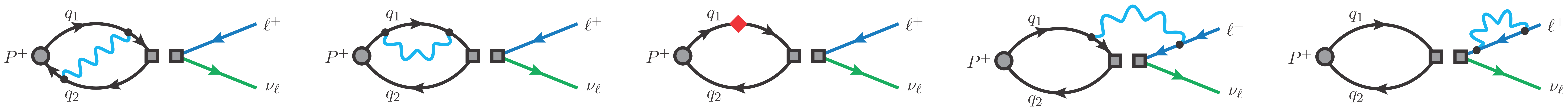


Isospin-breaking corrections to light-meson leptonic decays

Matteo Di Carlo

26th July 2024

CERN

Funded by the European Union

Mainz Institute for Theoretical Physics

Testing the Standard Model with flavour physics

Indirect searches of new physics using CKM matrix unitarity constraints

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

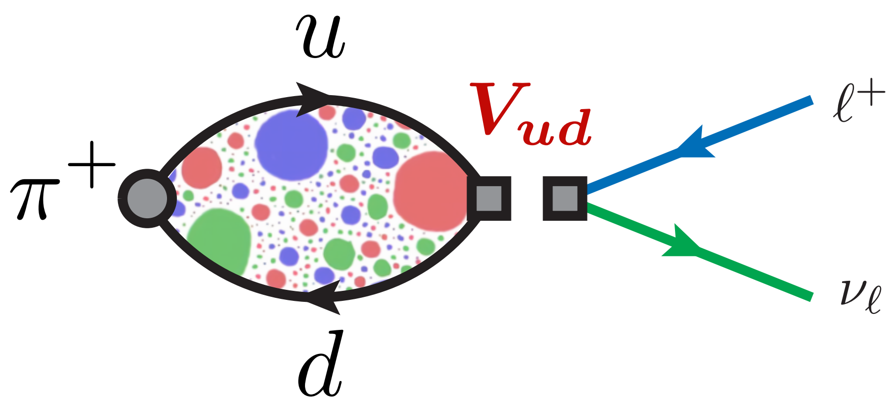
in the Standard Model:

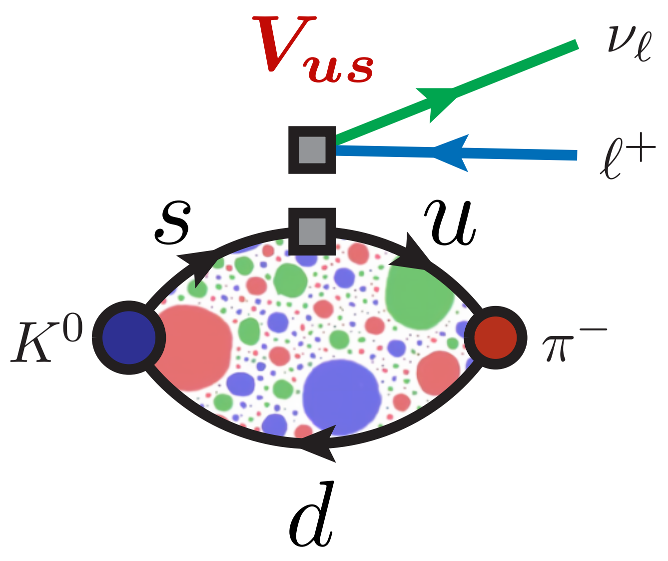
$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Matrix elements can be extracted e.g. from leptonic and semileptonic decays of mesons

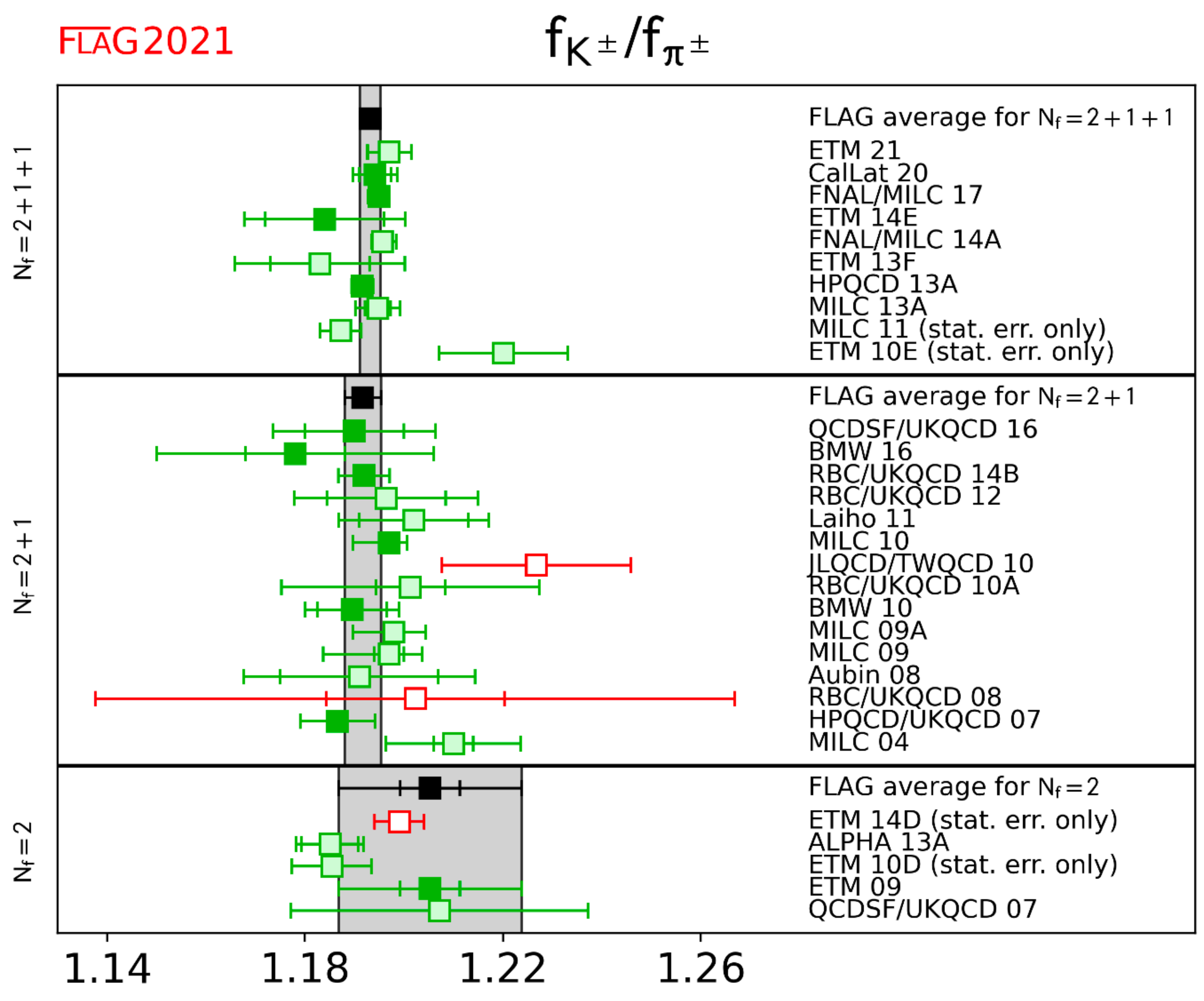
$$\underbrace{\frac{\Gamma\left[K \to \ell\nu_\ell(\gamma)\right]}{\Gamma\left[\pi \to \ell\nu_\ell(\gamma)\right]}}_{\text{experiments}} \propto \boxed{\left|\frac{V_{us}}{V_{ud}}\right|^2} \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{\text{QCD}}$$

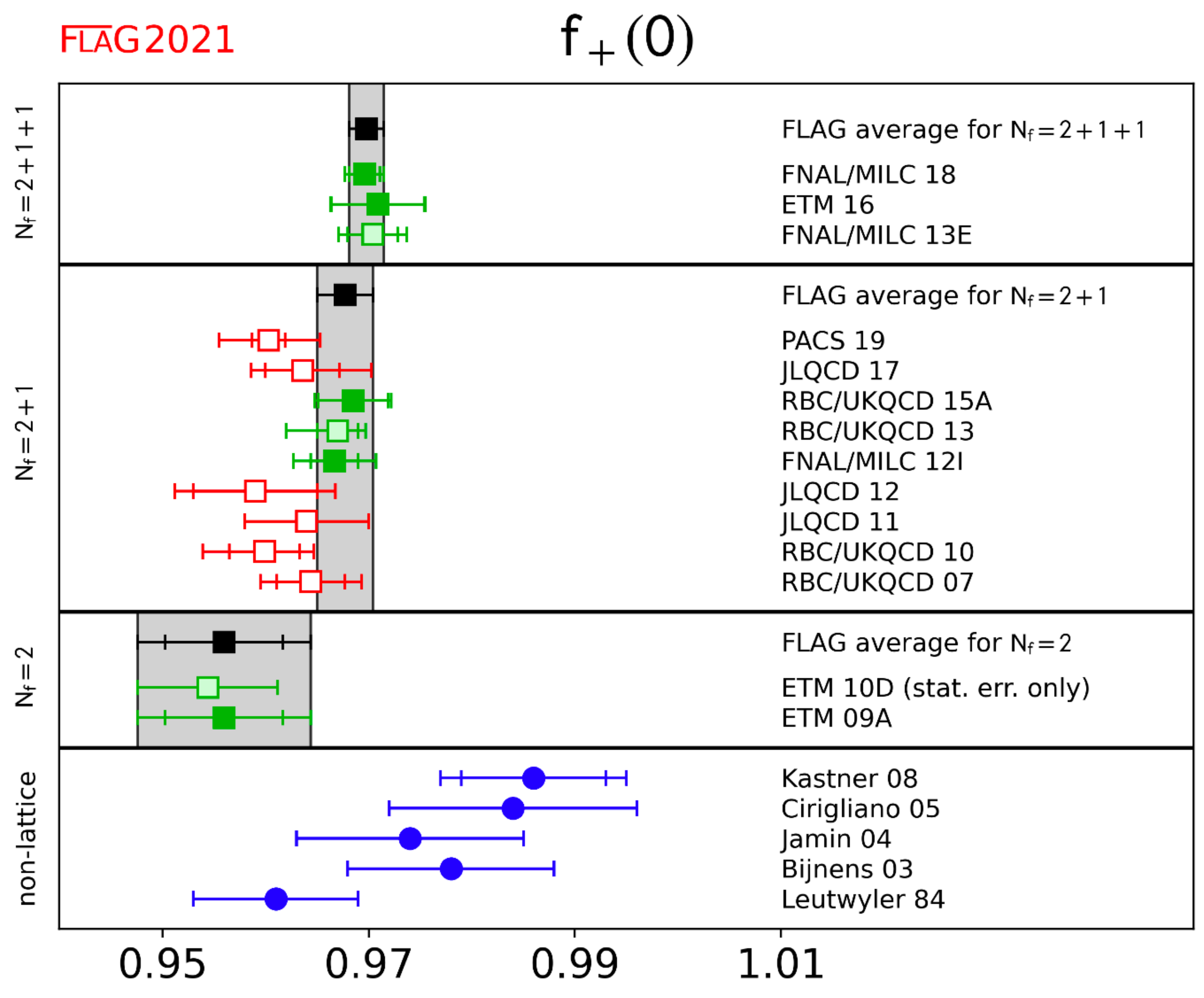
$$\underbrace{\Gamma\left[K \to \pi\ell\nu_\ell(\gamma)\right]}_{\text{experiments}} \propto \boxed{|V_{us}|^2} \underbrace{|f_+^{K\pi}(0)|^2}_{\text{QCD}}$$





Leptonic and semi-leptonic decays from lattice QCD





$$f_{K^\pm}/f_{\pi^\pm} = 1.1934\,(19)$$

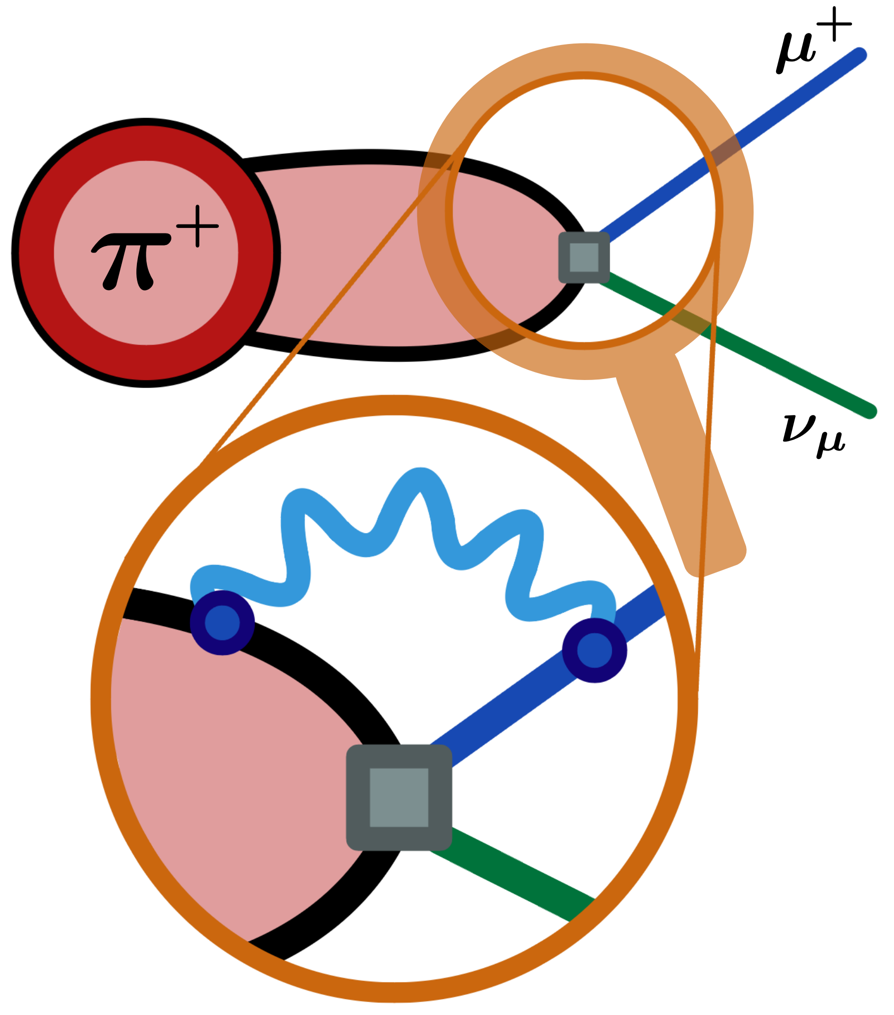
$$f_+^{K\pi}(0) = 0.9698\,(17)$$

FLAG 2021 Flavour Lattice Averaging Group

f_K/f_π and $f_+^{K\pi}(0)$ determined from lattice QCD with sub percent precision!

FLAG Review 2021. EPJC **82**, 869 (2022)

QED and isospin-breaking effects



Current level of precision requires the inclusion of isospin-breaking corrections due to

- strong effects $[m_u - m_d]_{\text{QCD}} \neq 0$
- electromagnetic effects $\alpha \neq 0$

$\sim \mathcal{O}(1\%)$

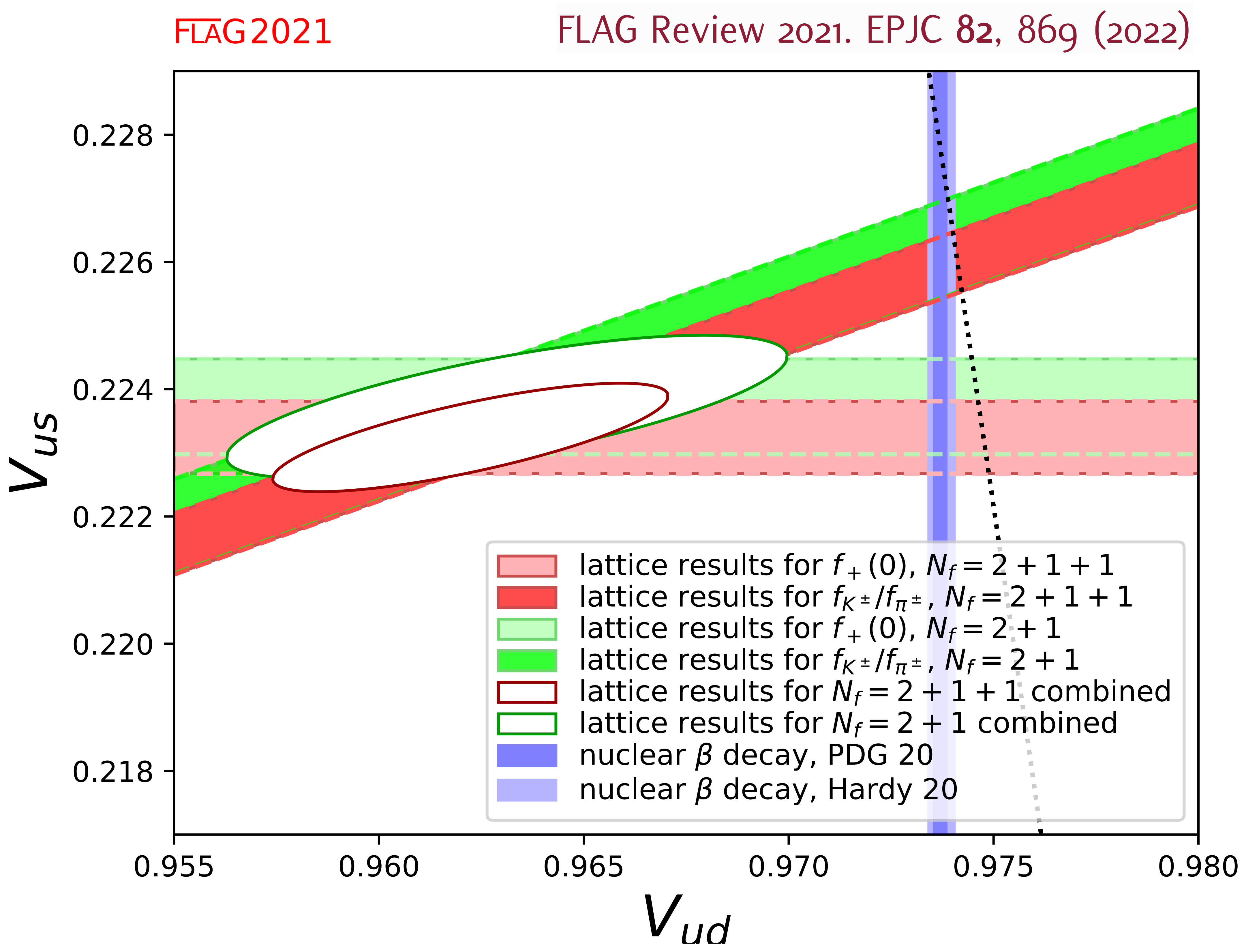
$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi}) \qquad \Gamma(K \to \pi\ell\nu_\ell) \propto |V_{us}|^2 \, |f_+^{K\pi}(0)|^2 \, (1 + \delta R_{K\pi}^\ell)$$

- results from χPT currently quoted in the PDG

V.Cirigliano & H.Neufeld, PLB **700** (2011)

- fully **non-perturbative** (i.e. structure dependent)
- can be obtained through first-principle **lattice calculations**

First-row CKM unitarity tests



Different **tensions** in the V_{us}-V_{ud} plane:

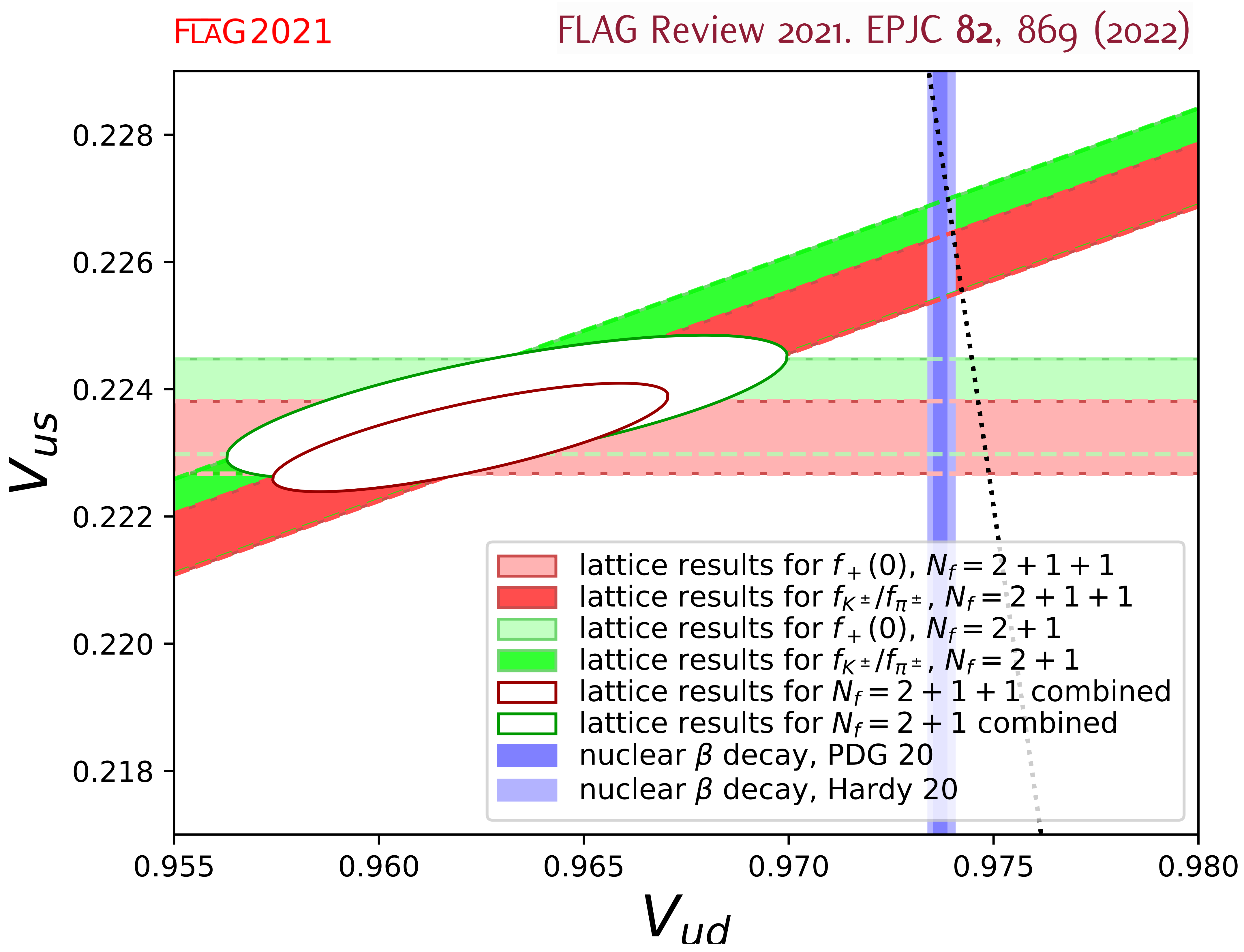
$$|V_u|^2 - 1 = 2.8\sigma$$

$$|V_u|^2 - 1 = 5.6\sigma \qquad |V_u|^2 - 1 = 3.3\sigma$$

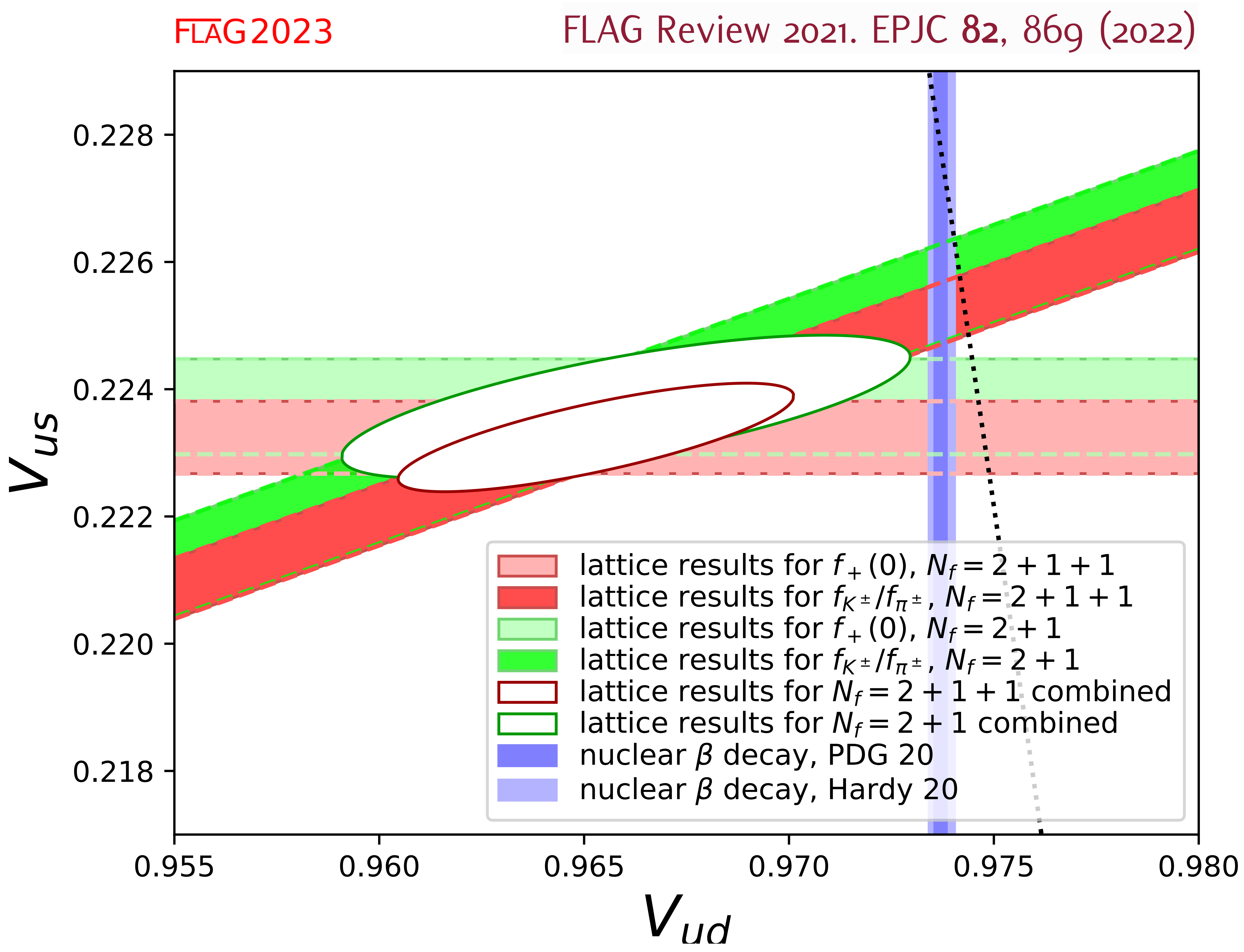
$$|V_u|^2 - 1 = 3.1\sigma \qquad |V_u|^2 - 1 = 1.7\sigma$$

