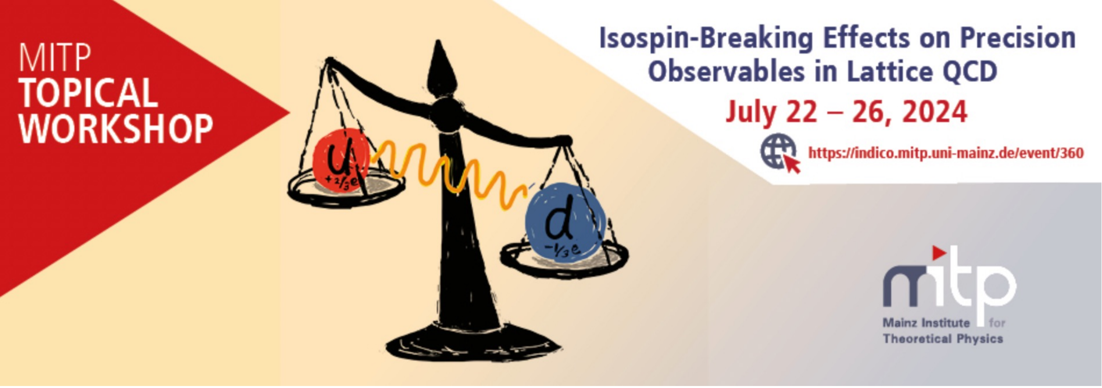


E&M corrections in radiative decays with IVR method

Xu Feng (Peking U.)

2024.07.25

Test of CKM unitarity

➢ In SM, CKM matrix is unitary, describing the strength of flavor-changing weak interaction

Cabibbo Kobayashi Maskawa

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = \begin{bmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{bmatrix} \begin{bmatrix} d \\ s \\ b \end{bmatrix}$$

➢ Most stringent test of CKM unitarity is given by the first row condition

$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

- $|V_{ub}|=3.82(24)\times10^{-3}$, tiny contribution [PDG 2022]
- $|V_{ud}|=0.97373(31)$, most precise determination from superallowed nuclear beta decays

  (also from neutron & π beta decays, but uncertainties are 3 and 10 times larger)
- $|V_{us}|$, most precise determination from kaon decays (K_{l3} + $K_{\mu2}/\pi_{\mu2}$) ➡ requires LQCD inputs

  (also from hyperon & tau decays, errors are about 3 and 2 times)

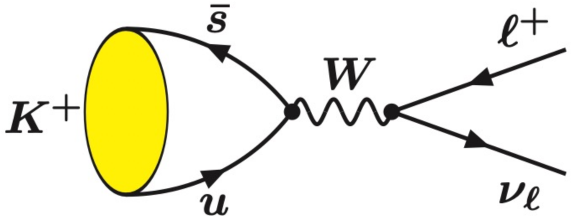
$|V_{us}|$

K/π systems provide idea laboratory for lattice QCD Study

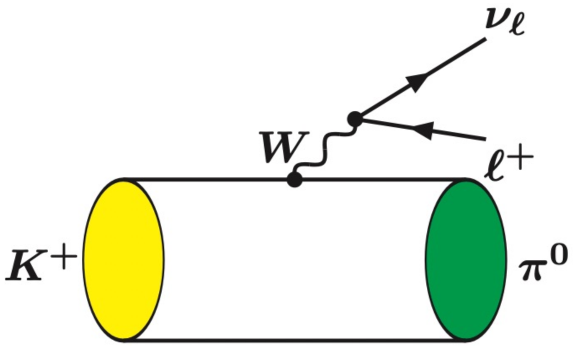
➢ Lattice QCD is powerful to study Kaon/pion decays

- Nearly no signal/noise problem
- Quark field contractions easily performed
- Simple final states: purely leptonic, 1 π , 2 π (K→ππ already very challenging!)
- Small recoil for hadronic particle in the final state
- Long-distance processes: much less low-lying intermediate states

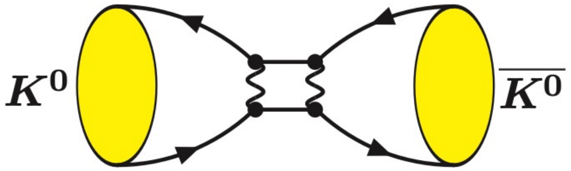
➢ Provide the hadronic matrix elements for precision SM tests



$K_{\ell 2} \rightarrow f_K$



$K_{\ell 3} \rightarrow f_+(0)$



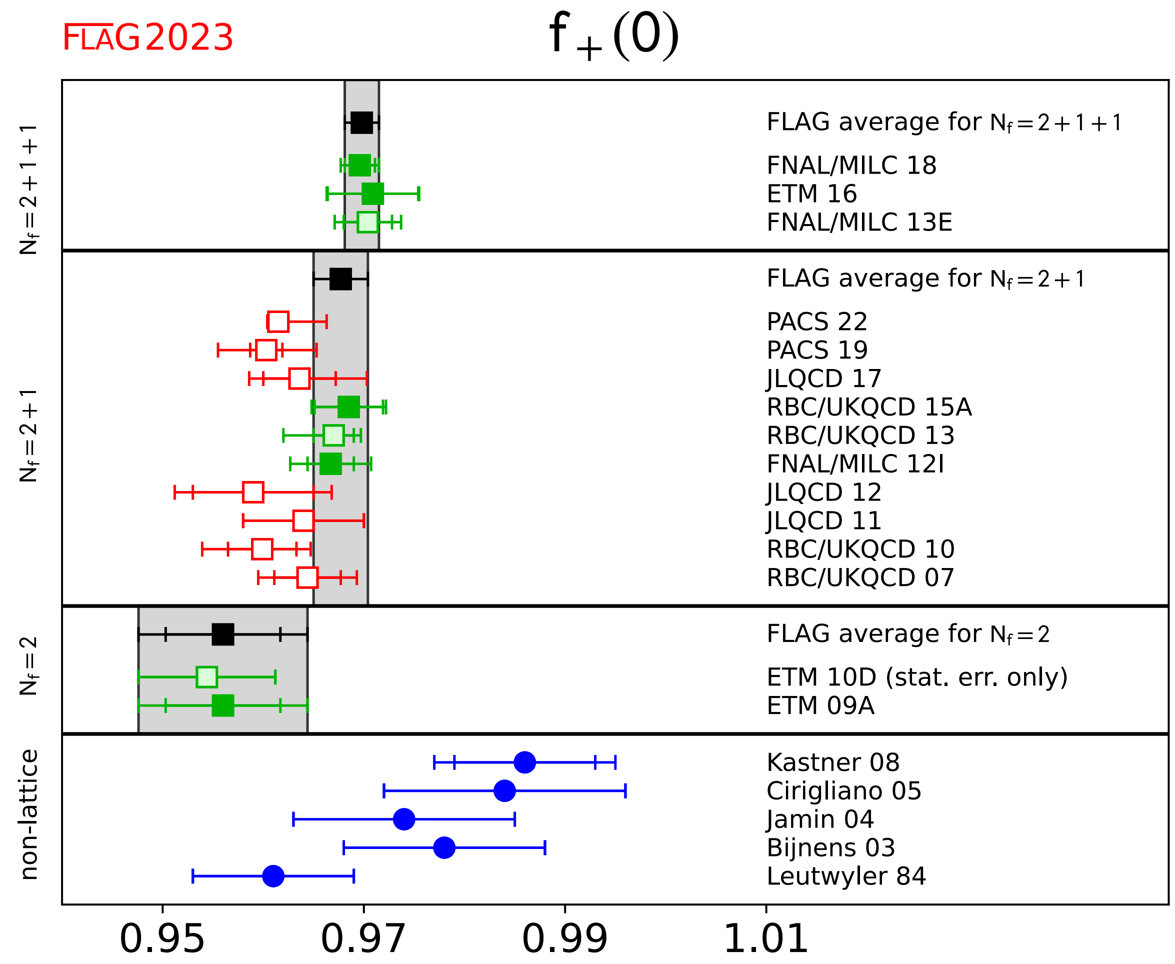
K^0-$\overline{K^0}$ mixing $\rightarrow B_K$

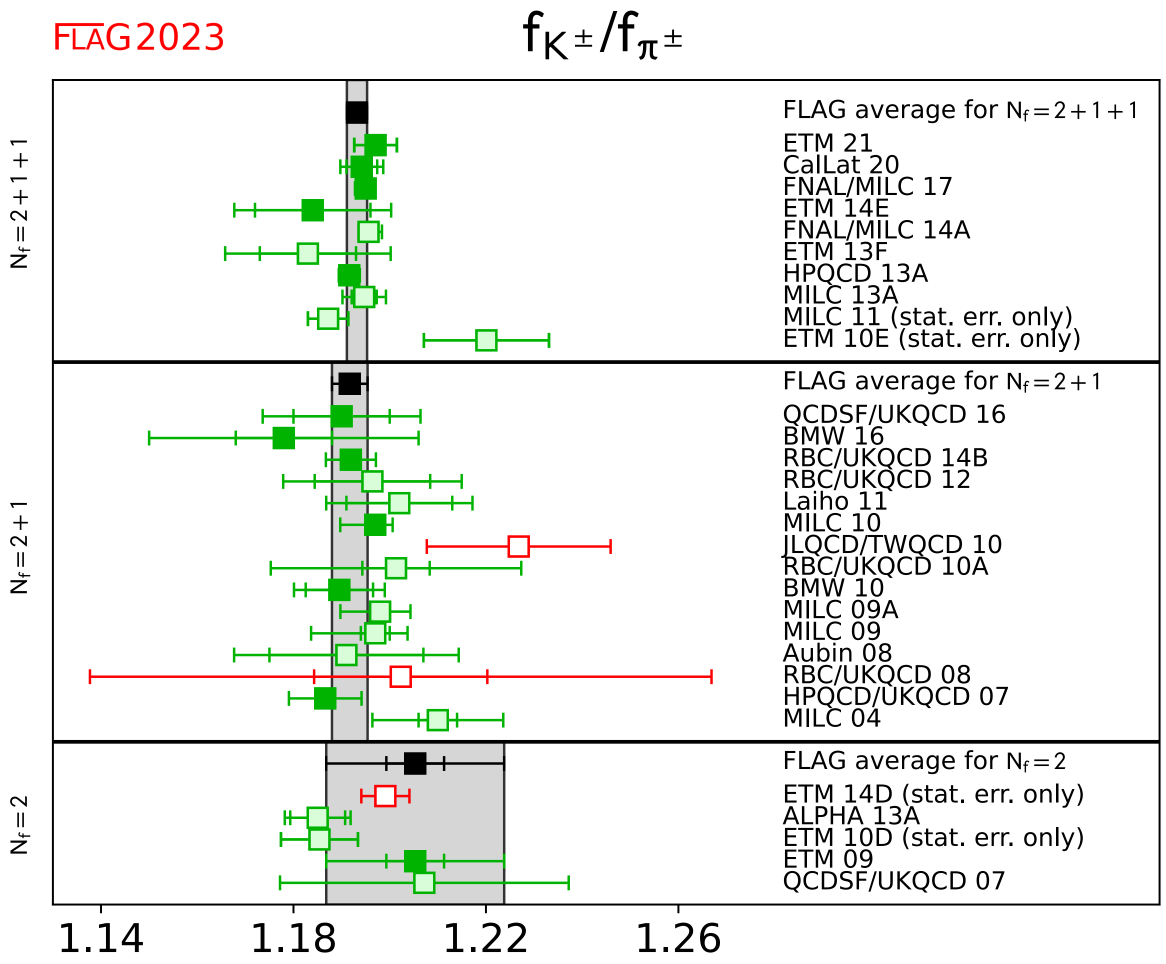
Leptonic and semileptonic decays

➤ Flavor Lattice Averaging Group (FLAG) average, updated on 2023

$$f_+^{K\pi}(0) = 0.9698(17) \quad \Rightarrow \quad 0.18\% \text{ error}$$

$$f_{K^\pm}/f_{\pi^\pm} = 1.1934(19) \quad \Rightarrow \quad 0.16\% \text{ error}$$



