Radiative leptonic decays

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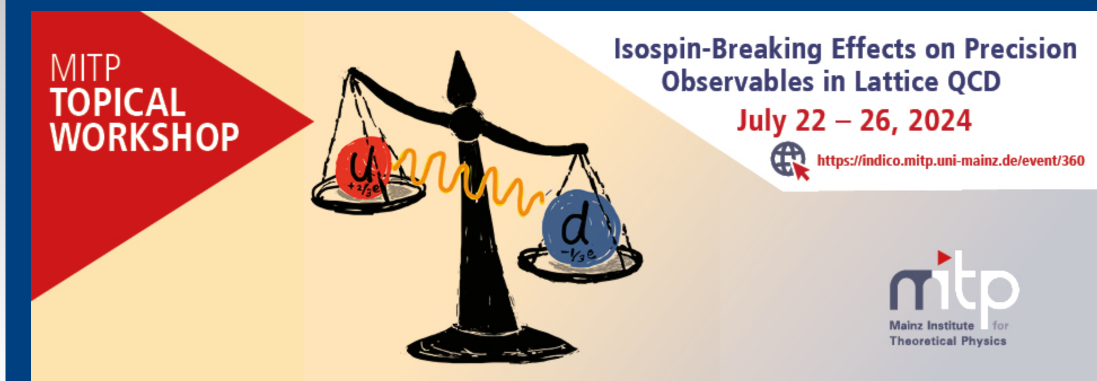


IBLATQCD2024

Mainz

25th July 2024

OUTLINE

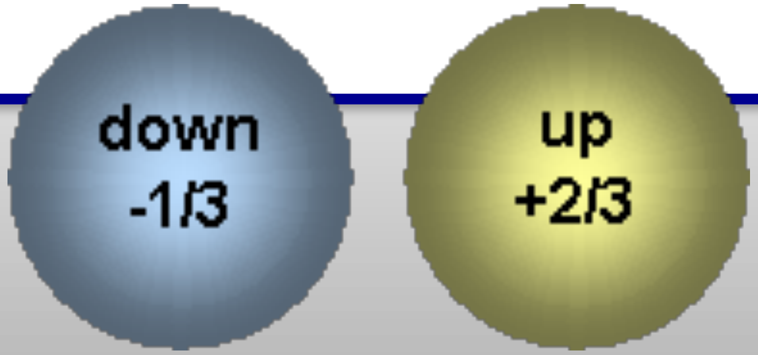
- Motivations
- Leptonic decays of hadrons $H \to \ell \nu_\ell(\gamma)$
- Outlook

In collaboration with

Christopher F. Kane, Christoph Lehner, Stefan Meinel, Amarjit Soni

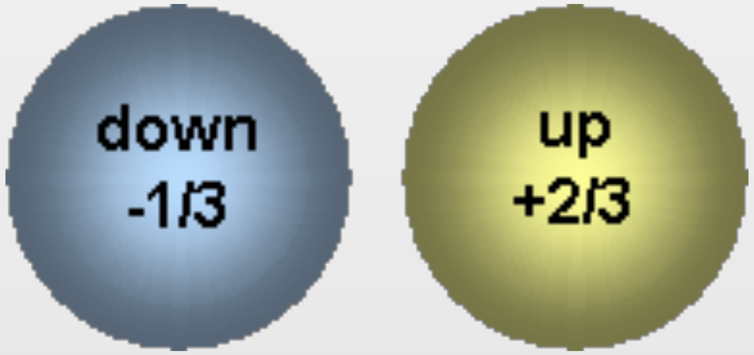
arXiv:1907.00279, arXiv:2110.13196, arXiv:2302.01298

Phenomenological motivations



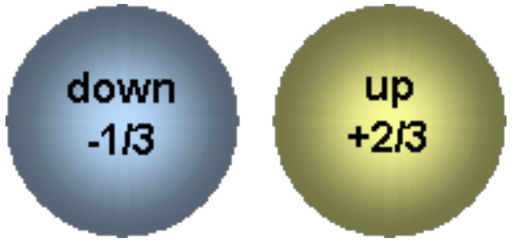
Phenomenological motivations

The determination of some hadronic observables in flavor physics has reached such an accurate degree of experimental and theoretical precision that electromagnetic and strong isospin-breaking effects cannot be neglected anymore



ISOSPIN-BREAKING EFFECTS

Isospin symmetry is an almost exact property of the strong interactions



Isospin-breaking effects are induced by:

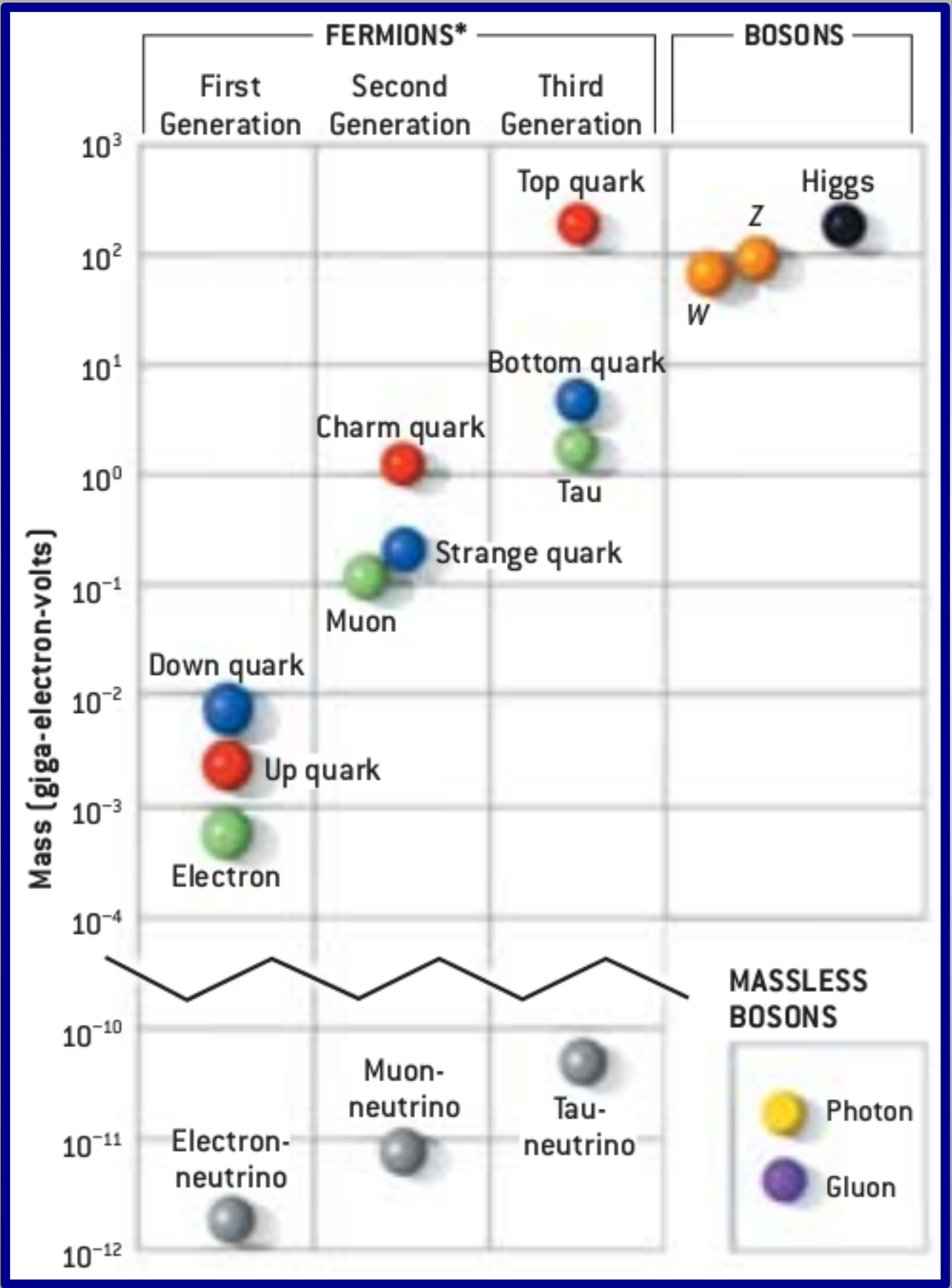
$m_u \neq m_d$: $O[(m_d-m_u)/\Lambda_{QCD}] \approx 1/100$ "Strong"

$Q_u \neq Q_d$: $O(\alpha_{em}) \approx 1/100$ "Electromagnetic"

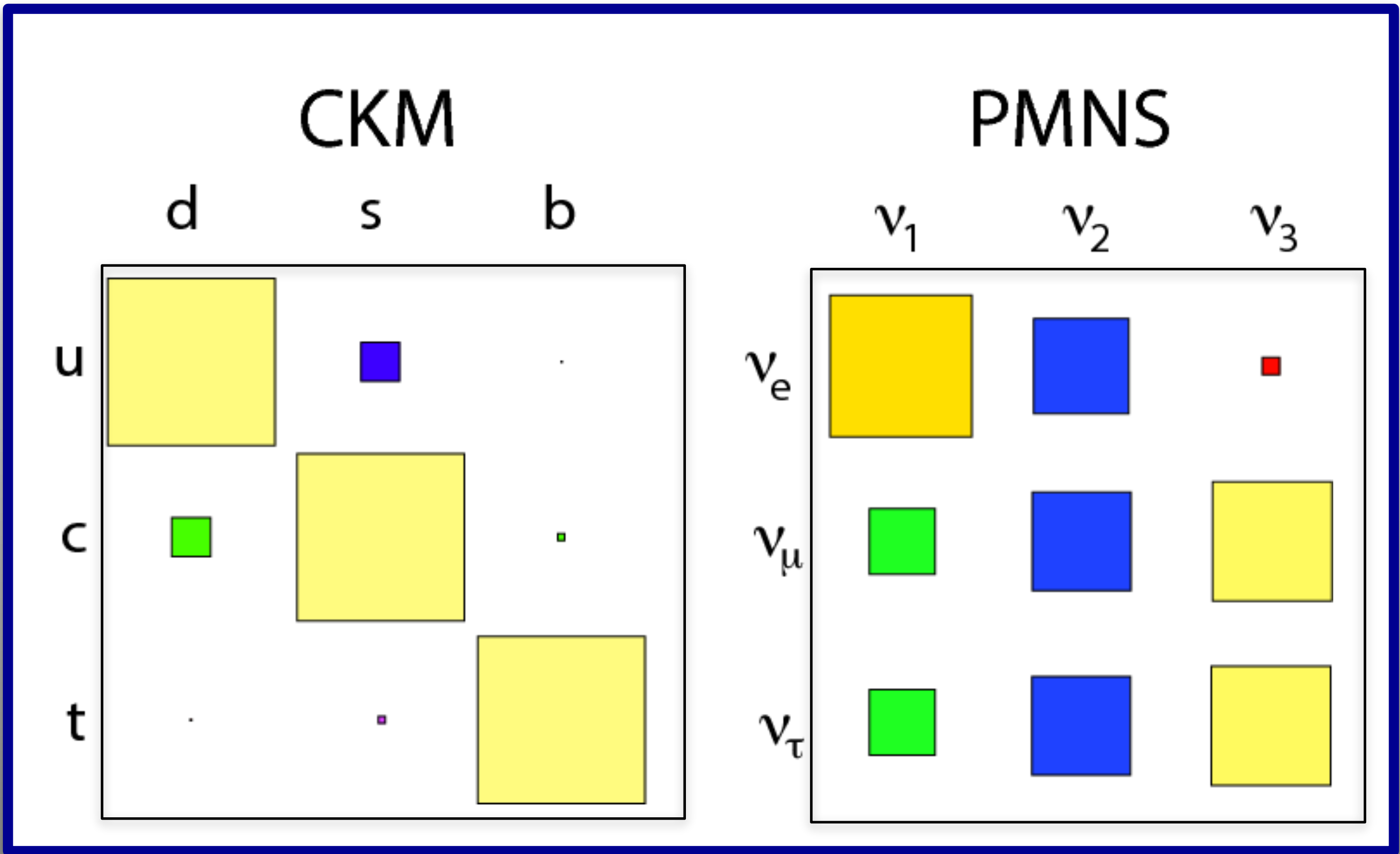
Since electromagnetic interactions renormalize quark masses the two corrections are intrinsically related

Though small, IB effects play often a very important role (quark masses, Mn - Mp, leptonic decay constants, vector form factor)

More fundamental motivations



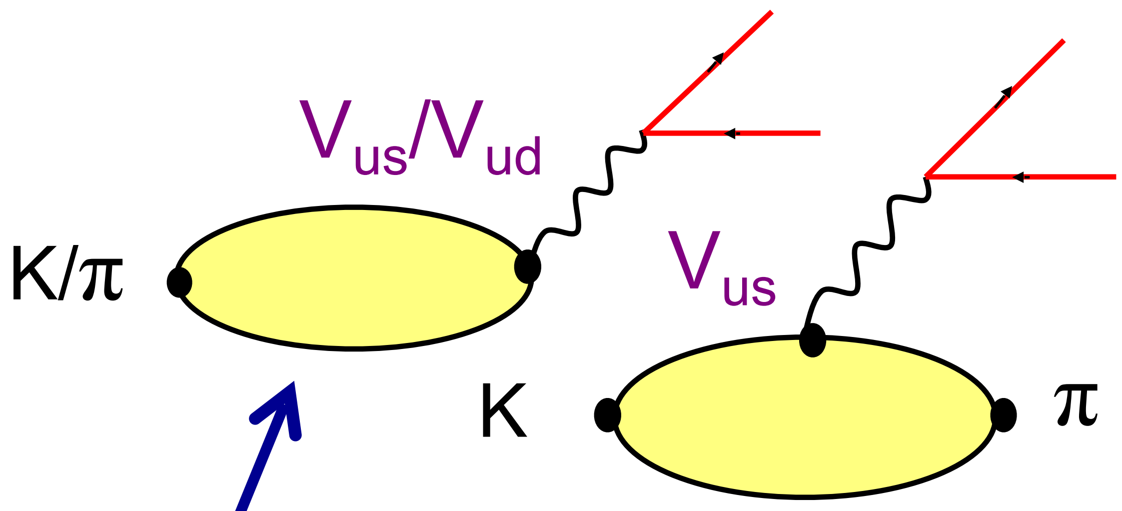
We aim to understand the theory of flavor and the structure of flavor couplings