Experimental and **theoretical** control of these quantities is of crucial importance to solve the issue

First-row CKM unitarity tests



with QED corrections
from lattice calculation



without QED corrections
from the lattice

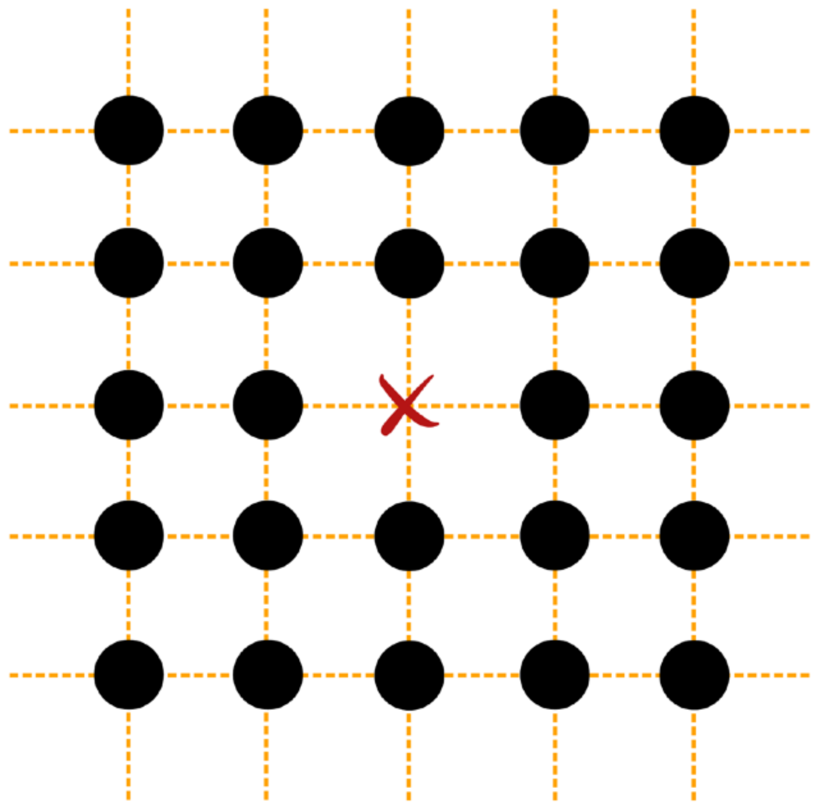
Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; j_0(t,\mathbf{x}) = \int_{\text{p.b.c.}} \mathrm{d}^3\mathbf{x}\; \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

Possible solutions:

$\mathrm{QED_L}$

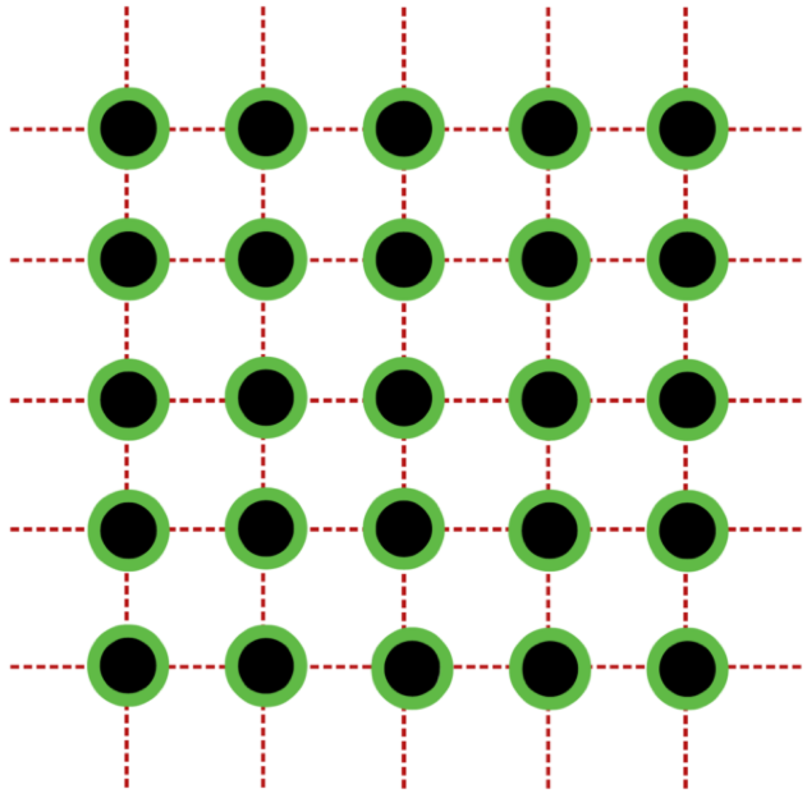


$\Omega_3 = 2\pi\mathbb{Z}^3/L$

remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP **120** (2008)

$\mathrm{QED_m}$

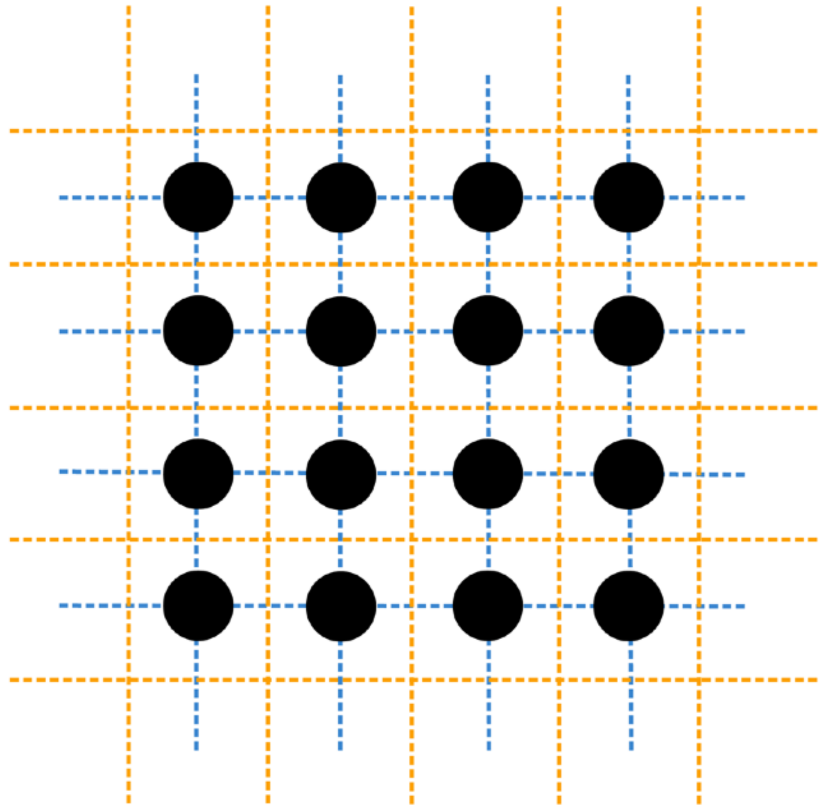


$\Omega_4 = 2\pi\{\mathbb{Z}^3/L, \mathbb{Z}/T\}$

use massive photon m_γ

M.G.Endres et al., [1507.08916]

$\mathrm{QED_{C^*}}$

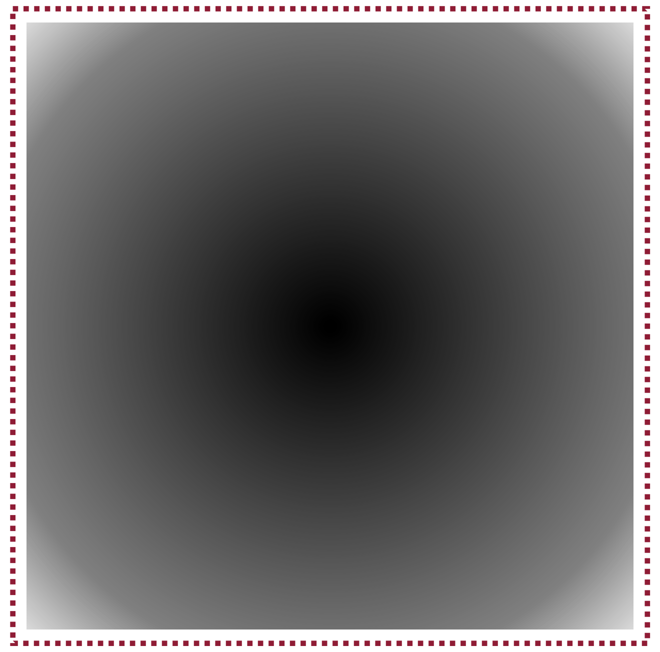


$\Omega_3 = 2\pi\mathbb{Z}^3/L$ $\quad \Omega_3' = (2\,\mathbb{Z}^3 + \bar{\mathbf{n}})\pi/L$

employ C* boundary conditions

A.S.Kronfeld & U.-J.Wiese, NPB **357** (1991)
B.Lucini et al., JHEP **02** (2016)

$\mathrm{QED_\infty}$



$\Omega_4 = \mathbb{R}^4$

infinite-volume reconstruction

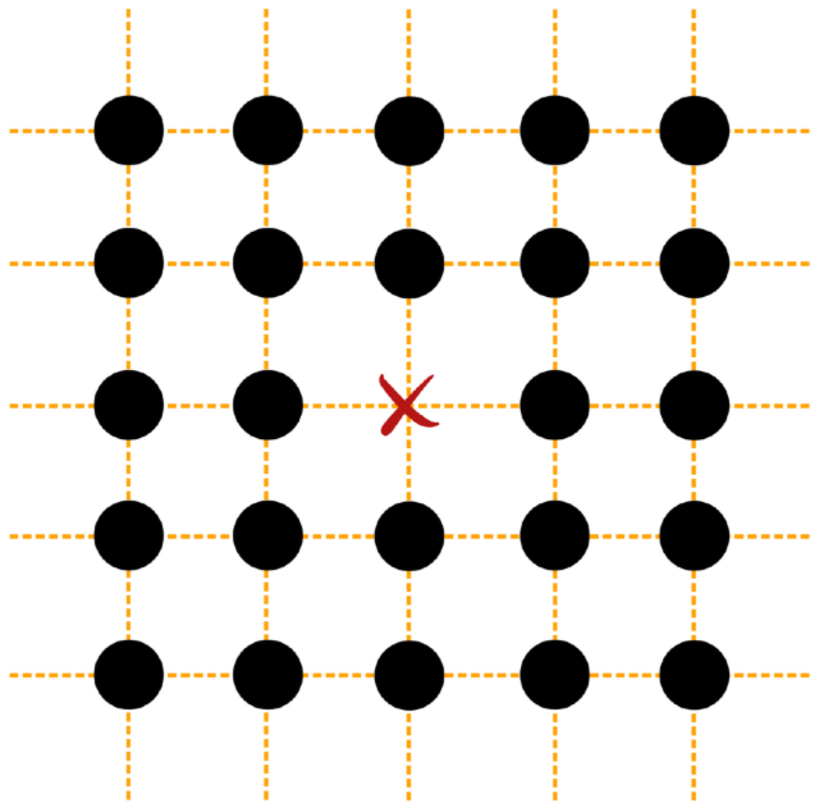
X.Feng & L.Jin, PRD **100** (2019)
N.Christ et al., [2304.08026]

Charged states in a finite box

Gauss law: only zero net charge is allowed in a finite volume with periodic boundary conditions

$$Q = \int_{\text{p.b.c.}} d^3\mathbf{x}\ j_0(t,\mathbf{x}) = \int_{\text{p.b.c.}} d^3\mathbf{x}\ \boldsymbol{\nabla}\cdot\boldsymbol{E}(t,\mathbf{x}) = 0$$

QED$_L$



remove spatial zero-mode of the photon field

M.Hayakawa & S.Uno, PTP **120** (2008)

- **Spatial zero-mode** of the photon field is **removed** at each k_0

$$\int d^3\mathbf{x}\ A_\mu(t,\mathbf{x}) = 0 \quad \longrightarrow \quad \Delta^\gamma_{\mu\nu}(x) = \frac{1}{V}\sum_{k_0}\sum_{\mathbf{k}\neq 0}\Delta^\gamma_{\mu\nu}(k)\,e^{ik\cdot x}$$

- **Long-distance** interaction translates into **power law finite-size effects**

$$\mathcal{O}(L) = \mathcal{O}(\infty) + \frac{\kappa_1}{L} + \frac{\kappa_2}{L^2} + \frac{\kappa_3}{L^3} + \dots \qquad \kappa_3 \propto c_0 = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n}\right) = -1$$

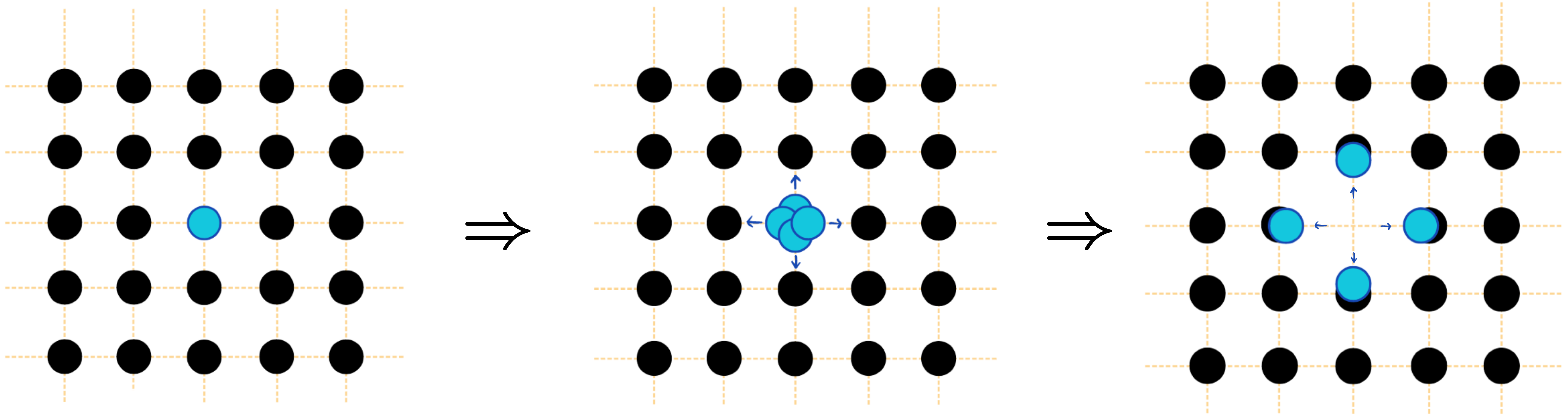
S.Borsanyi et al., Science **347** (2015) V.Lubicz et al., PRD **95** (2017) Z.Davoudi et al., PRD **99** (2019)
Z.Davoudi & M.Savage, PRD **90** (2014) N.Tantalo et al., [1612.00199] MDC et al., PRD **105** (2022)

QEDr regularization

Special case of "IR-improvement"

Z.Davoudi et al., PRD 99 (2019)

MDC, PoS LATTICE2023 (2024) [2401.07666]

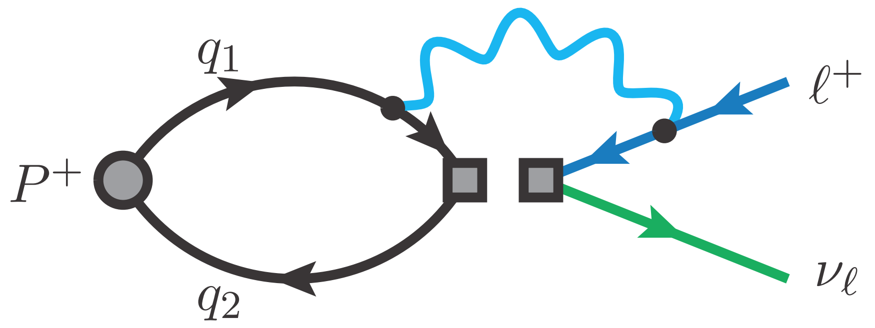


The spatial zero mode is not removed but redistributed over the neighbouring modes on a shell of radius $|\mathbf{p}| = \frac{2\pi}{L}|\mathbf{r}| \quad (\mathbf{r} \in \mathbb{Z}^3)$

QED$_\mathrm{L}$: $D_L^{\mu\nu}(k_0,\mathbf{k}) = \delta^{\mu\nu}\frac{1-\delta_{\mathbf{k},\mathbf{0}}}{k_0^2+\mathbf{k}^2}$ $\Rightarrow$ **QED$_\mathrm{r}$:** $D_{\mathbf{p}}^{\mu\nu}(k_0,\mathbf{k}) = \delta^{\mu\nu}\frac{1-\delta_{\mathbf{k},\mathbf{0}}}{k_0^2+\mathbf{k}^2} + \frac{\delta_{\mathbf{k}^2,\mathbf{p}^2}}{n(\mathbf{p}^2)}\frac{\delta^{\mu\nu}}{k_0^2+\mathbf{p}^2}$

Weak decays — some recent works

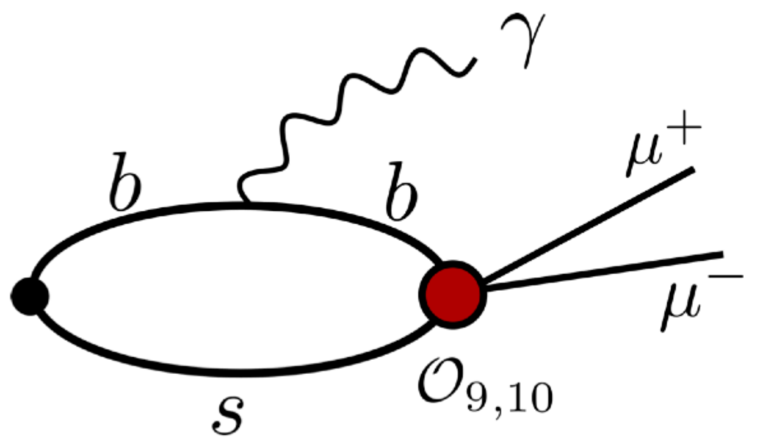
leptonic decays



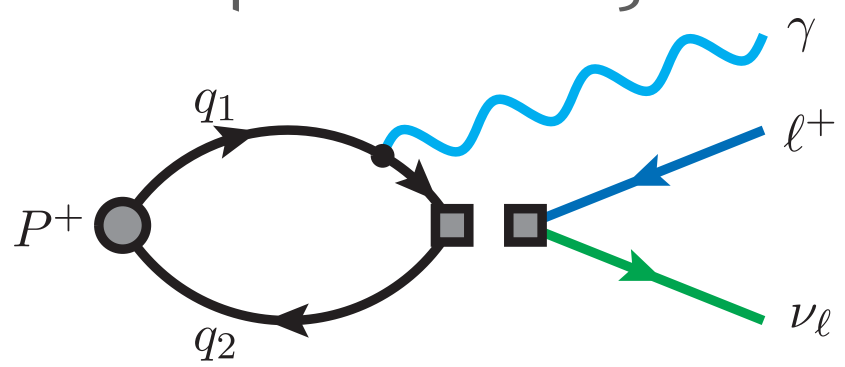
N. Carrasco et al., PRD 91 (2015)
V. Lubicz et al., PRD 95 (2017)
N.Tantalo et al., [1612.00199v2]
D. Giusti et al., PRL 120 (2018)
MDC et al., PRD 100 (2019)
MDC et al., PRD 105 (2022)
P.Boyle, MDC et al., JHEP 02 (2023)
N.Christ et al., [2304.08026]

R.Frezzotti et al., [2402.03262]

rare FCNC decays



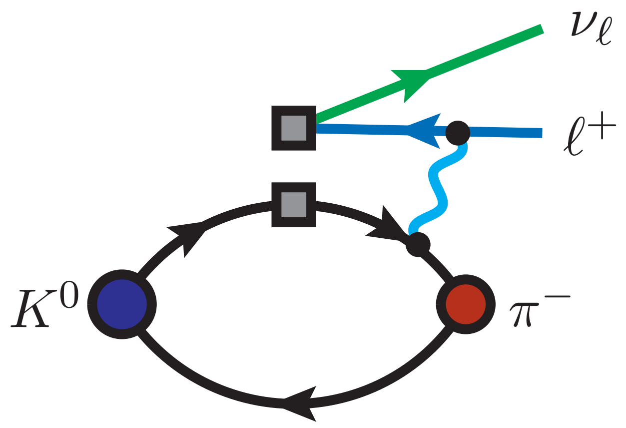
real photon emission
leptonic decays



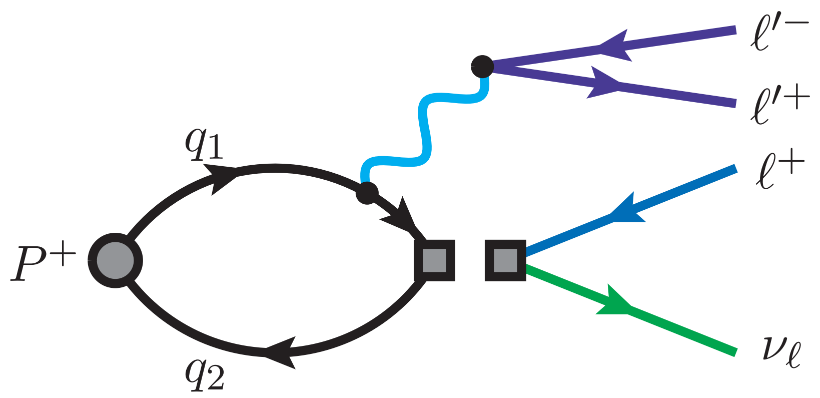
G.M. de Divitiis et al., [1908.10160]
C. Kane et al., [1907.00279 & 2110.13196]
R. Frezzotti et al., PRD 103 (2021)
A.Desiderio et al., PRD 102 (2021)
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