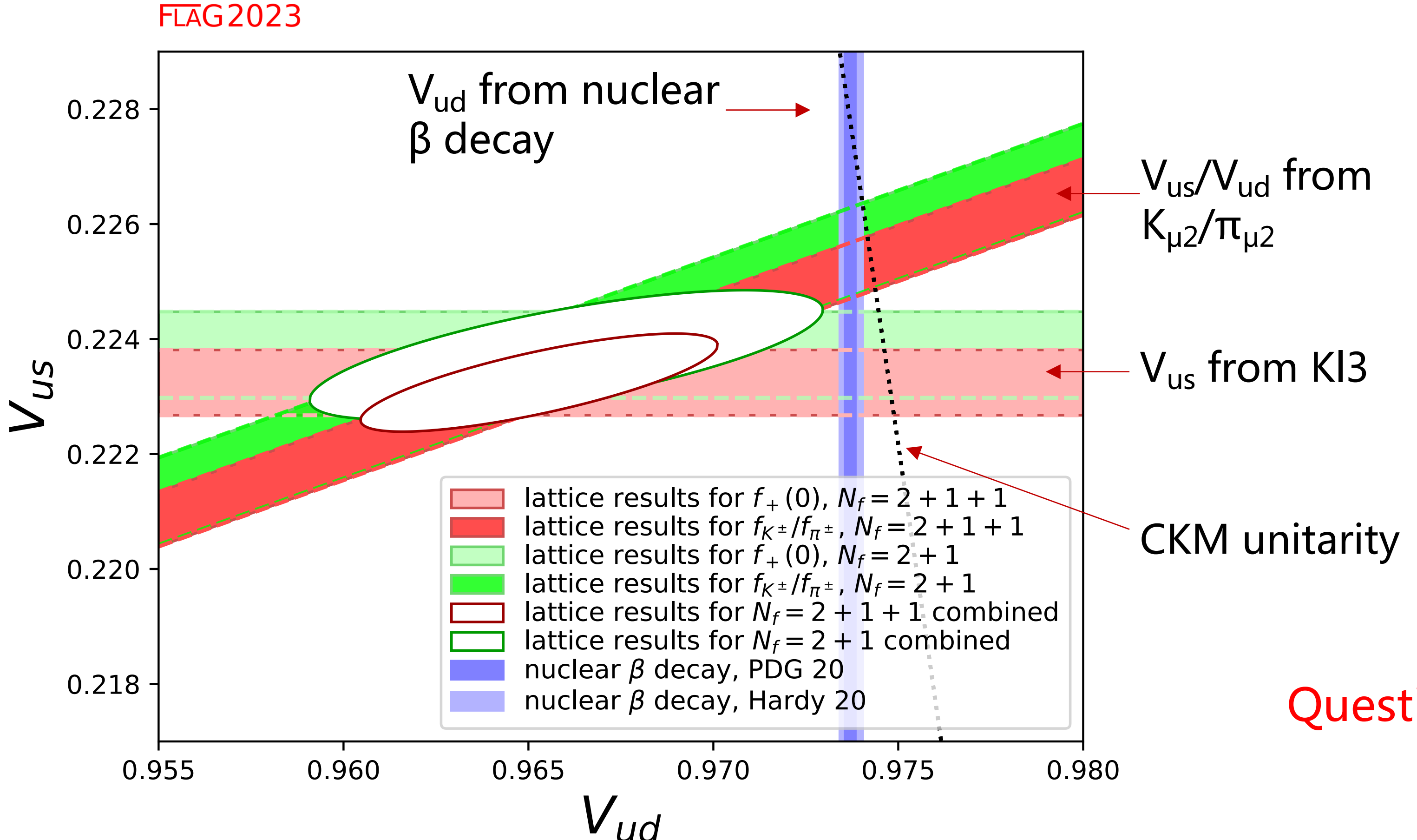


Extraction of V_{ud} and V_{us}

➢ Experimental information from kaon decays [arXiv:1411.5252, 1509.02220]

$$K_{\ell 3} \Rightarrow |V_{us}| f_+(0) = 0.2165(4) \quad \Rightarrow \quad |V_{us}| = 0.2232(6)$$

$$K_{\mu 2}/\pi_{\mu 2} \Rightarrow \left|\frac{V_{us}}{V_{ud}}\right| \frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.2760(4) \quad \Rightarrow \quad \left|\frac{V_{us}}{V_{ud}}\right| = 0.2313(5)$$



- Use $|V_{us}|$ from K_{l3} + $|V_{us}/V_{ud}|$ from $K_{\mu2}/\pi_{\mu2}$ (more accurate results from N_f=2+1+1)

$$|V_u|^2 = 0.9816(64) \quad \Rightarrow \quad 2.9\ \sigma$$

- Use $|V_{us}|$ from K_{l3} + $|V_{ud}|$ from β decays

$$|V_u|^2 = 0.99800(65) \quad \Rightarrow \quad 3.1\ \sigma$$

- $|V_{us}/V_{ud}|$ from $K_{\mu2}/\pi_{\mu2}$ + $|V_{ud}|$ from β decay

$$|V_u|^2 = 0.99888(67) \quad \Rightarrow \quad 1.7\ \sigma$$

Question: Deviation due to $|V_{ud}|$ from β decays, $|V_{us}|$ from K_{l3} or new physics?

CKM matrix elements quoted by PDG 2022

- Use $|V_{us}/V_{ud}|$ from $K_{\mu2}/\pi_{\mu2}$ + $|V_{ud}|$ from β decay to determine $|V_{us}|$

$$|V_{us}| = 0.2255(8)\ (N_f = 2+1,\ K_{\mu2}\ \text{decays})$$
$$= 0.2252(5)\ (N_f = 2+1+1,\ K_{\mu2}\ \text{decays})$$

- Use $|V_{us}|$ from K_{l3}

$$|V_{us}| = 0.2236(4)_{\text{exp+RC}}(6)_{\text{lattice}}\ (N_f = 2+1,\ K_{\ell3}\ \text{decays})$$
$$= 0.2231(4)_{\text{exp+RC}}(4)_{\text{lattice}}\ (N_f = 2+1+1,\ K_{\ell3}\ \text{decays})$$

2.7 σ

- Average yields

$$|V_{us}| = 0.2244(5) \quad N_f = 2+1$$
$$|V_{us}| = 0.2243(4) \quad N_f = 2+1+1$$

- Enlarge the error by a scale factor of 2.7 and average N_f=2+1 and N_f=2+1+1 values

$$|V_{us}| = 0.2243(8) \quad \Rightarrow \quad |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9985(6)(4).$$

Conservative estimate of $|V_{us}|$ due to the deviation between K_{l3} and $K_{\mu2}$ ⇒ 2.1 σ deviation

Inclusion of IB effects becomes important

- Flavor Lattice Averaging Group (FLAG) average, updated on 2023

$$f_+^{K\pi}(0) = 0.9698(17) \quad \Rightarrow \quad 0.18\% \text{ error}$$

$$f_{K^\pm}/f_{\pi^\pm} = 1.1934(19) \quad \Rightarrow \quad 0.16\% \text{ error}$$

$$\Gamma_{K\ell 3} = \frac{G_F^2 M_K^5}{192\pi^3} S_{EW} (1 + \delta_K^\ell + \delta_{SU2}) C^2 |V_{us}|^2 f_+^2(0) I_K^\ell$$

$$\frac{\Gamma(K_{\mu 2})}{\Gamma(\pi_{\mu 2})} = \left| \frac{V_{us}}{V_{ud}} \frac{f_K^{(0)}}{f_\pi^{(0)}} \right|^2 \frac{M_\pi^3}{M_K^3} \left(\frac{M_K^2 - m_\mu^2}{M_\pi^2 - m_\mu^2} \right)^2 (1 + \delta R_{K\pi})$$

Long-distance IB effects, ChPT provides a useful tool

Frontier for lattice QCD – inclusion of E&M effects

➢ For $K_{\mu 2}/\pi_{\mu 2}$ decays

- 1st calculation by RM123-SOTON collaboration @$m_\pi \approx 220$ MeV

LQCD

$$\delta R_{K\pi} = -1.26(14)\%$$

[PRL 2018, PRD 2019]

vs

ChPT

$$\delta R_{K\pi} = -1.12(21)\%$$

[Cirigliano & Neufeld, PLB 2011]

- 2nd calculation @m_π=139 MeV, $m_\pi L$=3.863

$$\delta R_{K\pi} = -0.0086\,(3)_{\text{stat.}}(^{+11}_{-4})_{\text{fit}}(5)_{\text{disc.}}(5)_{\text{quench.}}(39)_{\text{vol.}}$$

[P. Boyle et. al., JHEP 02 (2023) 242]

indicating large finite-volume effects

- $O(1/L)$: universal and analytical known
- $O(1/L^2)$: structure dependent, found to be small
- $O(1/L^3)$: structure dependent, potentially large

➢ For K_{l3} decays

[P. Ma, XF, M. Gorchtein, L. Jin, C. Seng, PRD103 (2021) 114503]

- So far only a combined analysis with LQCD and ChPT

$|V_{ud}|$

Role played by V_{ud}

➢ Interesting to review the deviation from CKM unitarity changes within recent years

$$\Delta_{\mathrm{CKM}} = |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 = 0$$

➢ PDG 2019 → PDG 2020 → PDG 2022

	PDG 2019	PDG 2020	PDG 2022
$\lvert V_{ud}\rvert$	0.97420(21)	0.97370(14)	0.97373(31)
$\lvert V_{us}\rvert$	0.2243(5)	0.2245(8)	0.2243(8)
$\lvert V_{ub}\rvert$	0.00394(36)	0.00382(24)	0.00382(20)
Δ_{CKM}	−0.00061(47)	−0.00149(45)	−0.00152(70)

- 2020 update: 3.3 σ deviation from CKM unitarity due to the update of EWR corrections
- 2022 update: 2.1 σ deviation only

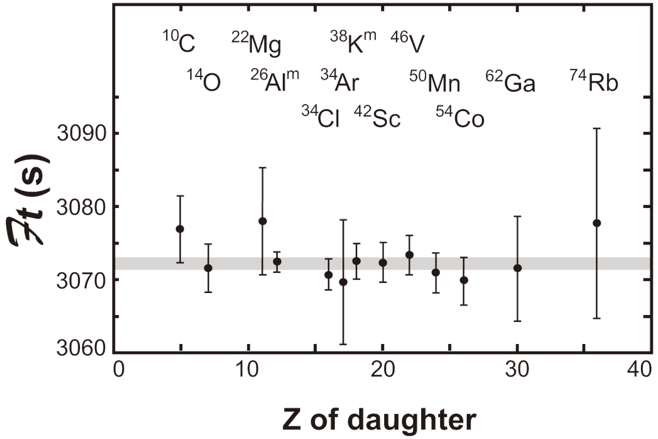
For V_{ud}, central value nearly unchanged, but uncertainty becomes twice larger

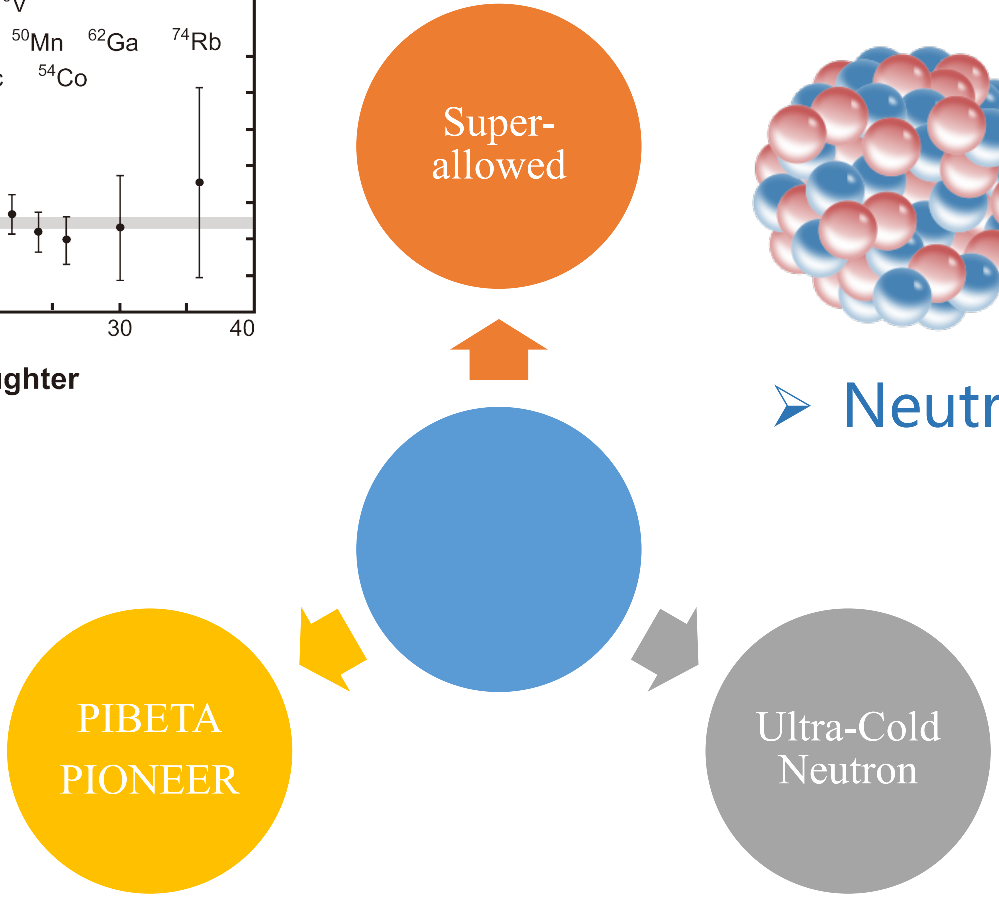
➡ A more conservative estimate of nuclear structure uncertainties

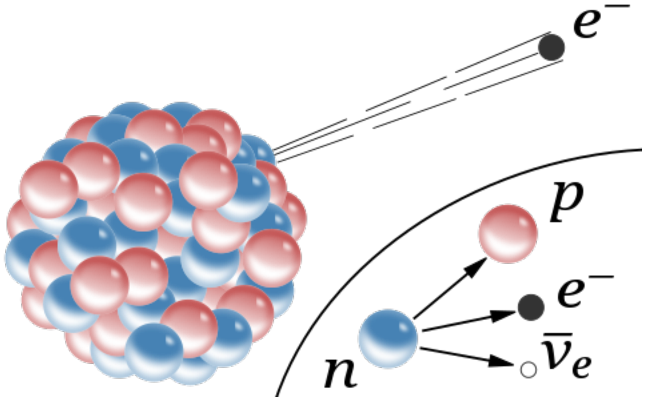
[M. Gorchtein, PRL123 (2019) 042503]