The determination of Vus and Vud

The relevant processes are leptonic and semileptonic

K and π decays

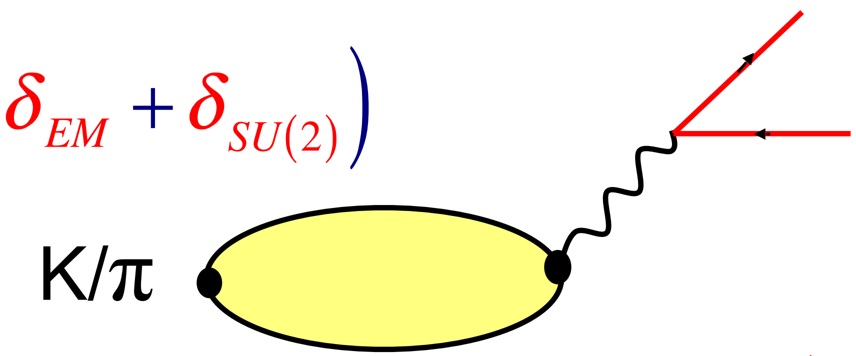


$$\frac{\Gamma\left(K^+ \to \ell^+ \nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+ \to \ell^+ \nu_\ell(\gamma)\right)} = \left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2 \frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$

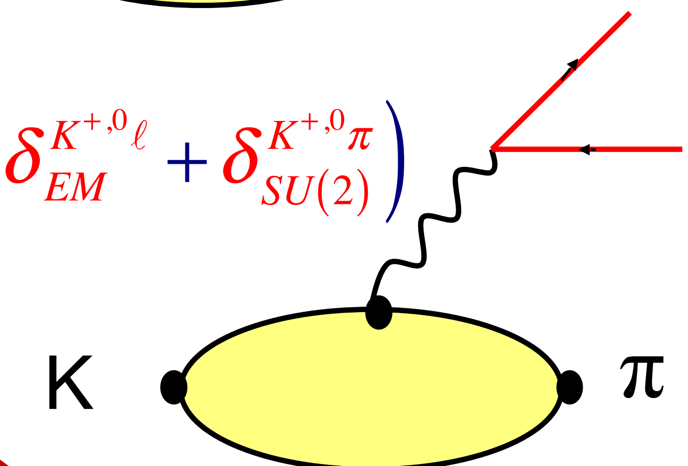
$$\Gamma\left(K^{+,0} \to \pi^{0,-} \ell^+ \nu_\ell(\gamma)\right) = \frac{G_F^2 M_{K^{+,0}}^5}{192\pi^3} C_{K^{+,0}}^2 \left|V_{us} f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)} S_{EW}\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)$$

Vus and Vud: experimental results

$$\frac{\Gamma\left(K^+\to\ell^+\nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+\to\ell^+\nu_\ell(\gamma)\right)}=\left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2\frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\left(1+\delta_{EM}+\delta_{SU(2)}\right)$$



$$\Gamma\left(K^{+,0}\to\pi^{0,-}\ell^+\nu_\ell(\gamma)\right)=\frac{G_F^2M_{K^{+,0}}^5}{192\pi^3}C_{K^{+,0}}^2\left|V_{us}f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)}S_{EW}\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)$$



$$\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}=0.27599(38)$$

$$|V_{us}|f_+(0)=0.21654(41)$$

< 0.2%

PDG

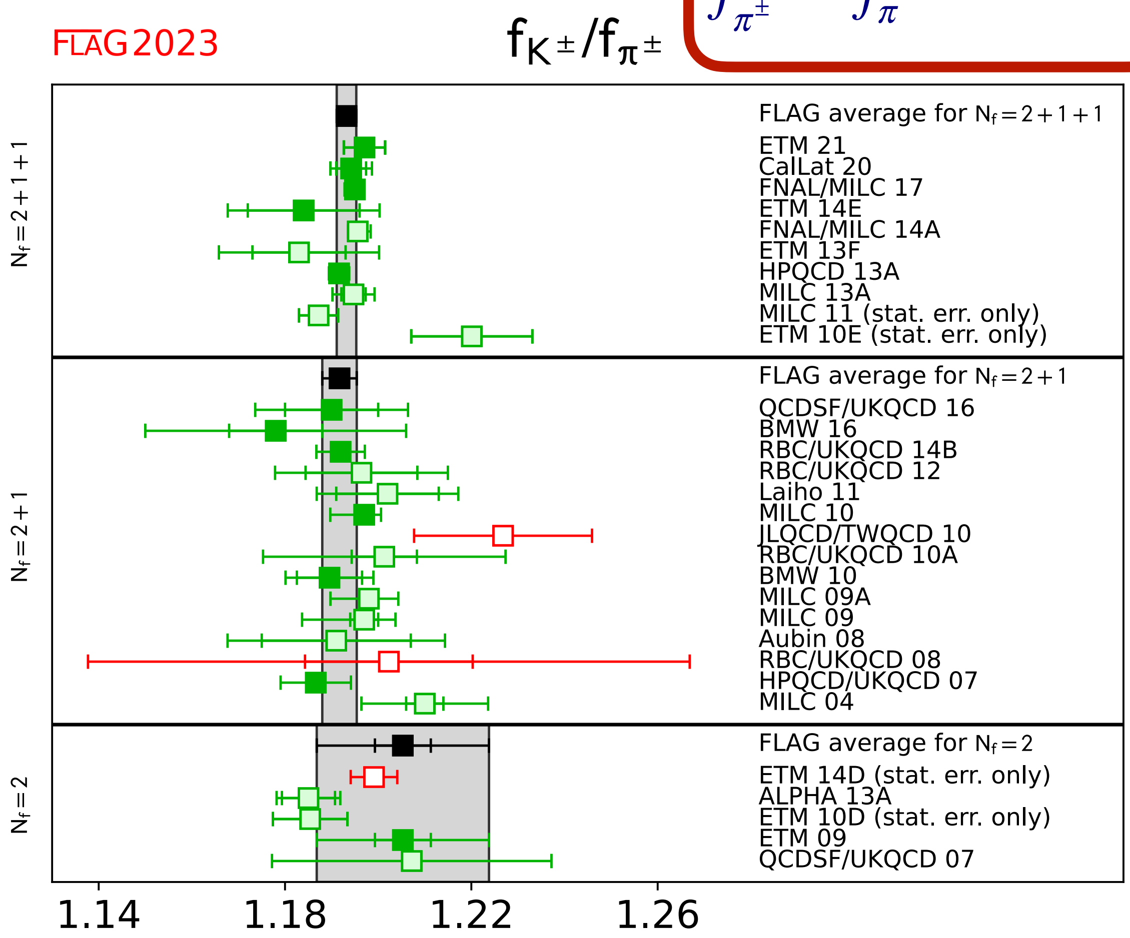
M. Moulson, arXiv:1704.04104

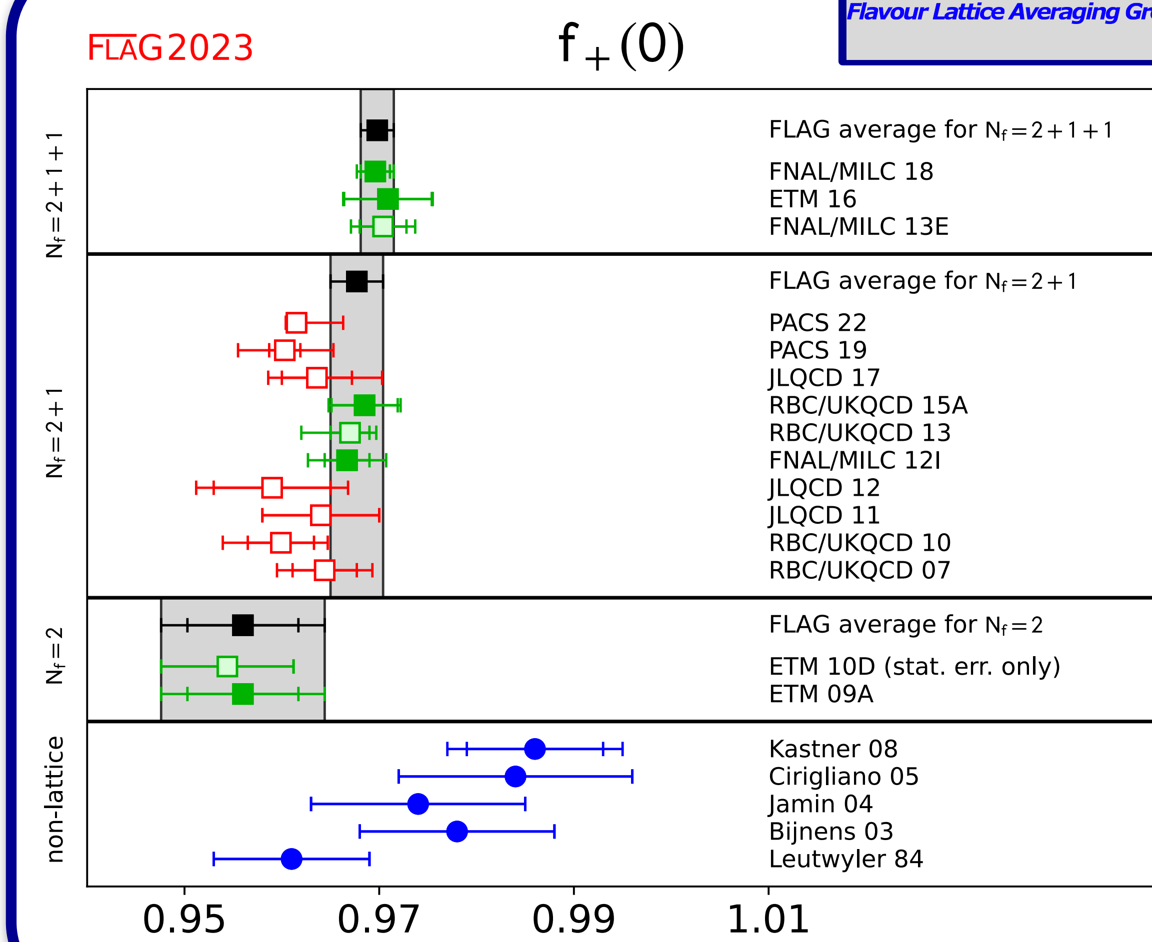
Vus and Vud: results from lattice QCD

$$\frac{f_{K^\pm}}{f_{\pi^\pm}} = \frac{f_K}{f_\pi}\sqrt{1+\delta_{SU(2)}}$$

FLAG 2021

Flavour Lattice Averaging Group





$f_{K\pm} / f_{\pi\pm} = 1.1934(19)$ Nf=2+1+1

$f_{K\pm} / f_{\pi\pm} = 1.1917(37)$ Nf=2+1

0.2%

$f_+(0) = 0.9698(17)$ Nf=2+1+1

$f_+(0) = 0.9677(27)$ Nf=2+1

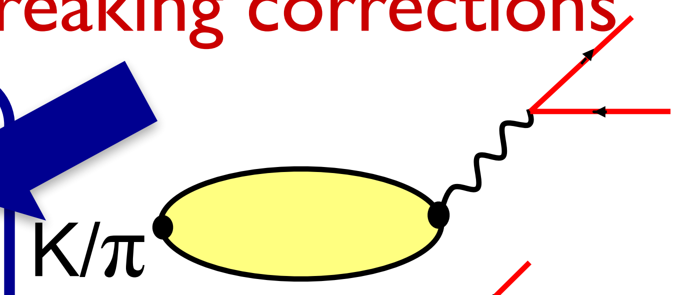
0.2%

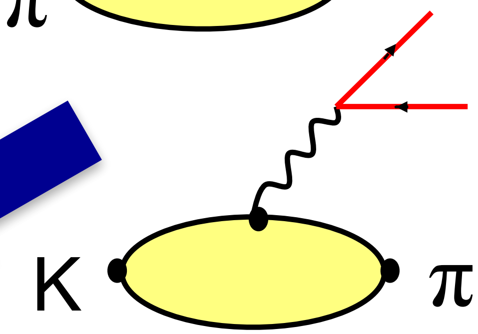
Electromagnetic and isospin-breaking effects

Given the present exper. and theor. (LQCD) accuracy, an important source of uncertainty are long distance electromagnetic and SU(2)-breaking corrections

$$\frac{\Gamma\left(K^+\to\ell^+\nu_\ell(\gamma)\right)}{\Gamma\left(\pi^+\to\ell^+\nu_\ell(\gamma)\right)}=\left(\frac{|V_{us}|}{|V_{ud}|}\frac{f_K}{f_\pi}\right)^2\frac{M_{K^+}\left(1-m_\ell^2/M_{K^+}^2\right)^2}{M_{\pi^+}\left(1-m_\ell^2/M_{\pi^+}^2\right)^2}\boxed{\left(1+\delta_{EM}+\delta_{SU(2)}\right)}$$

$$\Gamma\left(K^{+,0}\to\pi^{0,-}\ell^+\nu_\ell(\gamma)\right)=\frac{G_F^2M_{K^{+,0}}^5}{192\pi^3}C_{K^{+,0}}^2\left|V_{us}f_+^{K^0\pi^-}(0)\right|^2 I_{K\ell}^{(0)}S_{EW}\boxed{\left(1+\delta_{EM}^{K^{+,0}\ell}+\delta_{SU(2)}^{K^{+,0}\pi}\right)}$$





For $\Gamma_{Kl2}/\Gamma_{\pi l2}$ At leading order in ChPT both δ_{EM} and $\delta_{SU(2)}$ can be expressed in terms of physical quantities (e.m. pion mass splitting, f_K/f_π, …)

- $\boxed{\delta_{EM}=-0.0069(17)}$ 25% of error due to higher orders ➡ 0.2% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  M.Knecht *et al.*, 2000; V.Cirigliano and H.Neufeld, 2011