C.Sachrajda et al., [1910.07342]
N.Christ et al., [2304.08026]

semileptonic decays



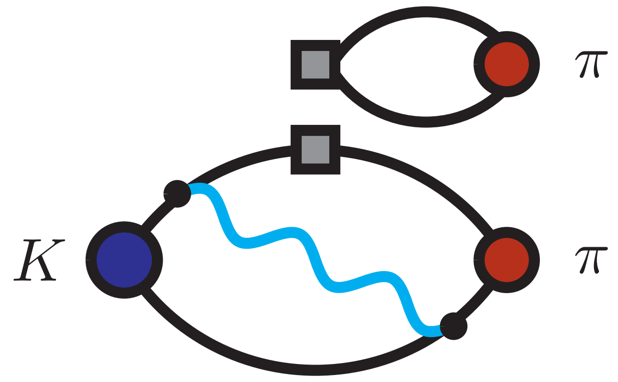
virtual photon emission
leptonic decays



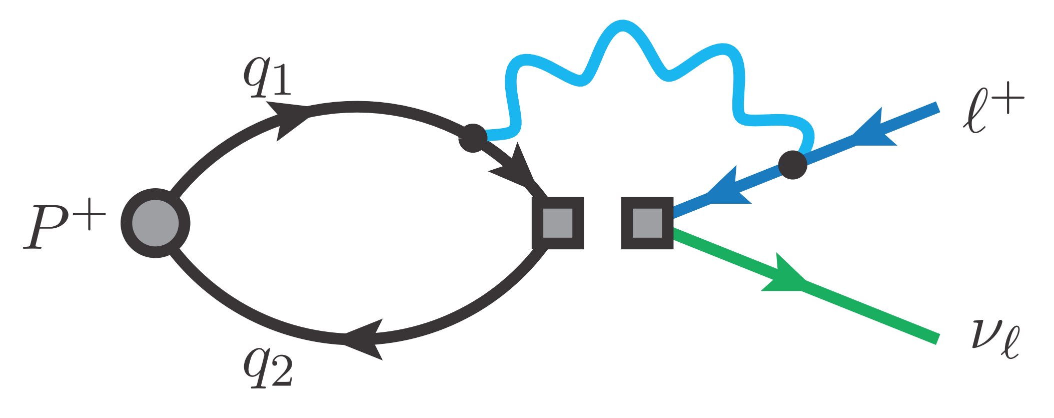
G.Gagliardi et al., Phys. Rev. D 105 (2022)
R.Frezzotti et al., [2306.07228]

R.Abbott et al., PRD 102 (2020)
Z.Bai et al., PRL 115 (2015)
N.Christ et al., PRD 106 (2022)
N.Christ & X.Feng, EPJ Web Conf. 175 (2018)
Y.Cai & Z.Davoudi, [1812.11015]

hadronic decays



leptonic decays of light pseudoscalar mesons



1904.08731

PHYSICAL REVIEW D **100**, 034514 (2019)

Editors' Suggestion

Light-meson leptonic decay rates in lattice QCD + QED

M. Di Carlo and G. Martinelli
Dipartimento di Fisica and INFN Sezione di Roma La Sapienza, Piazzale Aldo Moro 5, 00185 Roma, Italy

D. Giusti and V. Lubicz
Dip. di Matematica e Fisica, Università Roma Tre and INFN, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy

C. T. Sachrajda
Department of Physics and Astronomy, University of Southampton, Southampton SO17 1BJ, United Kingdom

F. Sanfilippo and S. Simula
Istituto Nazionale di Fisica Nucleare, Sezione di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy

N. Tantalo
Dipartimento di Fisica and INFN, Università di Roma "Tor Vergata," Via della Ricerca Scientifica 1, I-00133 Roma, Italy

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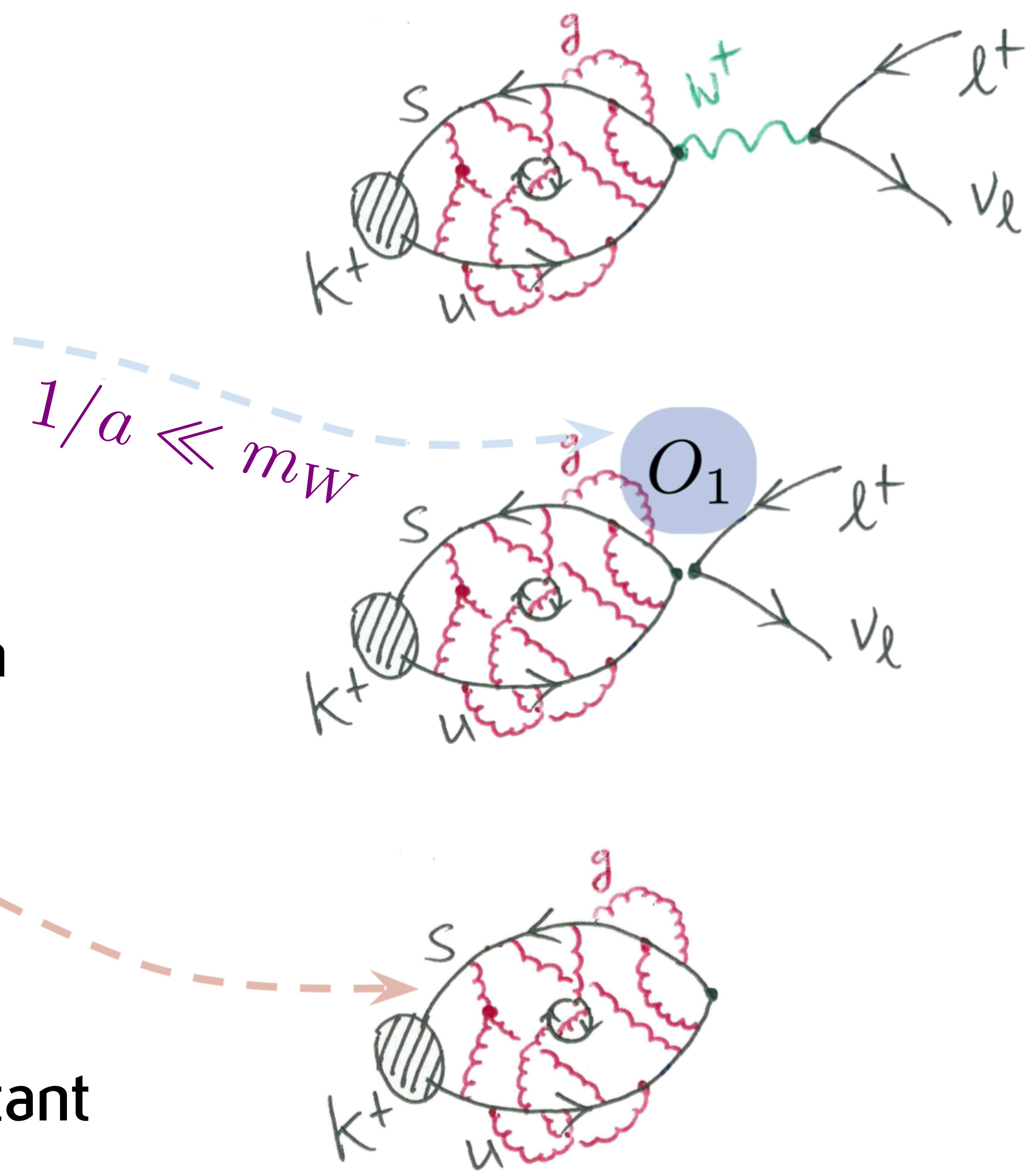
Isospin-breaking corrections to light-meson leptonic decays from lattice simulations at physical quark masses

Peter Boyle,[a,b] Matteo Di Carlo,[b] Felix Erben,[b] Vera Gülpers,[b] Maxwell T. Hansen,[b] Tim Harris,[b] Nils Hermansson-Truedsson,[c,d] Raoul Hodgson,[b] Andreas Jüttner,[e,f] Fionn Ó hÓgáin,[b] Antonin Portelli,[b] James Richings[b,e,g] and Andrew Zhen Ning Yong[b]

Leptonic decays of pseudoscalar mesons

Can be studied in an **effective Fermi theory** with the W-boson integrated out and the local interaction described by

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]$$



In the **PDG convention**, the tree-level decay rate takes the form

$$\Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2$$

with the non-perturbative dynamic encoded in the **decay constant**

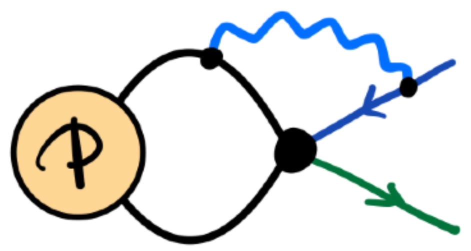
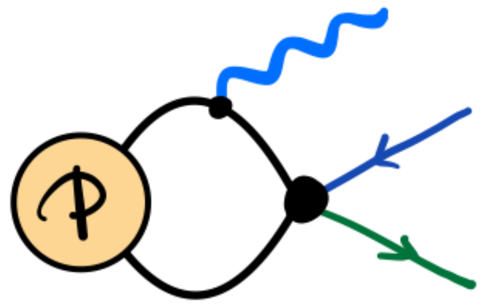
$$\mathcal{Z}_0 \langle 0 | \bar{q}_2\, \gamma_0 \gamma_5\, q_1 | P, \mathbf{0} \rangle^{(0)} = i\, m_{P,0}\, f_{P,0}$$

Leptonic decay rate at $\mathcal{O}(\alpha)$

- The **decay constant** $f_{P,0}$ becomes an ambiguous and unphysical quantity

- **IR divergences** appear in intermediate steps of the calculation

F. Bloch & A. Nordsieck, PR 52 (1937) 54

$$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\rm IR}\to 0} \Big\{$$  $$+$$  $$\Big\}$$

IR finite — IR divergent — IR divergent

- **UV divergences**: need to include QED corrections to the renormalization of the weak Hamiltonian

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(1 + \frac{\alpha_{\rm em}}{\pi} \ln\left(\frac{M_Z}{M_W}\right)\right) \underbrace{\left[\bar{q}_2\, \gamma_\mu (1-\gamma_5)\, q_1\right] \left[\bar{\nu}_\ell\, \gamma^\mu (1-\gamma_5)\, \ell\right]}_{O_1^{\text{W-reg}}(M_W)}$$

A.Sirlin, NPB **196** (1982)

E.Braaten & C.S.Li, PRD **42** (1990)

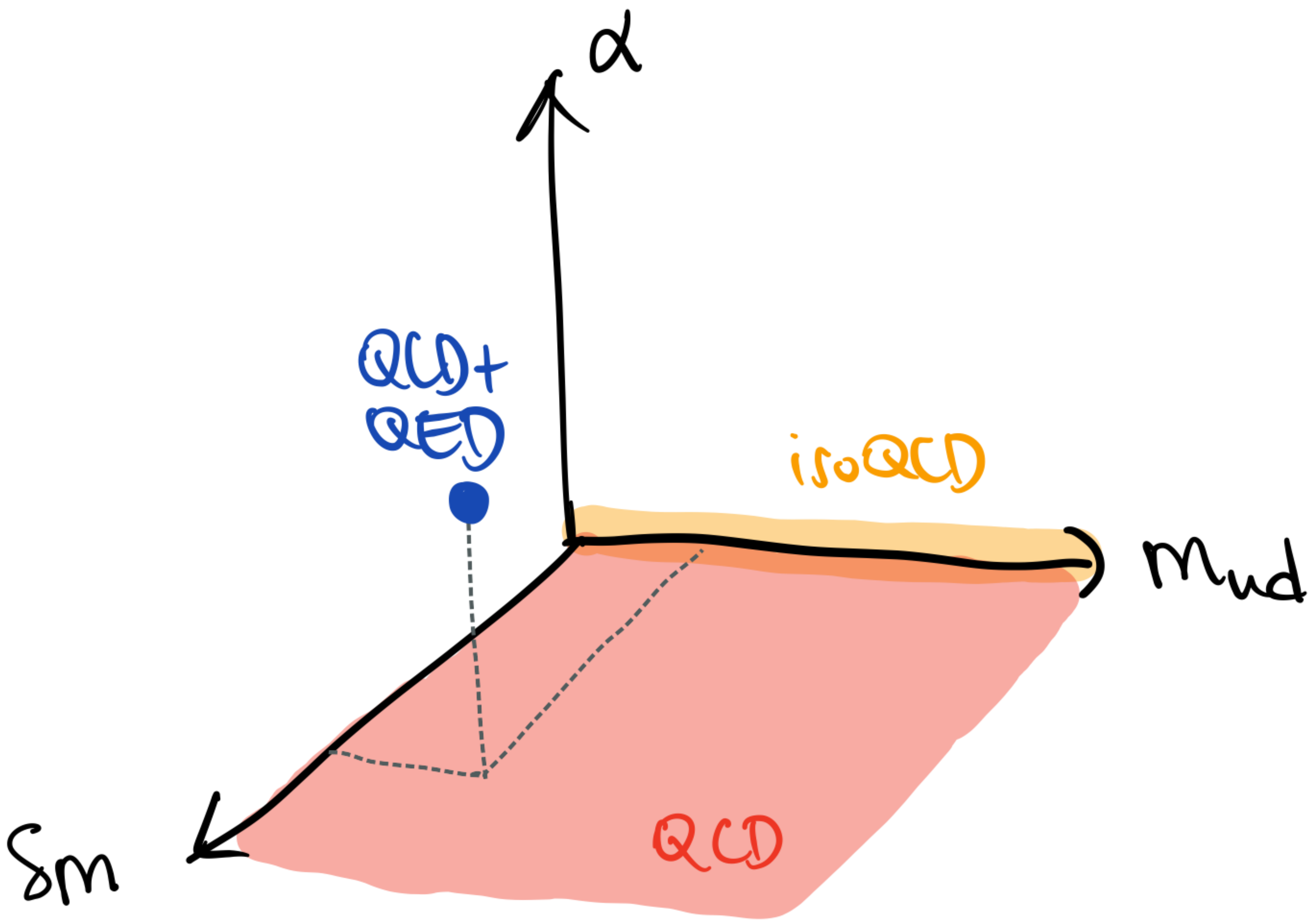
$$O_1^{\text{W-reg}}(M_W) = Z^{\text{W-S}}\left(\frac{M_W}{\mu}, \alpha_s(\mu), \alpha_{\rm em}\right) O_1^{\rm S}(\mu)$$

- perturbative @ 2 loops in QCD+QED
- non-perturbative in lattice QCD+QED

MDC et al., PRD **100** (2019)

Leptonic decay rate at $\mathcal{O}(\alpha)$

Defining the isospin symmetric world



- The full QCD+QED theory is unambiguously defined after **matching** a set of observables to the real world

$$\left[\frac{\hat{\mathrm{M}}_j}{\hat{\Lambda}}\right]^2(g, e^\phi, \hat{\mathbf{m}}^\phi) = \left(\frac{\mathrm{M}_j^\phi}{\Lambda^\phi}\right)^2 \longrightarrow \hat{\mathbf{m}}^\phi(g)$$

$$j = 1, \ldots, N_f$$

- The definition of QCD or isoQCD requires a prescription, i.e. some renormalization conditions to **fix the bare parameters** of the action

$$\boldsymbol{\sigma}^{\mathrm{QCD}} = (g^{\mathrm{QCD}}, 0, \hat{\mathbf{m}}^{\mathrm{QCD}}) \qquad \hat{\mathbf{m}}^{\mathrm{QCD}} = (\hat{m}_{ud}^{\mathrm{QCD}}, \delta\hat{m}^{\mathrm{QCD}}, \hat{m}_s^{\mathrm{QCD}}, \ldots)$$

$$\boldsymbol{\sigma}^{(0)} = (g^{(0)}, 0, \hat{\mathbf{m}}^{(0)}) \qquad \hat{\mathbf{m}}^{(0)} = (\hat{m}_{ud}^{(0)}, 0, \hat{m}_s^{(0)}, \ldots)$$

BMW hadronic scheme in **RBC-UKQCD (2022)** compatible with **GRS quark mass** scheme in **RM123S (2019)**

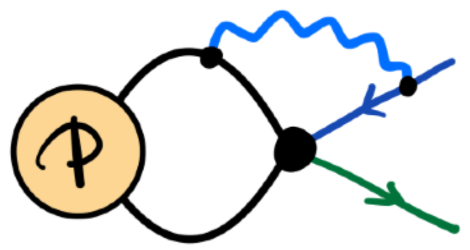
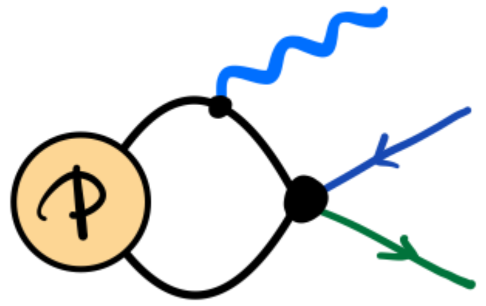
BMW, PRL 111 (2013), PRL 117 (2016)

Gasser, Rusetsky & Scimemi, EPJC 32 (2003)

Leptonic decay rate at $\mathcal{O}(\alpha)$

The RM123+Soton approach

F. Bloch & A. Nordsieck, PR **52** (1937)
N. Carrasco et al., PRD **91** (2015)
D. Giusti et al., PRL **120** (2018)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) = \lim_{\Lambda_{\mathrm{IR}} \to 0} \Big\{$  $+$  $\Big\}$

IR finite

IR divergent

IR divergent

Leptonic decay rate at $\mathcal{O}(\alpha)$

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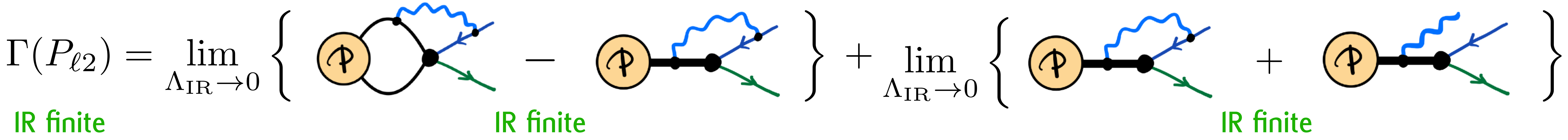

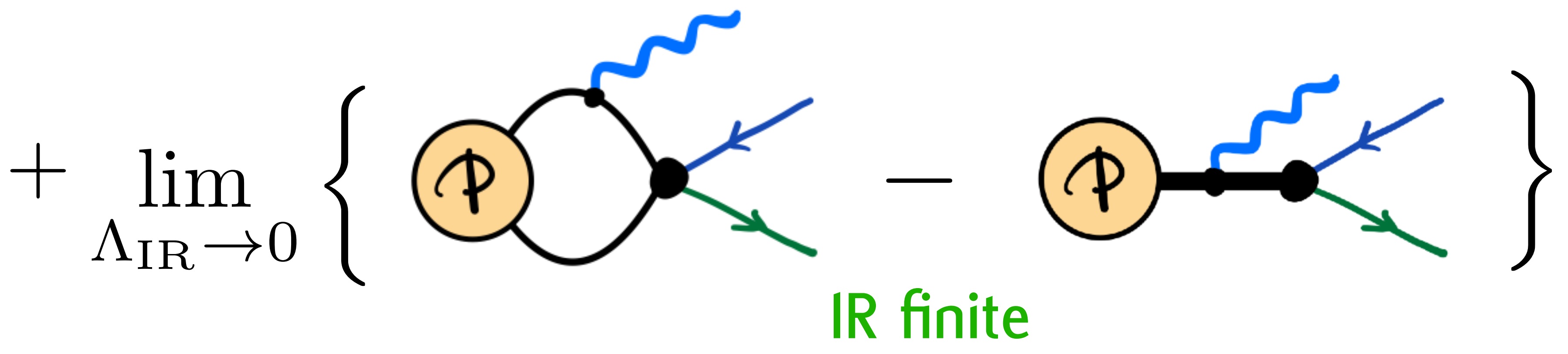

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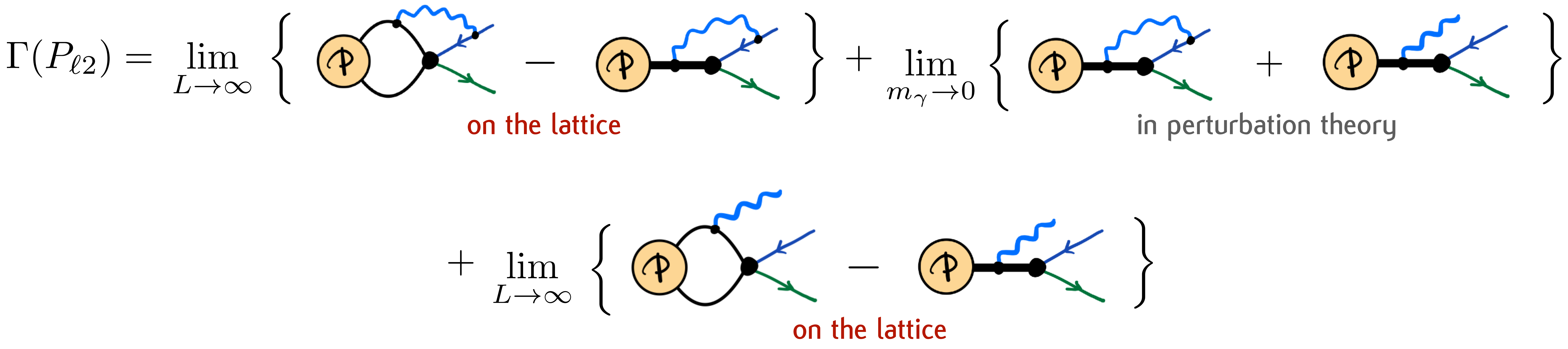

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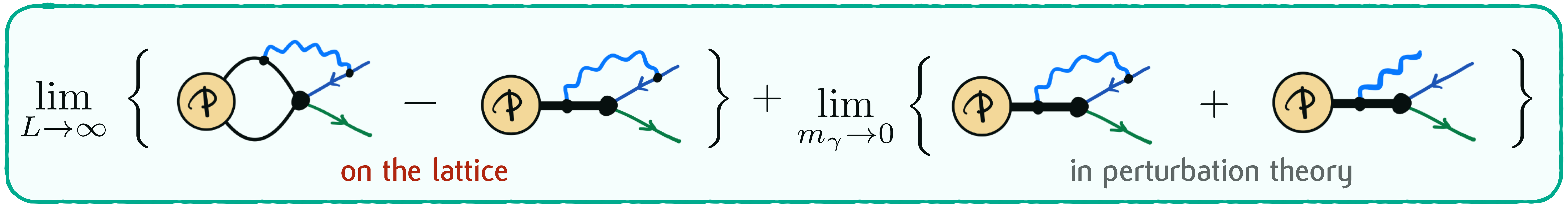
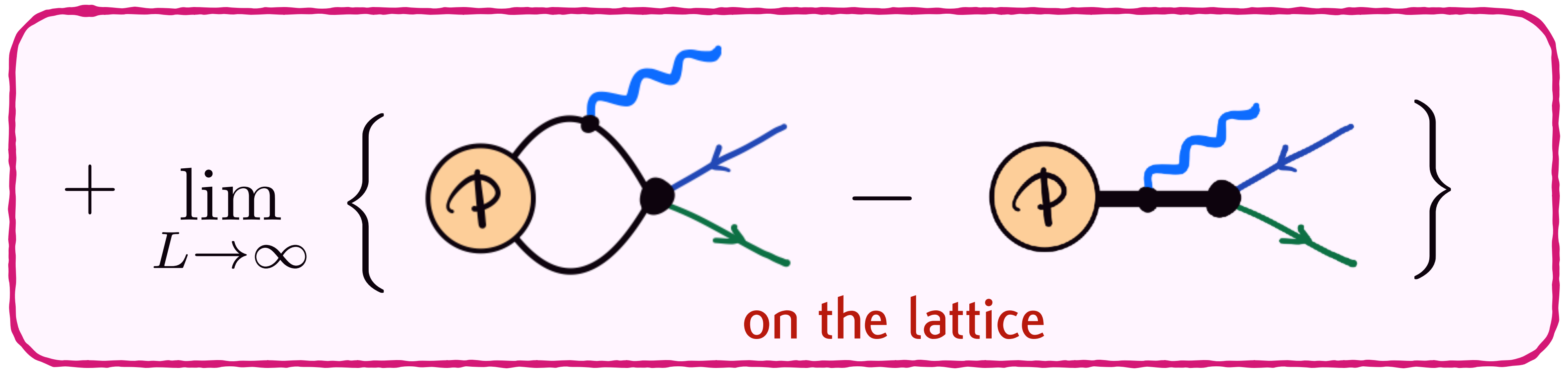
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