V_{ud} from different measurements

➢ Superallowed nuclear β decays

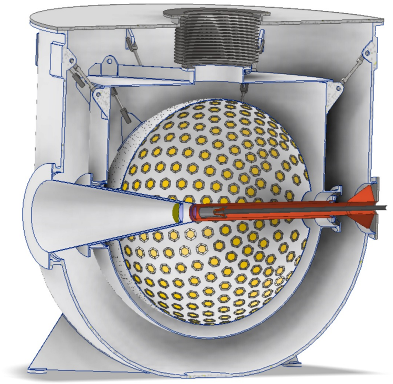






➢ Neutron β decays

➢ Pion β decays



Status for V_{ud}

- Superallowed β decays $|V_{ud}|=0.9737(3)$
  - $0^+ \to 0^+$ nuclear β decays : $J=0 \to 0$, transition must be V_0 or A_0; Parity unchanged, must be V_0
  - Vector current transition (Fermi transition) at leading order
  - Estimate of nuclear structure uncertainties is important

- Neutron β decays $|V_{ud}|=0.9737(9)$
  - Free from nuclear structure uncertainties
  - Nuclear-structure independent radiative correction (RC) is same as superallowed nuclear β decay

- Pion semileptonic β decays $|V_{ud}|=0.9739(29)$
  - More difficult to measure pion decays
  - Theoretically simpler, especially for lattice QCD

◆ Summary

- To extract V_{ud} from superallowed decay or neutron β decay

  ⇨ Need a well determined EW radiative corrections

Important uncertainty from γW box diagram

Superallowed nuclear β decays

$$|V_{ud}|^2 = 0.97154(22)(54)_{\mathrm{NS}}/(1+\Delta_{\mathrm{R}}^{\mathrm{V}})$$

Neutron β decays

$$|V_{ud}|^2 = \frac{5024.7\ \mathrm{s}}{\tau_n(1+3g_A^2)(1+\Delta_R^V)}$$

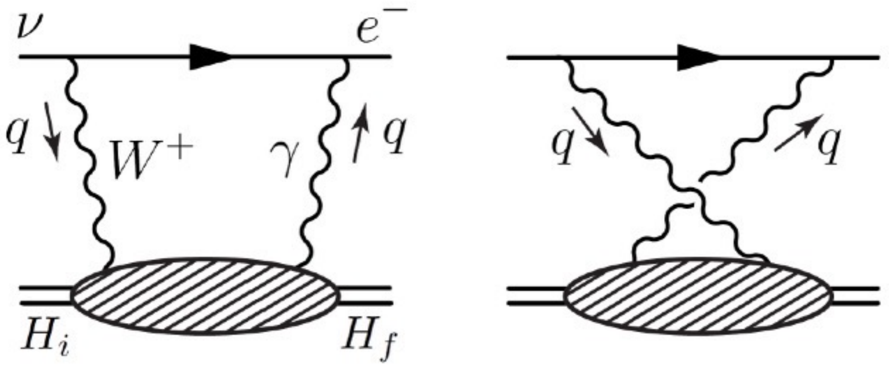
Universal electroweak radiative corrections (EWR)

Nuclear structure uncertainties

Axial vector current transition absorbed in g_A
Measured by experiment, different from lattice

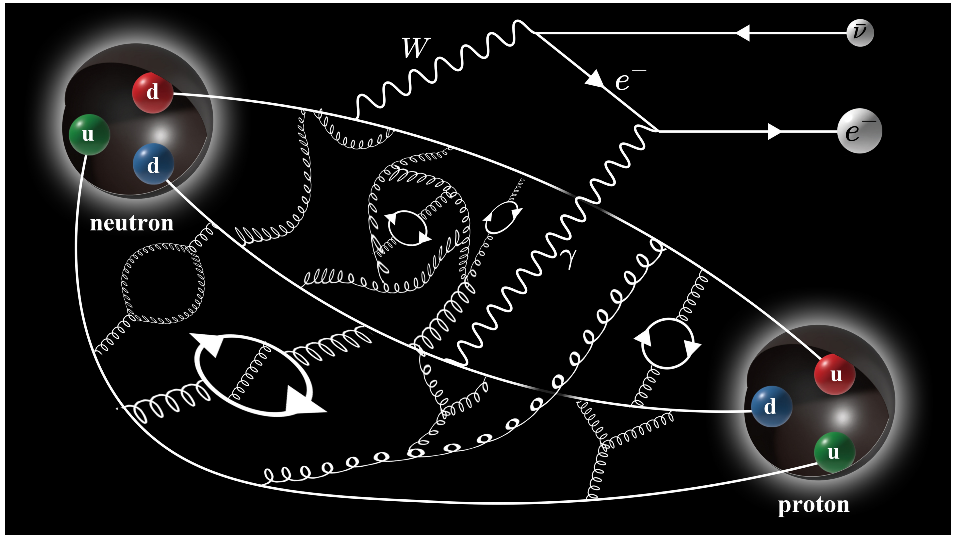
- Based current algebra, only axial γW box diagram is sensitive to hadronic scale

[A. Sirlin, Rev. Mod. Phys. 07 (1978) 573]



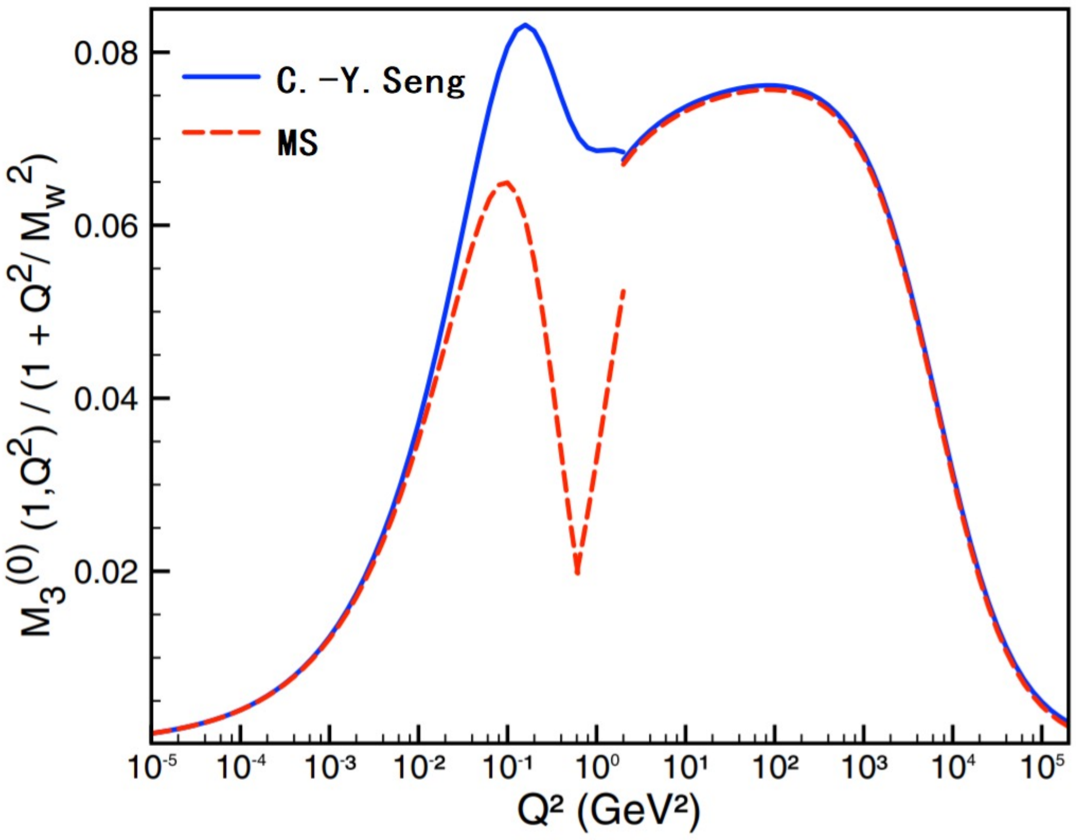
$$\square_{\gamma W}^{VA} = \frac{3\alpha_e}{2\pi}\int \frac{dQ^2}{Q^2}\frac{m_W^2}{m_W^2+Q^2}M_n(Q^2).$$

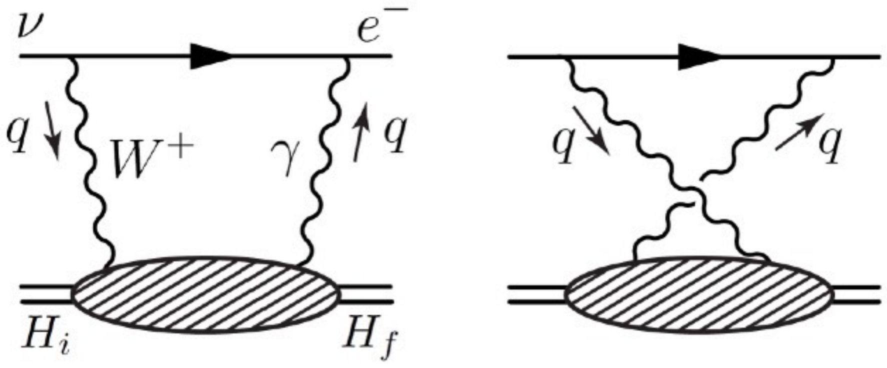
It dominates the uncertainties in EWR



Important uncertainty from γW box diagram

$$\Box^{VA}_{\gamma W} = \frac{3\alpha_e}{2\pi} \int \frac{dQ^2}{Q^2} \frac{m_W^2}{m_W^2 + Q^2} M_n(Q^2).$$





- PDG 2019 → PDG 2020

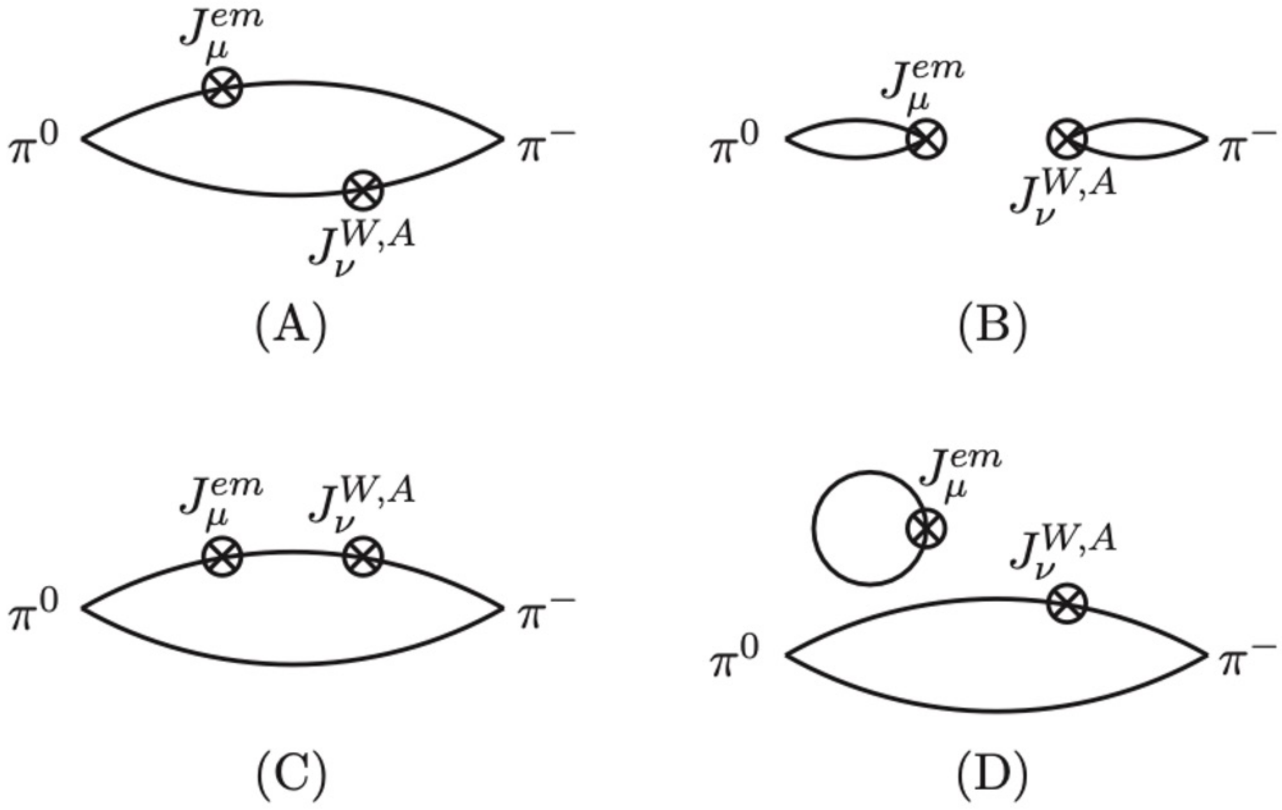
	PDG 2019	PDG 2020
$\lvert V_{ud}\rvert$	0.97420(21)	0.97370(14)
$\lvert V_{us}\rvert$	0.2243(5)	0.2245(8)
$\lvert V_{ub}\rvert$	0.00394(36)	0.00382(24)
Δ_{CKM}	−0.00061(47)	−0.00149(45)

It is responsible for the update of PDG and 3.3 σ deviation in CKM unitarity

[1] Marciano & Sirlin, PRL96, 032002 (2006)
[2] Seng et.al. PRL 121, 241804 (2018)

Quark contractions for the γW-box diagram

$$\mathcal{H}_{\mu\nu}^{VA}(x) = \langle \pi^0(p) | T\left[J_\mu^{em}(x) J_\nu^{W,A}(0)\right] | \pi^-(p) \rangle$$



① Coulomb gauge fixed wall source used for pion

② For type (A) & (B), double FFT to achieve spacetime translation average over $L^3 \times T$ measurements

③ Type (C) is most important contribution, with one current as source and the other as sink using 1024-2048 point-source prop per conf.