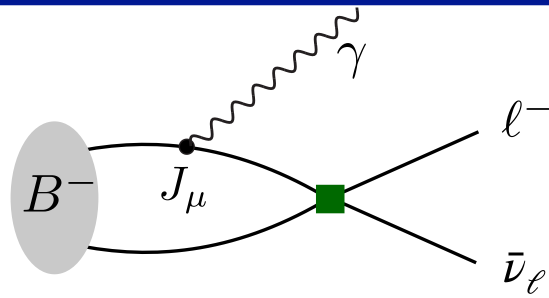
- $\boxed{\delta_{SU(2)}=\left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}\right)^2-1=-0.0044(12)}$ 25% of error due to higher orders ➡ 0.1% on $\Gamma_{Kl2}/\Gamma_{\pi l2}$

  J.Gasser and H.Leutwyler, 1985; V.Cirigliano and H.Neufeld, 2011

ChPT is not applicable to D and B decays

Radiative corrections to leptonic B-meson decays

$B^- \to \ell^- \bar{\nu}_\ell \gamma$



- The emission of a real hard photon removes the $(m_\ell/M_B)^2$ helicity suppression
- This is the simplest process that probes (for large E_γ) the first inverse moment of the B-meson LCDA

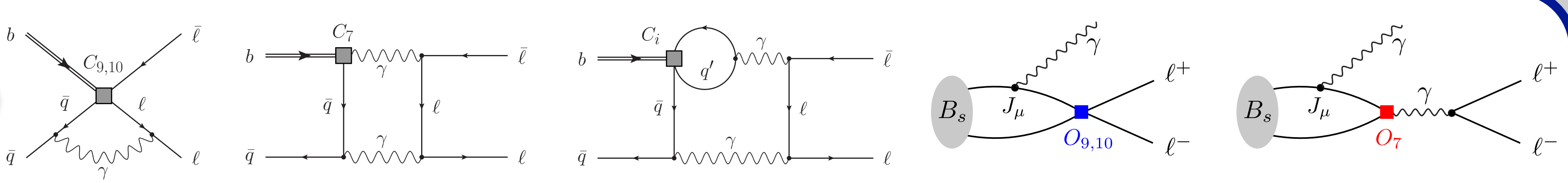
$$\frac{1}{\lambda_B(\mu)} = \int_0^\infty \frac{d\omega}{\omega} \Phi_{B+}(\omega,\mu)$$

λ_B is an important input in QCD-factorization predictions for non-leptonic B decays but is poorly known

M. Beneke, V. M. Braun, Y. Ji, Y.-B. Wei, 2018

- Belle 2018: $\mathcal{B}(B^- \to \ell^- \bar{\nu}_\ell \gamma, E_\gamma > 1 \text{ GeV}) < 3.0 \cdot 10^{-6}$ $\longrightarrow$ $\lambda_B > 0.24$ GeV
- QCD sum rules in HQET: $\lambda_B(1 \text{ GeV}) = 0.46\,(11)$ GeV

$B_q \to \ell^+ \ell^- (\gamma)$



- Enhancement of the virtual corrections by a factor M_B/Λ_{QCD} and by large logarithms

M. Beneke, C. Bobeth, R. Szafron, 2019

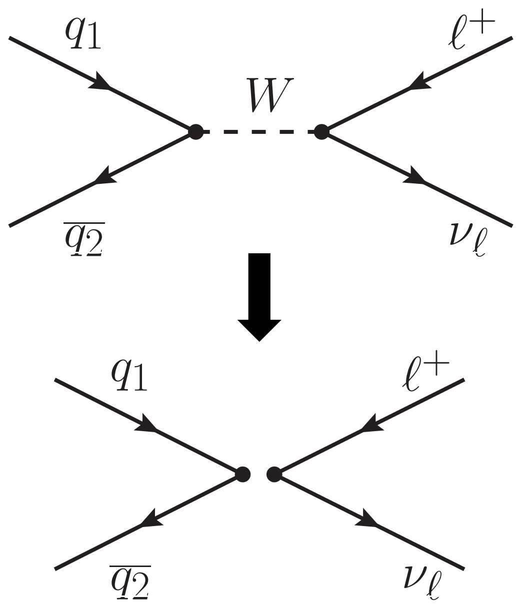
- The real photon emission process is a clean probe of NP: sensitiveness to C_9, C_{10}, C_7

Radiative corrections to leptonic decay rates

$$H \to \ell \nu_\ell (\gamma)$$

Leptonic decays at tree level

Since the masses of the pion and kaon are much smaller than M_W we use the effective Hamiltonian



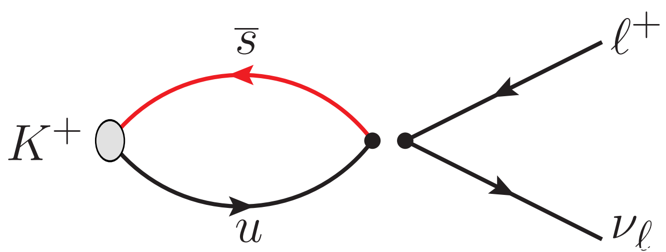
$$H_{eff} = \frac{G_F}{\sqrt{2}} V^*_{q_1 q_2} \left(\overline{q_2}\gamma^\mu (1-\gamma_5) q_1\right)\left(\overline{\nu_\ell}\gamma_\mu (1-\gamma_5)\ell\right)$$

This replacement is necessary in a lattice calculation, since $1/a \ll M_W$

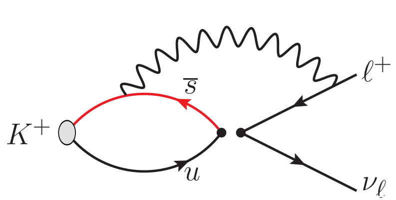
The rate is:

$$\Gamma^{(tree)}_{P^\pm}\left(P^\pm \to \ell^\pm \nu_\ell\right) = \frac{G_F^2}{8\pi}\left|V_{q_1 q_2}\right|^2 \left[f_P^{(0)}\right]^2 M_{P^\pm} m_\ell^2 \left(1 - \frac{m_\ell^2}{M_{P^\pm}^2}\right)^2$$

In the absence of electromagnetism, the non-perturbative QCD effects are contained in a single number, the pseudoscalar decay constant



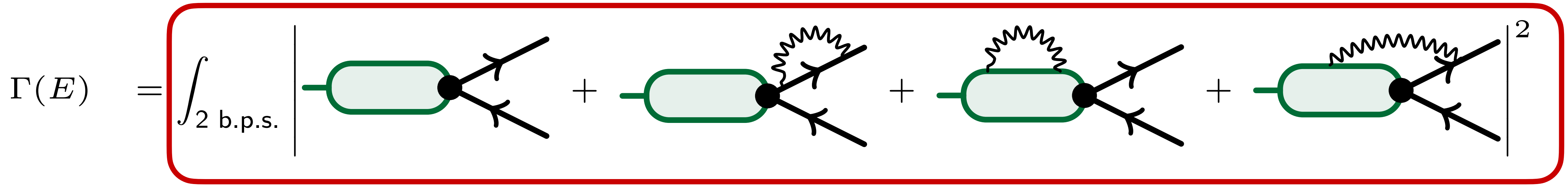
$$A_P^{(0)} \equiv \langle 0 | \overline{q}_2 \gamma_4 \gamma_5 q_1 | P^{(0)} \rangle = f_P^{(0)} M_P^{(0)}$$



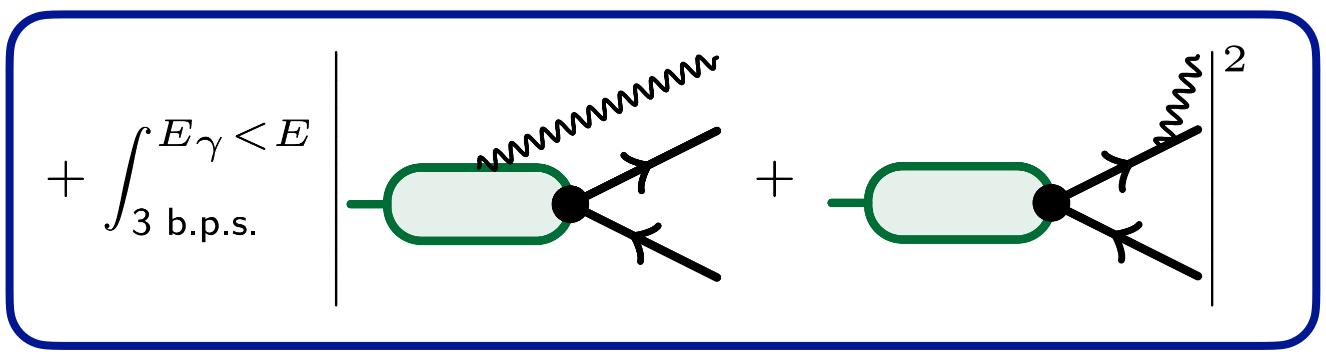
In the presence of electromagnetism it is not even possible to give a physical definition of f_P

J. Gasser and G.R.S. Zarnauskas, PLB 693 (2010) 122

Leptonic decays at O(α)



Γ_0



$\Gamma_1(E)$

$\Gamma(E) = \Gamma_0 + \Gamma_1(E)$ with $0 \leq E_\gamma \leq E$ is infrared finite

F. Bloch and A. Nordsieck, 1937

- Both Γ_0 and $\Gamma_1(E)$ can be evaluated in a fully non-perturbative way in lattice simulations
- The first lattice calculations of $\Gamma[\pi, K \to \ell\nu(\gamma)]$ have been finalized

N. Carrasco *et al.* arXiv:1502.00257 V. Lubicz *et al.* arXiv:1611.08497 DG *et al.* arXiv:1711.06537 M. Di Carlo *et al.* arXiv:1904.08731 P. Boyle *et al.* arXiv:2211.12865

Real photon emission amplitude

$$J_\mu^{em} = \sum_q Q_q \bar{q}\gamma_\mu q \qquad J_\nu^{weak} = \bar{q}_1\gamma_\nu(1-\gamma_5)\,q_2$$

$$\mathcal{A}(H^- \to \gamma\ell\bar{\nu}) = \frac{G_F V_{q_1q_2}}{\sqrt{2}}\left[e(\epsilon^*)^\mu \bar{\ell}\gamma^\nu(1-\gamma_5)\nu \cdot T_{\mu\nu}(p_H,p_\gamma) - ieQ_\ell f_H \cdot \bar{\ell}\not{\epsilon}^*(1-\gamma_5)\nu\right]$$

$$\boxed{T_{\mu\nu}(p_H,p_\gamma) = -i\int \mathrm{d}t_{\mathrm{em}} \int \mathrm{d}^3x\; e^{ip_\gamma\cdot x}\,\langle 0|\,\mathbf{T}\big(J_\mu^{\mathrm{em}}(t_{\mathrm{em}},\vec{x})J_\nu^{\mathrm{weak}}(0)\big)\,|H(\vec{p}_H)\rangle}$$

- The hadronic tensor can be written as the sum $T_{\mu\nu} = T_{\mu\nu}^{<} + T_{\mu\nu}^{>}$ of the contributions from the two time orderings of the currents

Real photon emission amplitude

By setting $p_\gamma^2 = 0$, at fixed meson mass, the form factors depend on $p_H \cdot p_\gamma$ only. Moreover, by choosing a *physical* basis for the polarization vectors, *i.e.* $\epsilon_r(\mathbf{p}_\gamma) \cdot p_\gamma = 0$, one has

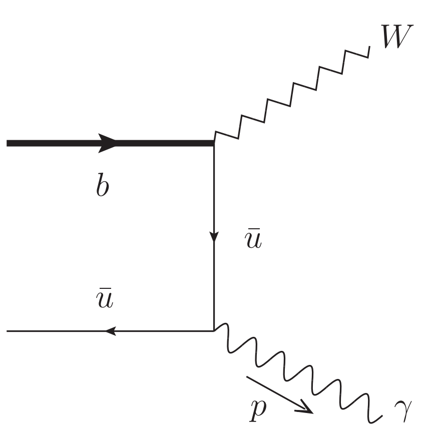
$$\epsilon^r_\mu(\mathbf{p}_\gamma)\, T^{\mu\nu}(p_\gamma, p_H) = \epsilon^r_\mu(\mathbf{p}_\gamma) \left\{ \varepsilon^{\mu\nu\tau\rho}(p_\gamma)_\tau v_\rho F_V + i \left[-g^{\mu\nu}(p_\gamma \cdot v) + v^\mu p_\gamma^\nu \right] F_A - i \frac{v^\mu v^\nu}{p_\gamma \cdot v} m_H f_H \right\}$$

In the case of off-shell photons $(p_\gamma^2 \neq 0) \longrightarrow \Gamma[H \to \ell \nu_\ell \ell^+ \ell^-]$ expressed in terms of 4 form factors

For large photon energies and in the B-meson rest frame the form factors can be written as

$$F_V(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) + \Delta\xi(E_\gamma)$$

$$F_A(E_\gamma) = \frac{e_u M_B f_B}{2E_\gamma \lambda_B(\mu)} R(E_\gamma, \mu) + \xi(E_\gamma) - \Delta\xi(E_\gamma)$$



M. Beneke and J. Rohrwild, 2011

Form factors: results

PHYSICAL REVIEW D **103,** 014502 (2021)

arXiv:2006.05358

First lattice calculation of radiative leptonic decay rates of pseudoscalar mesons

A. Desiderio,[1] R. Frezzotti,[1] M. Garofalo,[2] D. Giusti,[3,4] M. Hansen,[5] V. Lubicz,[2] G. Martinelli,[6] C. T. Sachrajda,[7] F. Sanfilippo,[4] S. Simula,[4] and N. Tantalo[1]

$$F^P_{A,V}(x_\gamma) = C^P_{A,V} + D^P_{A,V} x_\gamma$$

F_A

$C_A^\pi = 0.010 \pm 0.003\,;$ $D_A^\pi = 0.0004 \pm 0.0006\,;$ $\rho_{C_A^\pi, D_A^\pi} = -0.419\,;$

$C_A^K = 0.037 \pm 0.009\,;$ $D_A^K = -0.001 \pm 0.007\,;$ $\rho_{C_A^K, D_A^K} = -0.673\,;$

$C_A^D = 0.109 \pm 0.009\,;$ $D_A^D = -0.10 \pm 0.03\,;$ $\rho_{C_A^D, D_A^D} = -0.557\,;$