$\Gamma(P_{\ell 2}) =$





enough for $K_{\mu 2}$ and $\pi_{\mu 2}$

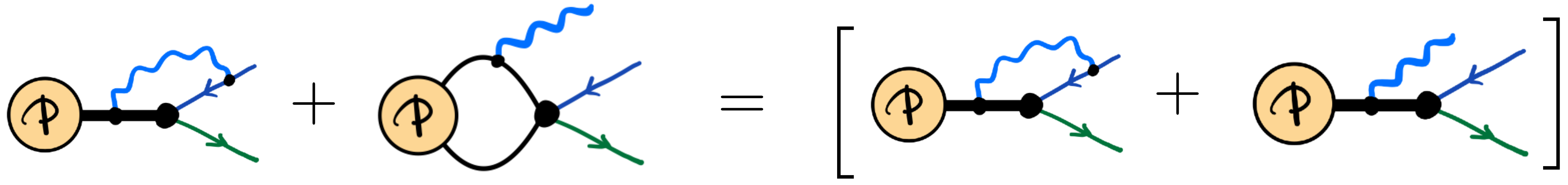
leading **finite-volume scaling** well studied

V.Lubicz et al., PRD **95** (2017) N.Tantalo et al., [1612.00199v2]
MDC et al., PRD **105** (2022)

relevant for K_{e2} and π_{e2}
& decays of heavier mesons

G.M. de Divitiis et al., [1908.10160]
R. Frezzotti et al., PRD **103** (2021)
A. Desiderio et al., PRD **102** (2021)
C. Kane et al., [1907.00279 & 2110.13196]
D. Giusti et al., [2302.01298]
R.Frezzotti et al., [2306.05904]

Real photon emission and structure dependence

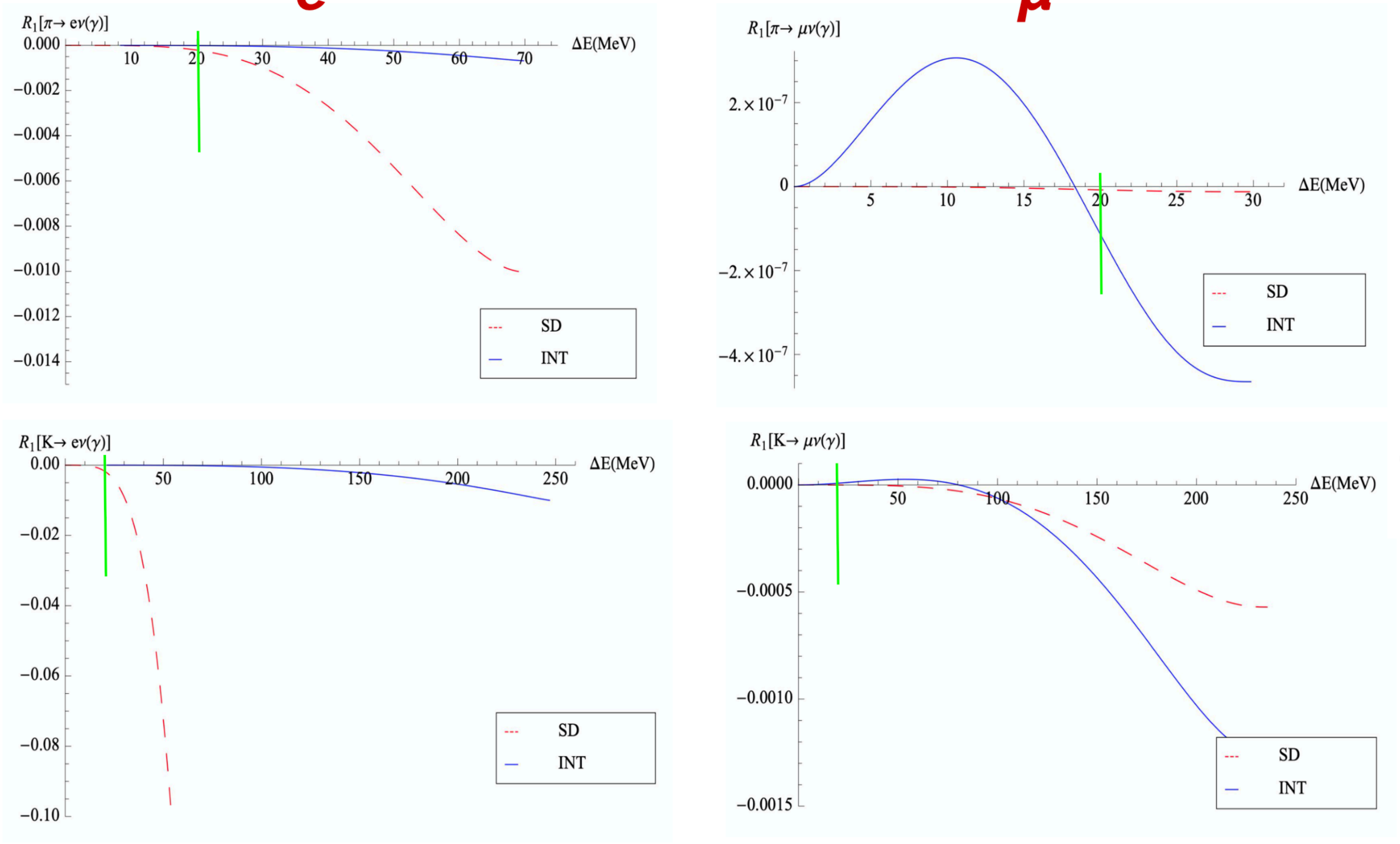


$$\left(1 + R_1^{\mathrm{SD}}(\Delta E) + R_1^{\mathrm{INT}}(\Delta E)\right)$$

e *μ*

π

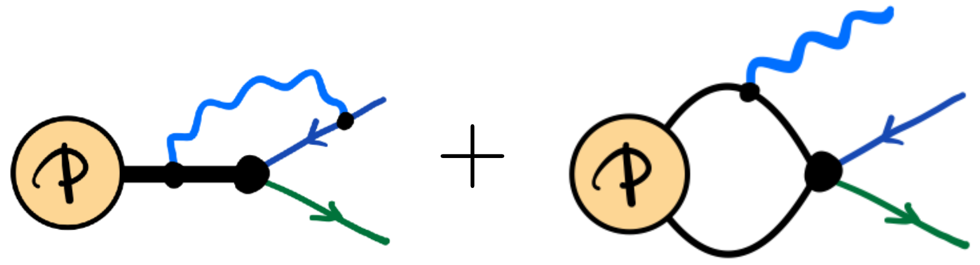
K



Calculation at O(p4) in χPT

N. Carrasco et al., PRD 91 (2015)

Real photon emission and structure dependence



$$= \delta R_{\text{pt}}(\Delta E) + \delta R_1^{\text{SD}}(\Delta E) + \delta R_1^{\text{INT}}(\Delta E)$$

	$\pi_{e2[\gamma]}$	$\pi_{\mu 2[\gamma]}$	$K_{e2[\gamma]}$	$K_{\mu 2[\gamma]}$
δR_0	$(*)$	$0.0411\ (19)$	$(*)$	$0.0341\ (10)$
$\delta R_{\text{pt}}(\Delta E_\gamma^{max})$	-0.0651	-0.0258	-0.0695	-0.0317
$\delta R_1^{\text{SD}}(\Delta E_\gamma^{max})$	$5.4\ (1.0) \times 10^{-4}$	$2.6\ (5) \times 10^{-10}$	$1.19\ (14)$	$2.2\ (3) \times 10^{-5}$
$\delta R_1^{\text{INT}}(\Delta E_\gamma^{max})$	$-4.1\ (1.0) \times 10^{-5}$	$-1.3\ (1.5) \times 10^{-8}$	$-9.2\ (1.3) \times 10^{-4}$	$-6.1\ (1.1) \times 10^{-5}$
ΔE_γ^{max} (MeV)	69.8	29.8	246.8	235.5

$^{(*)}$ *Not yet evaluated by numerical lattice QCD+QED simulations.*

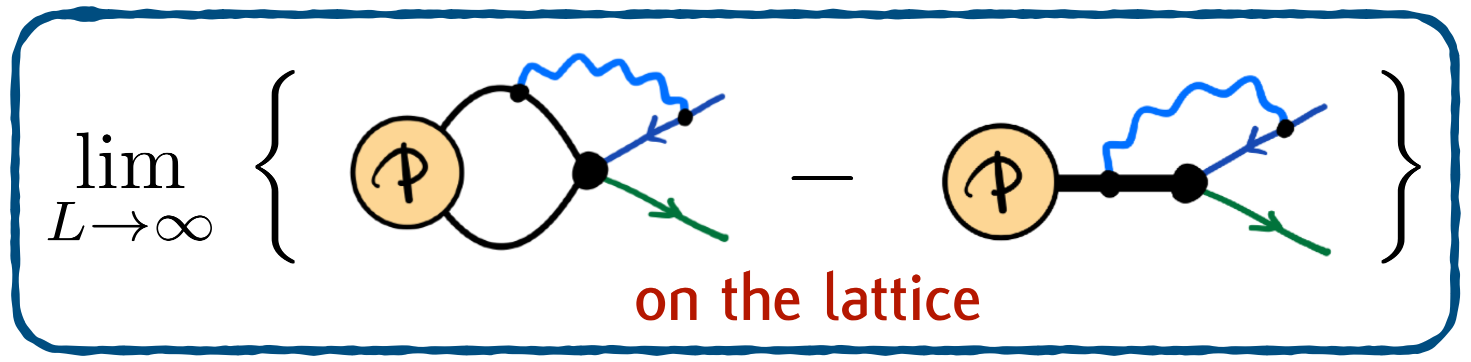
Confirmed by numerical lattice calculation

A. Desiderio et al., PRD 102 (2021)

R. Frezzotti et al., PRD 103 (2021)

Leptonic decay rate at $\mathcal{O}(\alpha)$

Virtual decay rate



$$\Gamma(P_{\ell 2}) = \Gamma_P^{\text{tree}}(1+\delta R_P) \quad \blacktriangleright \quad \Gamma_P^{\text{tree}} = \frac{G_F^2}{8\pi} m_\ell^2 \left(1-\frac{m_\ell^2}{m_P^2}\right)^2 m_P \left[f_{P,0}\right]^2 \quad \blacktriangleright \quad \delta R_P = 2\left(\frac{\delta\mathcal{A}_P}{\mathcal{A}_{P,0}} - \frac{\delta m_P}{m_{P,0}} + \frac{\delta\mathcal{Z}}{\mathcal{Z}_0}\right)$$

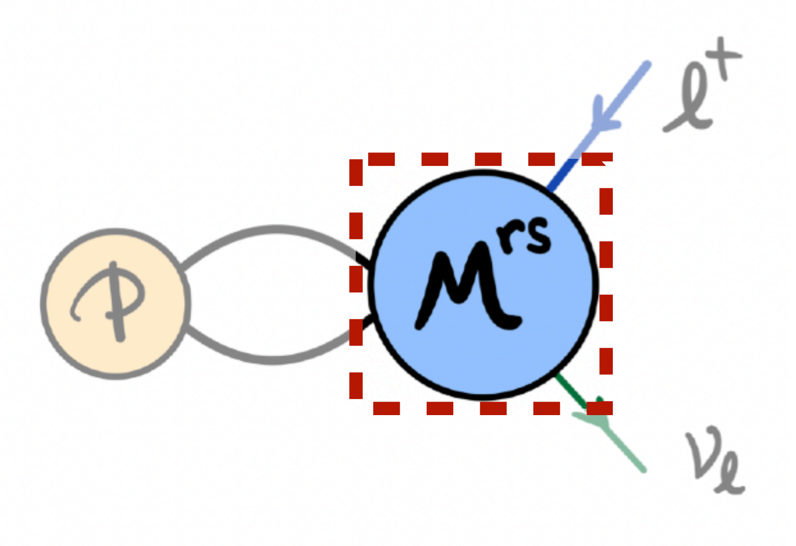
PDG convention

- $\boldsymbol{\delta\mathcal{A}_P}$ from the correction to the (bare) matrix element $\mathcal{M}_P^{rs}(\mathbf{p}_\ell) = \langle \ell^+, r, \mathbf{p}_\ell;\, \nu_\ell, s, \mathbf{p}_\nu |\, O_W \,| P^+, \mathbf{0}\rangle$
- $\boldsymbol{\delta m_P}$ correction to the meson mass
- $\boldsymbol{\delta\mathcal{Z}}$ correction to the renormalization of the weak operator O_W MDC et al., PRD **100** (2019)

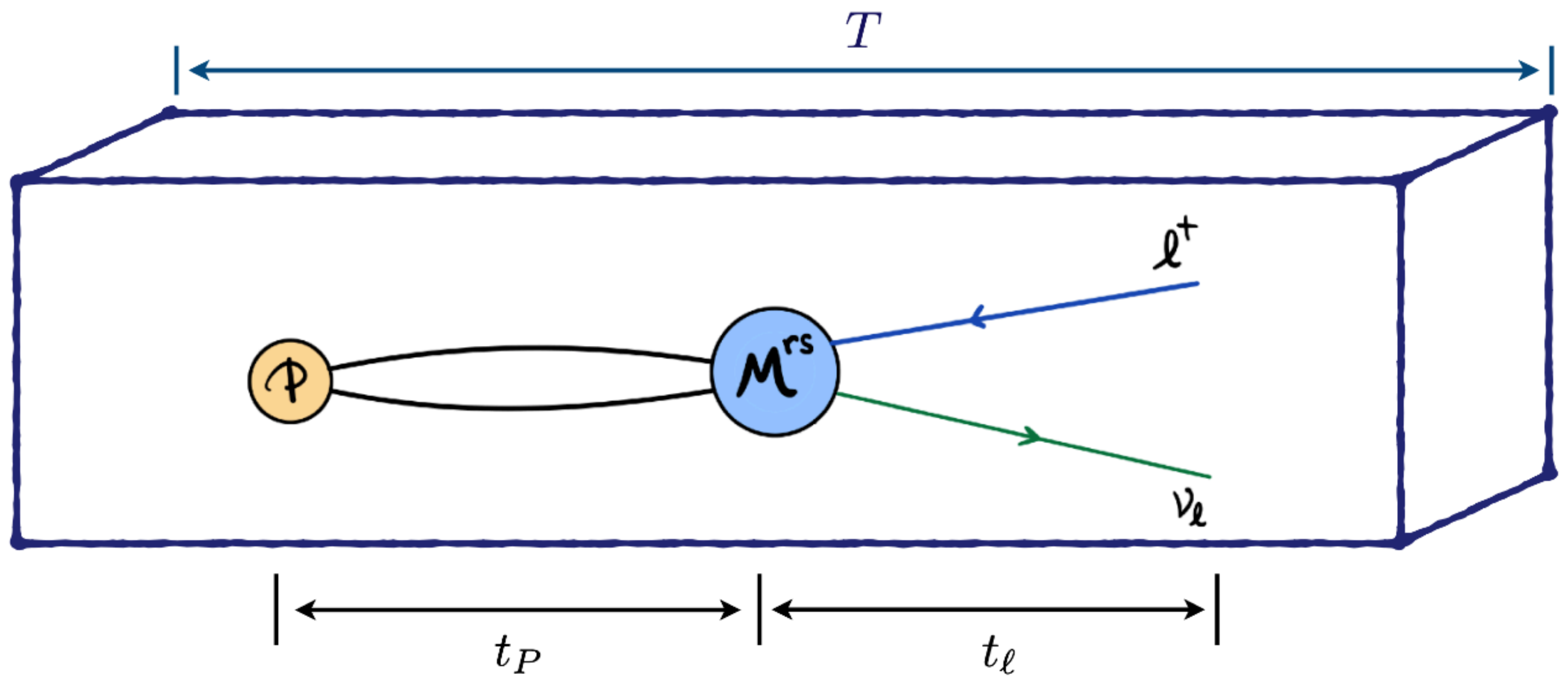
$$\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})} \rightarrow \delta R_{K\pi} = 2\left(\frac{\delta\mathcal{A}_K}{\mathcal{A}_{K,0}} - \frac{\delta m_K}{m_{K,0}}\right) - 2\left(\frac{\delta\mathcal{A}_\pi}{\mathcal{A}_{\pi,0}} - \frac{\delta m_\pi}{m_{\pi,0}}\right)$$

From correlators to matrix elements

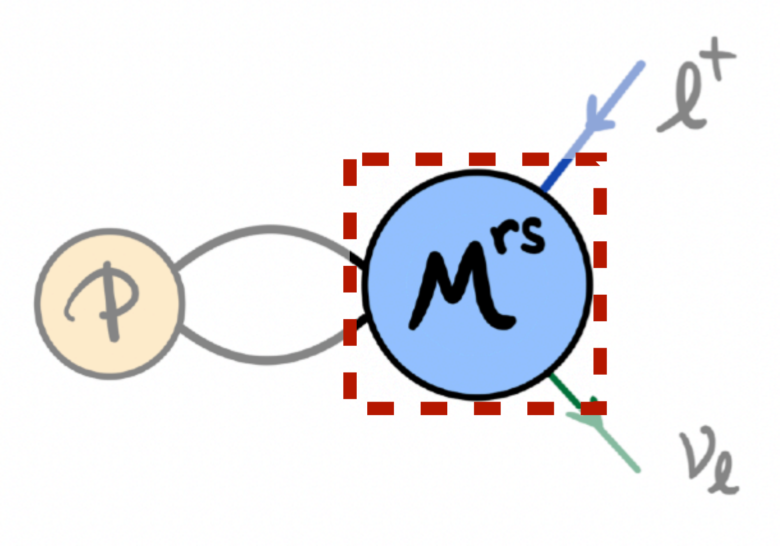
Our goal:

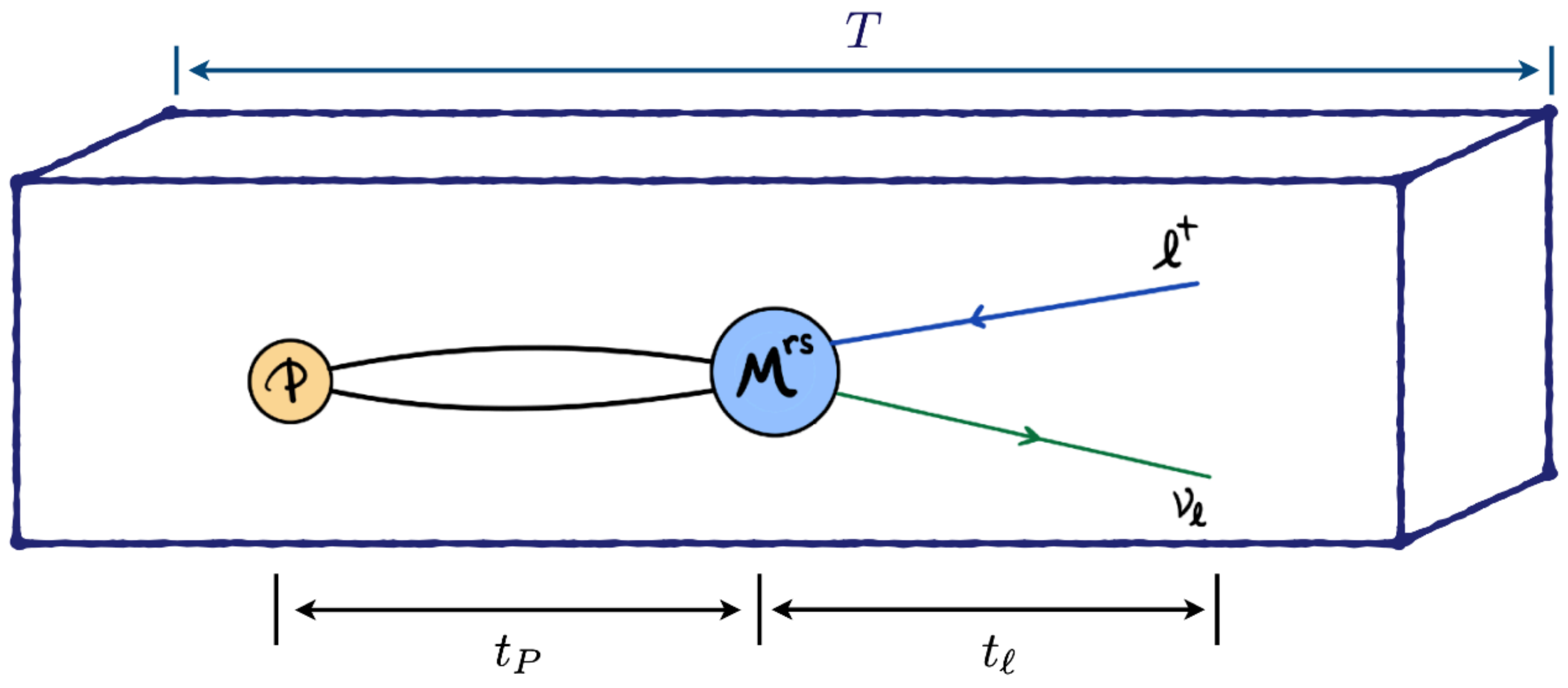


How we realise it:

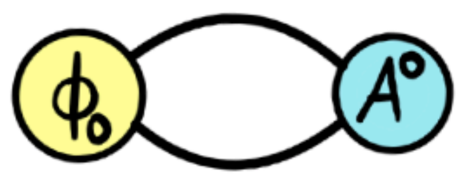


From correlators to matrix elements

Our goal: 

How we realise it: 

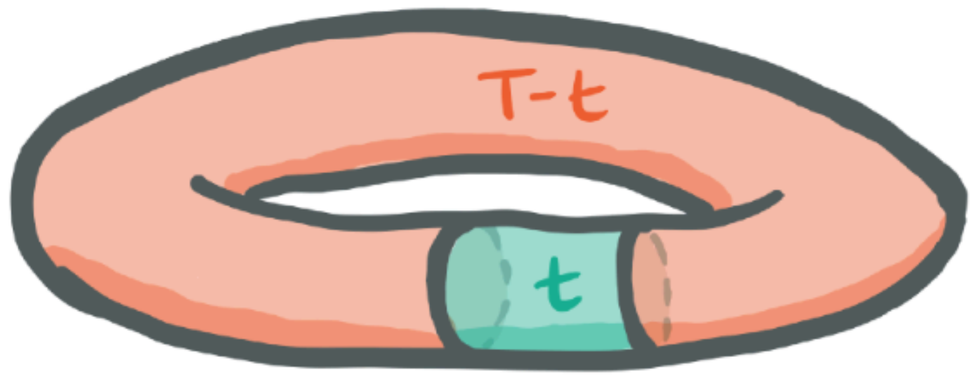
Tree-level decay amplitude: $|\mathcal{M}_0(\mathbf{p}_\ell)|^2 = |\mathcal{A}_{P,0}|^2 |\mathcal{L}_0(\mathbf{p}_\ell)|^2 \qquad \mathcal{A}_{P,0} = \langle 0|A^0|P\rangle_0 = i m_{P,0}\,[f_{P,0}]$



$$= \langle 0|A^0(0)\phi^\dagger(-t)|0\rangle_T \to \frac{Z_{P,0}\mathcal{A}_{P,0}}{2m_{P,0}}\left\{\mathrm{e}^{-m_{P,0}t} - \mathrm{e}^{-m_{P,0}(T-t)}\right\}$$

$$Z_{P,0} = \langle P, \mathbf{p}=\mathbf{0}|\phi^\dagger|0\rangle_0$$

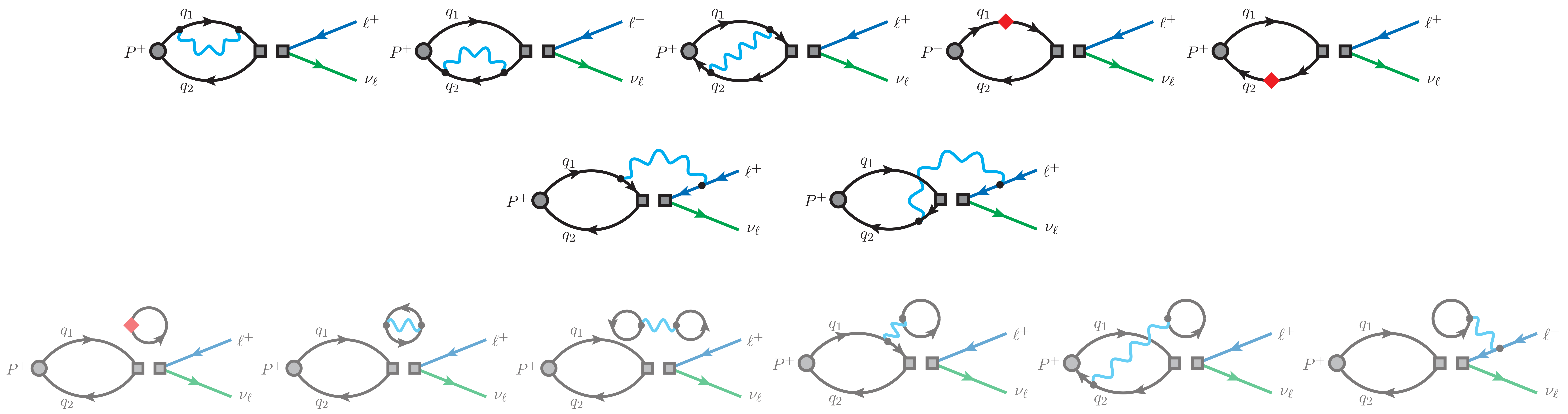
$$= \langle 0|\phi(0)\phi^\dagger(-t)|0\rangle_T \to \frac{Z_{P,0}^2}{2m_{P,0}}\left\{\mathrm{e}^{-m_{P,0}t} + \mathrm{e}^{-m_{P,0}(T-t)}\right\}$$