④ Type (D) vanishes in the flavor SU(3) limit

Five gauge ensembles at physical pion mass

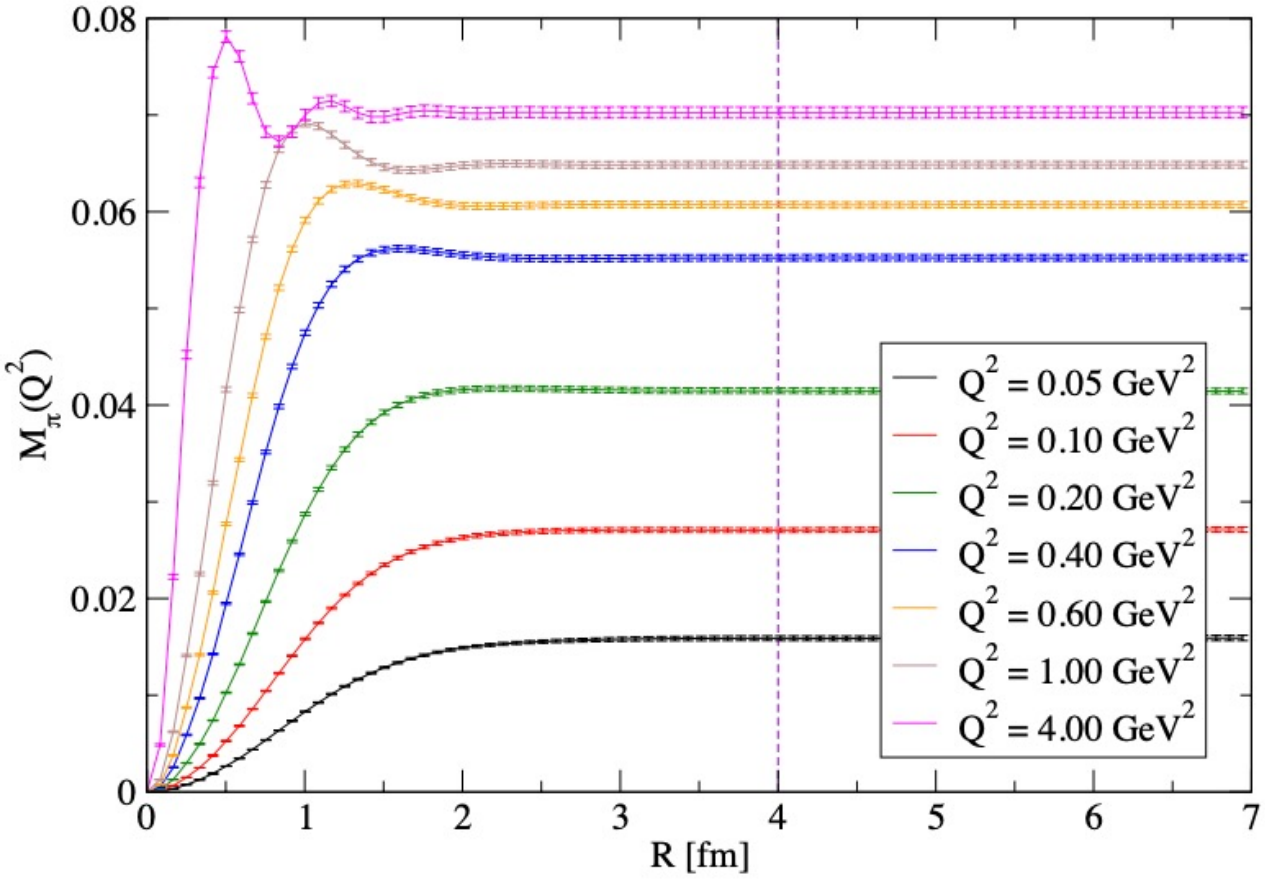
ensemble	M_π/MeV	$L^3 \times T$	a/fm	L·a/fm	N_{conf}	N_r	$N_{conf} \times N_r$
24D	141.2(4)	$24^3 \times 64$	0.1944	4.665	46	1024	47104
32D	141.4(3)	$32^3 \times 64$	0.1944	6.221	32	2048	65536
32D-fine	143.0(3)	$32^3 \times 64$	0.1432	4.582	71	1024	72704
48I	135.5(4)	$48^3 \times 96$	0.1140	5.474	28	1024	28672
64I	135.3(2)	$64^3 \times 128$	0.0836	5.353	62	1024	63488

- Gauge ensembles generated by RBC-UKQCD Collaborations using 2+1 flavor domain wall fermion
- 24D, 32D, 32D-fine use Iwasaki+DSDR action; while 48I, 64I use Iwasaki gauge action

Lattice results for the hadronic functions

Construct the Lorentz scalar function $M_\pi(Q^2)$ from $\mathcal{H}^{VA}_{\mu\nu}(x)$

$$M_\pi(Q^2) = -\frac{1}{6\sqrt{2}}\frac{\sqrt{Q^2}}{m_\pi}\int d^4x\,\omega(Q,x)\epsilon_{\mu\nu\alpha 0}x_\alpha\mathcal{H}^{VA}_{\mu\nu}(x)$$



At large spacetime separation, integral converges very quickly!

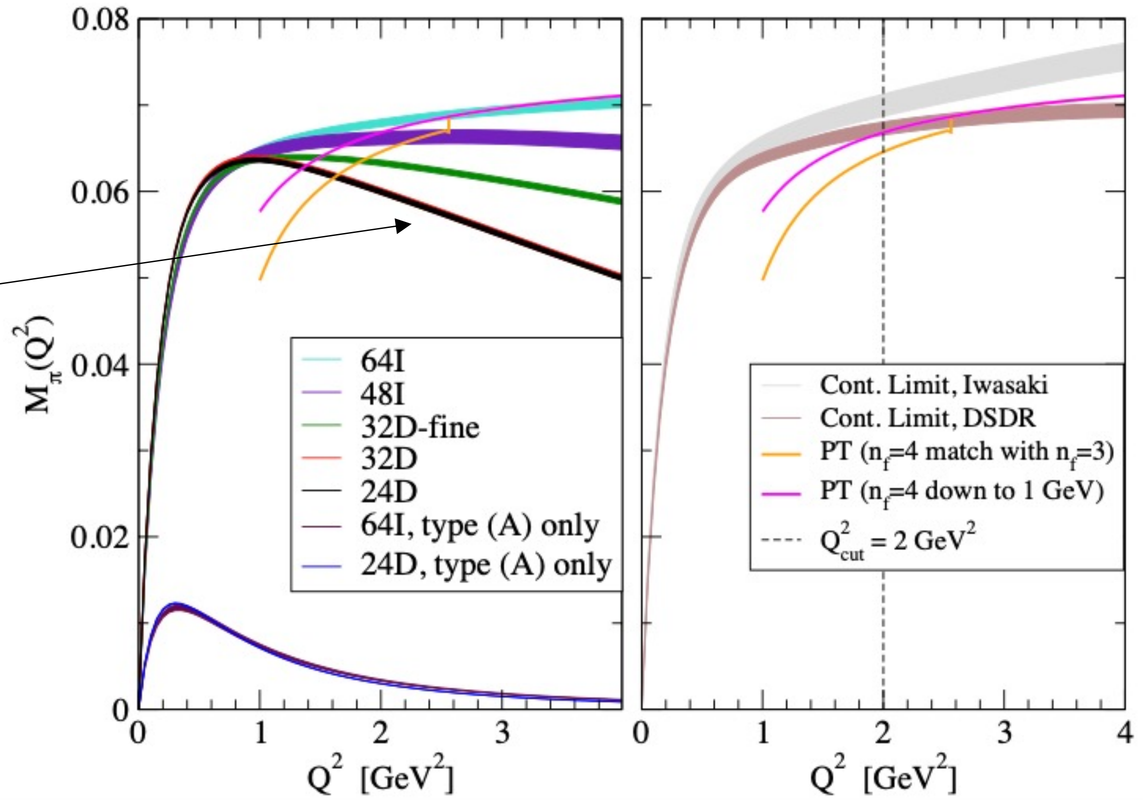
Combine lattice results with pQCD

Radiative correction requires the momentum integral from $0 < Q^2 < \infty$

$$\Box_{\gamma W}^{VA} = \frac{3\alpha_e}{2\pi} \int \frac{dQ^2}{Q^2} \frac{m_W^2}{m_W^2 + Q^2} M_\pi(Q^2)$$

- Lattice data used for low-Q^2 region
- OPE and perturbative Wilson coefficients used for high-Q^2 region

24D & 32D with diff. volume on top of each other



➢ OPE with Wilson coefficients at 4-loop accuracy

$$\frac{1}{2}\int d^4x e^{-iQx} T\left[J_\mu^{em}(x) J_\nu^{W,A}(0)\right]$$
$$= \frac{i}{2Q^2}\{C_a(Q^2)\delta_{\mu\nu}Q_\alpha - C_b(Q^2)\delta_{\mu\alpha}Q_\nu$$
$$-C_c(Q^2)\delta_{\nu\alpha}Q_\mu\} J_\alpha^{W,A}(0)$$
$$+\frac{1}{6Q^2}C_d(Q^2)\epsilon_{\mu\nu\alpha\beta}Q_\alpha J_\beta^{W,V}(0) + \cdots.$$

Only last term contributes to pion & superallowed β decays

Error analysis

Use the momentum scale $Q^2_{\rm cut}$ to separate the LD and SD contributions

$$\Box^{VA}_{\gamma W} = \begin{cases} 2.816(9)_{\rm stat}(24)_{\rm PT}(18)_{\rm a}(3)_{\rm FV} \times 10^{-3} & \text{using } Q^2_{\rm cut} = 1\ \text{GeV}^2 \\ 2.830(11)_{\rm stat}(9)_{\rm PT}(24)_{\rm a}(3)_{\rm FV} \times 10^{-3} & \text{using } Q^2_{\rm cut} = 2\ \text{GeV}^2 \\ 2.835(12)_{\rm stat}(5)_{\rm PT}(30)_{\rm a}(3)_{\rm FV} \times 10^{-3} & \text{using } Q^2_{\rm cut} = 3\ \text{GeV}^2 \end{cases}$$

- When $Q^2_{\rm cut}$ increase, the lattice artifacts become larger
- When $Q^2_{\rm cut}$ decrease, systematic effects in pQCD become larger
- For 1 GeV$^2 \le Q^2_{\rm cut} \le 3$ GeV^2, all results are consistent within uncertainties

➢ Independent calculation by Los Alamos group using Wilson-clover fermion

[J. Yoo, T. Bhattacharya, R. Gupta et.al. PRD 108 (2023) 034508]

$$\Box^{VA}_{\gamma W}|_\pi = 2.810(26) \times 10^{-3}$$

Pion semileptonic β decay

Decay width measured by PIBETA experiment

$$\Gamma_{\pi\ell 3} = \frac{G_F^2 |V_{ud}|^2 m_\pi^5 |f_+^\pi(0)|^2}{64\pi^3}(1+\delta) I_\pi$$

- ChPT [Cirigliano et.al. (2002), Czarnecki, Marciano, Sirlin (2019)]

$$\delta = 0.0334(10)_{\mathrm{LEC}}(3)_{\mathrm{HO}}$$

- Sirlin's parametrization [A. Sirlin, Rev. Mod. Phys. 07 (1978) 573]

$$\begin{aligned}\delta &= \frac{\alpha_e}{2\pi}\left[\bar{g} + 3\ln\frac{m_Z}{m_p} + \ln\frac{M_Z}{M_W} + \tilde{a}_g\right] + \delta_{\mathrm{HO}}^{\mathrm{QED}} + 2\Box_{\gamma W}^{VA} \\ &= 0.0332(1)_{\gamma W}(3)_{\mathrm{HO}}\end{aligned}$$

where $\frac{\alpha_e}{2\pi}\bar{g} = 1.051 \times 10^{-2}$, $\frac{\alpha_e}{2\pi}\tilde{a}_g = -9.6 \times 10^{-5}$, $\delta_{\mathrm{HO}}^{\mathrm{QED}} = 0.0010(3)$

- Hadronic uncertainty reduced by a factor of 10, which results in

$$|V_{ud}| = 0.9739(28)_{\mathrm{exp}}(5)_{\mathrm{th}} \quad \Rightarrow \quad |V_{ud}| = 0.9739(28)_{\mathrm{exp}}(1)_{\mathrm{th}}$$

Interplay between theory and experiment

- V_{ud} from π β decay

$$|V_{ud}| = 0.9740(28)_{\text{exp}}(1)_{\text{th}}$$

XF, M. Gorchtein, L. Jin, et.al.
PRL124 (2020) 19, 192002

- Main uncertainty arises from exp. measurements

which is normalized using the very precisely measured $BR(\pi^+ \to e^+\nu_e(\gamma)) = 1.2325(23) \times 10^{-4}$ [7], rather than the theoretical branching ratio of $1.2350(2) \times 10^{-4}$, which if used, would increase $|V_{ud}|$ to 0.9749(27). Theoretical uncertainties in pion beta decay are very small [21], leaving open more than an order of magnitude improvement of its experimental branching ratio before theory uncertainties become a problem. Although challenging, improved measurements of pion beta decay currently under discussion would allow this decay mode to compete with superallowed beta decays and future neutron decay efforts for the most precise direct $|V_{ud}|$ determination.