$C_A^{D_s} = 0.092 \pm 0.006\,;$ $D_A^{D_s} = -0.07 \pm 0.01\,;$ $\rho_{C_A^{D_s}, D_A^{D_s}} = -0.745\,.$

F_V

$C_V^\pi = 0.023 \pm 0.002\,;$ $D_V^\pi = -0.0003 \pm 0.0003\,;$ $\rho_{C_V^\pi, D_V^\pi} = -0.570\,;$

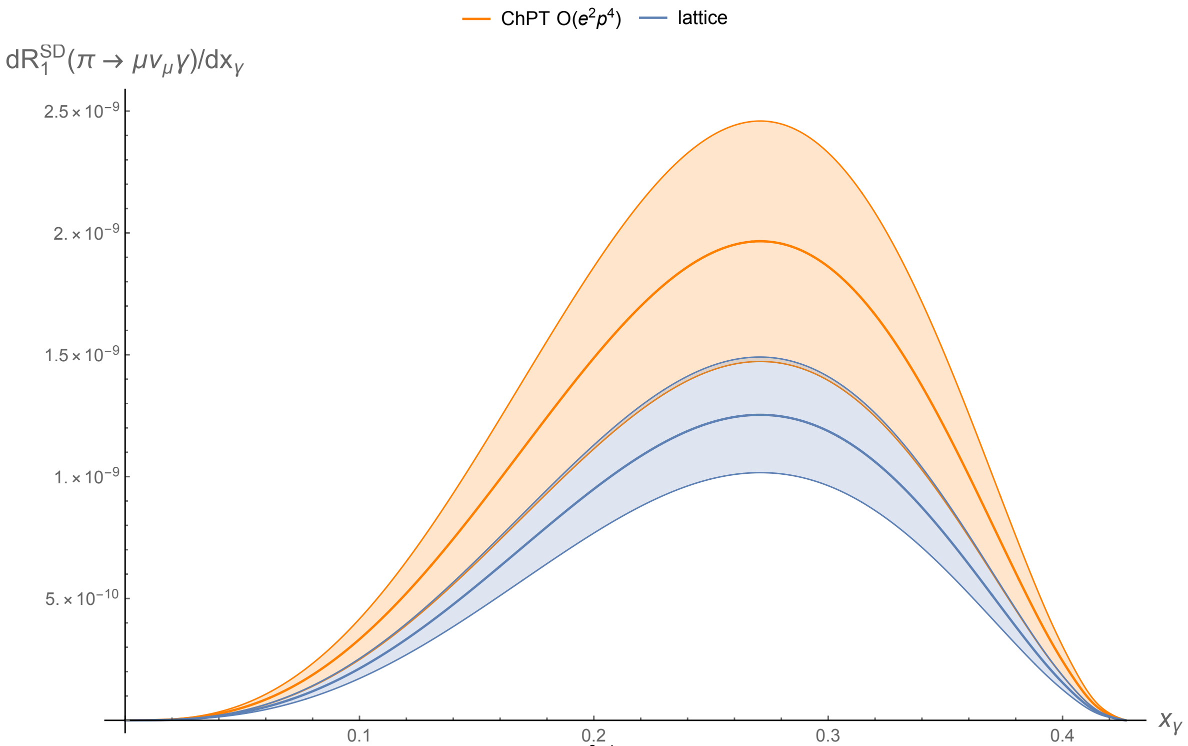
$C_V^K = 0.12 \pm 0.01\,;$ $D_V^K = -0.02 \pm 0.01\,;$ $\rho_{C_V^K, D_V^K} = -0.714\,;$

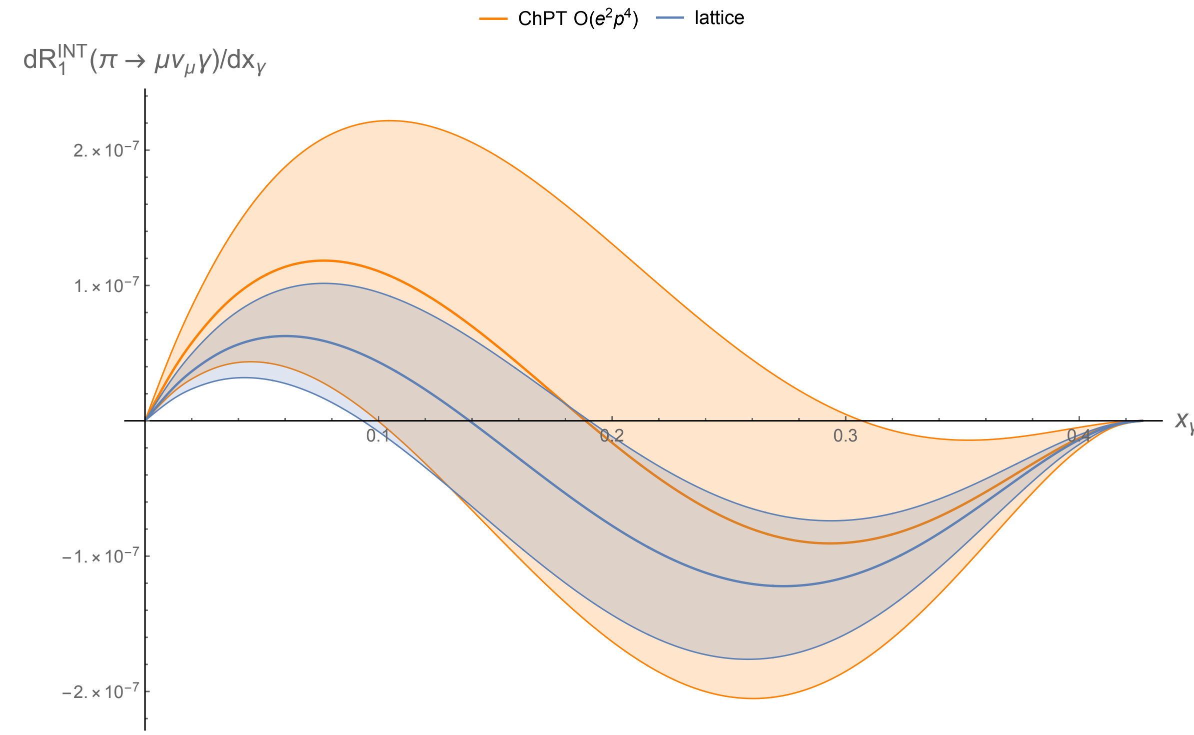
$C_V^D = -0.15 \pm 0.02\,;$ $D_V^D = 0.12 \pm 0.04\,;$ $\rho_{C_V^D, D_V^D} = -0.580\,;$

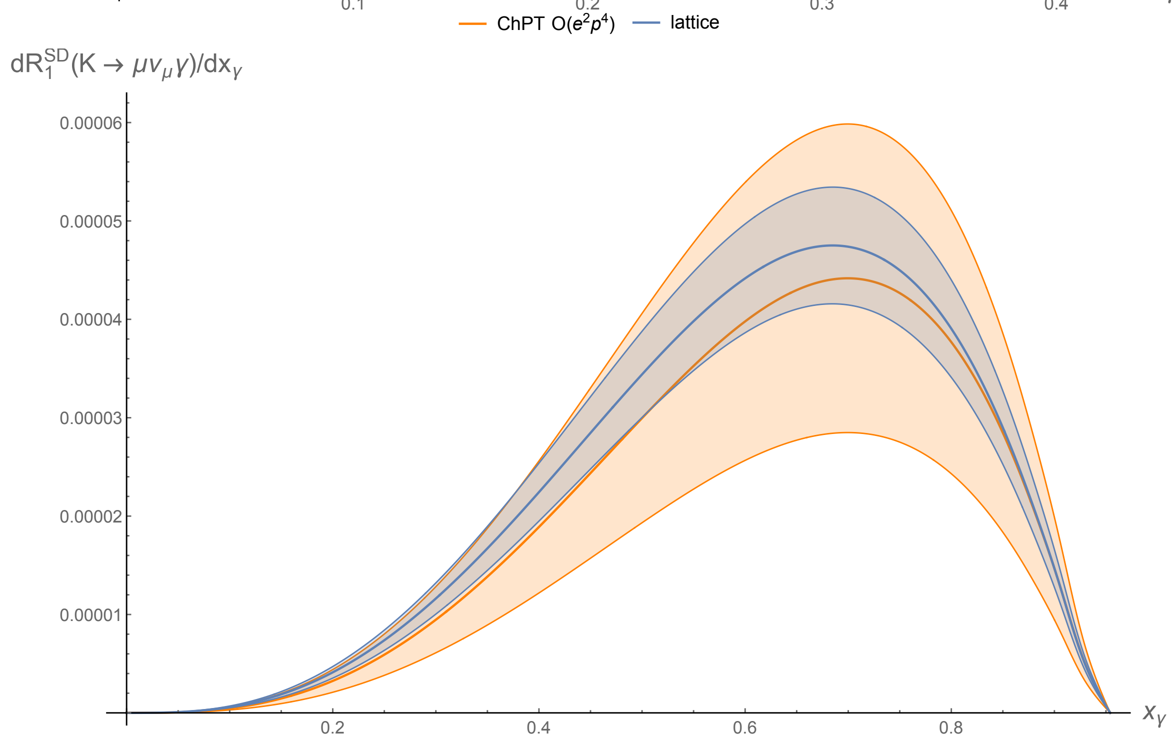
$C_V^{D_s} = -0.12 \pm 0.02\,;$ $D_V^{D_s} = 0.16 \pm 0.03\,;$ $\rho_{C_V^{D_s}, D_V^{D_s}} = -0.900\,.$

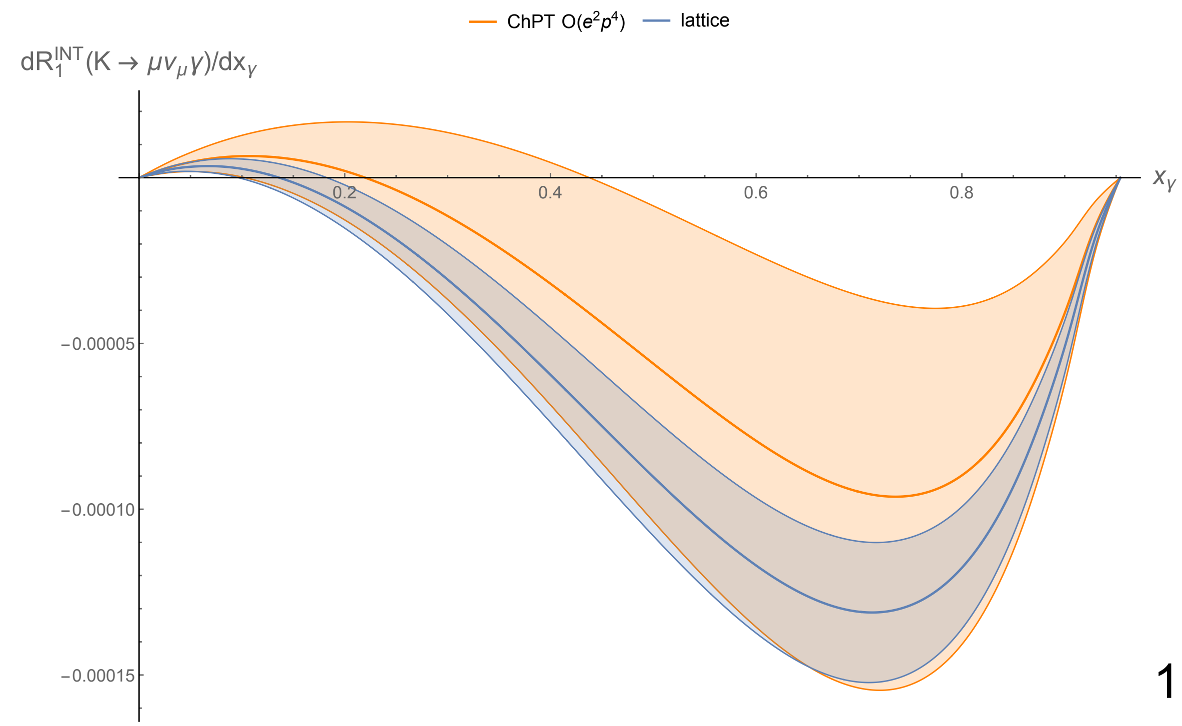
$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\,\frac{d\Gamma_1^{\rm SD}}{dx_\gamma}=\frac{m_P^2}{6f_P^2\,r_\ell^2\,(1-r_\ell^2)^2}\,[F_V(x_\gamma)^2+F_A(x_\gamma)^2]\,f^{\rm SD}(x_\gamma)$$

$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\,\frac{d\Gamma_1^{\rm INT}}{dx_\gamma}=-\frac{2\,m_P}{f_P\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)\,f_V^{\rm INT}(x_\gamma)+F_A(x_\gamma)\,f_A^{\rm INT}(x_\gamma)\right]$$









Leptonic decays at O(α): RESULTS

$$\Gamma(\Delta E) = \Gamma^{(tree)}\left[1 + \delta R_0 + \delta R_{pt}(\Delta E) + \delta R_1^{SD}(\Delta E) + \delta R_1^{INT}(\Delta E)\right]$$

	$\pi_{e2[\gamma]}$	$\pi_{\mu 2[\gamma]}$	$K_{e2[\gamma]}$	$K_{\mu 2[\gamma]}$
δR_0	$(*)$	0.0411 (19)	$(*)$	0.0341 (10)
$\delta R_{\rm pt}(\Delta E_\gamma^{max})$	-0.0651	-0.0258	-0.0695	-0.0317
$\delta R_1^{\rm SD}(\Delta E_\gamma^{max})$	$5.4\ (1.0) \times 10^{-4}$	$2.6\ (5) \times 10^{-10}$	1.19 (14)	$2.2\ (3) \times 10^{-5}$
$\delta R_1^{\rm INT}(\Delta E_\gamma^{max})$	$-4.1\ (1.0) \times 10^{-5}$	$-1.3\ (1.5) \times 10^{-8}$	$-9.2\ (1.3) \times 10^{-4}$	$-6.1\ (1.1) \times 10^{-5}$
ΔE_γ^{max} (MeV)	69.8	29.8	246.8	235.5

