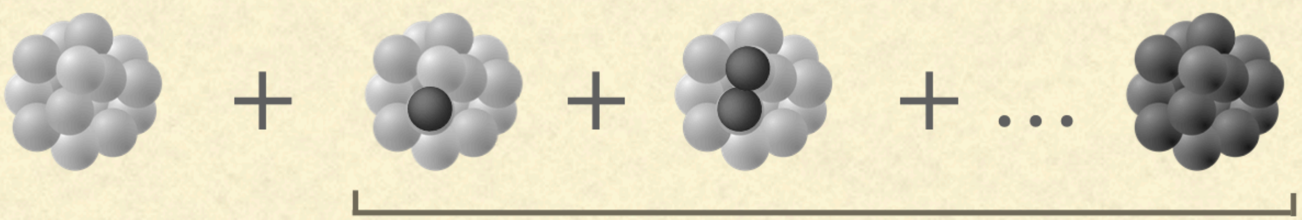
For $m_1 > 0$

Irreps of $G_1 = SU(N_f - 1)_L \times SU(N_f - 1)_R \times U(1)_{H_1} \times U(1)_B$



Massless ($H_1 = 0$)

Massive ($H_1 \neq 0$)

PMC$[N_f, 1]$

$$0 = \tilde{\ell}(r_i, N_f - 1)$$
$$= \sum_r \ell(r, N_f) k(r \to r_i)$$

One PMC equation for each irrep r_i of G_1 with $H_1 \neq 0$

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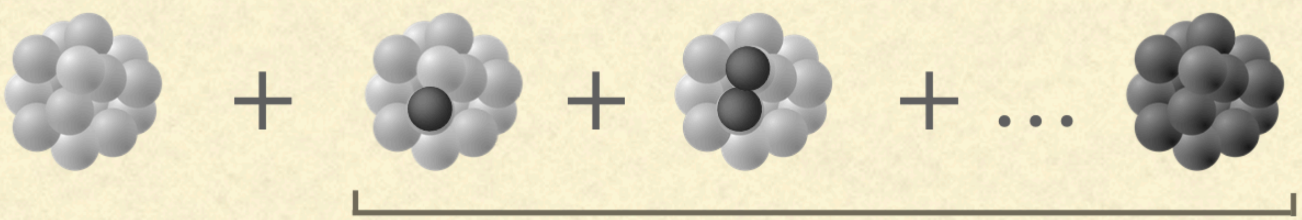
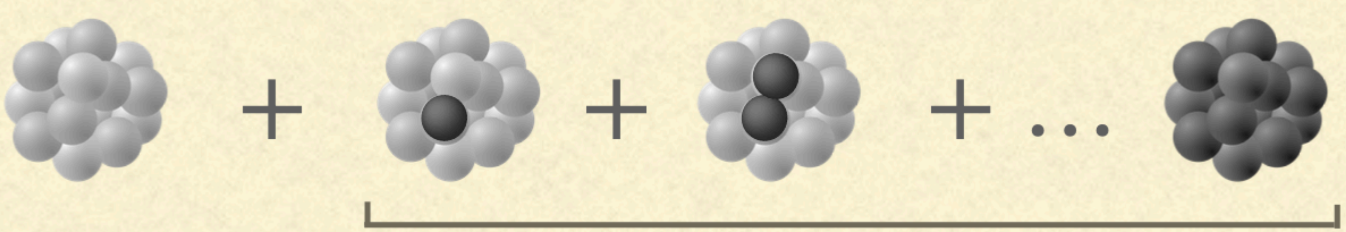


Massless ($H_1 = 0$) Massive ($H_1 \neq 0$)

For $m_2 \neq m_1 > 0$

Irreps of $G_2 = SU(N_f - 2)_L \times SU(N_f - 2)_R \times U(1)_{H_1} \times U(1)_{H_2} \times U(1)_B$