PDG 2022, reviewed by E. Blucher & W. J. Marciano

- Past Experiment - PIBETA

D. Pocanic et.al. PRL 93 (2004) 181803

  - Precision 0.6%

- New Experiment - PIONEER

M. Hoferichter, arXiv:2403.18889

Phase I : π leptonic decays

Phase II+III : π β decays

  - Ultimate precision 3×10^{-4} , 20 times better than PIBETA

Future exp. uncertainty comparable to theoretical one !

From π to K sector

- For π and neutron β decays， initial/final-state hadron has nearly the same mass

  ➡ only axial γW box diagram is sensitive to hadronic scale

- For K_{l3} decays, LQCD needs to calculate all the diagrams, not only just γW box diagram!

- Idea is to combine LQCD with ChPT [C. Seng, XF, M. Gorchtein, L. Jin, U.-G. Meißner, JHEP 10 (2020) 179]

  - Use ChPT to determine EWR correction

$$\delta_{\rm em}^{K^\pm} = 2e^2\left[-\frac{8}{3}X_1 - \frac{1}{2}\tilde{X}_6^{\rm phys}(M_\rho) - 2K_3^r(M_\rho) + K_4^r(M_\rho) + \frac{2}{3}K_5^r(M_\rho) + \frac{2}{3}K_6^r(M_\rho)\right]$$

$$\delta_{\rm em}^{K^0} = 2e^2\left[\frac{4}{3}X_1 - \frac{1}{2}\tilde{X}_6^{\rm phys}(M_\rho)\right] + \cdots$$

  ➡ still requires LECs X_1 and $\tilde{X}_6^{\rm phys}$

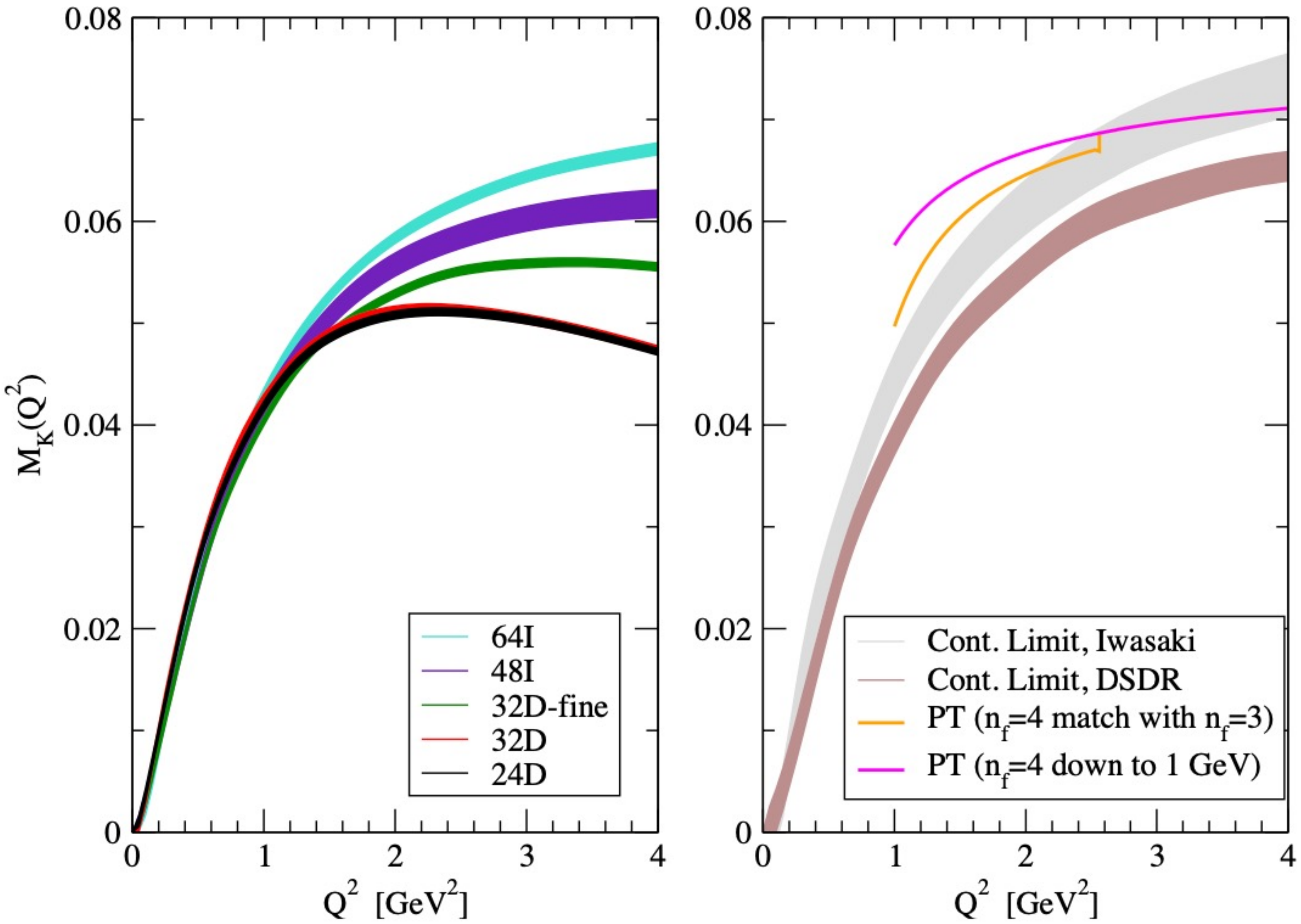
  - Use LQCD to calculate EWR at flavor SU(3) limit by decreasing m_s with $m_s=m_u=m_d$

  ➡ provide LECs, which are independent of quark masses

Axial γW-box diagram contribution to $K^0 \to \pi^+$ decays

$$\square_{\gamma W}^{VA}\Big|_H = \frac{3\alpha_e}{2\pi}\int \frac{dQ^2}{Q^2}\frac{m_W^2}{m_W^2+Q^2}M_H(Q^2)$$

Calculation is performed in the flavor $SU(3)$ limit with $m_K = m_\pi$



Lattice results

[P. Ma, XF, M. Gorchtein, L. Jin, C. Seng, PRD103 (2021) 114503]

- After combining the lattice data and PT results, we have

$$\Box_{\gamma W}^{VA}\big|_{K^0} = \begin{cases} 2.460(18)_{\mathrm{stat}}(42)_{\mathrm{PT}}(22)_a(1)_{\mathrm{FV}} \times 10^{-3} & Q^2_{\mathrm{cut}} = 1\ \mathrm{GeV}^2 \\ 2.443(20)_{\mathrm{stat}}(15)_{\mathrm{PT}}(36)_a(1)_{\mathrm{FV}} \times 10^{-3} & Q^2_{\mathrm{cut}} = 2\ \mathrm{GeV}^2 \\ 2.433(22)_{\mathrm{stat}}(7)_{\mathrm{PT}}(45)_a(1)_{\mathrm{FV}} \times 10^{-3} & Q^2_{\mathrm{cut}} = 3\ \mathrm{GeV}^2 \end{cases}$$

- The relation between box contribution and the LECs is given by

$$-\frac{8}{3}X_1 + \bar{X}_6^{\mathrm{phys}}(M_\rho) = -\frac{1}{2\pi\alpha}\left(\Box_{\gamma W}^{VA}\big|_{K^0} - \frac{\alpha}{8\pi}\ln\frac{M_W^2}{M_\rho^2}\right) + \frac{1}{8\pi^2}\left(\frac{5}{4} - \tilde{a}_g\right)$$

- This results in

$$-\frac{8}{3}X_1 + \tilde{X}_6^{\mathrm{phys}} = 0.0197(10)$$

- ChPT quoted the minimal resonance model as input

$$X_1 = -3.7(3.7)\times 10^{-3} \quad \text{and} \quad \tilde{X}_6^{\mathrm{phys}} = 10.4(10.4)\times 10^{-3}$$

$$-\frac{8}{3}X_1 + \tilde{X}_6^{\mathrm{phys}} = 0.0203(143)$$

Consistent between lattice and ChPT, but error from lattice is much smaller

Determination of LECs

- Combine the $SU(3)$ K^0 decay

$$-\frac{8}{3}X_1 + \tilde{X}_6^{\rm phys} = 0.0197(10) \quad \text{for } K^0 \to \pi^+$$

with semileptonic pion decay

$$\frac{4}{3}X_1 + \tilde{X}_6^{\rm phys} = 0.0110(6) \quad \text{for } \pi^- \to \pi^0$$

- We have

$$X_1 = -2.2(4) \times 10^{-3}, \quad \tilde{X}_6^{\rm phys} = 13.9(7) \times 10^{-3}$$

- This is comparable with the minimal resonance model

$10^3\,X_1$	$10^3\,X_2^r$	$10^3\,X_3^r$	$10^3\,\tilde{X}_6^{eff}$	$10^3(X_6^{eff})_{\alpha_s}$	$10^3\,X_6^{eff}$
-3.7	3.6	5.00	10.4	3.0	-231.5

Axial γW-box diagram contribution to $K^0 \to \pi^+$ decays

- Use lattice input to update the EWR correction

$$\delta^{e}_{K^0} = 0.99(19)_{e^2p^4}(11)_{\mathrm{LEC}} \quad \to \quad 1.00(19)$$

$$\delta^{\mu}_{K^0} = 1.40(19)_{e^2p^4}(11)_{\mathrm{LEC}} \quad \to \quad 1.41(19)$$

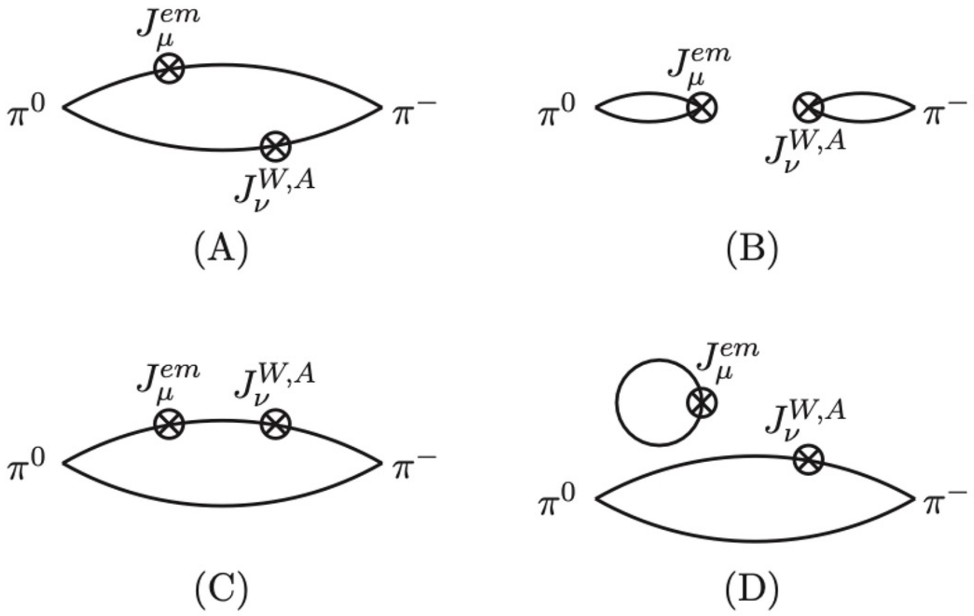
$$\delta^{e}_{K^\pm} = 0.10(19)_{e^2p^4}(16)_{\mathrm{LEC}} \quad \to \quad -0.01(19)$$

$$\delta^{\mu}_{K^\pm} = 0.02(19)_{e^2p^4}(16)_{\mathrm{LEC}} \quad \to \quad -0.09(19)$$