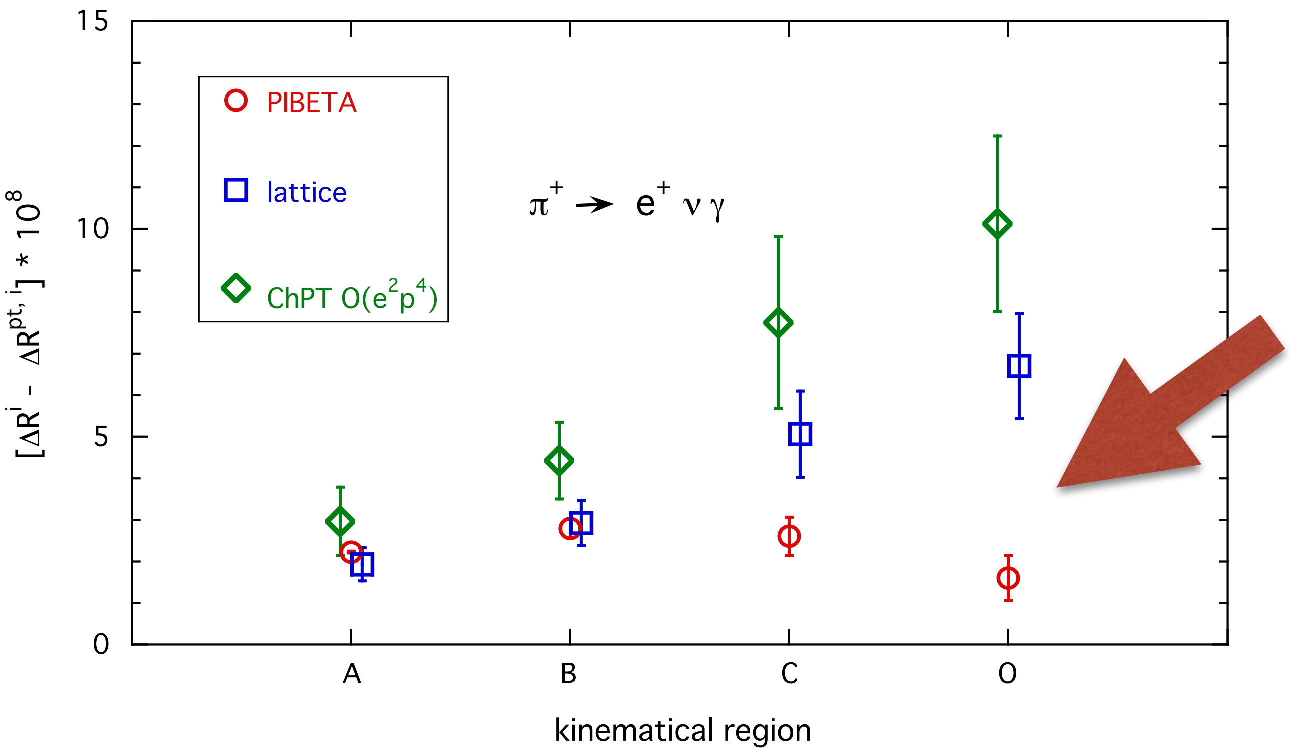
$^{(*)}$ *Not yet evaluated by numerical lattice QCD+QED simulations.*

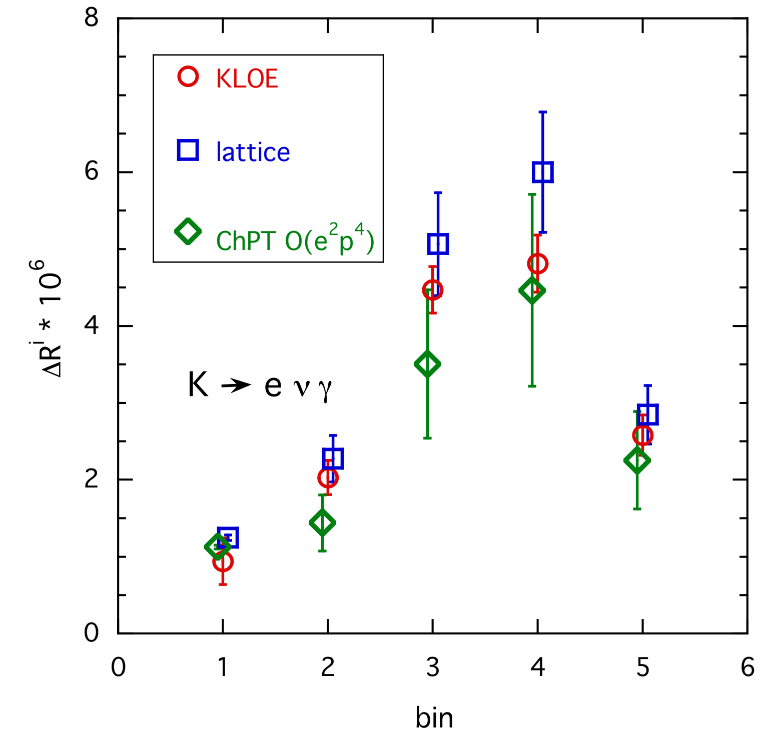
Large SD effects $\longrightarrow$

$\Gamma^{(tree)} \propto (m_\ell / m_P)^2$ helicity suppression

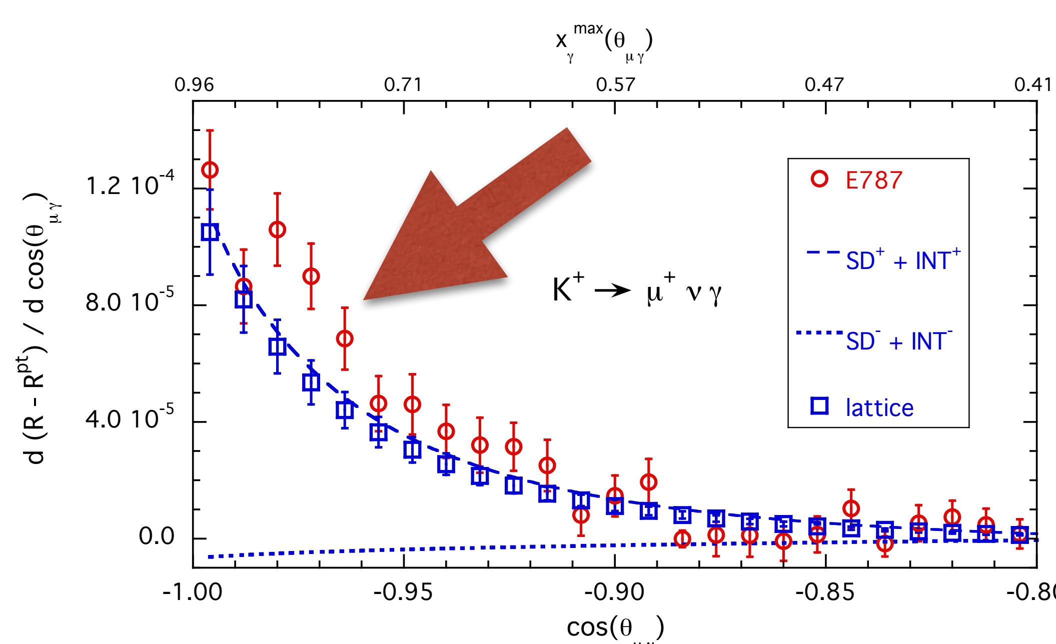
$\delta R_1^{SD} \propto (m_P / m_\ell)^2$ remove the suppression

Comparison with experimental data





region	E_γ	E_e	$\theta_{e\gamma}$	$\Delta R^{\mathrm{exp},i}$	$\Delta R^{\mathrm{pt},i}$	$(\Delta R^{\mathrm{exp},i} - \Delta R^{\mathrm{pt},i})$	$\Delta R^{\mathrm{SD},i}$	$(\Delta R^{\mathrm{th},i} - \Delta R^{\mathrm{pt},i})$	ChPT
A	> 50	> 50	$> 40°$	2.614 ± 0.021	0.385	2.229 ± 0.021	1.94 ± 0.40	1.93 ± 0.40	2.97 ± 0.82
B	> 50	> 10	$> 40°$	14.46 ± 0.22	11.66	2.80 ± 0.22	3.01 ± 0.54	2.93 ± 0.54	4.43 ± 0.92
C	> 10	> 50	$> 40°$	37.69 ± 0.46	35.08	2.61 ± 0.46	5.07 ± 1.03	5.07 ± 1.04	7.75 ± 2.07
O	> 10	$> m_e$	$> 40°$	73.86 ± 0.54	72.26	1.60 ± 0.54	6.87 ± 1.26	6.70 ± 1.26	10.13 ± 2.11



Tensions may be due to the presence of NP, such as flavour changing interactions beyond the V-A couplings and non-universal corrections to lepton couplings

Frezzotti *et al.*, arXiv:2012.02120

Lattice calculation

PHYSICAL REVIEW D **107,** 074507 (2023)

arXiv:2302.01298

Methods for high-precision determinations of radiative-leptonic decay form factors using lattice QCD

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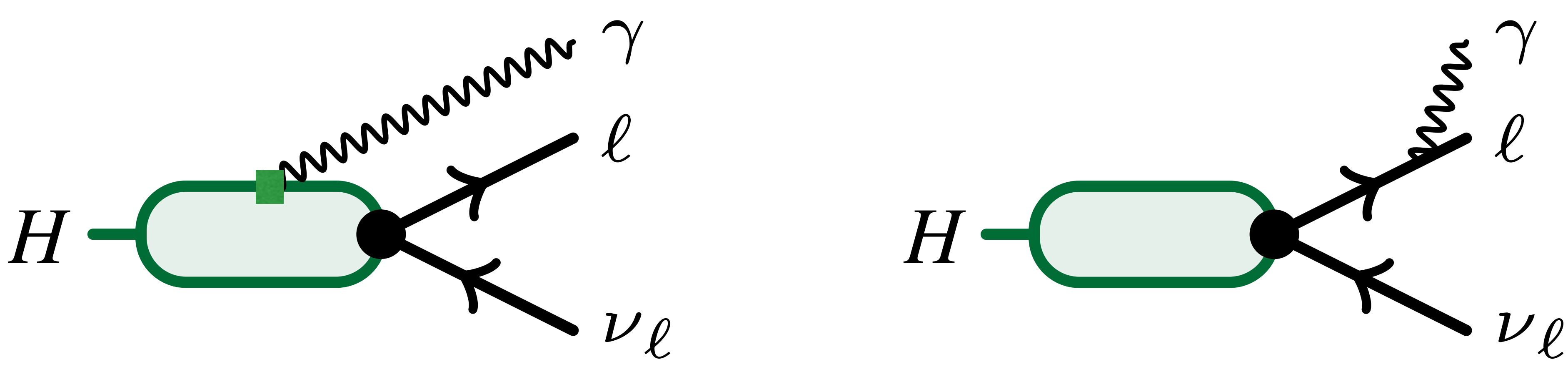
Hadronic tensor and form factors



$$T_{\mu\nu} = -\,i\int d^4x\; e^{ip_\gamma\cdot x}\,\langle 0\,|\,\mathbf{T}\left(J_\mu^{em}(x)\,J_\nu^{weak}(0)\right)|\,H(\vec{p}_H)\rangle \qquad (p_H = m_H v)$$

$$= \varepsilon_{\mu\nu\tau\rho}p_\gamma^\tau v^\rho F_V + i\left[-g_{\mu\nu}(p_\gamma\cdot v) + v_\mu (p_\gamma)_\nu\right] F_A - i\frac{v_\mu v_\nu}{p_\gamma\cdot v} m_H f_H + (p_\gamma)_\mu - \text{terms}$$

$$F_A = F_{A,SD} + (-Q_\ell f_H/E_\gamma^{(0)}), \qquad E_\gamma^{(0)} = p_\gamma\cdot v$$

Goal: Calculate $F_V, F_{A,SD}$ as a function of $E_\gamma^{(0)}$

Euclidean correlation function

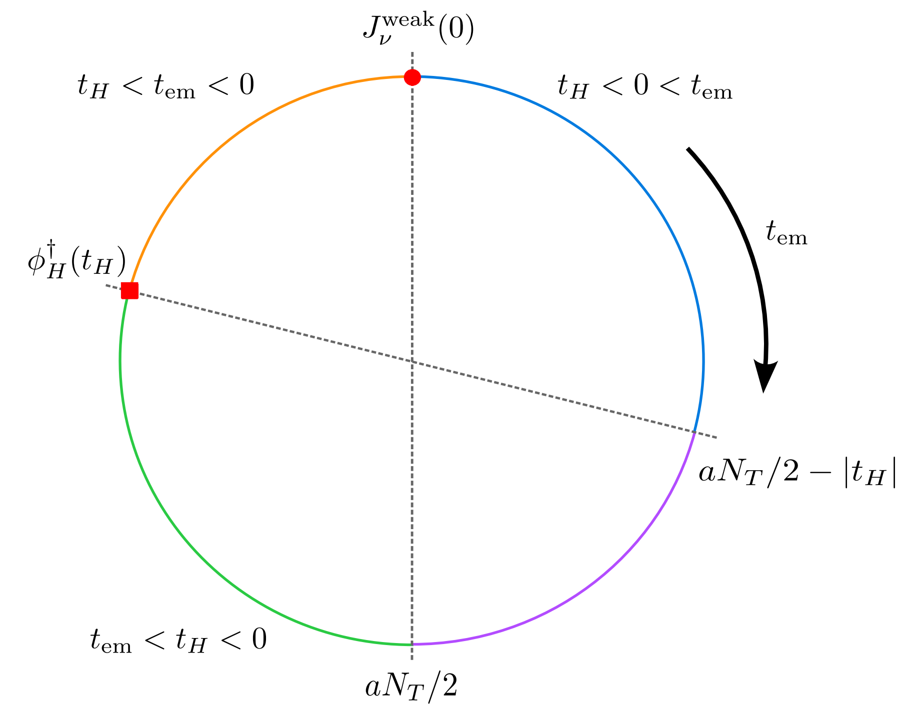
$$C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y \; e^{-i\vec{\mathbf{p}}_\gamma\cdot\vec{\mathbf{x}}} e^{i\vec{\mathbf{p}}_H\cdot\vec{\mathbf{y}}} \langle J_\mu^{\text{em}}(t_{em}, \vec{\mathbf{x}}) J_\nu^{\text{weak}}(0) \phi_H^\dagger(t_H, \vec{\mathbf{y}}) \rangle$$

$$\phi_H^\dagger \sim \bar{Q}\gamma_5 u$$

$$I_{\mu\nu}^{<}(T, t_H) = \int_{-T}^{0} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

$$I_{\mu\nu}^{>}(T, t_H) = \int_{0}^{T} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