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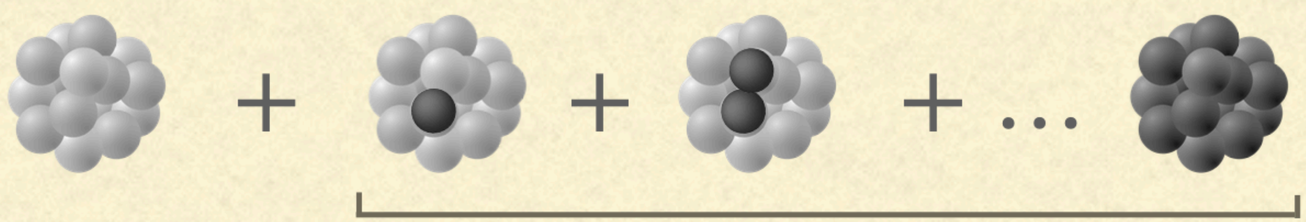

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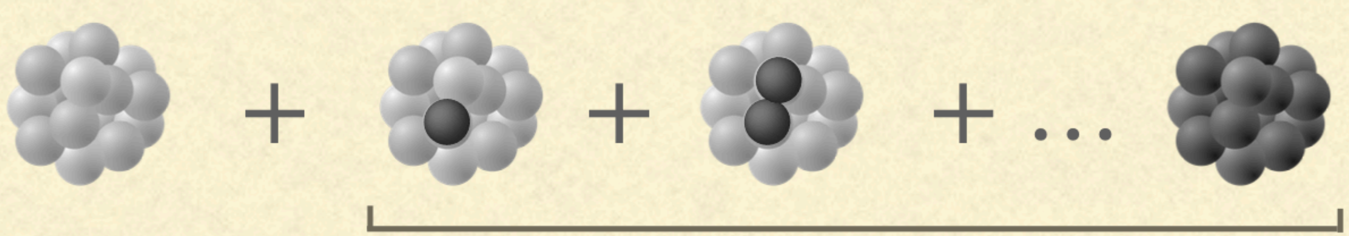

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One PMC equation for each irrep r_i of G_1 with $H_1 \neq 0$

PMC$[N_f, 2]$

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One PMC equation for each irrep r_i of G_2 with $H_2 \neq 0$

Chiral symmetry and PMC equations

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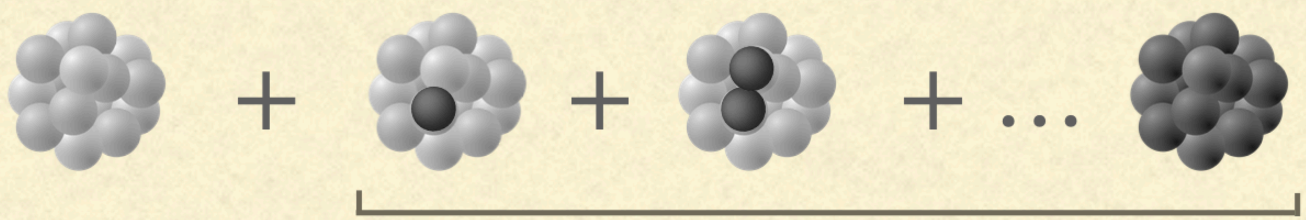

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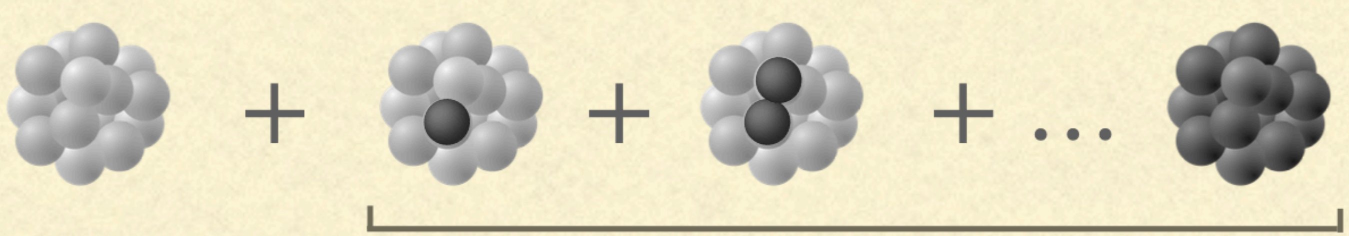

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⋮

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PMC[N_f,2]

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One PMC equation <u>for each</u> irrep r_i of G_2 with $H_2 \neq 0$