IB corrections to the decay amplitude

G.M.de Divitiis et al. [RM123], PRD **87** (2013)

RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_u - m_d = 0$

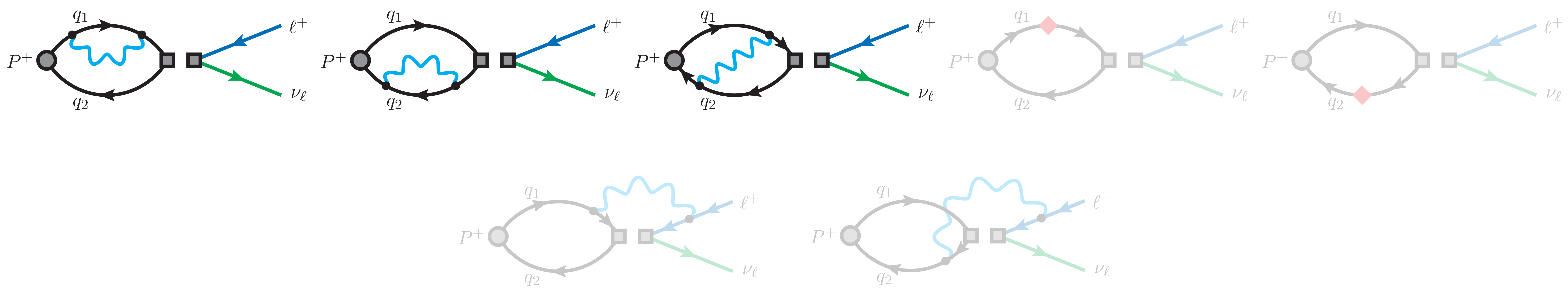


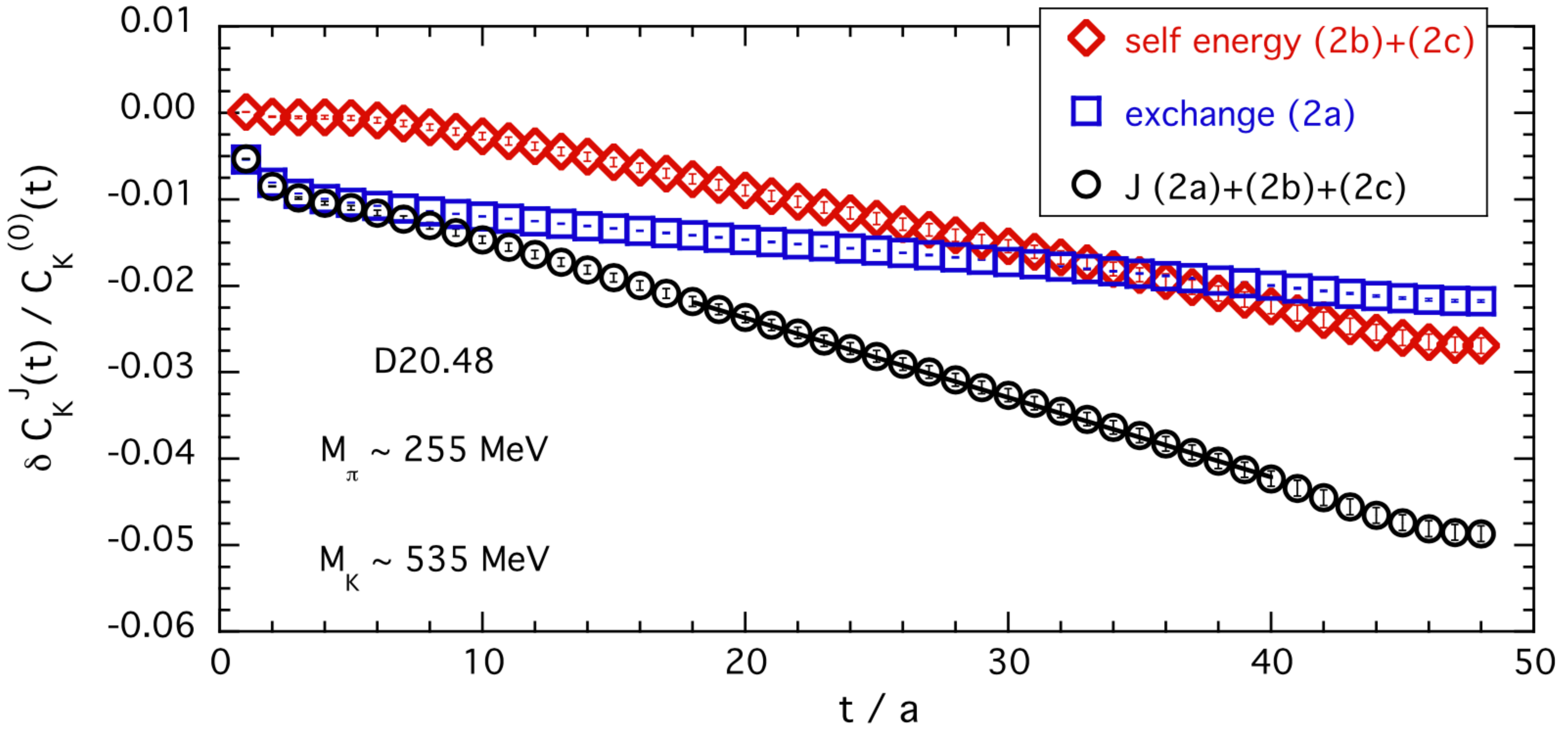
Both RM123S and RBC-UKQCD calculations are performed in the **electro-quenched approximation: sea quarks electrically neutral**

IB corrections to the decay amplitude

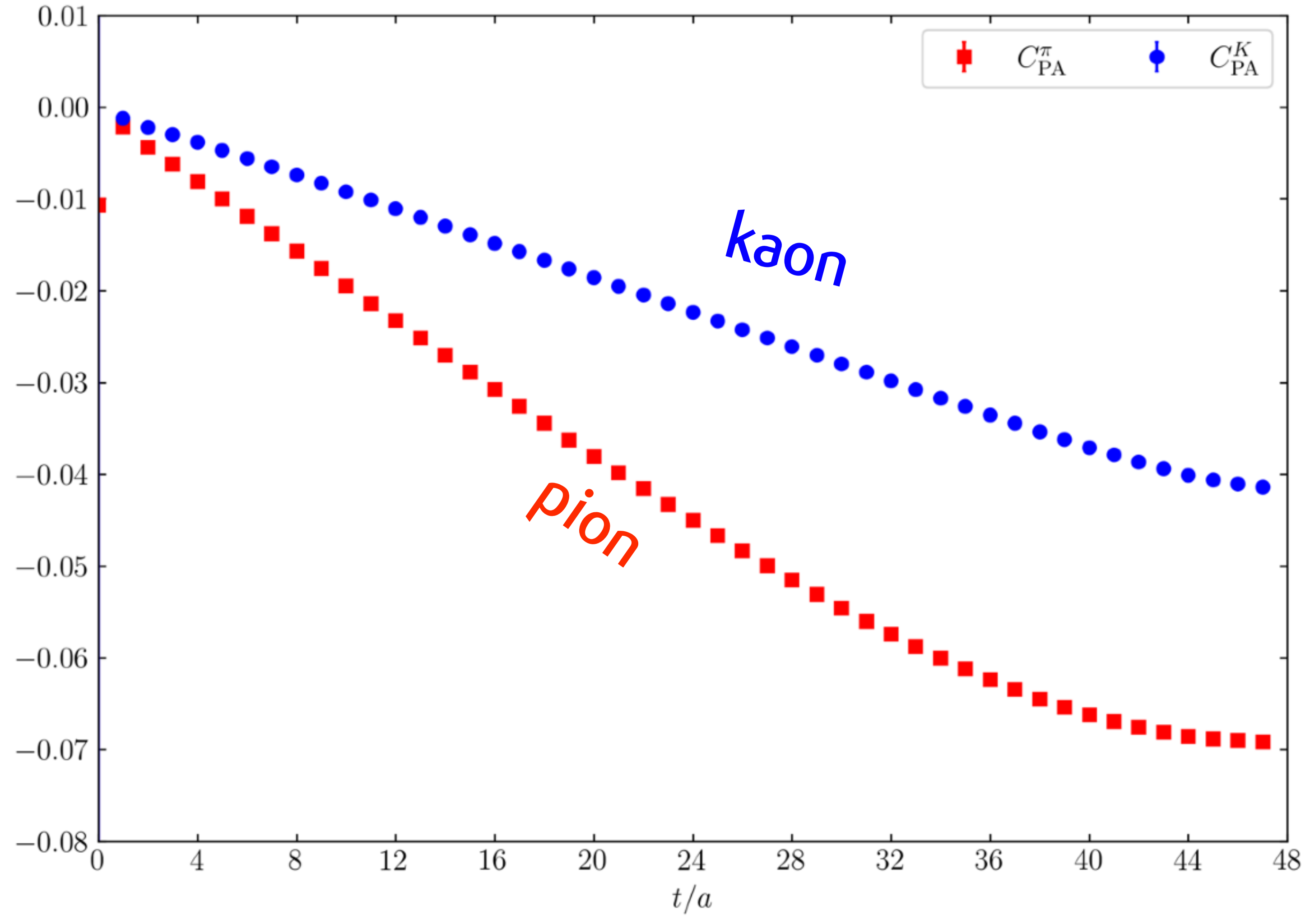
G.M.de Divitiis et al. [RM123], PRD **87** (2013)

RM123 perturbative method: expand lattice path-integral around isosymmetric point $\alpha = m_{\rm u} - m_{\rm d} = 0$





MDC et al., PRD 100 (2019)



P.Boyle, MDC et al., JHEP 02 (2023)

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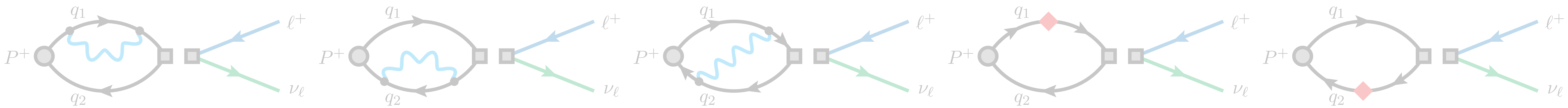
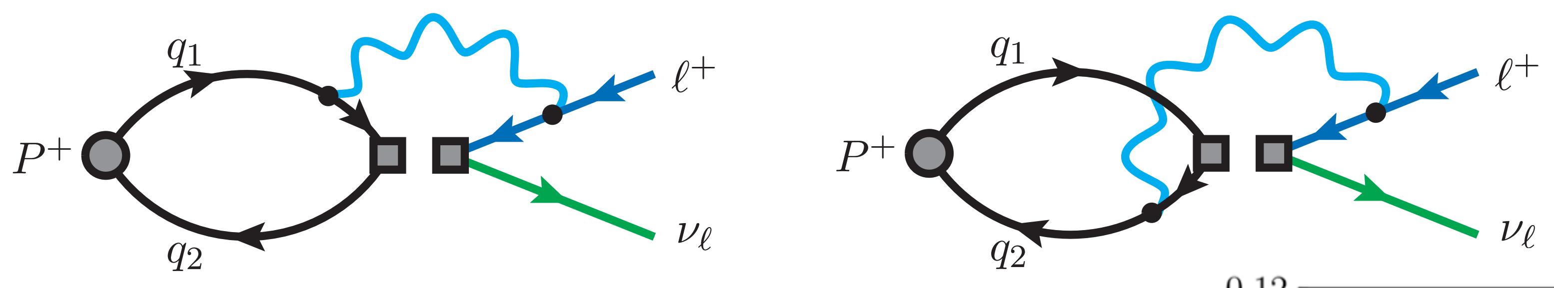
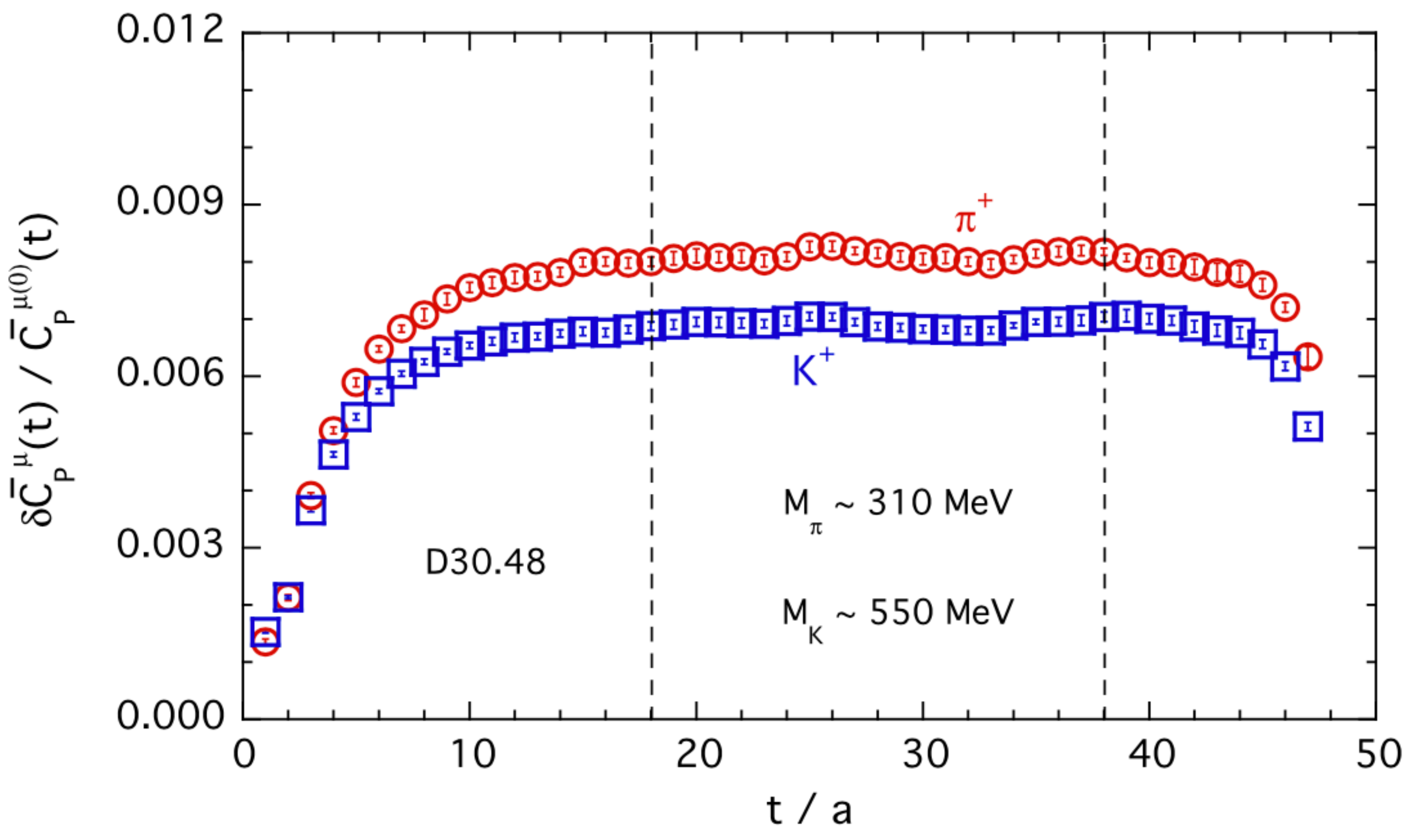
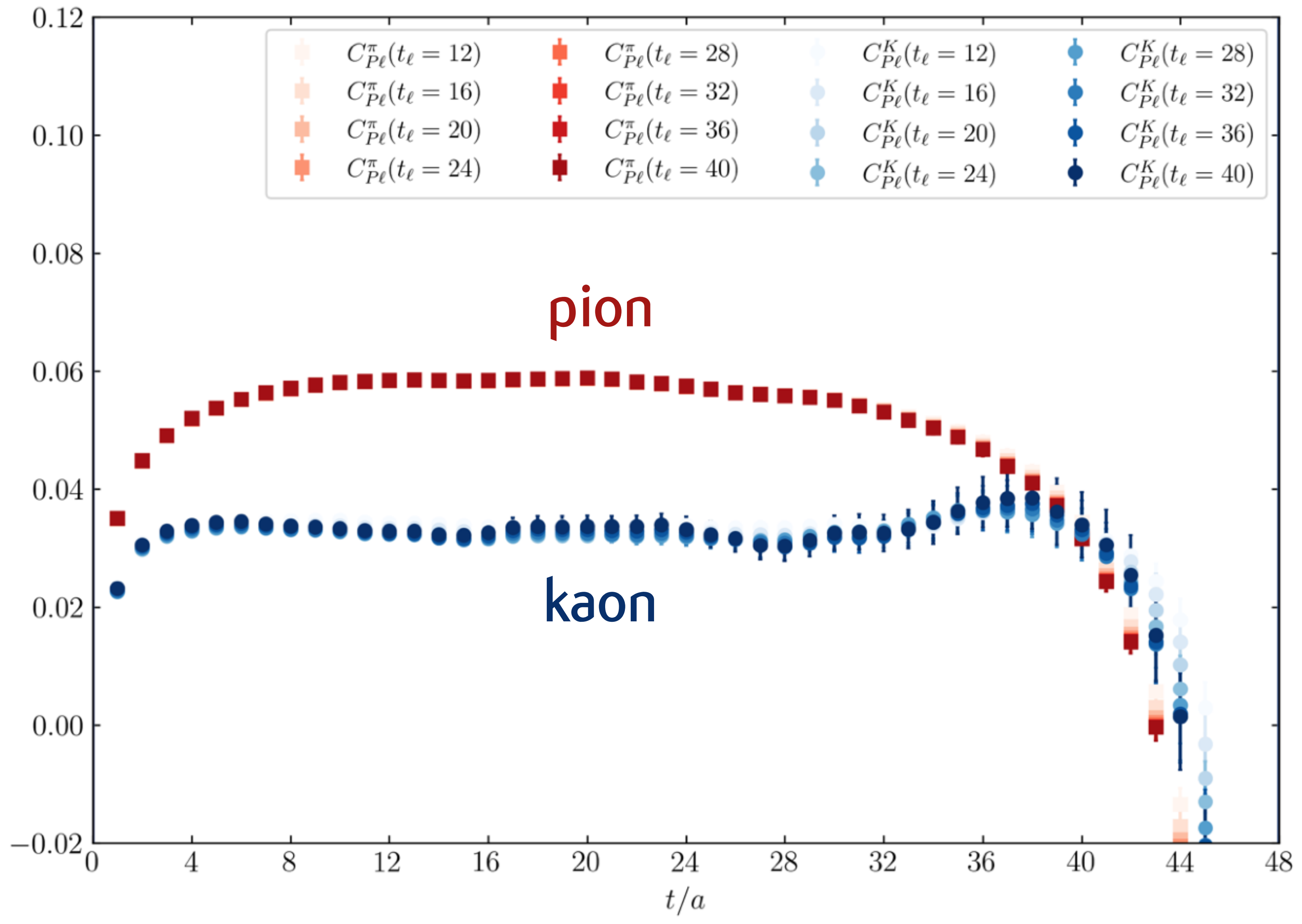






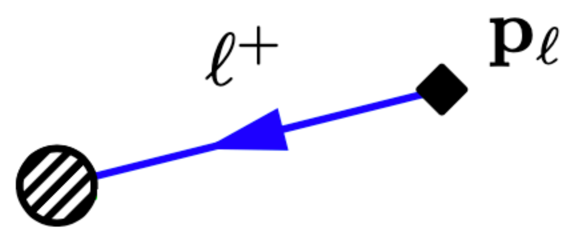
MDC et al., PRD 100 (2019)



P.Boyle, MDC et al., JHEP 02 (2023)

Non-factorisable QED corrections

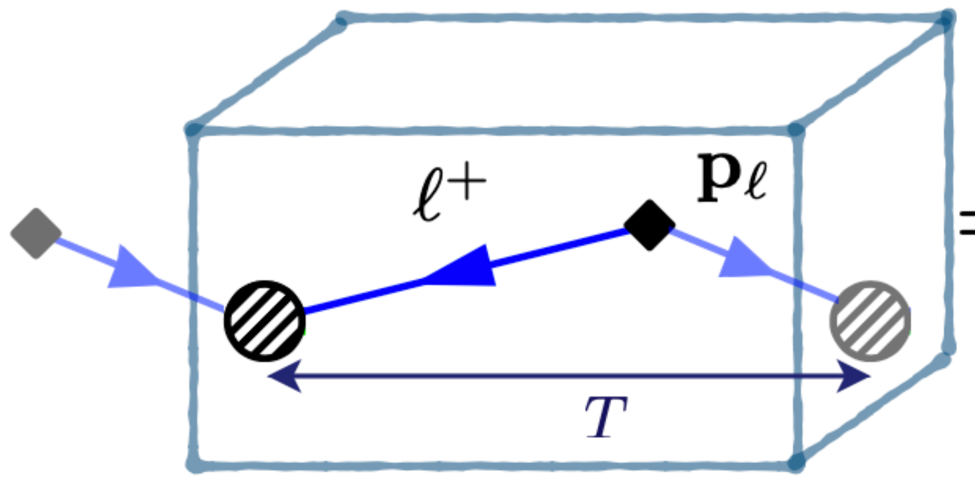
The lepton in a finite volume



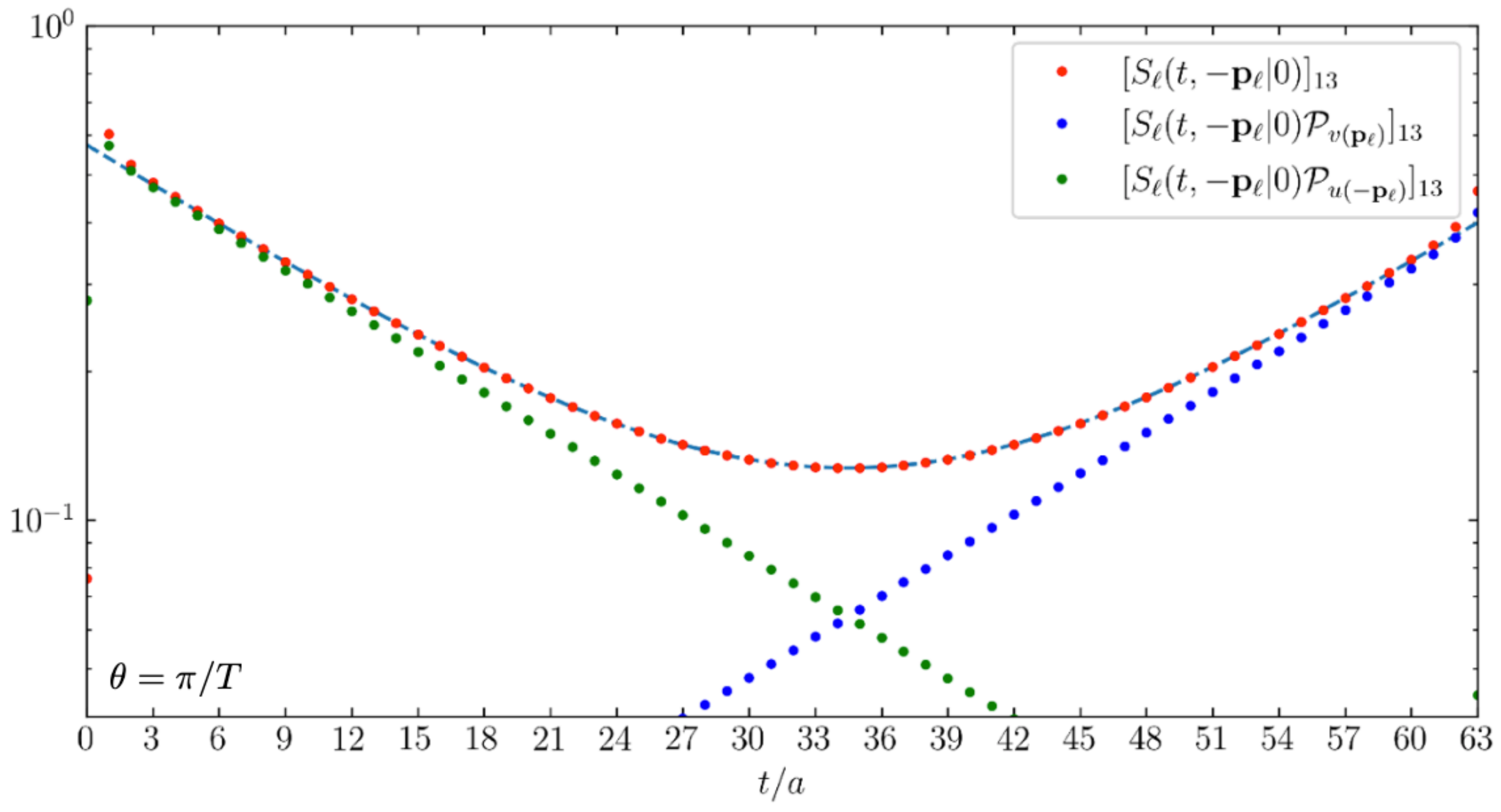
$$= \quad S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -\mathrm{e}^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} \right\}$$