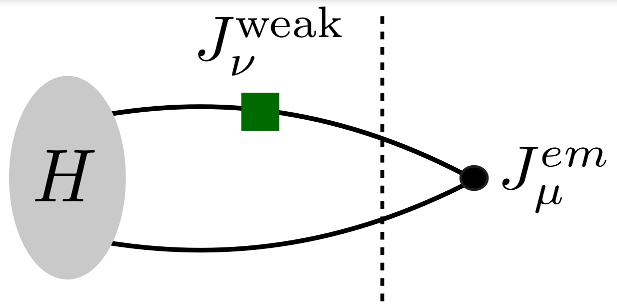
$$I_{\mu\nu}(T, t_H) = I^{<}(T, t_H) + I^{>}(T, t_H)$$



Show relation between $I_{\mu\nu}(T, t_H)$ and $T_{\mu\nu}$
$\rightarrow$ compare spectral decompositions of both time orderings of $I_{\mu\nu}$ and $T_{\mu\nu}$

Analytic continuation from Minkowski to Euclidean spacetime

Time ordering: $t_{em} > 0$



$$T^{>}_{\mu\nu} = -\sum_n \frac{\langle 0|\, J_\mu^{em}(0)\, |n(\vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_\gamma)|\, J_\nu^{weak}(0)\, |H(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})}$$

$$I^{>}_{\mu\nu}(t_H, T) = \int_0^T dt_{em}\; e^{E_\gamma t_{em}} C_{\mu\nu}(t_{em}, t_H)$$

$$= -\sum_m e^{E_m t_H} \frac{\langle m(\vec{\mathbf{p}}_H)|\, \phi_H^\dagger(0)\, |0\rangle}{2E_{m,\vec{\mathbf{p}}_H}}$$

$$\times \sum_n \frac{\langle 0|\, J_\mu^{em}(0)\, |n(\vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_\gamma)|\, J_\nu^{weak}(0)\, |m(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})} \left[1 - e^{(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})T}\right]$$

$t_H \to -\infty$ to achieve ground state saturation

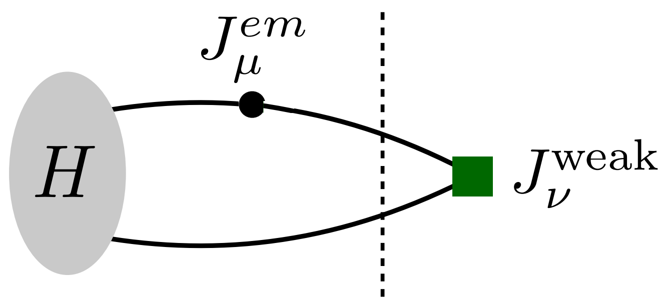
$T \to \infty$ to remove unwanted exponentials that come with intermediate states

The unwanted exponential $e^{(E_\gamma - E_{n,\vec{p}_\gamma})\,T}$ goes to zero for large T if $E_{n,\vec{p}_\gamma} > E_\gamma$.

Because the states $|n(\vec{p}_\gamma)\rangle$ have a nonzero mass, this is always satisfied.

Analytic continuation from Minkowski to Euclidean spacetime [2]

Time ordering: $t_{em} < 0$



$$T_{\mu\nu}^{<} = -\sum_n \frac{\langle 0|\, J_\nu^{\mathrm{weak}}(0)\,|n(\vec{p}_H - \vec{p}_\gamma)\rangle \langle n(\vec{p}_H - \vec{p}_\gamma)|\, J_\mu^{\mathrm{em}}(0)\,|H(\vec{p}_H)\rangle}{2E_{n,\vec{p}_H-\vec{p}_\gamma}(E_\gamma + E_{n,\vec{p}_H-\vec{p}_\gamma} - E_{H,\vec{p}_H})}$$

$$I_{\mu\nu}^{<}(t_H, T) = \int_{-T}^{0} dt_{\mathrm{em}}\; e^{E_\gamma t_{\mathrm{em}}} C_{3,\mu\nu}(t_{\mathrm{em}}, t_H)$$

$$= \sum_{l,n} \frac{\langle 0|\, J_\nu^{\mathrm{weak}}(0)\,|n(\vec{p}_H - \vec{p}_\gamma)\rangle \langle n(\vec{p}_H - \vec{p}_\gamma)|\, J_\mu^{\mathrm{em}}(0)\,|l(\vec{p}_H)\rangle \langle l(\vec{p}_H)|\, \phi_H^\dagger(0)\,|0\rangle}{2E_{n,\vec{p}_H-\vec{p}_\gamma} 2E_{l,\vec{p}_H}(E_\gamma + E_{n,\vec{p}_H-\vec{p}_\gamma} - E_{l,\vec{p}_H})}$$

$$\times\, e^{E_{l,\vec{p}_H} t_H}\left[1 - e^{-(E_\gamma - E_{l,\vec{p}_H} + E_{n,\vec{p}_H-\vec{p}_\gamma})T}\right]$$

Since the electromagnetic current operator cannot change the flavor quantum numbers of a state, the lowest-energy state appearing in the sum over n is the meson H.

The unwanted exponential vanishes if $|\vec{p}_\gamma| + \sqrt{m_H^2 + (\vec{p}_H - \vec{p}_\gamma)^2} > \sqrt{m_H^2 + \vec{p}_H^2}$, which is always true for $|\vec{p}_\gamma| > 0$

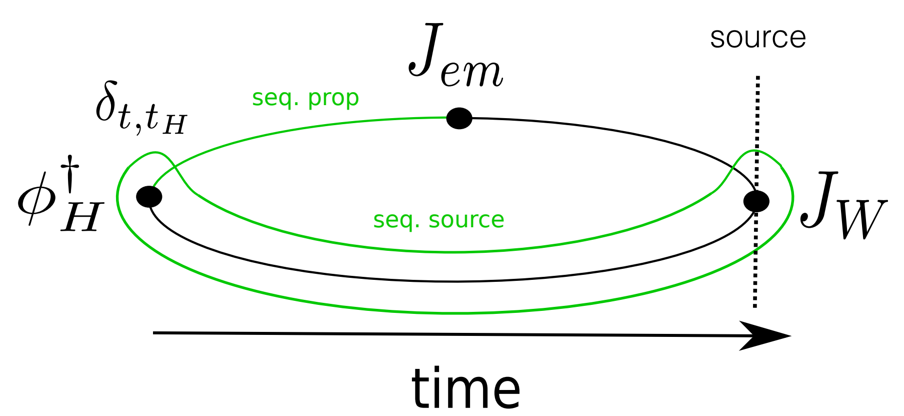
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_H\to-\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{\mathbf{p}}_H)| \phi_H^\dagger |0\rangle} \underbrace{\int_{-T}^{T} dt_{em}\ e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)}_{I_{\mu\nu}(T,t_H)}$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed t_H get all t_{em} for free

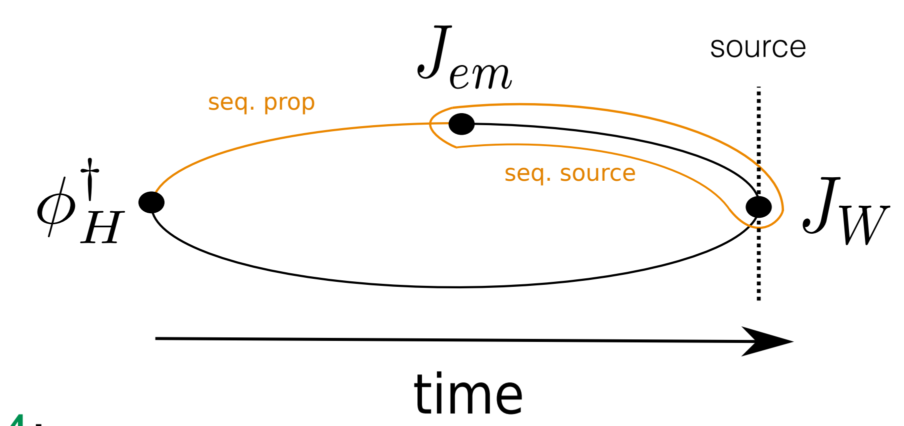
arXiv:1907.00279; arXiv:2110.13196 & arXiv:2302.01298



2: 4d sequential propagator through J_μ^{em} → calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed T get all t_H for free

RM123 & Soton Coll., arXiv:2006.05358 & arXiv:2306.05904:

Set $T = N_T/2$ and fit to constant in t_H where data has plateaued



Simulation details

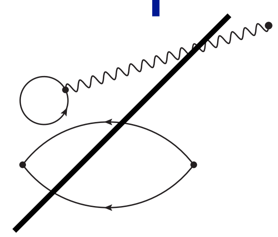
- $N_f = 2+1$ DWF, 3 RBC/UKQCD ensembles $M_\pi \simeq 139 \div 340$ MeV, $a \simeq 0.08 \div 0.11$ fm, charm valence quarks: Möbius DW with "stout" smear.

- 25 gauge configurations

Method	Source	Meson Momentum	Photon Momentum
3d	$\mathbb{Z}_2$-wall	$\vec{p}_{D_s} = (0,0,0)$	$\lvert\vec{p}_\gamma\rvert^2 \in (2\pi/L)^2\,\{1,2,3,4\}$
3d	point	$p_{D_s,z} \in 2\pi/L\,\{0,1,2\}$	all
4d	$\mathbb{Z}_2$-wall	$p_{D_s,z} \in 2\pi/L\,\{-1,0,1,2\}$	$p_{\gamma,z} = 2\pi/L$
$4d^{>,<}$	$\mathbb{Z}_2$-wall	$p_{D_s,z} \in 2\pi/L\,\{-1,0,1,2\}$	$p_{\gamma,z} = 2\pi/L$

- $\mathbb{Z}_2$ random wall sources & randomly placed point sources

- Disconnected diagrams are neglected



- Local electromagnetic current + mostly non-perturbative RCs

- Two datasets: $J^{weak}(0)$ or $J^{em}(0)$

- For point sources use translational invariance to fix em/weak operator at $\mathbf{0}$

use an "infinite-volume approximation" to generate data for arbitrary photon momenta (only exp. small FVEs are introduced)

$$C_{3,\mu\nu} = \int d^3x\, d^3y\, e^{-i\vec{p}_\gamma\cdot\vec{x}} \langle J_\mu^{em}(t_{em},\vec{x}) J_\nu^{weak}(0) \phi_H^\dagger(t_H,\vec{y})\rangle \qquad \vec{p}_H = 0, \text{ several } \vec{p}_\gamma$$

Fit form: 3d method

Include terms to fit
(1) unwanted exponential from first intermediate state
(2) first excited state

Fit form factors F_V and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

Time ordering $t_{em} < 0$:

$$F^<(t_H, T) = F^< + B_F^<(1 + B_{F,\mathrm{exc}}^< \overbrace{e^{\Delta E(T+t_H)}}) \overbrace{e^{-(E_\gamma - E_H + E^<)T}} + C_F^< \overbrace{e^{\Delta E t_H}}$$