An important observation on PMC equations

$$\mathrm{PMC}[N_f]$$

$$\mathrm{PMC}[N_f,1]$$

$$\mathrm{PMC}[N_f,2]$$

$$\mathrm{PMC}[N_f,3]$$

$$\vdots$$

$$\mathrm{PMC}[N_f, N_f - 2]$$

An important observation on PMC equations

$\mathrm{PMC}[N_f]$	$\mathrm{PMC}[N_f+1]$
$\mathrm{PMC}[N_f,1]$	$\mathrm{PMC}[N_f+1,1]$
$\mathrm{PMC}[N_f,2]$	$\mathrm{PMC}[N_f+1,2]$
$\mathrm{PMC}[N_f,3]$	$\mathrm{PMC}[N_f+1,3]$
$\vdots$	$\mathrm{PMC}[N_f+1,4]$
$\mathrm{PMC}[N_f,N_f-2]$	$\vdots$
	$\mathrm{PMC}[N_f+1,N_f-1]$

An important observation on PMC equations

$\mathrm{PMC}[N_f]$	$\mathrm{PMC}[N_f + 1]$
$\mathrm{PMC}[N_f,1]$	$\mathrm{PMC}[N_f + 1,1]$
$\mathrm{PMC}[N_f,2]$	$\mathrm{PMC}[N_f + 1,2]$
$\mathrm{PMC}[N_f,3]$	$\mathrm{PMC}[N_f + 1,3]$
$\vdots$	$\mathrm{PMC}[N_f + 1,4]$
$\mathrm{PMC}[N_f, N_f - 2]$	$\vdots$
	$\mathrm{PMC}[N_f + 1,N_f - 1]$

The PMC equations connected by the diagonal lines can be identified,

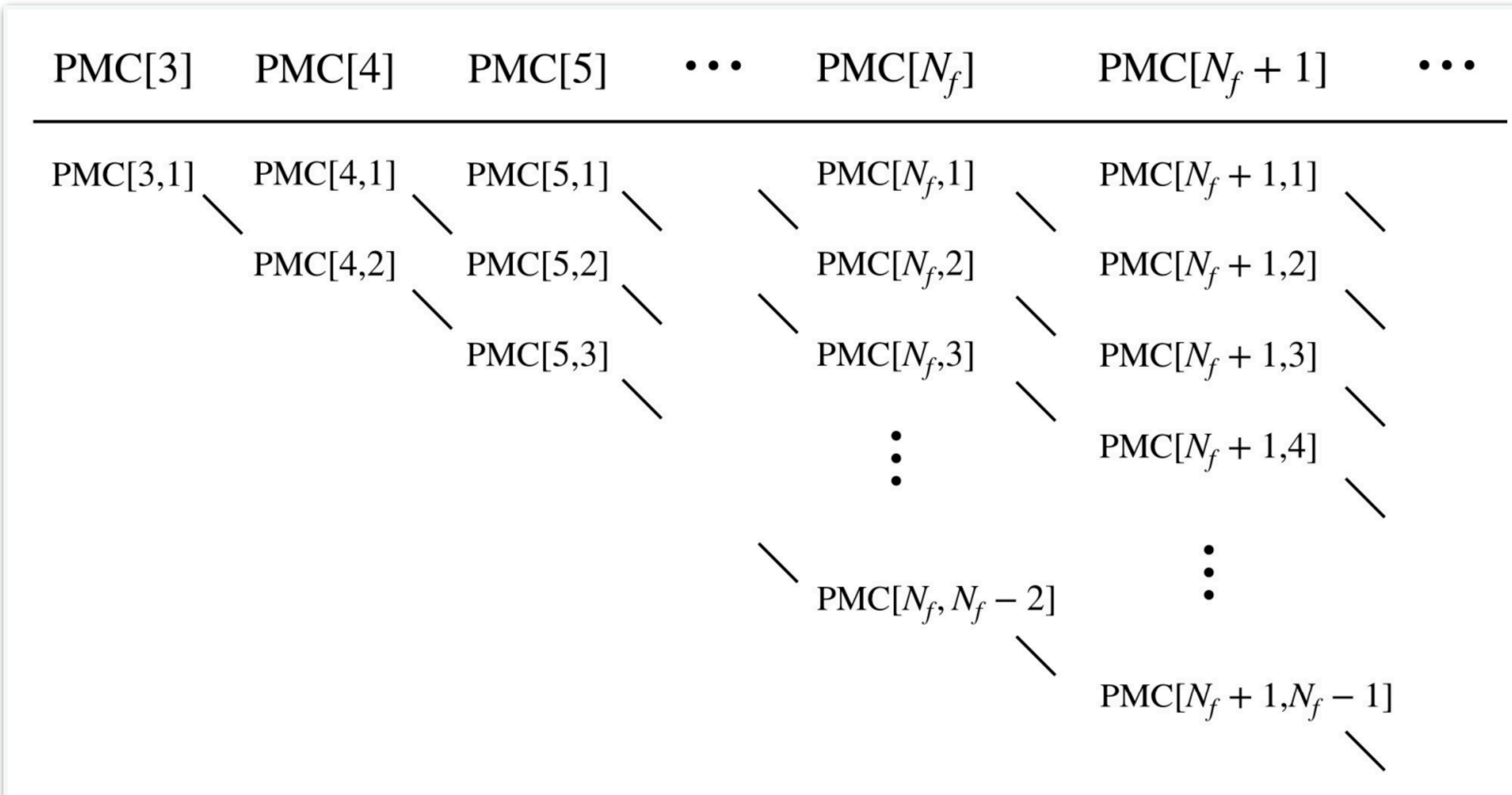
An important observation on PMC equations