Non-factorisable QED corrections

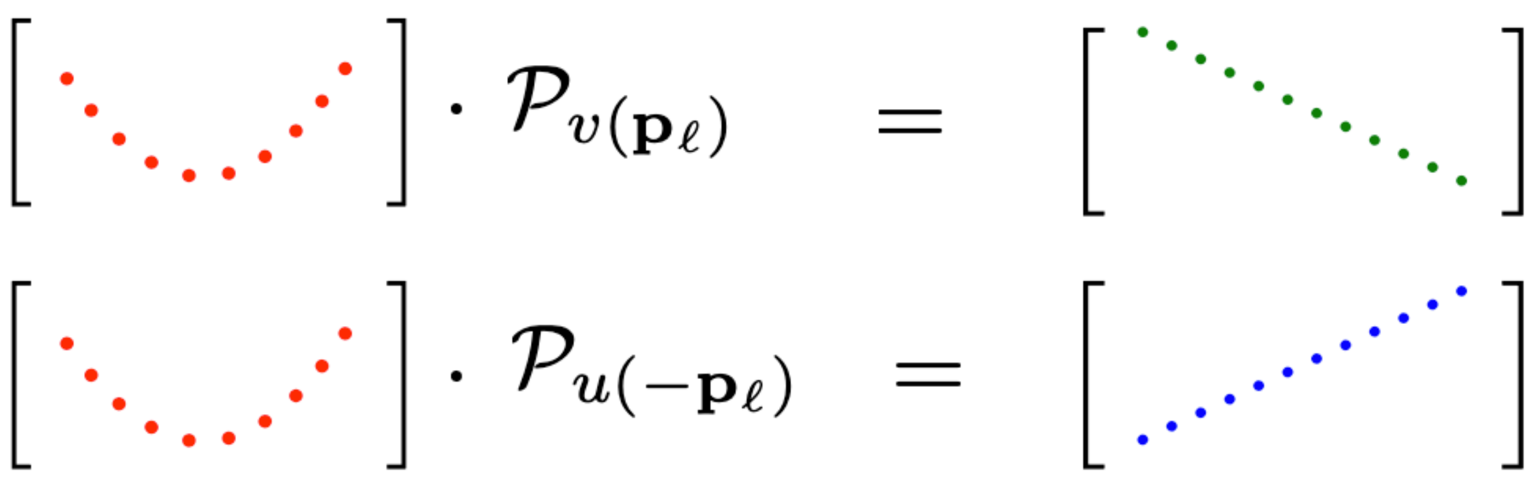
The lepton in a finite volume



$$= \; S(0|t,\mathbf{p}_\ell) = \sum_r \left\{ -e^{-tE_\ell} \frac{v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)}{2\Omega_\ell} + e^{i\theta T} e^{-(T-t)E_\ell} \frac{u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)}{2\Omega_\ell} \right\} \times \frac{1}{1 - e^{-TE_\ell} e^{i\theta T}}$$



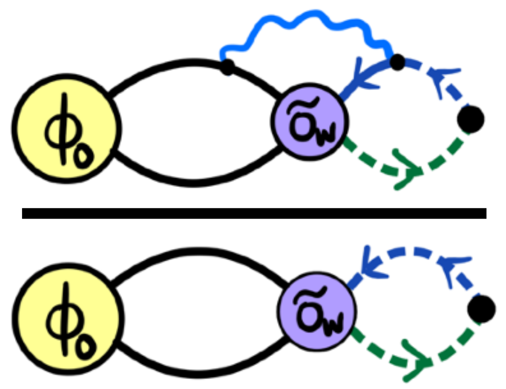
We can select specific components using projectors:



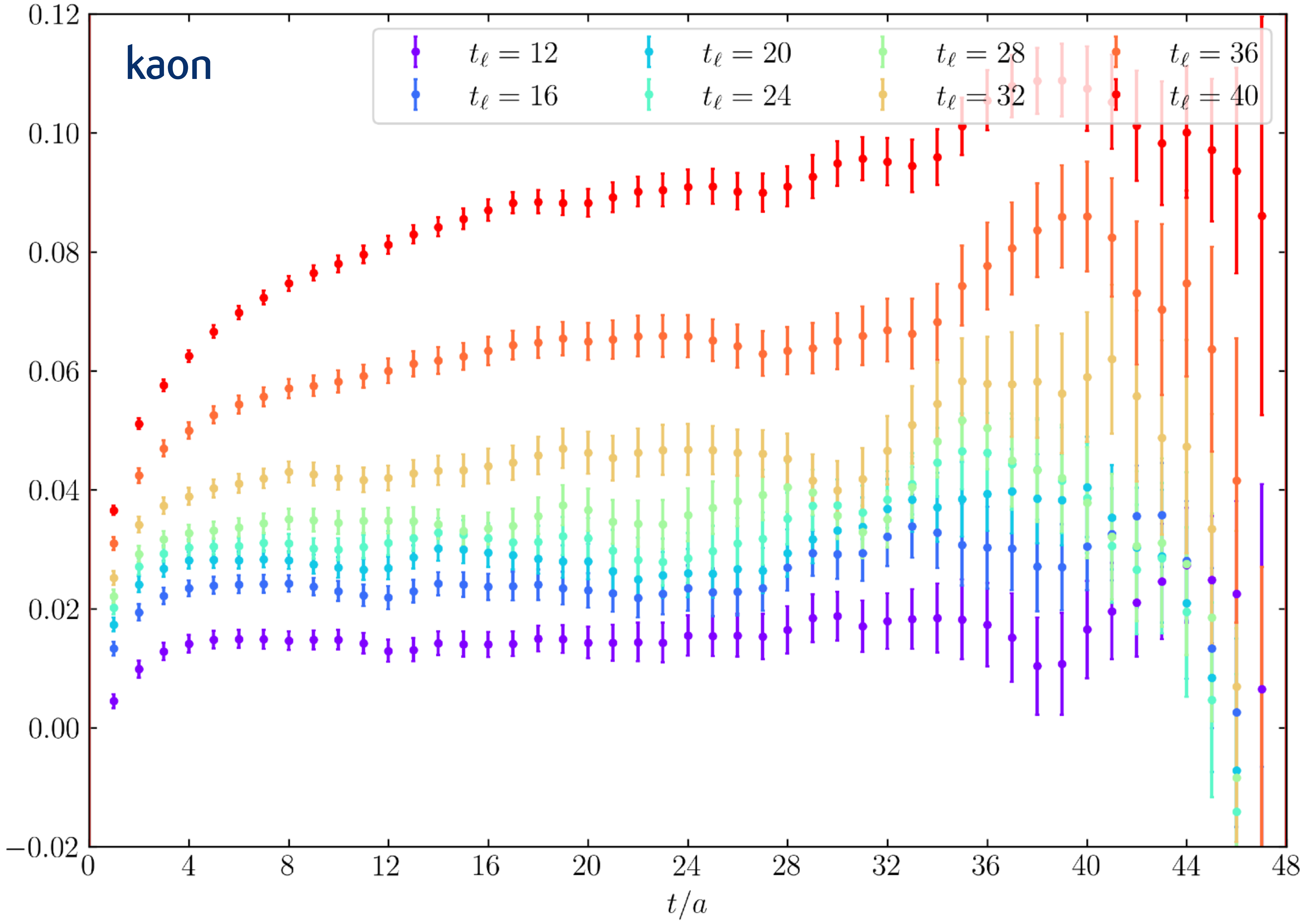
$$\mathcal{P}_{v(\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1} [v_r(\mathbf{p}_\ell)\bar{v}_r(\mathbf{p}_\ell)]$$

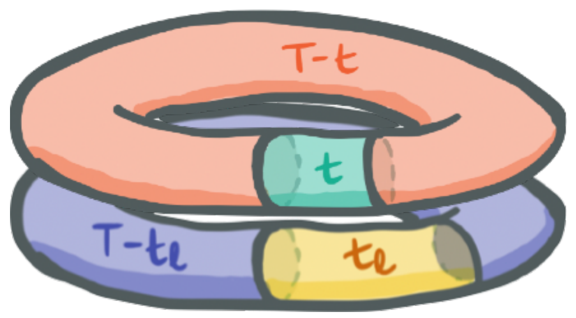
$$\mathcal{P}_{u(-\mathbf{p}_\ell)} = \{u_t(-\mathbf{p}_\ell)\bar{u}_t(-\mathbf{p}_\ell) + v_s(\mathbf{p}_\ell)\bar{v}_s(\mathbf{p}_\ell)\}^{-1} [u_r(-\mathbf{p}_\ell)\bar{u}_r(-\mathbf{p}_\ell)]$$

Non-factorisable QED corrections

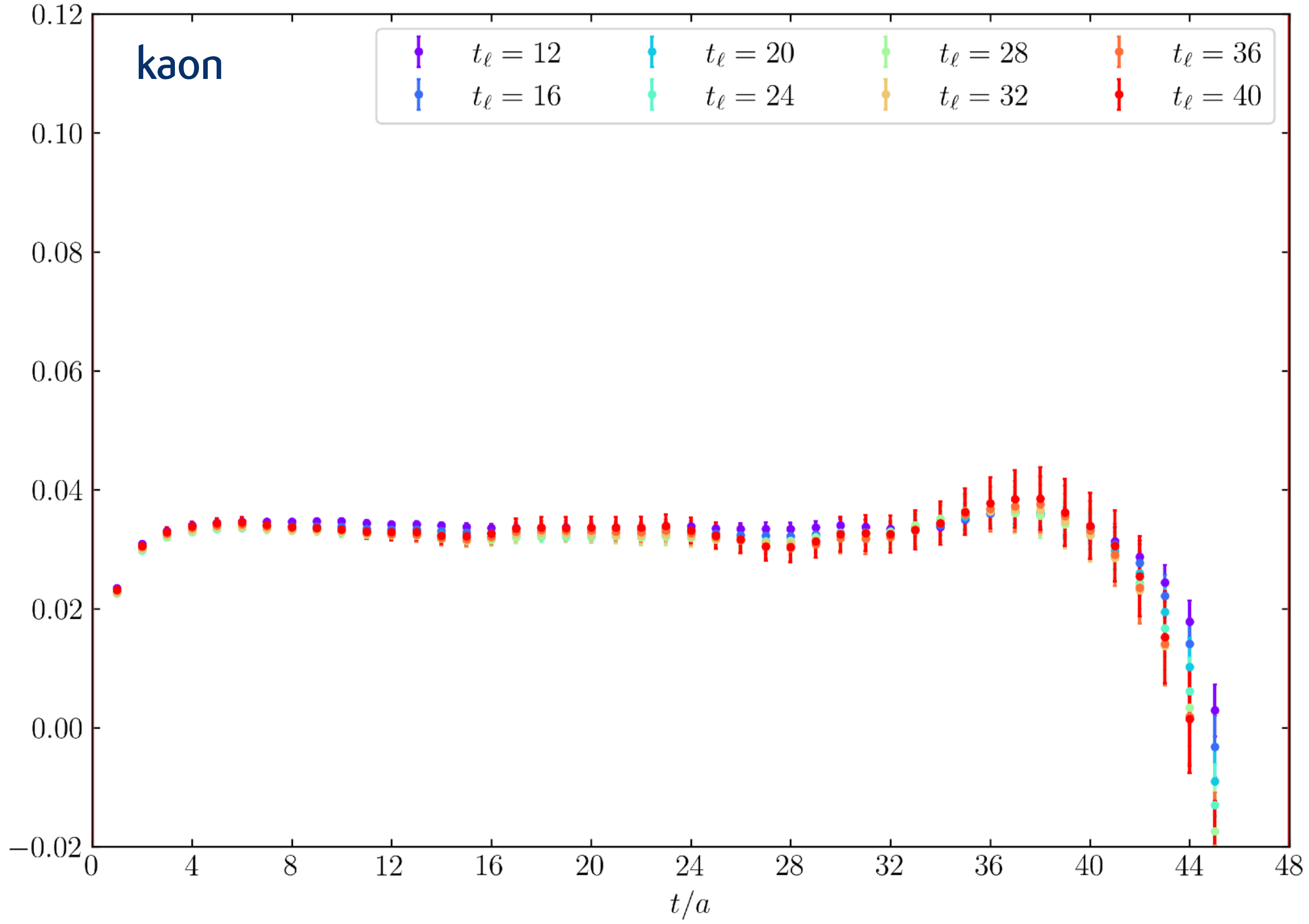


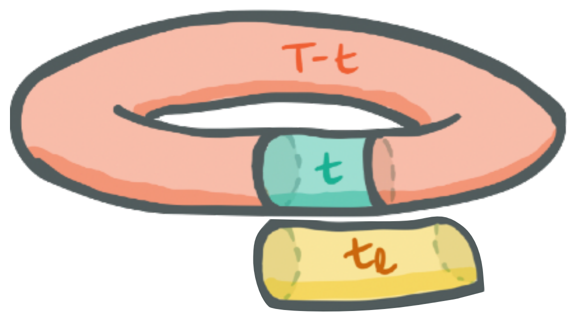
$$\rightarrow \frac{\delta_{\text{non-fact}}\mathcal{A}_P}{\mathcal{A}_{P,0}} \quad f_{\mathrm{P}\ell}(t,T)$$





without projection



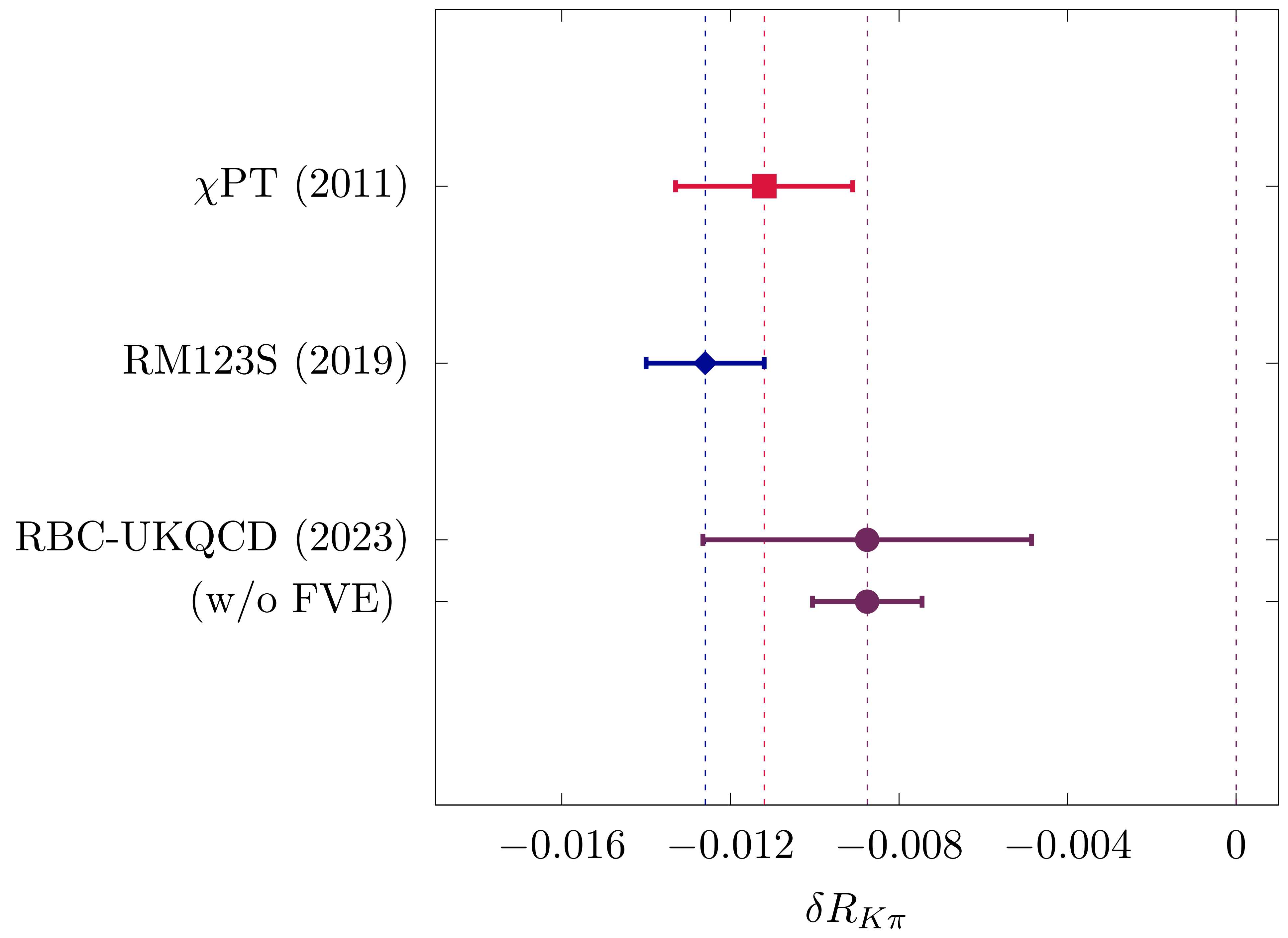


with projection

Results for $\delta R_{K\pi}$

V. Cirigliano et al., PLB **700** (2011)
MDC et al., PRD **100** (2019)
P.Boyle, MDC et al., JHEP **02** (2023)

- ■ $\delta R_{K\pi} = -0.0112\,(21)$
- ◆ $\delta R_{K\pi} = -0.0126\,(14)$
- ● $\delta R_{K\pi} = -0.0086\,(13)(39)_{\mathrm{vol.}}$



$$\frac{\Gamma(K \to \ell\nu_\ell)}{\Gamma(\pi \to \ell\nu_\ell)} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \left(\frac{f_K}{f_\pi}\right)^2 (1 + \delta R_{K\pi})$$

- Strong evidence that $\delta R_{K\pi}$ **can be computed from first principles non-perturbatively on the lattice!**
- **RBC-UKQCD error** dominated by a large systematic uncertainty related to **finite-volume effects**
- **Errors on** $|V_{us}|/|V_{ud}|$ from **theoretical** inputs can become comparable with those from **experiments**

Prospects for $|V_{us}/V_{ud}|$

An exercise on the error budget

$$\left|\frac{V_{us}}{V_{ud}}\right|^2 = \left[\frac{\Gamma(K_{\ell 2})}{\Gamma(\pi_{\ell 2})}\frac{M_{K^+}^3}{M_{\pi^+}^3}\frac{(M_{K^+}^2 - M_{\mu^+}^2)^2}{(M_{\pi^+}^2 - M_{\mu^+}^2)^2}\right]_{\text{exp}} \cdot \left[\frac{f_{K,0}}{f_{\pi,0}}\right]^2 (1 + \delta R_{K\pi})$$

- Using our new result $\delta R_{K\pi} = -0.0086\,(13)(39)_{\text{vol.}}$

$[f_{K,0}/f_{\pi,0}]$		$\lvert V_{us}/V_{ud}\rvert$
FLAG21 2+1 average	1.1930 (33)	$0.23154\ (28)_{\text{exp}}\ (15)_{\delta R}\ (45)_{\delta R,\text{vol.}}\ (65)_{f_P}$

- Using RM123S result $\delta R_{K\pi} = -0.0126\,(14)$

$[f_{K,0}/f_{\pi,0}]$		$\lvert V_{us}/V_{ud}\rvert$
FLAG19 2+1+1 average	1.1966 (18)	$0.23131\ (28)_{\text{exp}}\ (17)_{\delta R}\ (35)_{f_P}$

QED finite-volume effects

In **finite-volume** (massless) QED the photon zero modes require a regularisation

$$\Delta g(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathcal{G}(k_0, \mathbf{k})}{k_0^2 + |\mathbf{k}|^2} \qquad D^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1}{k_0^2 + |\mathbf{k}|^2}$$

M. Hayakawa & S. Uno, PTP 120 (2008) $\downarrow$ QED$_L$

$$\Delta' g(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{\mathrm{d}k_0}{2\pi} \frac{\mathcal{G}(k_0, \mathbf{k})}{k_0^2 + |\mathbf{k}|^2} \qquad D_L^{\mu\nu}(k_0, \mathbf{k}) = \delta^{\mu\nu} \frac{1 - \delta_{\mathbf{k},\mathbf{0}}}{k_0^2 + |\mathbf{k}|^2}$$

QED finite-volume effects

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Hadron masses

Mass corrections can be obtained from **Compton amplitude** using **Cottingham formula**

$$\Delta m_P(L) = m_P(L) - m_P(\infty) = \frac{e^2}{4m_P}\Delta_{\mathbf{k}} \frac{M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k})}{|\mathbf{k}|} \qquad M^{\mu\mu}(-\mathrm{i}|\mathbf{k}|,\mathbf{k}) = \frac{Z_{1\mathrm{P}}(0)}{|\mathbf{k}|} + \mathcal{M}(|\mathbf{k}|)$$

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$$\Delta m_P(L) = \frac{e^2}{4m_P} \left[c_2(\boldsymbol{\theta}) \frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta}) \frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta}) \frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty} \frac{(2\pi)^{\ell+1}}{L^{4+\ell}} \frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!} \mathcal{M}^{(\ell+2)}(0) \right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n} \right) \frac{1}{|\mathbf{n}|^s}$$

QED finite-volume effects

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$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$