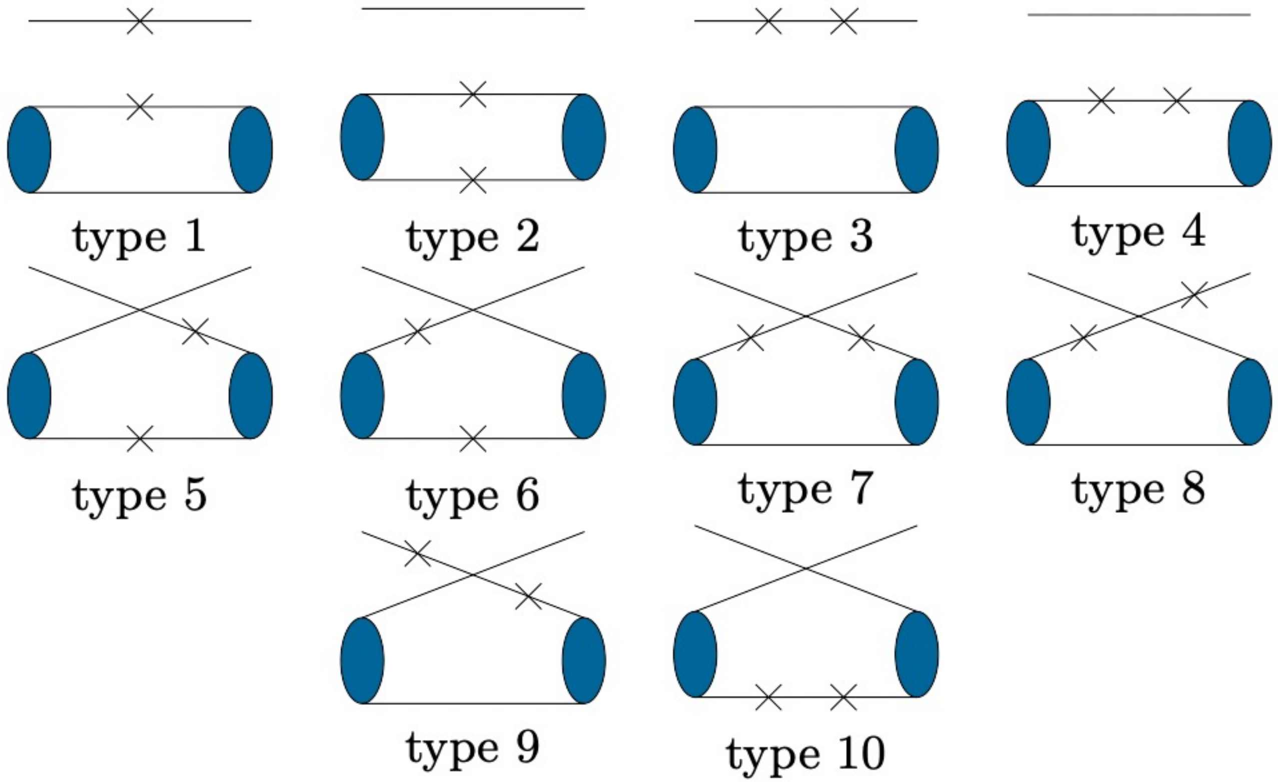
Uncertainty from LECs are negligible, but uncertainty from ChPT $O(e^2p^4)$ terms are still large …

Challenges for moving to nucleon sector (I)

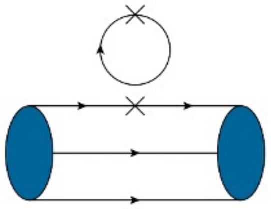
- π γW box diagram



- Nucleon γW box diagram
  - Connected diagram (8 of 10)



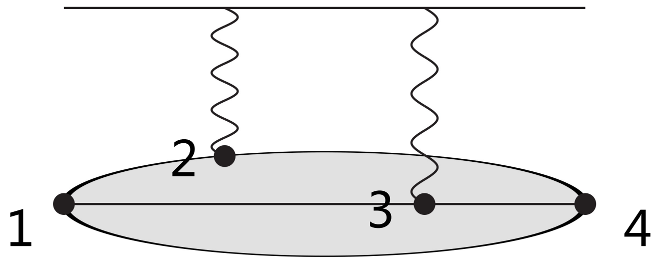
  - Disconnected diagram



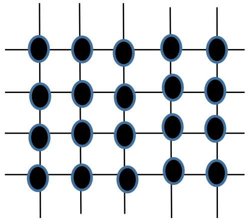
Vanish in flavor SU(3) limit, so far neglected

Challenges for moving to nucleon sector (II)

- Hadronic part from a typical 4-point function

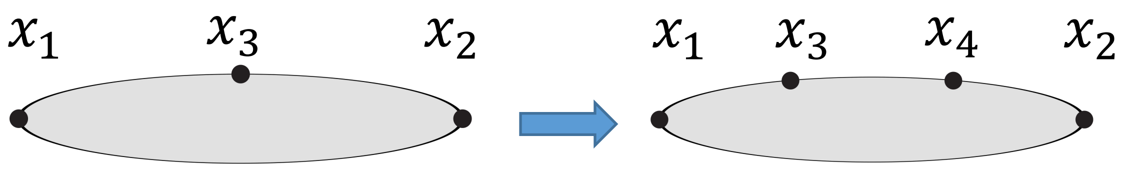


- Perform the volume summation for each point



$$\sum_{x \in L^3}$$

- From 3-point to 4-point function



3-point: L^6 summation 4-point: L^9 summation

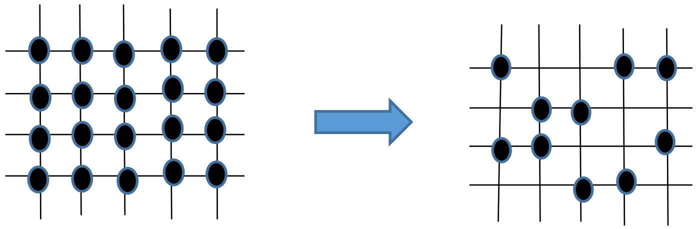
Increasing each point, computational cost increases by 10^4-10^5 times!

Solution： Field sparsening method

【Y. Li, S. Xia, XF, L. Jin, C. Liu, PRD 103 (2021) 014514】

【W. Detmold, D. Murphy, et. al. PRD 104 (2021) 034502】

【See also HLbL calculation & M. Bruno's talk】



- Less summation points may lead to lower precision
- It is not the case because of high correlation in lattice data

→ 10^2-10^3 times less points yields similar precision

- Used for pion, proton, g_A to verify its application

Utilize field sparsening method

- Reduce the computational cost by a factor of 10^2-10^3 with almost no loss of precision!

Challenges for moving to nucleon sector (III)

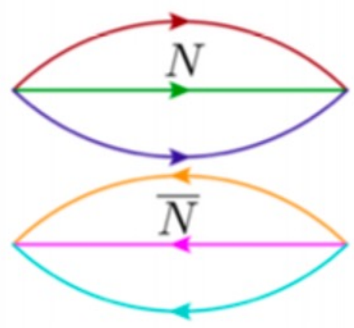
➢ Nucleon system – severe signal/noise (S/N) problem

- Statistics tells us that variance is given by <O^2>-<O>^{2}

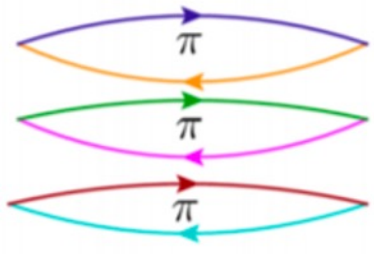
Square of signal

Variance is dominated by <O^2>

<O>^{2} =



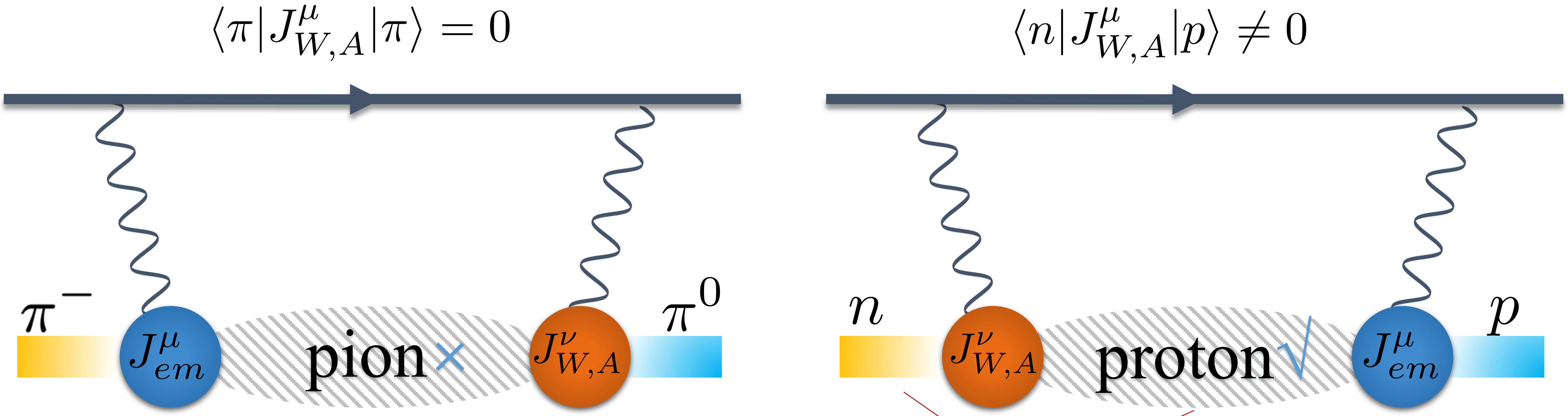
<O^2> =



- S/N is $\exp\left[-(M_N - \frac{3}{2}M_\pi)t\right]$ ➡ γW box diagram requires 4-pt correlation function and thus large *t* separation

It is essentially a sign problem! ➡ It is the main reason for fake plateau and excited-state contamination!

Challenges for moving to nucleon sector (IV)



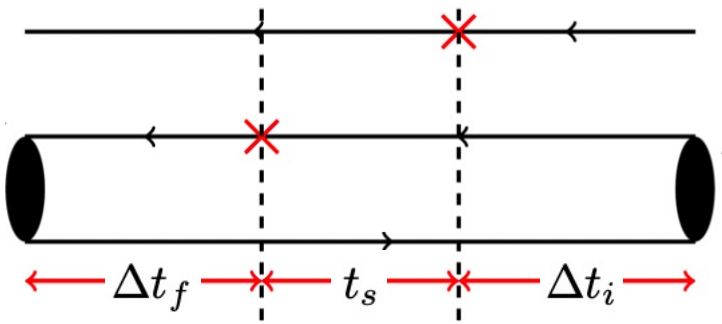
$$\int dt H(t) e^{-(E_X - m)t}$$

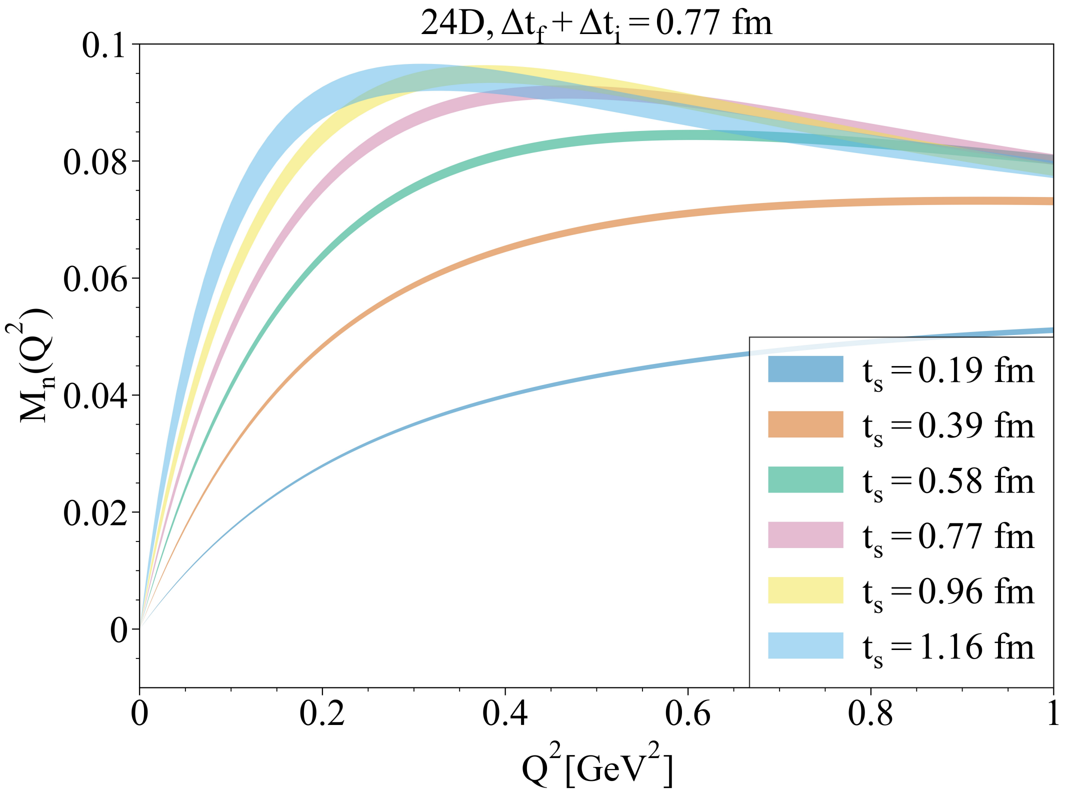
Slow convergence in the temporal integral

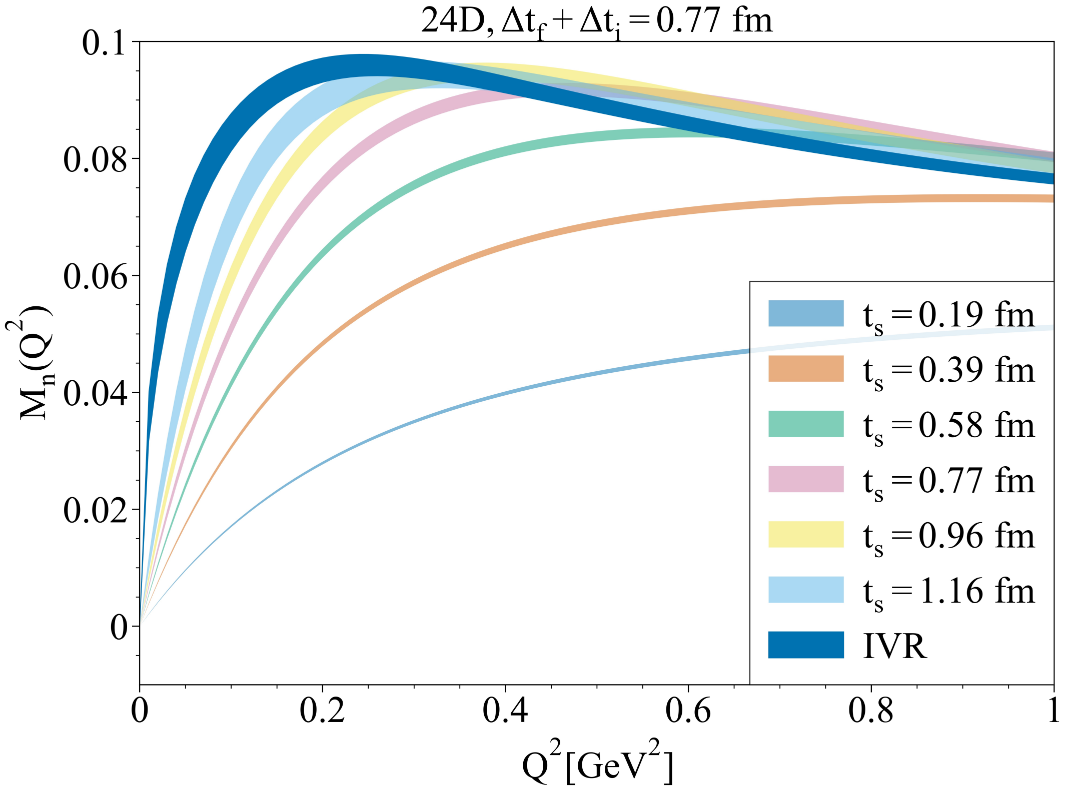
Numerical results

- Ensemble information

Ensemble	m_π [MeV]	L	T	a^{-1} [GeV]	$N_{\rm conf}$
24D	142.6(3)	24	64	1.023(2)	207
32D-fine	143.6(9)	32	64	1.378(5)	69







t_s = 1.16 fm, $\Delta t_i + \Delta t_f + t_s$ = 1.93 fm

Is time separation sufficient?