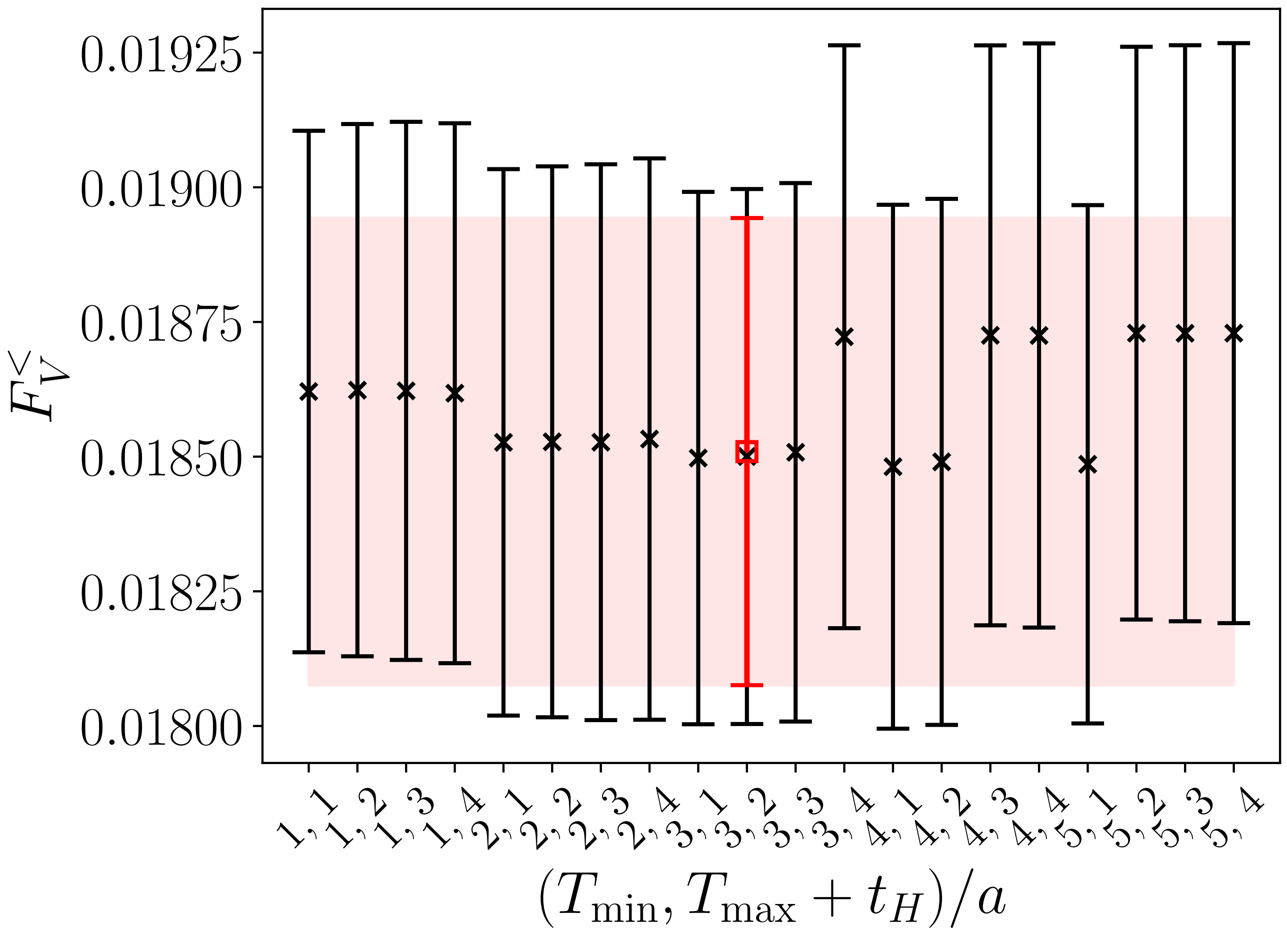
■ fit parameters

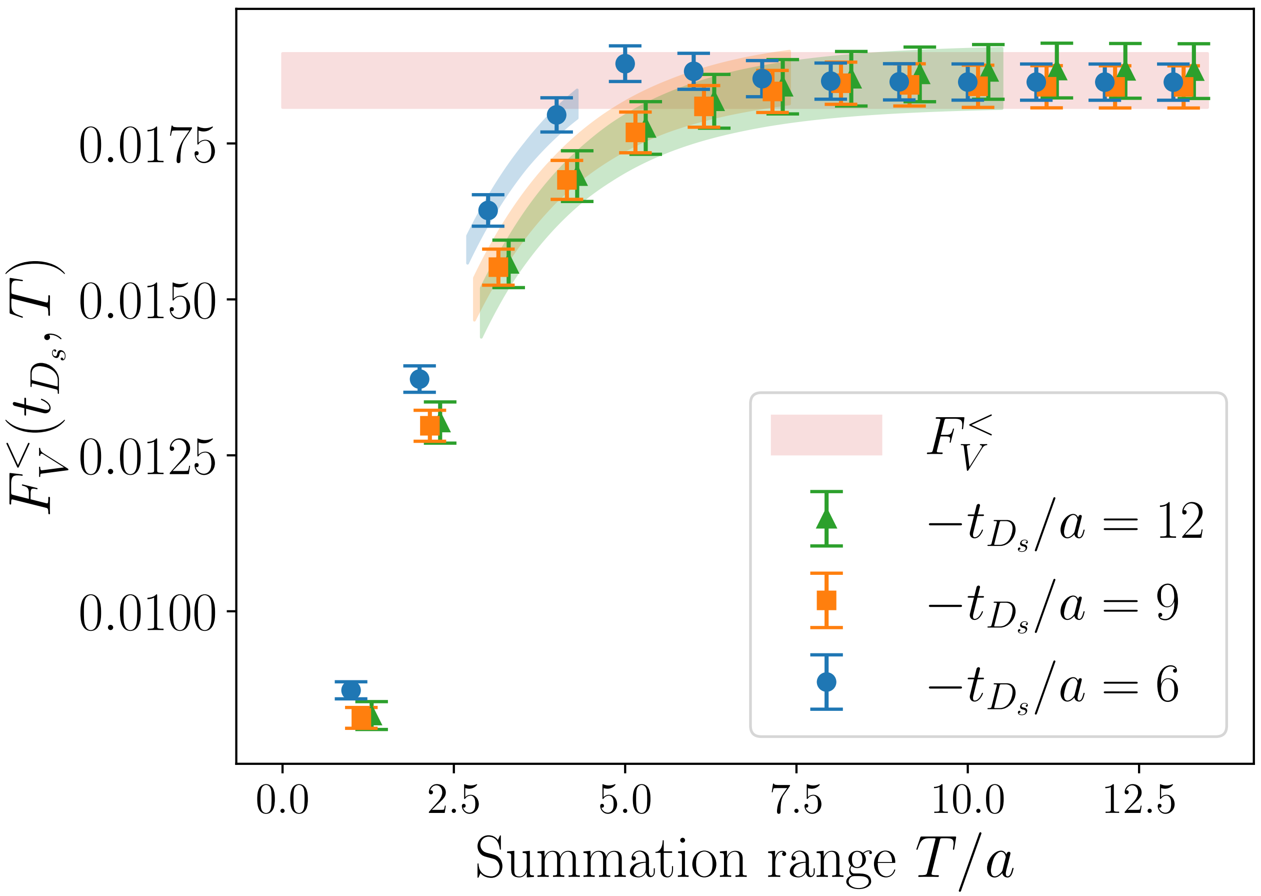
Only have three values of t_H, fitting multiple exponentials not possible
$\rightarrow$ Determine ΔE from the pseudoscalar two-point correlation function
$\rightarrow$ use result as Gaussian prior in form factor fits

$D_s \to \ell\nu_\ell\gamma$: 3d method

Time ordering $t_{em} < 0$:

$$F_V^<(t_H, T) = F_V^< + B_{F_V}^<\left(1 + B_{F_V,\text{exc}}^< \, e^{\Delta E(T+t_H)}\right) e^{-(E_\gamma - E_H + E^<)T} + C_{F_V}^< \, e^{\Delta E t_H}$$





Fit form: 4d method

Use fit ranges where data has plateaued in t_H, i.e. $t_H \to -\infty$

Include terms to fit
(1) unwanted exponential from first intermediate state

Sum of both time orderings $I_{\mu\nu}(T, t_H) = I^{<}_{\mu\nu}(T, t_H) + I^{>}_{\mu\nu}(T, t_H)$

$$F(t_H, T) = F + B_F^{<} \underbrace{e^{-(E_\gamma - E_H + E^{<})T}}_{t_{em}<0} + B_F^{>} \underbrace{e^{(E_\gamma - E^{>})T}}_{t_{em}>0}$$

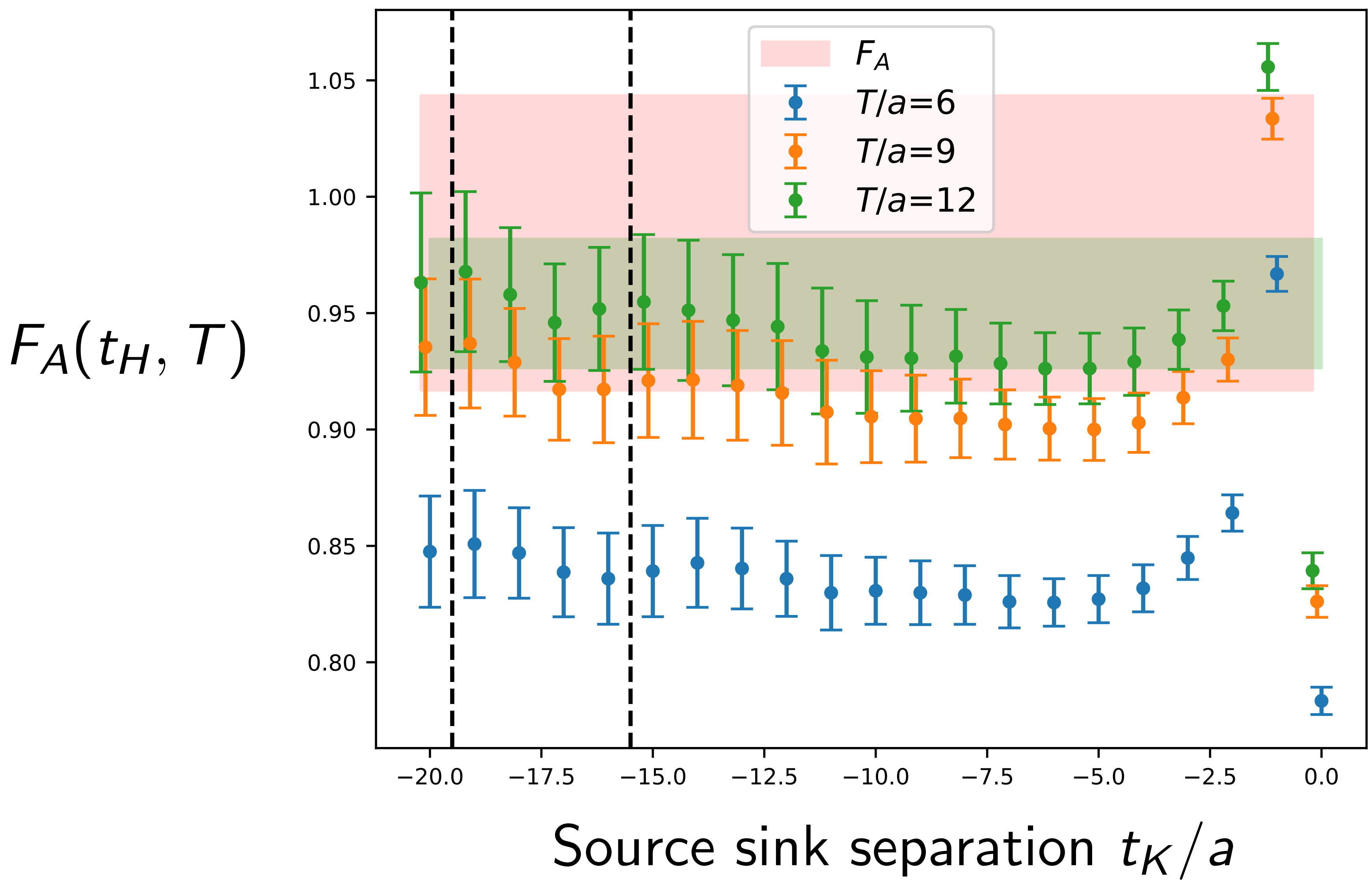
■ fit parameters

Only have three values of T, fitting multiple exponentials not possible
$\to$ Use broad Gaussian prior on $E^{>}$

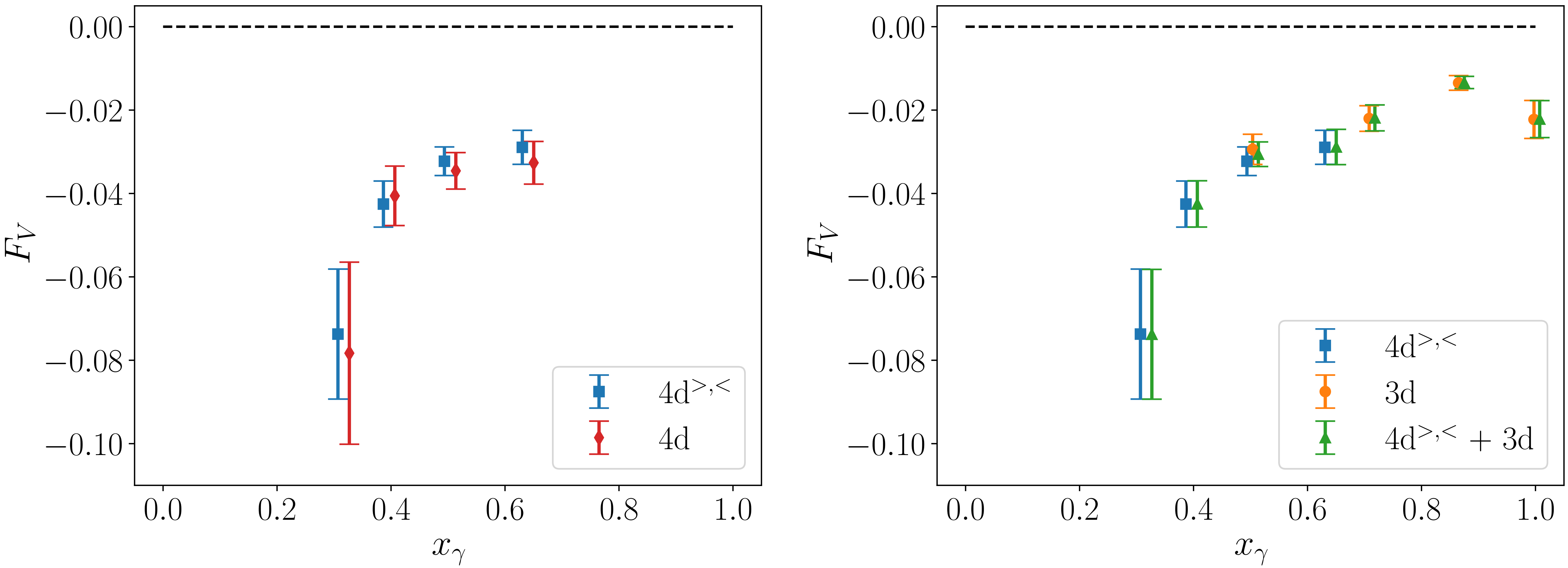
$K \to \ell\nu_\ell\gamma$: 4d method

Sum of both time orderings $t_{em} < 0 + t_{em} > 0$:

$$F_A(t_H, T) = F_A + B^{<}_{F_A} e^{-(E_\gamma - E_K + E^{<}_{A})T} + B^{>}_{F_A} e^{(E_\gamma - E^{>}_{A})T}$$