PMC$[N_f]$		PMC$[N_f+1]$
PMC$[N_f,1]$		PMC$[N_f+1,1]$
	＼	
PMC$[N_f,2]$		PMC$[N_f+1,2]$
	＼	
PMC$[N_f,3]$	＼	PMC$[N_f+1,3]$
⋮		PMC$[N_f+1,4]$
PMC$[N_f,N_f-2]$		⋮
	＼	
		PMC$[N_f+1,N_f-1]$

The PMC equations connected by the diagonal lines can be identified, since each irrep of $G[N_f] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ can be identified with that of $G[N_f+1,1] = SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_{H_1}$ with zero $U(1)_{H_1}$ charge.

The bird's-eye view on PMC

- Therefore, we obtain the coherent structure of PMC for theories with different N_f by analyzing the symmetries and their correspondences:



- In particular, we have the identifications PMC[N_f, i] $\sim$ PMC[$N_f-1,\ i-1$] given the identifications of irreps.

Our proof

QCD with $N_c = 3$

Prime factor

No solution of AMC[$N_f = 3$],

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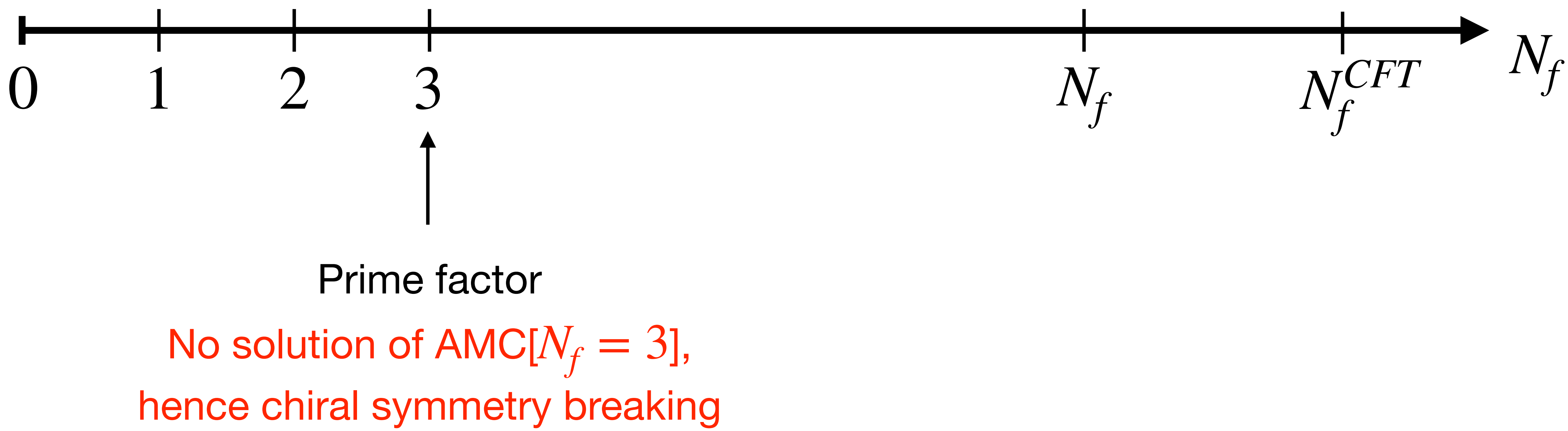