Examine the ground-state dominance

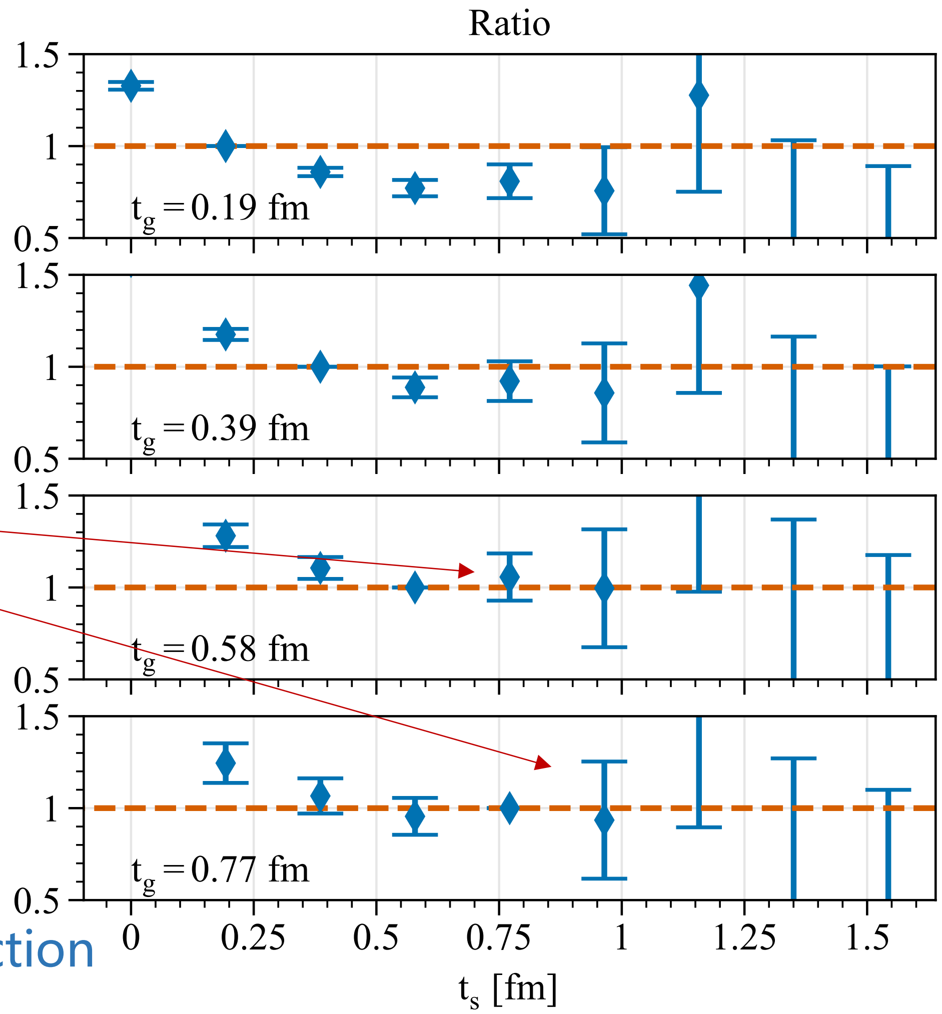
➤ Infinite-volume reconstruction method: [XF, L. Jin, PRD100 (2019) 094509]

Use $\mathcal{H}_{\mu\nu}(\vec{x}, t = t_g)$ to reconstruct the ground state contribution $\mathcal{H}^{GS}_{\mu\nu}(\vec{x}, t)$

➤ Construct a ratio to examine the ground-state dominance

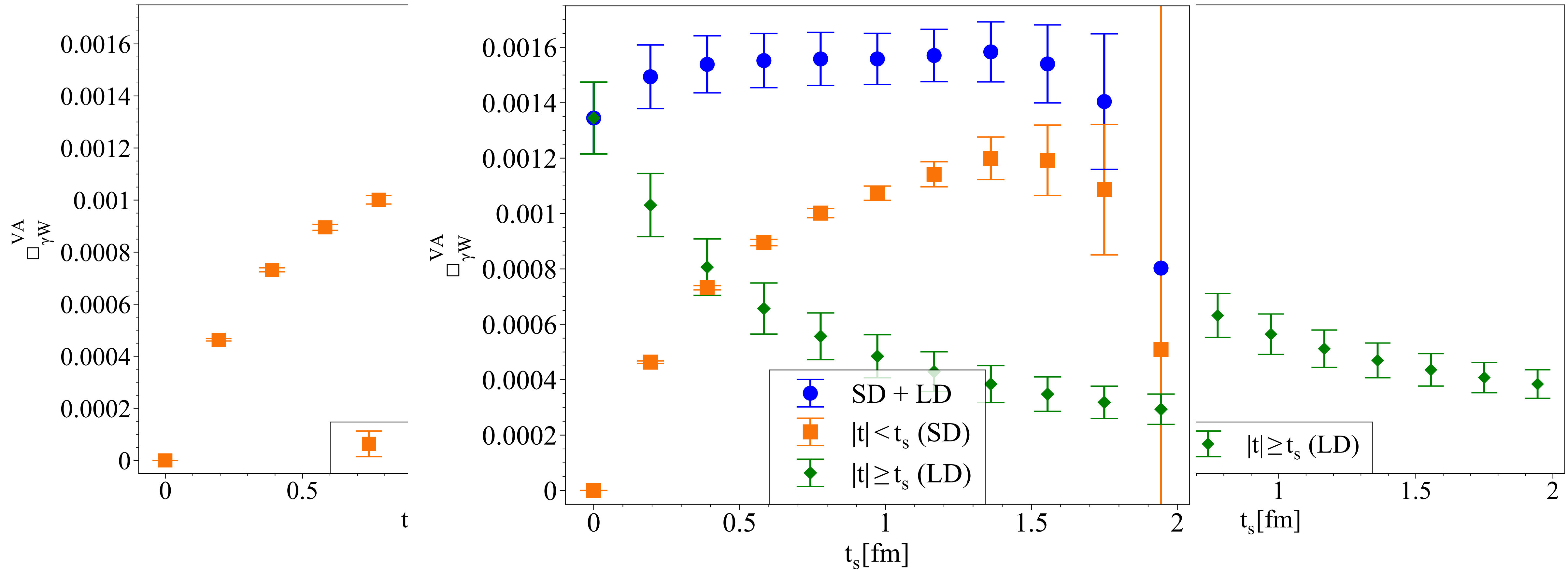
$$R(t) = \frac{\text{Lattice data}}{\text{GS contribution from IVR}}$$