$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s}$$

universal terms fixed by Ward identities

QED finite-volume effects

Hadron masses

using the notation of
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$$c_s(\boldsymbol{\theta}) = \left(\sum_{\mathbf{n}\in\Omega_{\boldsymbol{\theta}}} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s}$$

universal terms fixed by Ward identities

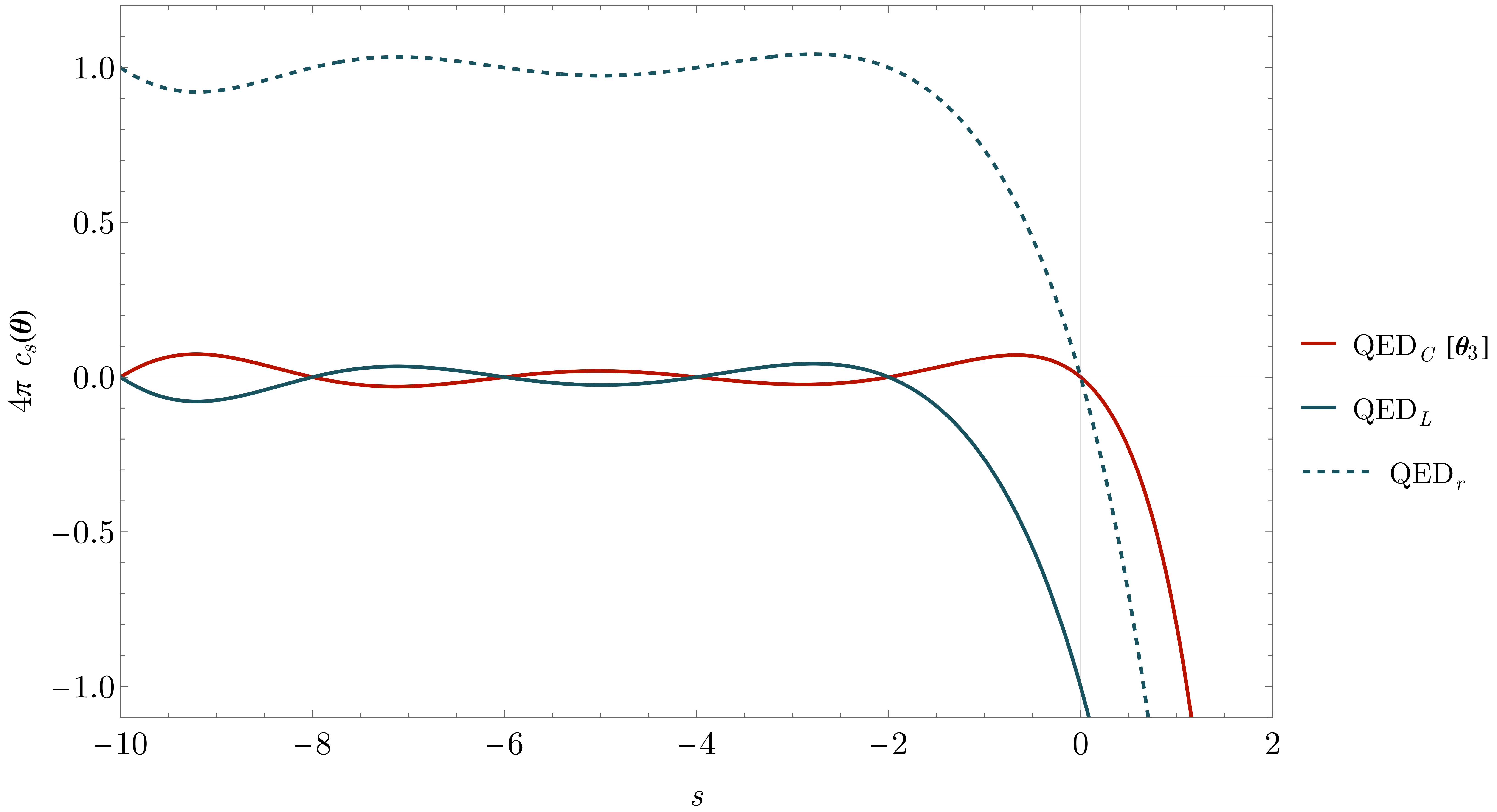
structure + multi-particle dependence

QED finite-volume effects

using the notation of
B.Lucini et al., JHEP 1602 (2016)

Hadron masses

$$\Delta m_P(L) = \frac{e^2}{4m_P}\left[c_2(\boldsymbol{\theta})\frac{Z_{1\mathrm{P}}(0)}{4\pi^2 L} + c_1(\boldsymbol{\theta})\frac{\mathcal{M}(0)}{2\pi L^2} + c_0(\boldsymbol{\theta})\frac{\mathcal{M}'(0)}{L^3} + \sum_{\ell=0}^{\infty}\frac{(2\pi)^{\ell+1}}{L^{4+\ell}}\frac{c_{-1-\ell}(\boldsymbol{\theta})}{(\ell+2)!}\mathcal{M}^{(\ell+2)}(0)\right]$$



QED finite-volume effects

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]

Leptonic decay amplitude

$$\Delta Y_P(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$

$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$

universal

$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A(0)}{f_P}\,\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, C_\ell^{(1)} + c_0(\mathbf{v}_\ell)\, C_\ell^{(2)}\right]$$

$$+ \cdots$$

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- **Collinear** divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

QED finite-volume effects

Leptonic decay amplitude

V. Lubicz et al., PRD 95 (2017)
N. Tantalo et al., [1612.00199v2]
MDC et al., PRD 105 (2022)
MDC et al., [2310.13358]

$$\Delta Y_P(L) = \frac{3}{4} + 4\log\left(\frac{m_\ell}{m_W}\right) + 2\log\left(\frac{m_W L}{4\pi}\right) - 2A_1(\mathbf{v}_\ell)\left[\log\frac{m_P L}{2\pi} + \log\frac{m_\ell L}{4\pi} - 1\right] + \frac{c_3 - 2(c_3(\mathbf{v}_\ell) - B_1(\mathbf{v}_\ell))}{2\pi}$$

$$- \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\, c_2 - 4r_\ell^2 c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$

} **universal**

$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A(0)}{f_P}\frac{4\pi m_P[(1+r_\ell)^2\, c_1 - 4r_\ell^2\, c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi[(1+r_\ell^2)c_1 - 2c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

$$+ \frac{1}{(m_P L)^3}\left[\frac{32\pi^2 c_0\,(2+r_\ell^2)}{(1+r_\ell^2)^3} + c_0\, C_\ell^{(1)} + c_0(\mathbf{v}_\ell)\, C_\ell^{(2)}\right]$$

$$+ \cdots$$

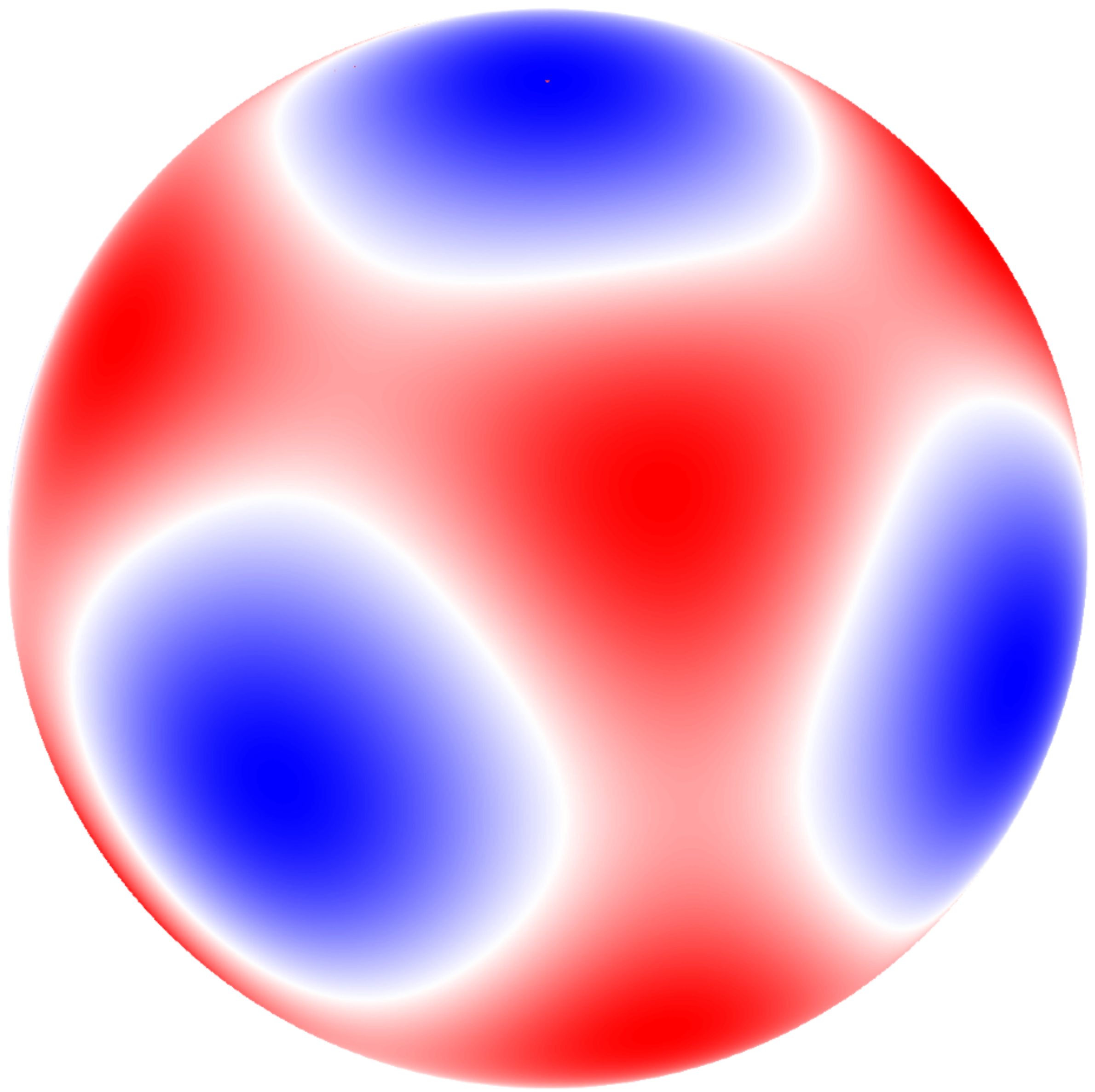
can QED_r help removing this unknown term?

$$c_s(\mathbf{v}_\ell) = \left(\sum_{\mathbf{n}\neq 0} - \int \mathrm{d}^3\mathbf{n}\right)\frac{1}{|\mathbf{n}|^s\,(1-\mathbf{v}_\ell\cdot\hat{\mathbf{n}})}$$

- **Collinear** divergent terms as $|\mathbf{v}| \to 1$ and $\mathbf{v} \parallel \mathbf{k}$
- Dependence on the direction $\hat{\mathbf{v}}$ due to **rotational symmetry breaking**