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- From these solutions, one constructs integral solutions of AMC[3] & PMC[3]. (Suppose this step is done, as I will discuss how next.)
- But there is not any integral solution of AMC[3]. Contradiction!

The final pillar

Downlifting — The following theorem holds true:

Let $\{\ell(r)\}$ *be a solution of* $AMC[N_f] \cup PMC[N_f]$*; then* $\{\tilde{\ell}(r')\}$ *is a solution of* $AMC[N_f - 1] \cup PMC[N_f - 1]$ *for*

$$\tilde{\ell}(r') \equiv \sum_{r \in \mathcal{R}[N_f]} \ell(r)\, k\,(r \to r') \quad \forall\, r' \in \mathcal{R}[N_f - 1]\,. \quad (7)$$

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- We are interested in r' with zero $U(1)_{H_1}$ charge in particular, their indices solve PMC[N_f, i] with $2 \leq i \leq N_f - 2$ by further decomposition step by step.

- Since each r' can be identified with an irrep of $G[N_f - 1] = SU(N_f - 1)_L \times SU(N_f - 1)_R \times U(1)_B$, this is the chiral symmetry group of QCD$[N_c, N_f - 1]$.

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- According to the important observation on the identification of PMC, the indices $\ell(r')$ given by the ansatz automatically solve PMC$[N_f-1,\ i-1] \sim$ PMC$[N_f, i]$ where $2 \leq i \leq N_f-2$. All these equations of PMC$[N_f-1,\ i-1]$ are just PMC$[N_f-1]$.

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- So far, we have shown the ansatz successfully solves PMC$[N_f]$. Next, we show the same ansatz also solves AMC$[N_f]$.

- One can evaluate anomaly coefficients of $SU(N_f)_{L,R}$ on the $SU(N_f - 1)_{L,R}$ Lie subalgebra. Following the rule of decomposition, we have

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- Plugging this equation into AMC[N_f] and switching the order of sums, we have

$$\begin{aligned} A_{\text{UV}} &= \sum_{r \in \mathcal{R}[N_f]} \ell(r) \left(\sum_{\text{All } r'} k(r \to r')\ A(r') \right) \\ &= \sum_{\text{All } r'} \left(\sum_{r \in \mathcal{R}[N_f]} \ell(r)\ k(r \to r') \right) A(r') \ . \end{aligned}$$

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- This equation can be viewed as AMC[$N_f - 1$], whose solution is happily the ansatz!

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- When apply to QCD with $N_c = 3$, chiral symmetry breaking is proven for $3 \leq N_f < N_f^{CFT}$.
- Many groundbreaking works are needed to coherently derive the phase structure of QCD.

Please feel free to send me emails if anything is unclear

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Danke Schoen

Backup slides

Some general comments on ’t Hooft anomaly

- ’t Hooft anomaly does not imply that the theory is inconsistent.
- Rather, ’t Hooft anomaly is a powerful probe of non-perturbative physics of strongly-coupled QFTs.
- ’t Hooft anomaly implies that theory cannot be trivially gapped:
  — If there are ’t Hooft anomalies identified in the UV, it implies that the symmetries have to act in the IR, such that the same ’t Hooft anomalies are reproduced.
  — Even though symmetries in the UV and IR may not be the same.

Brief account of past works

- In the seminal Cargese lectures, 't Hooft worked out the cases $N_c = 3$ and $N_c = 5$ with no integral solution found for $N_f > 2$. Only massless baryons are considered.
- Frishman et al extended the analysis of only baryons to $N_c > 5$. They assumed that all mixed representations have vanishing indices, and found no solutions for $N_f > 2$.
- A more detailed analysis was performed by Cohen and Frishman, they notice that the analysis must be different for the cases $N_f > N_c$ and $N_f \leq N_c$ for baryons. (Hence it implies that 'N_f independence' is not in general valid.)
- Farrar considered exotics (bound states with antiquark constituents) and was above to prove chiral symmetry breaking through 'N_f independence'.
- Schwimmer provided another proof using superalgebra $SU(N_f|N_f)$, which contains the chiral symmetry as a subalgebra.
- Coleman and Witten proved chiral symmetry breaking in the large N_c limit.