Within statistical uncertainty, reconstruction works well

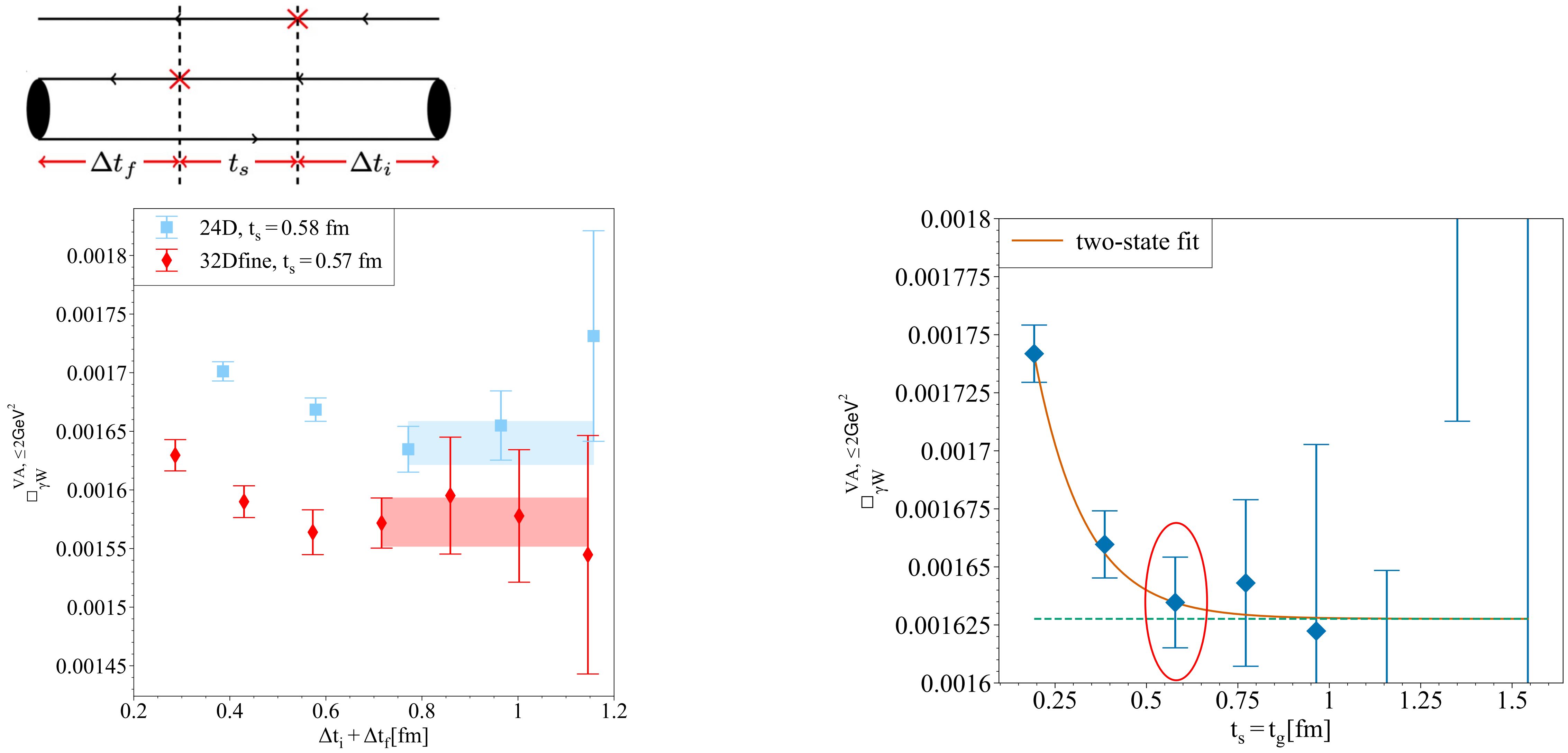


Confirm t_g=0.6 fm is a safe choice for reconstruction

Results from IVR

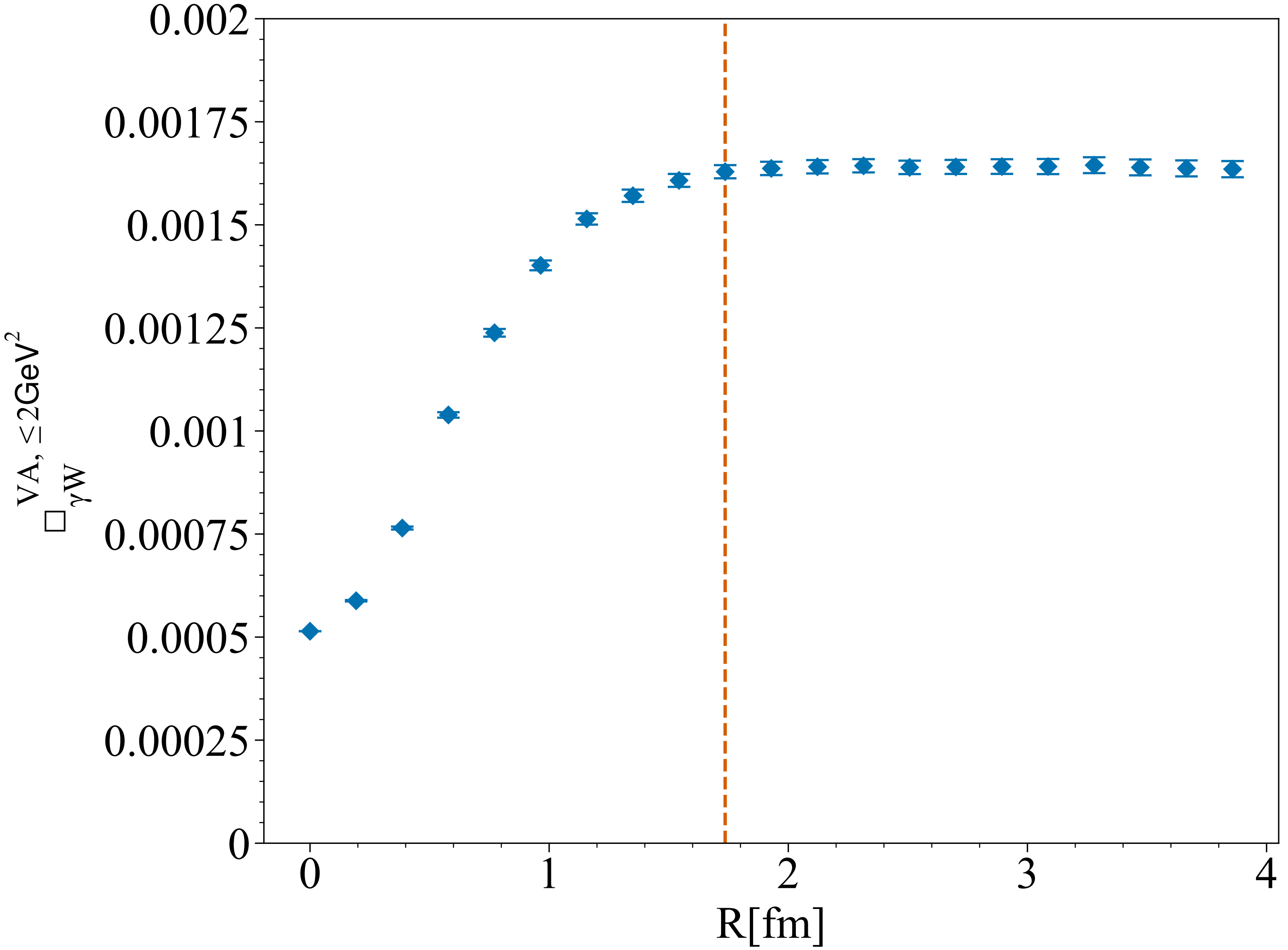


Examine excited-state contamination



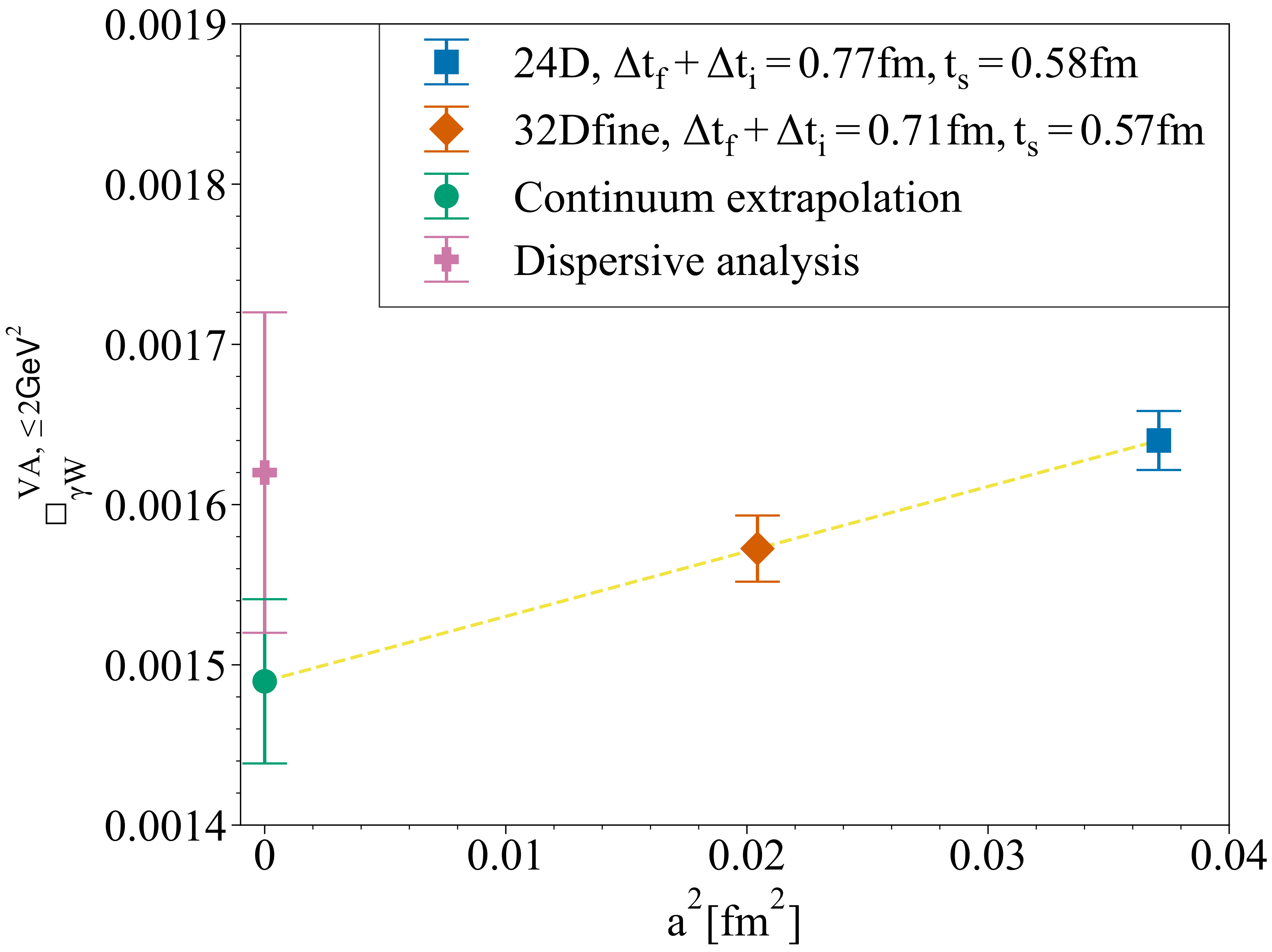
- Nucleon excited-state contamination is very strong for axial-vector current, $A_\mu \to \pi$
- Axial γW is essentially a V_0 transition. Excited-state contamination is not that significant

Examine finite-volume effects



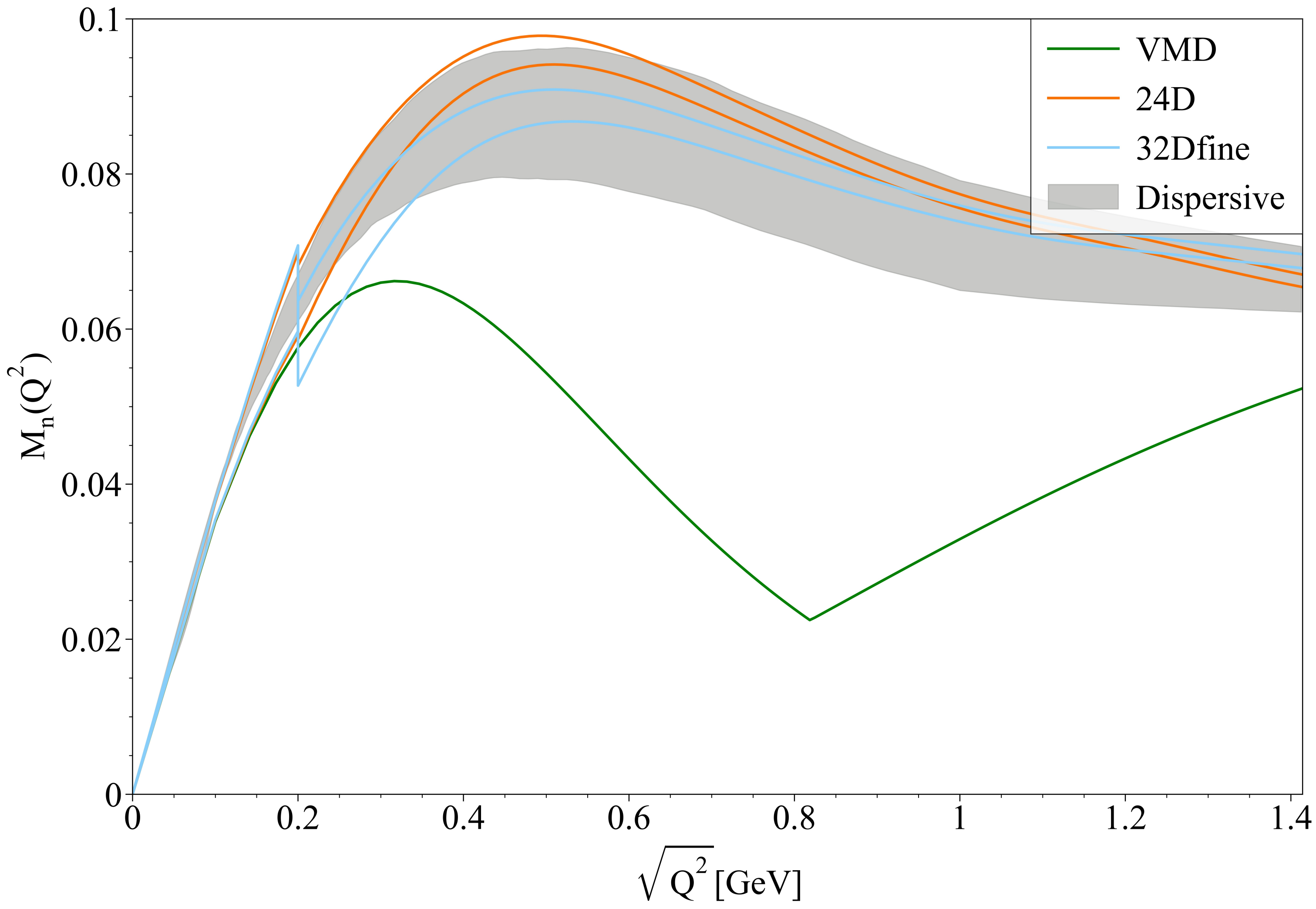
- Good convergence in the spatial integral when using substitution method
- Better to study FV effects using more ensembles and multiple volumes

Continuum extrapolation



Currently only use 2 lattice spacings, more ensembles with finer lattice spacing shall be beneficial!

Comparison with dispersive analysis



P. Ma, XF, M. Gorchtein, L. Jin, C. Seng, Z. Zhang, PRL132 (2024) 191901

Using lattice input, deviation from CKM unitarity: 2.1 σ → 1.8 σ

Low-Q^2 behavior of the hadronic function

$$\Box_{\gamma W}^{VA} = \frac{3\alpha_e}{2\pi}\int \frac{dQ^2}{Q^2}\frac{m_W^2}{m_W^2+Q^2}M_n(Q^2) \quad \text{with} \quad M_n^{\rm LD}(Q^2,t_s,t_g) = -\frac{1}{6}\frac{\sqrt{Q^2}}{m_N}\int d^3\vec{x}\,\tilde{\omega}(t_s,t_g,\vec{x})\bar{H}(t_g,\vec{x}),$$

- Due to $1/Q^2$ factor, $\Box_{\gamma W}^{VA}$ encounters a notably increased noise at small Q^2
- For ground-state dominance at large t_g, we have

$$\int d^3\vec{x}\,\bar{H}(t_g,\vec{x}) = -3\mathring{g}_A(\mathring{\mu}_p+\mathring{\mu}_n) \quad \text{with}$$

$$\bar{H}(t,\vec{x}) = [H(t,\vec{x})+H(-t,\vec{x})]/2$$
$$H(t,\vec{x}) = \epsilon_{\mu\nu\alpha 0}x_\alpha \mathcal{H}_{\mu\nu}^{VA}(t,\vec{x})$$
$$\mathcal{H}_{\mu\nu}^{VA}(t,\vec{x}) \equiv \langle H_f|T\left[J_\mu^{em}(t,\vec{x})J_\nu^{W,A}(0)\right]|H_i\rangle$$

	24D	32Dfine	Cont.	PDG
$-3g_A(\mu_p+\mu_n)$	−3.31(49)	−3.02(53)	−2.65(1.31)	−3.366(3)

- Make substitution

$$M_n^{\rm LD} = -\frac{1}{6}\frac{\sqrt{Q^2}}{m_N}\int d^3\vec{x}\,[\tilde{\omega}(t_s,\vec{x})-\tilde{\omega}_0]\,\bar{H}(t_g,\vec{x}) + \frac{1}{2}\frac{\sqrt{Q^2}}{m_N}\tilde{\omega}_0 g_A(\mu_p+\mu_n).$$

Help to reduce uncertainties

But introduce additional experimental inputs

Conclusion

- Test of first-row CKM unitarity
  - $|V_{ud}|$ Theory: EWR, Nuclear structure
  - $f_+(0)$: More lattice calculations for average

- Inclusion of isospin breaking effects
  - An interesting frontier
  - Beta decay or other semileptonic decay → More studies + new method

- γW box diagrams
  - More studies with different discretization and more ensembles to control systematic effects