Velocity-dependent coefficients in QED_r

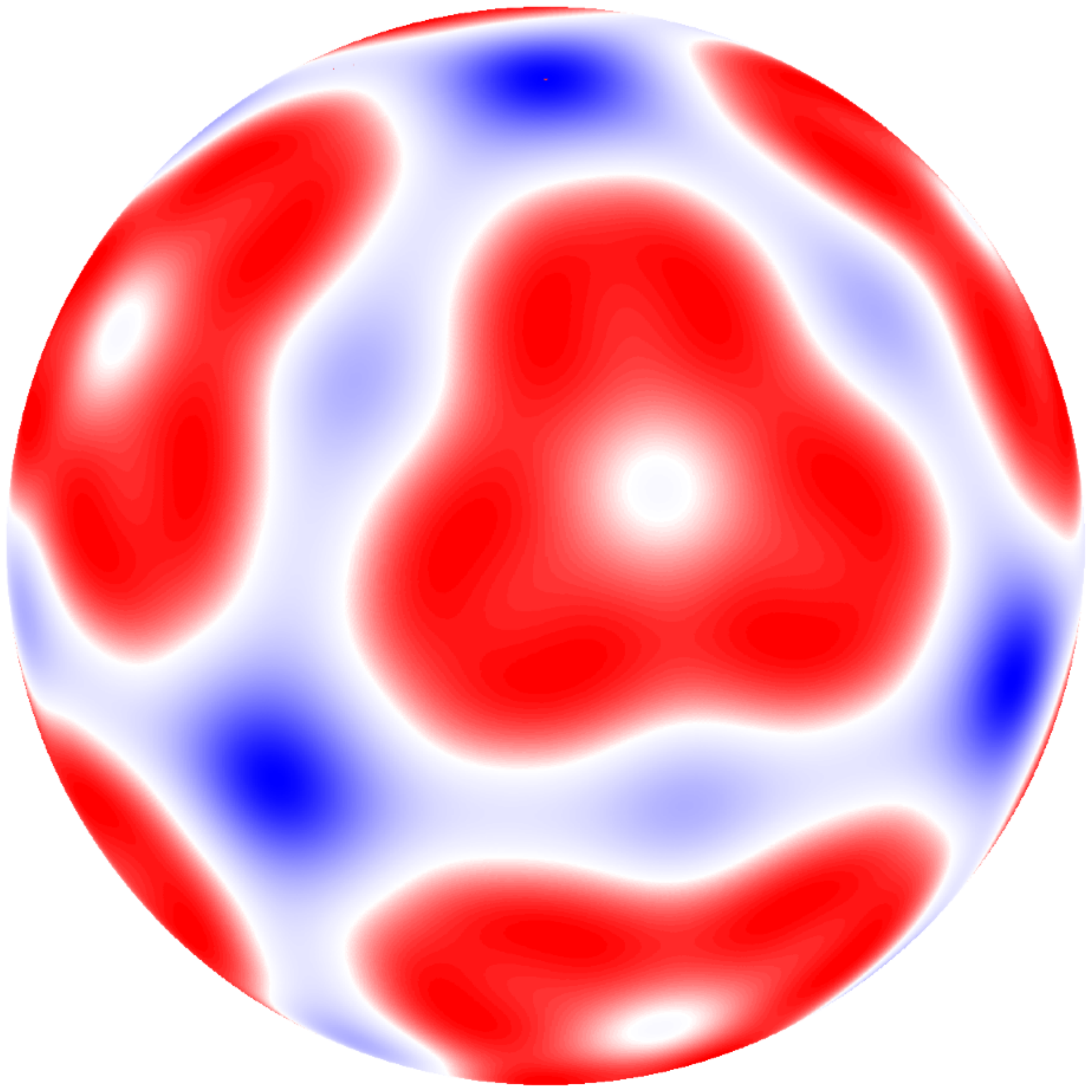
|v| = 0.40



$\max \bar{c}_0(\mathbf{v}) = 0.0171$

$\min \bar{c}_0(\mathbf{v}) = -0.0114$

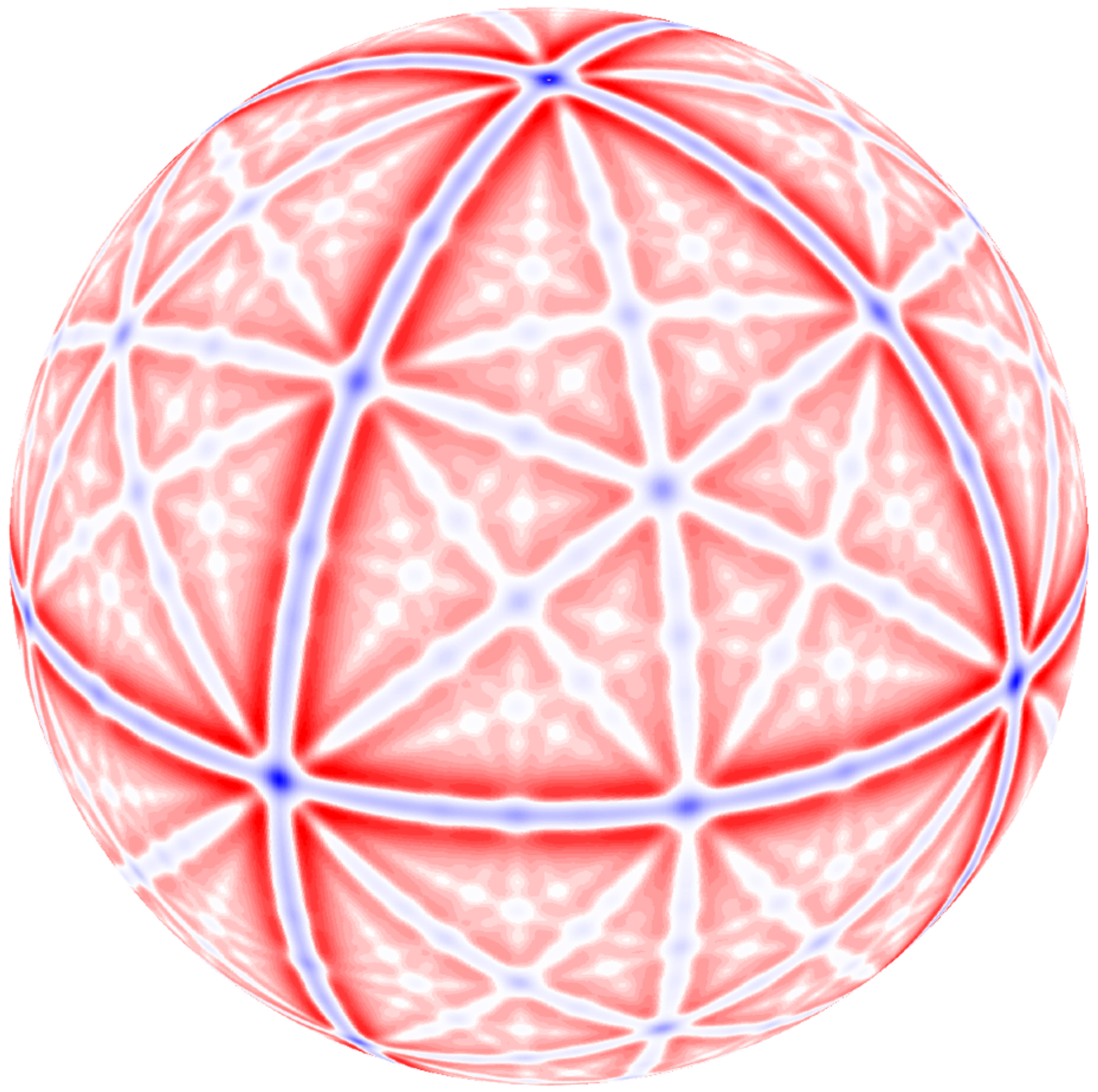
|v| = 0.95



$\max \bar{c}_0(\mathbf{v}) = 15.2832$

$\min \bar{c}_0(\mathbf{v}) = -2.8258$

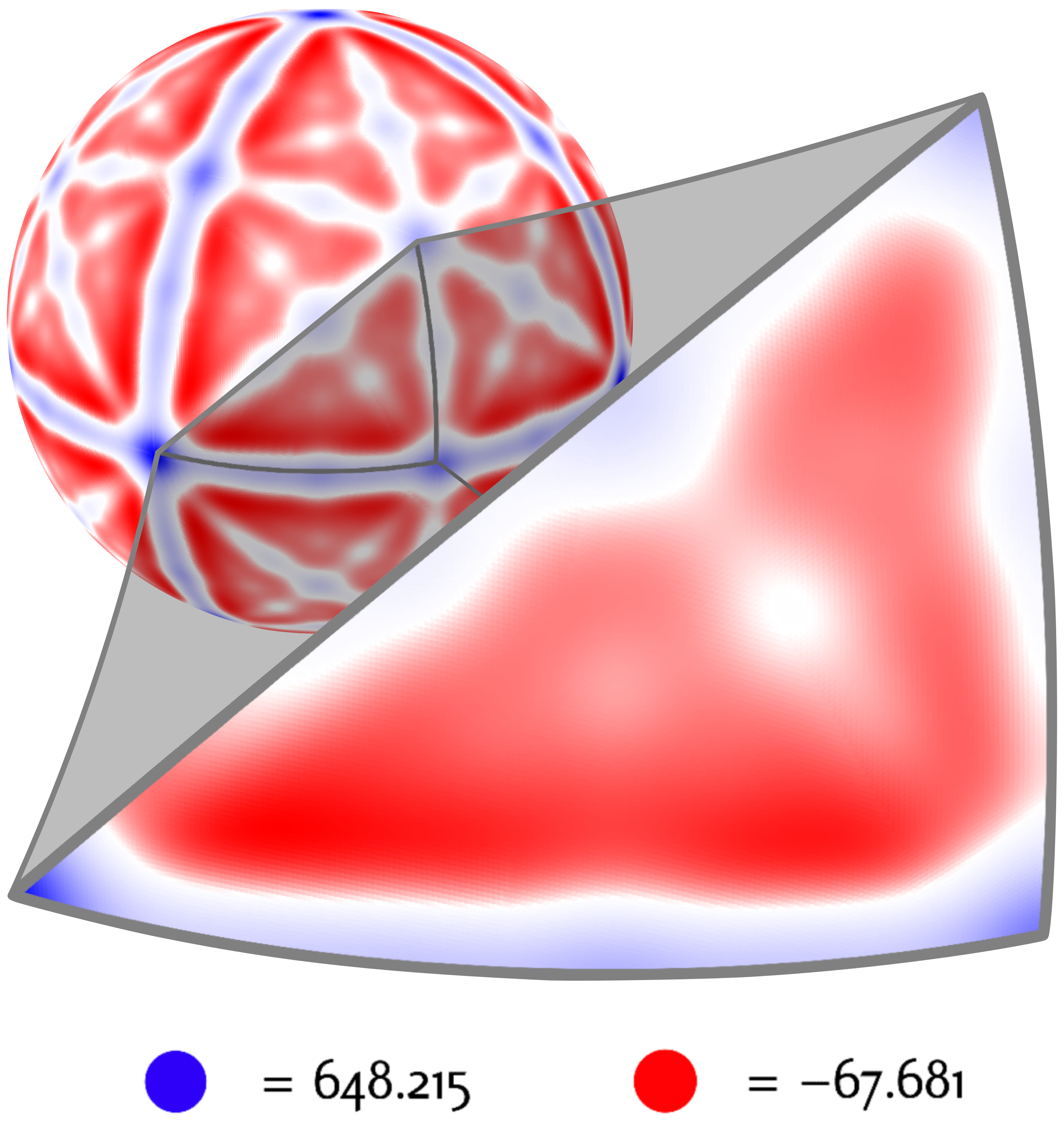
|v| = 0.999

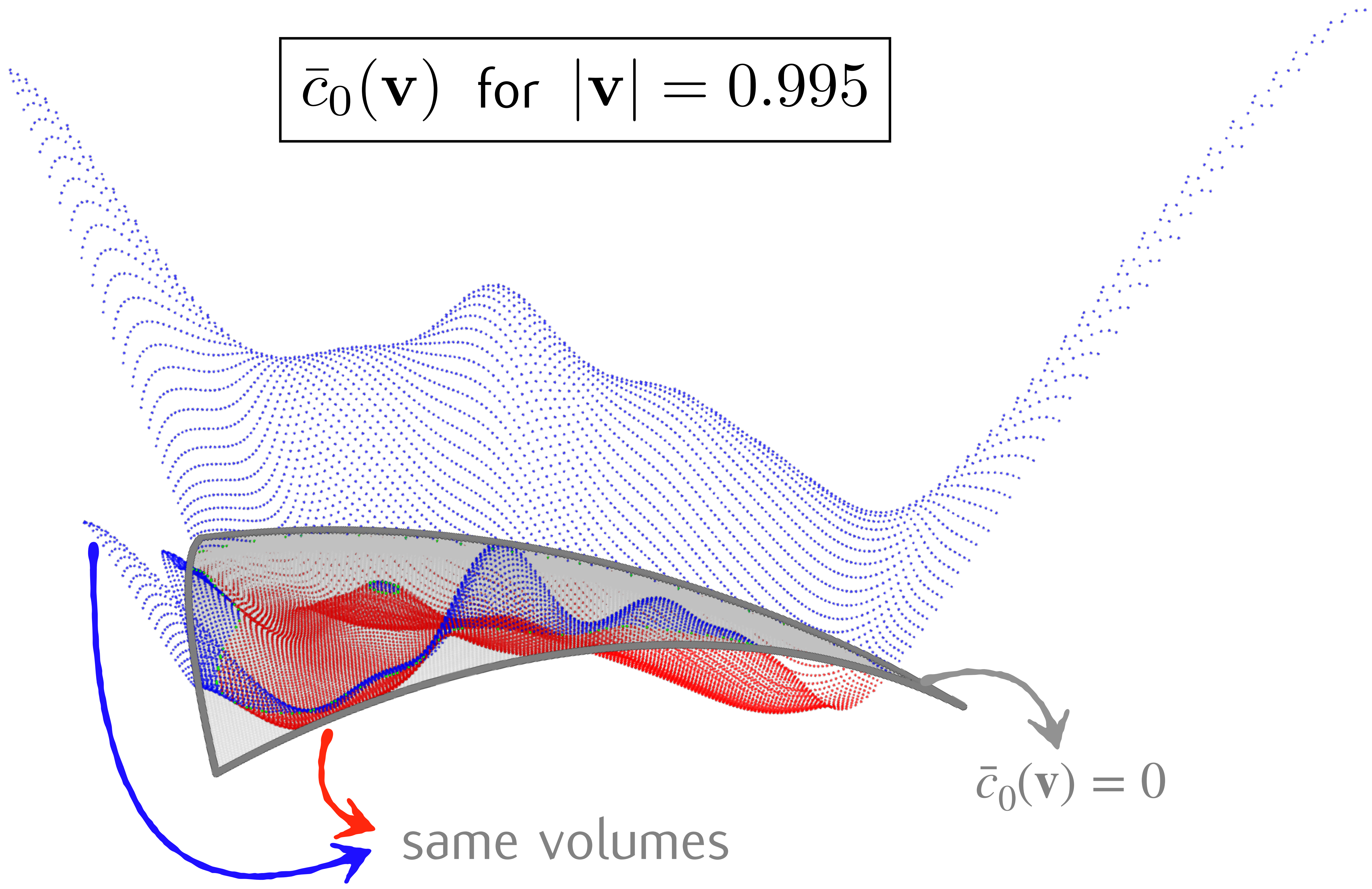


$\max \bar{c}_0(\mathbf{v}) = 9002.2317$

$\min \bar{c}_0(\mathbf{v}) = -807.4018$

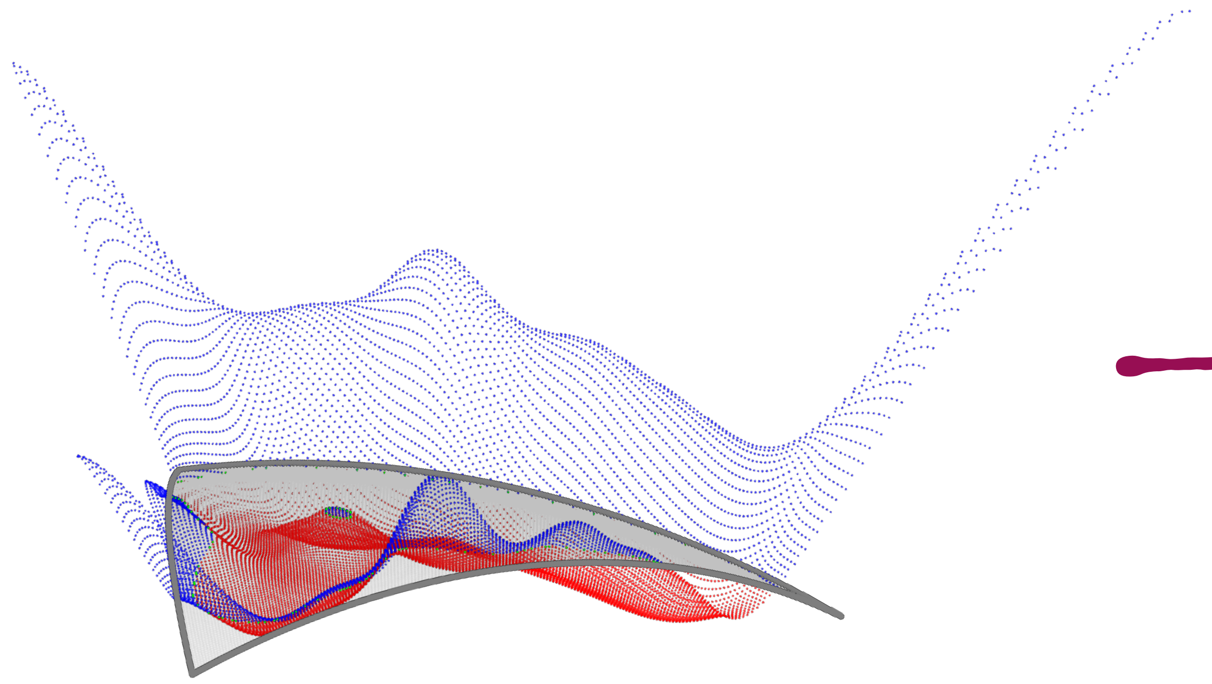
Velocity-dependent coefficients in QED_r



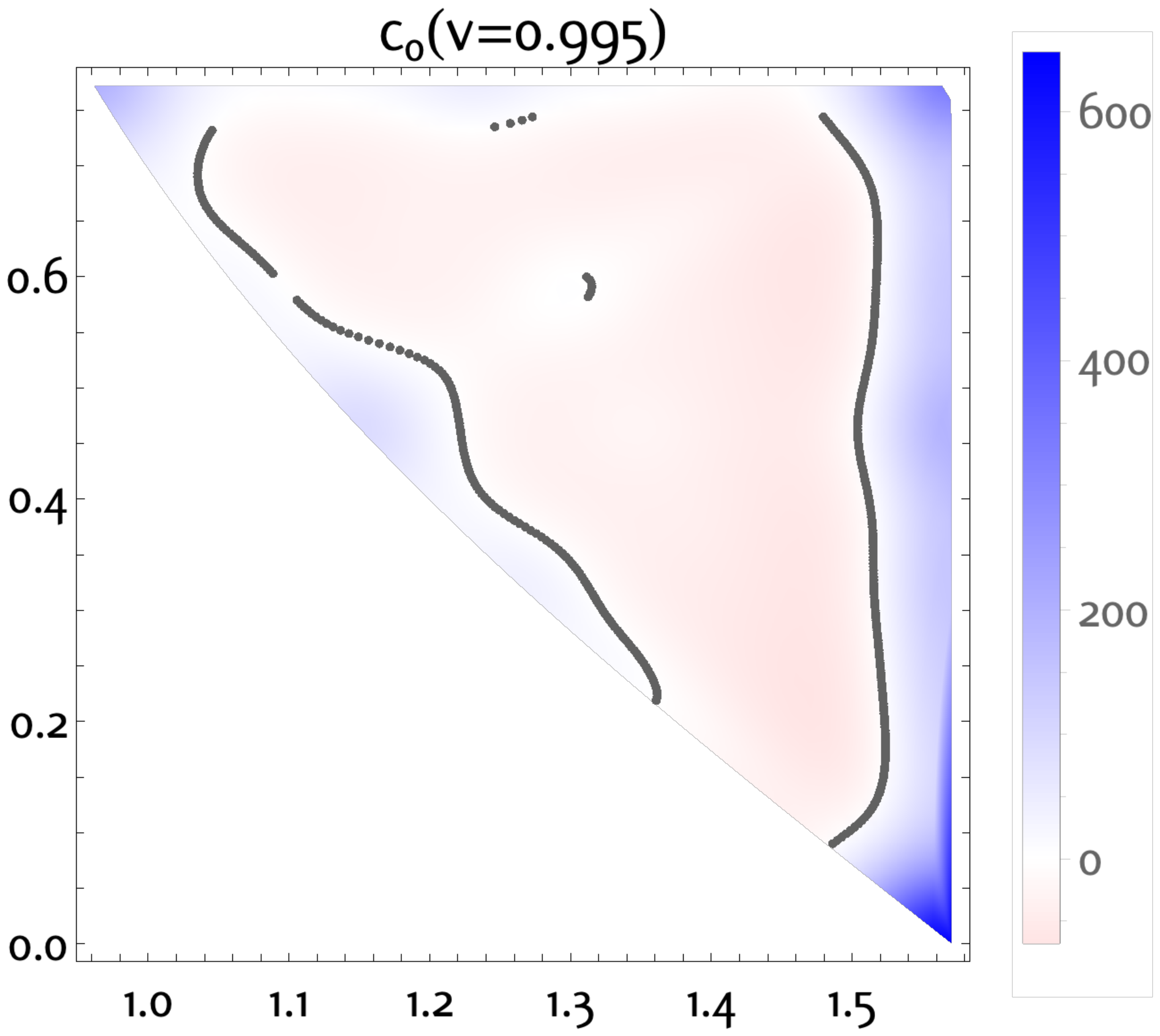


Velocity-dependent coefficients in QED$_r$

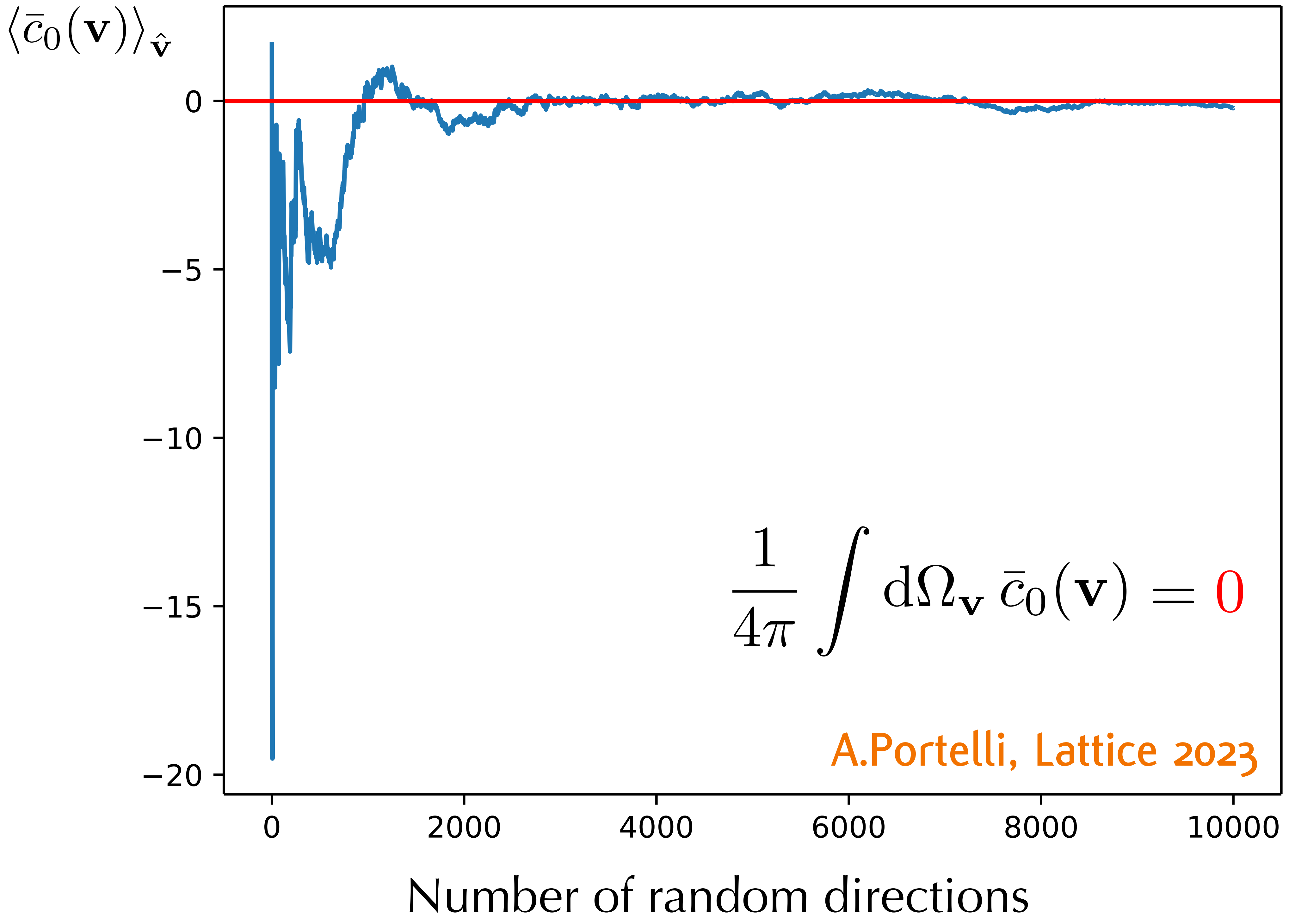
work in progress
in both directions



"magic angles"



Stochastic direction average



Take home messages on finite-volume effects?

- **Finite-volume expansions** studied for **masses** and **leptonic decays**
- **Unknown structure-dependent** contributions start at $O(1/L^3)$
- QED_r **regularisation** could help pushing unknown effects to $O(1/L^4)$?
- **Velocity-dependent** effects potentially problematic for heavy meson decays
- **Asymptotic series** need further study: up to what order subtracting FV effects is beneficial?

Infinite volume reconstruction

X.Feng & L.Jin, PRD **100** (2019)

QED$_\infty$

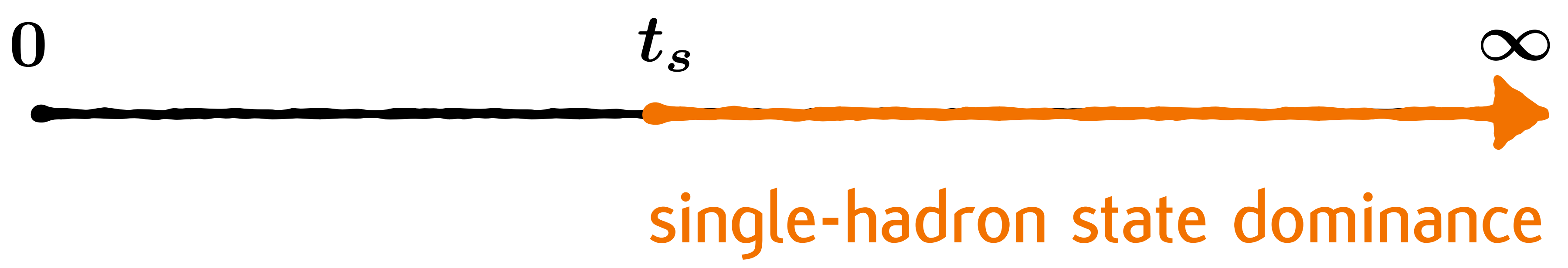
- An alternative approach is to compute radiative corrections as a **convolution** of **hadronic correlators** with **infinite-volume QED kernels**

$$\Delta\mathcal{O} = \int \mathrm{d}t \int \mathrm{d}^3\mathbf{x}\ \mathcal{H}(t,\mathbf{x})\, f_{\mathrm{QED}}(t,\mathbf{x}) = \Delta\mathcal{O}^{(s)} + \Delta\mathcal{O}^{(l)}$$

Separate correlator into **short** and **long** distance parts:

$$\Delta\mathcal{O}^{(s)} \approx \frac{1}{2}\int_{-t_s}^{t_s} \mathrm{d}t \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t,\mathbf{x}) f_{\mathrm{QED}}(t,\mathbf{x})$$

$$\Delta\mathcal{O}^{(l)} \approx \int_{L^3} \mathrm{d}^3\mathbf{x}\ \mathcal{H}^L(t_s,\mathbf{x})\, \mathcal{F}_{\mathrm{QED}}(t_s,\mathbf{x})$$

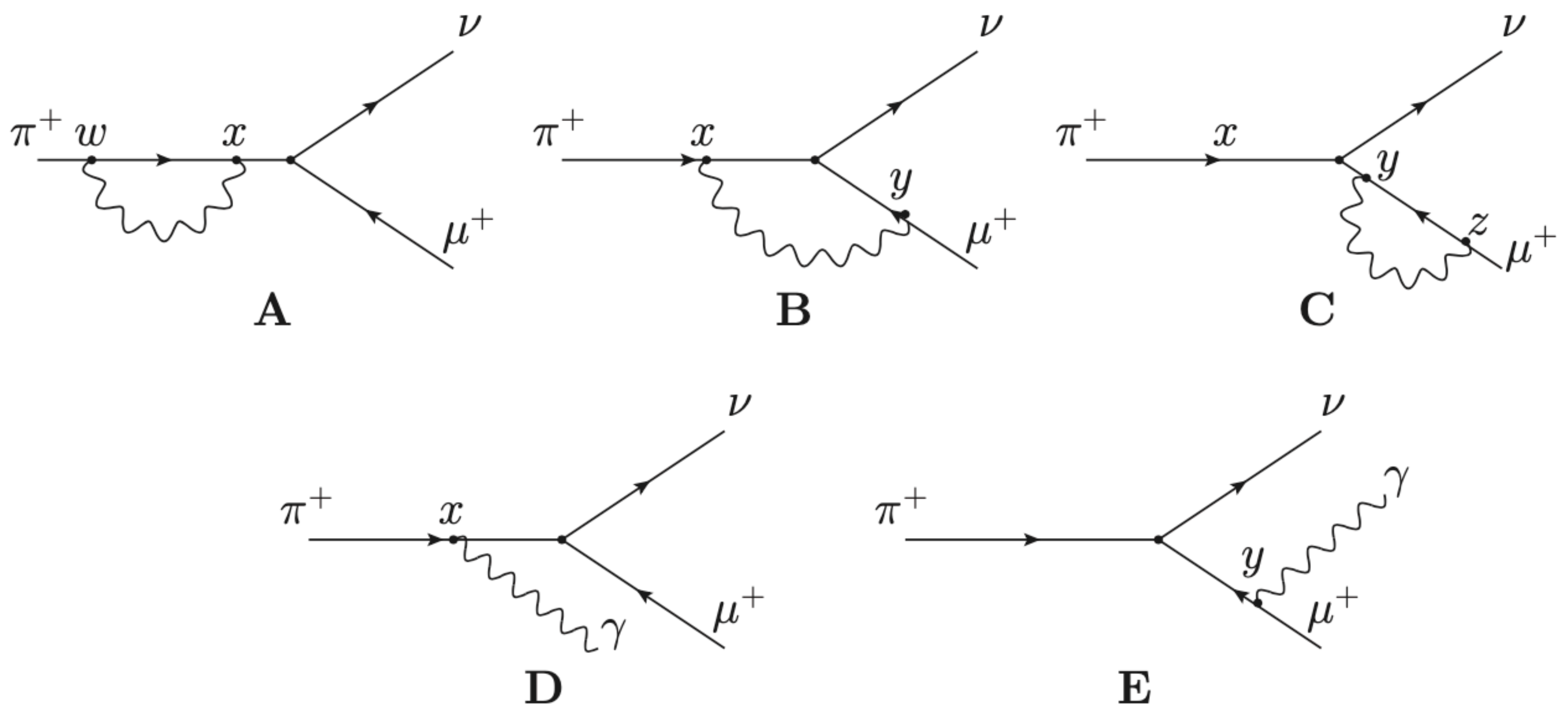


Exponentially suppressed (a) finite-volume effects (b) contributions of states with higher energy

Infinite volume reconstruction

QED$_\infty$

N.Christ et al., [2304.08026]



- Diagram A:

$$H^{(2)}_{\mu,\rho,\sigma}(t_1, t_2, \vec{x}) = \int d^3\vec{w}\, \langle 0|T\{J^W_\mu(0) J^{EM}_\rho(t_1, \vec{w}+\vec{x}) J^{EM}_\sigma(t_2, \vec{w})\}|\pi(\vec{0})\rangle$$

- Diagram B and D:

$$H^{(1)}_{\mu,\rho}(x) = H^{(1)}_{\mu,\rho}(x_t, \vec{x}) = \langle 0|T\{J^W_\mu(0) J^{EM}_\rho(x)\}|\pi(\vec{0})\rangle$$

- Diagram C and E ($f_\pi \approx 130$ MeV):

$$H^{(0)}_\mu = H^{(0)}_t \delta_{\mu,t} = \langle 0|J^W_\mu(0)|\pi(\vec{0})\rangle = -i m_\pi f_\pi \delta_{\mu,t}$$

from Luchang Jin's talk @ Edinburgh May 30, 2023

Strategy proposed for **leptonic decay rates**:

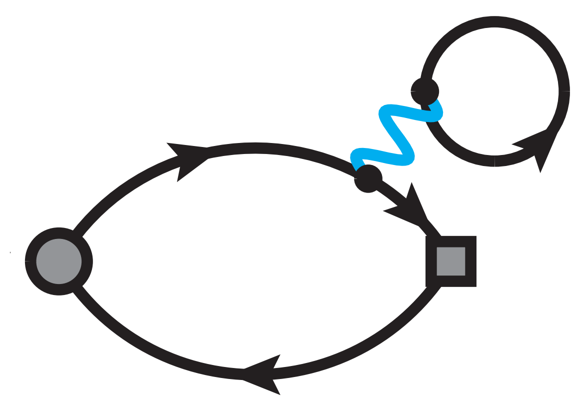
- Logarithmic IR divergences appear
- **but** they cancel analytically between diagrams
- Numerical calculation still **ongoing**...

... systematics under control?

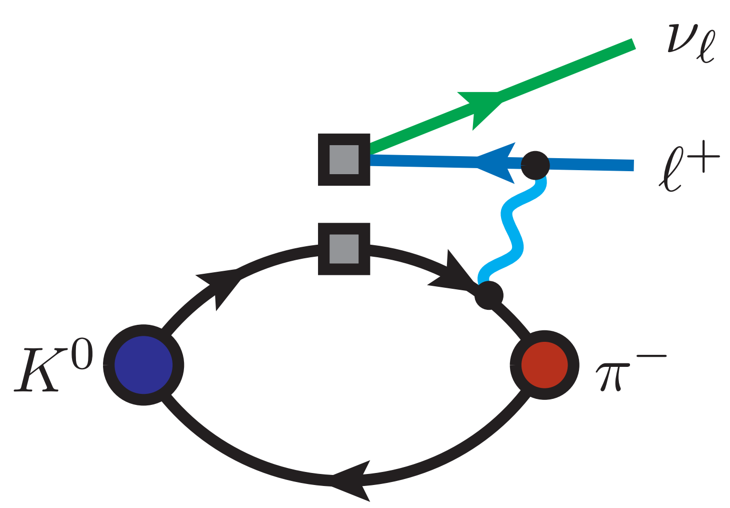
Where do we stand

- Current **tensions** in CKM unitarity require a combined effort of **theory** and **experiments**
- **Two lattice calculations** of IB and QED corrections to **light-meson leptonic decay rates**
- **Finite volume QED effects** have to be carefully investigated

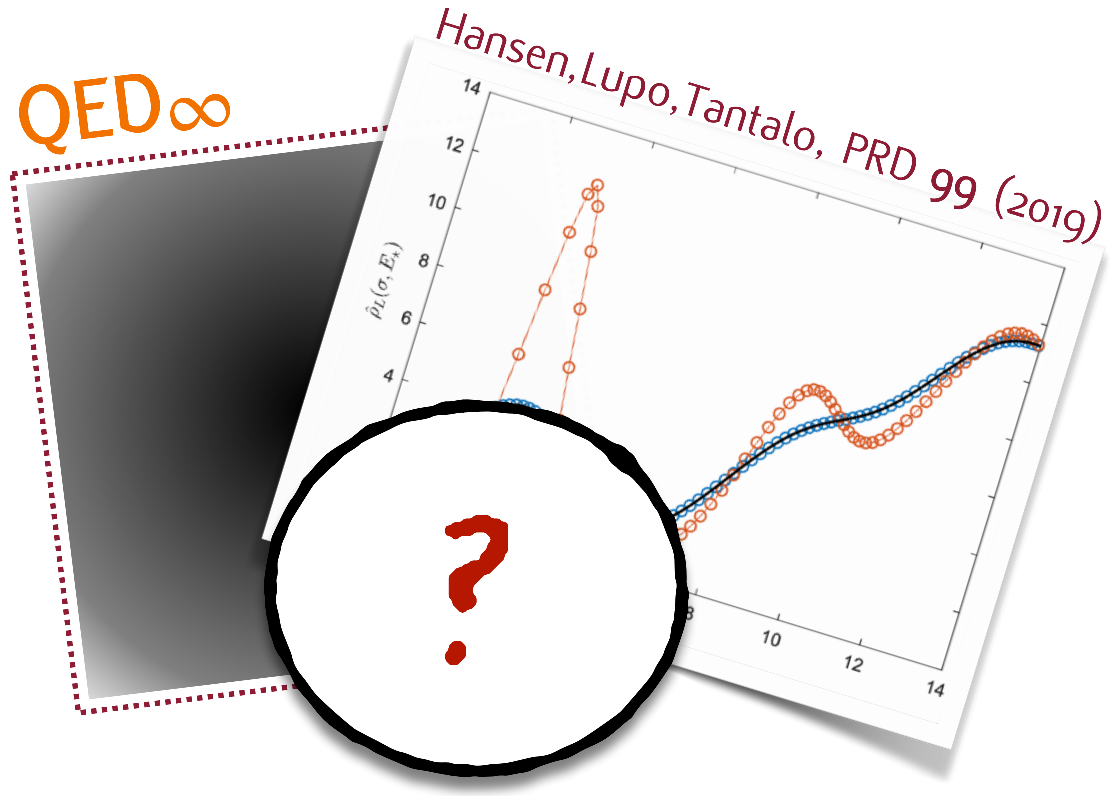
… and where to go?



move to unquenched calculations



tackle different weak processes



develop and apply new techniques

Thank you

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