What we found instead...

See hep-th/2212.02930, 2404.02971 for many details

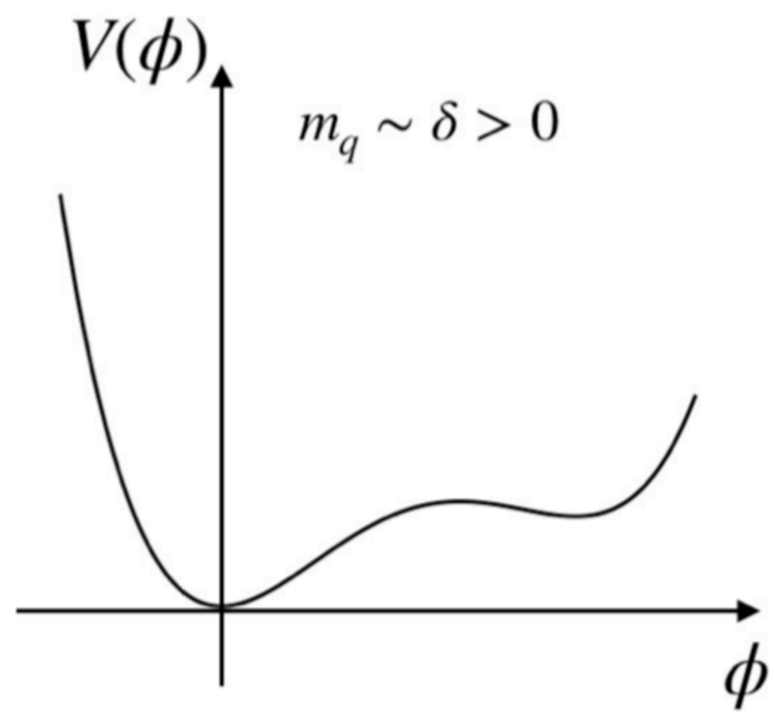
- N_f independence is false in general, it is only valid for special cases where the putative bound states satisfy the condition

$$n + \bar{n} < N_f$$

- Even though one can show that each irrep of superalgebra $SU(N_f|N_f)$ gives a solution to PMC, it is unclear whether *all* the PMC can be captured by a collection of superalgebra irreps. It would be interesting to *prove* this.

Comments on continuity

- Useful for the case $N_c = 3$ and $N_f = 2$ in QCD; in general for N_f smaller than the smallest nontrivial prime factor p of N_c
- Let's consider a theory with N_f massless flavors and $(p - N_f)$ massive flavors. Suppose that the chiral symmetry $SU(N_f)_L \times SU(N_f)_R$ is unbroken by *the vacuum* for *any* value of the massive quark masses near the origin.
- This implies that the effective potential $V(\phi)$ has a global minimum at $\phi = 0$, where ϕ is the VEV of any color-singlet operator which is charged under $SU(N_f)_L \times SU(N_f)_R$.



Comments on continuity

- *Continuity* of $V(\phi)$ with respect to the quark masses implies that an $SU(p)_L \times SU(p)_R$ preserving vacuum exists in the limit where all the masses vanish.

- This is because the vectorlike $SU(p)_V$ symmetry cannot be spontaneously broken, *so the unbroken chiral symmetry has to be enhanced to* $SU(p)_L \times SU(p)_R$ in order to accommodate both $SU(N_f)_L \times SU(N_f)_R$ and $SU(p)_V$ symmetries.

- If the theory with p flavors confines, it contracts the fact that AMC[p] do not have integral solutions! Hence the inital assumption is false and $SU(N_f)_L \times SU(N_f)_R$ is broken.

- As a last step, one can send the quark masses to infinity for the massive $(p - N_f)$ flavors. With *the assumption that there is no phase transition,* chiral symmetry breaking persists.

- Notice that, however, this is not a rigorous proof for $N_f < p$.