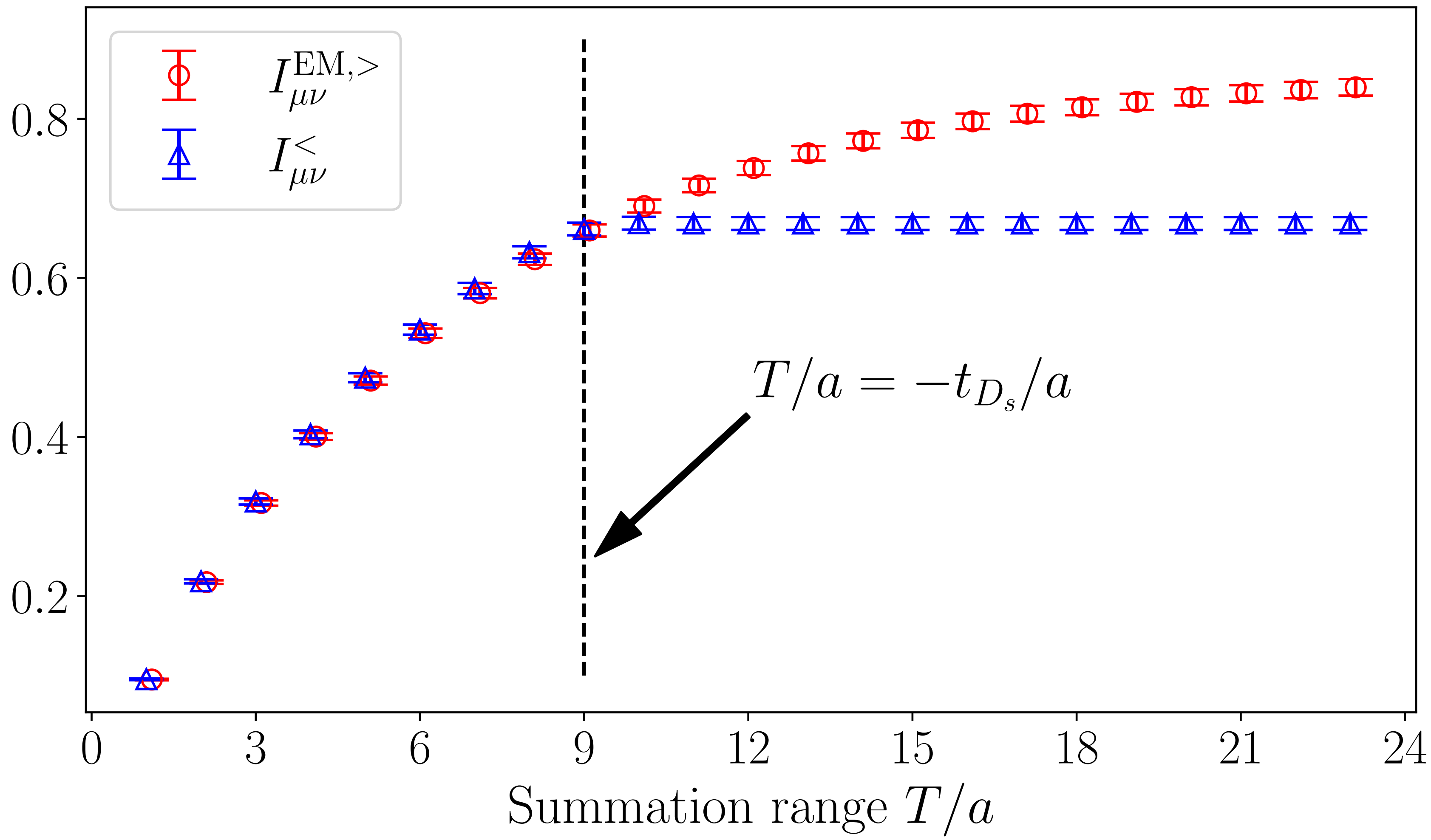
$D_s \to \ell\nu_\ell\gamma$: 3d vs 4d analysis results



- 4d method cannot resolve the sum of the unwanted exponentials of the separate time orderings
- 3d method offers good control over the unwanted exponentials for a significantly cheaper computational cost

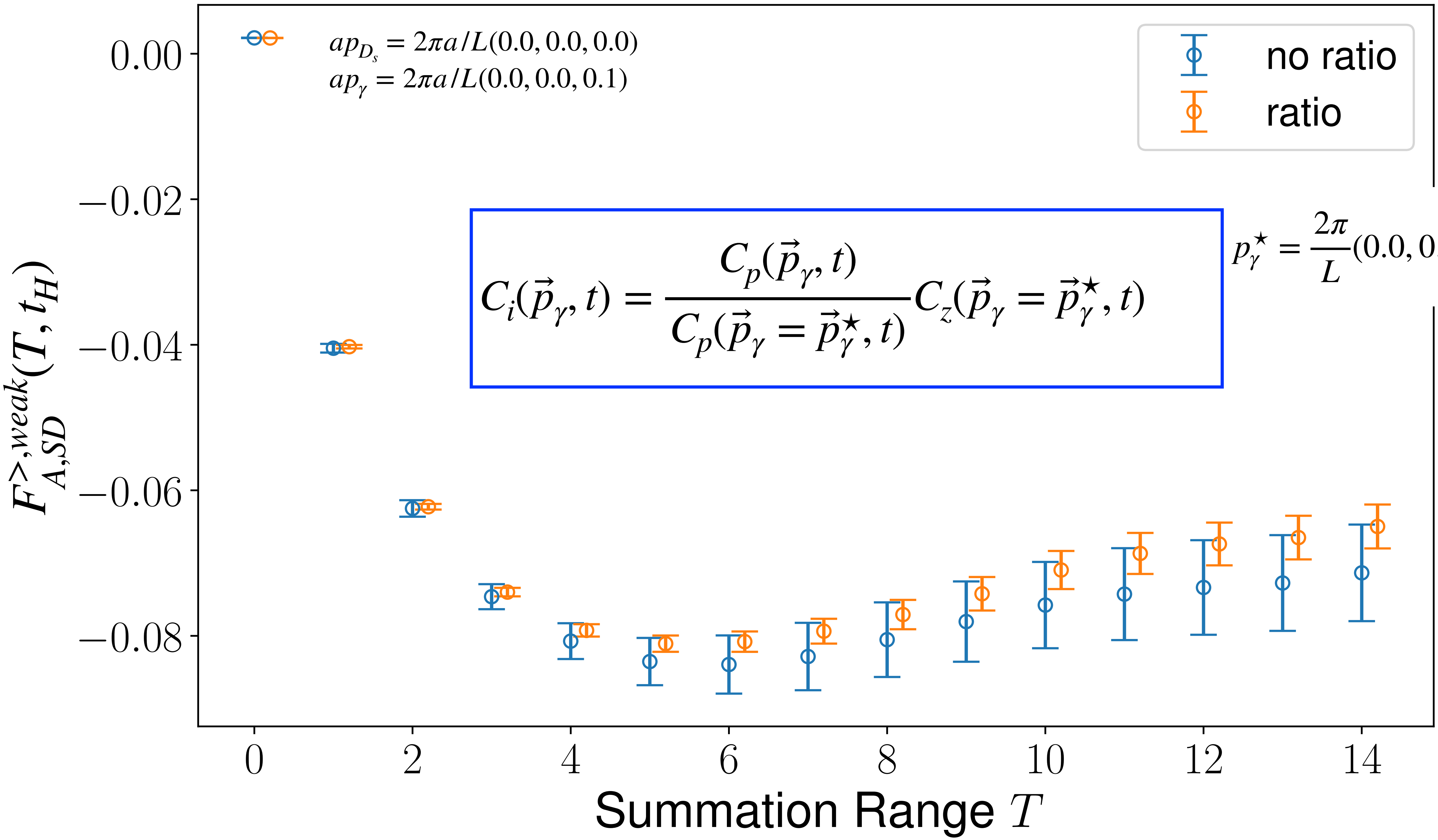
3pt function with e.m. current at origin

$$C^{\mathrm{EM}}_{3,\mu\nu}(t_W, t_H) = e^{E_H t_W} \int d^3x \int d^3y\, e^{i(\vec{p}_\gamma - \vec{p}_H)\cdot\vec{x}} e^{i\vec{p}_H\cdot\vec{y}} \langle J^{\mathrm{em}}_\mu(0) J^{\mathrm{weak}}_\nu(t_W, \vec{x}) \phi^\dagger_H(t_H, \vec{y})\rangle$$



The spectral decomposition of the $t_W > 0$ time ordering of $I^{\mathrm{EM}}_{\mu\nu}$ and the $t_{em} < 0$ time ordering of $I_{\mu\nu}$ are equal up to excited state effects

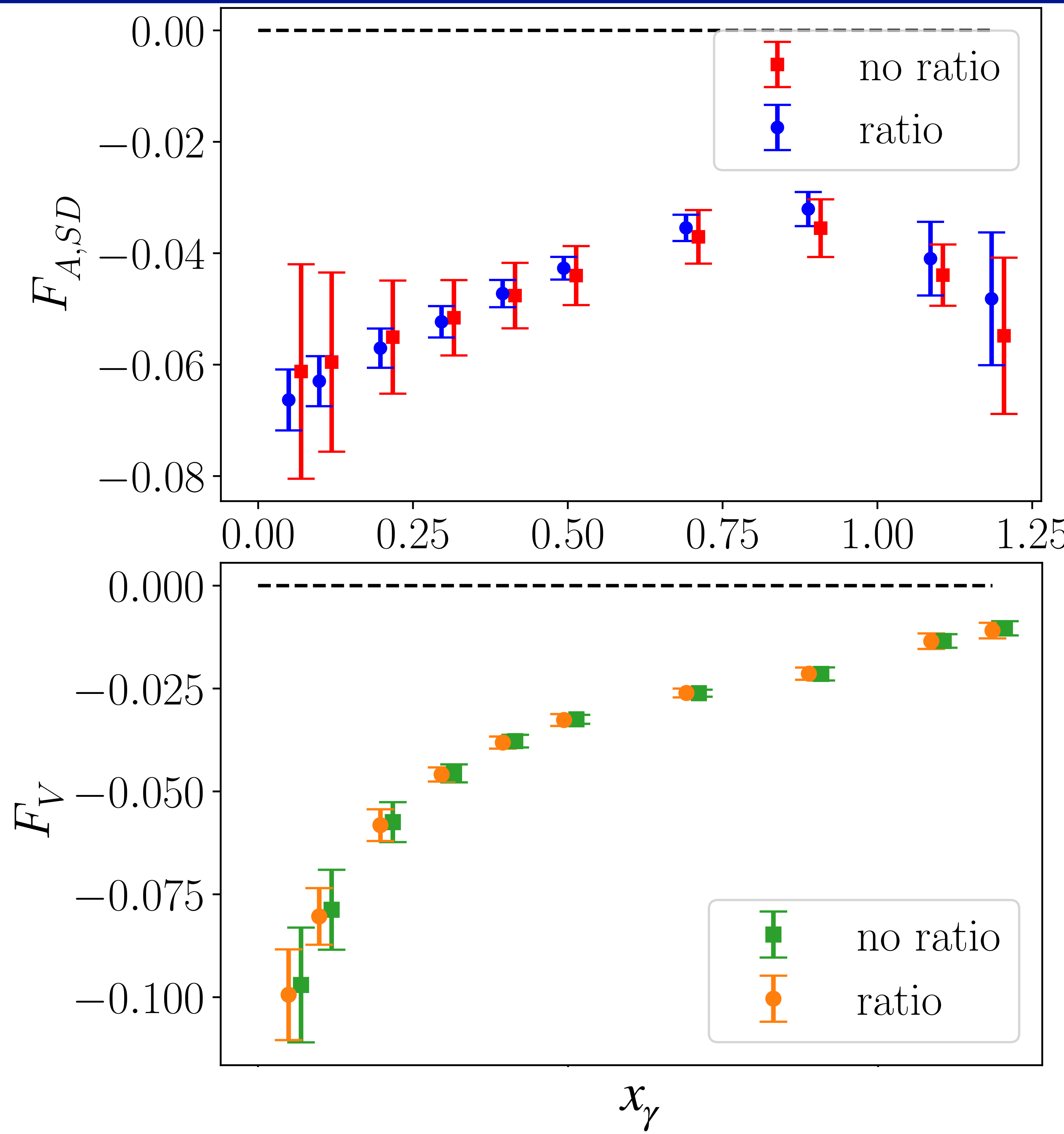
Improved form factors estimators



$$C_i(\vec{p}_\gamma, t) = \frac{C_p(\vec{p}_\gamma, t)}{C_p(\vec{p}_\gamma = \vec{p}^\star_\gamma, t)} C_z(\vec{p}_\gamma = \vec{p}^\star_\gamma, t)$$

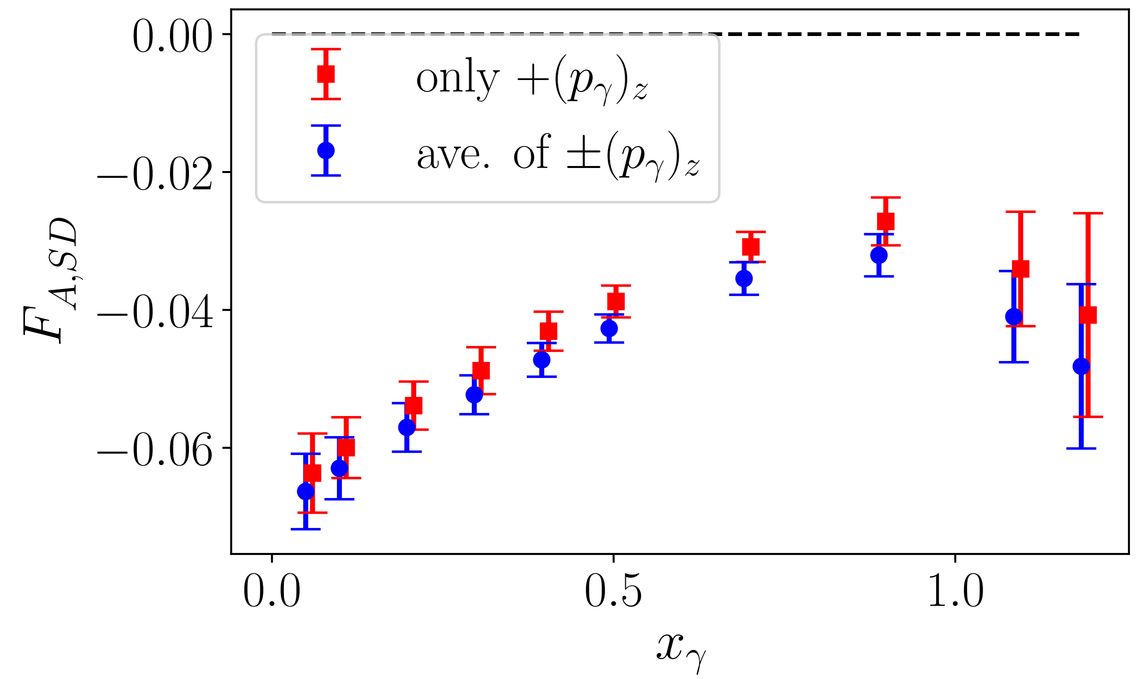
$$p^\star_\gamma = \frac{2\pi}{L}(0.0, 0.0, 0.0)$$

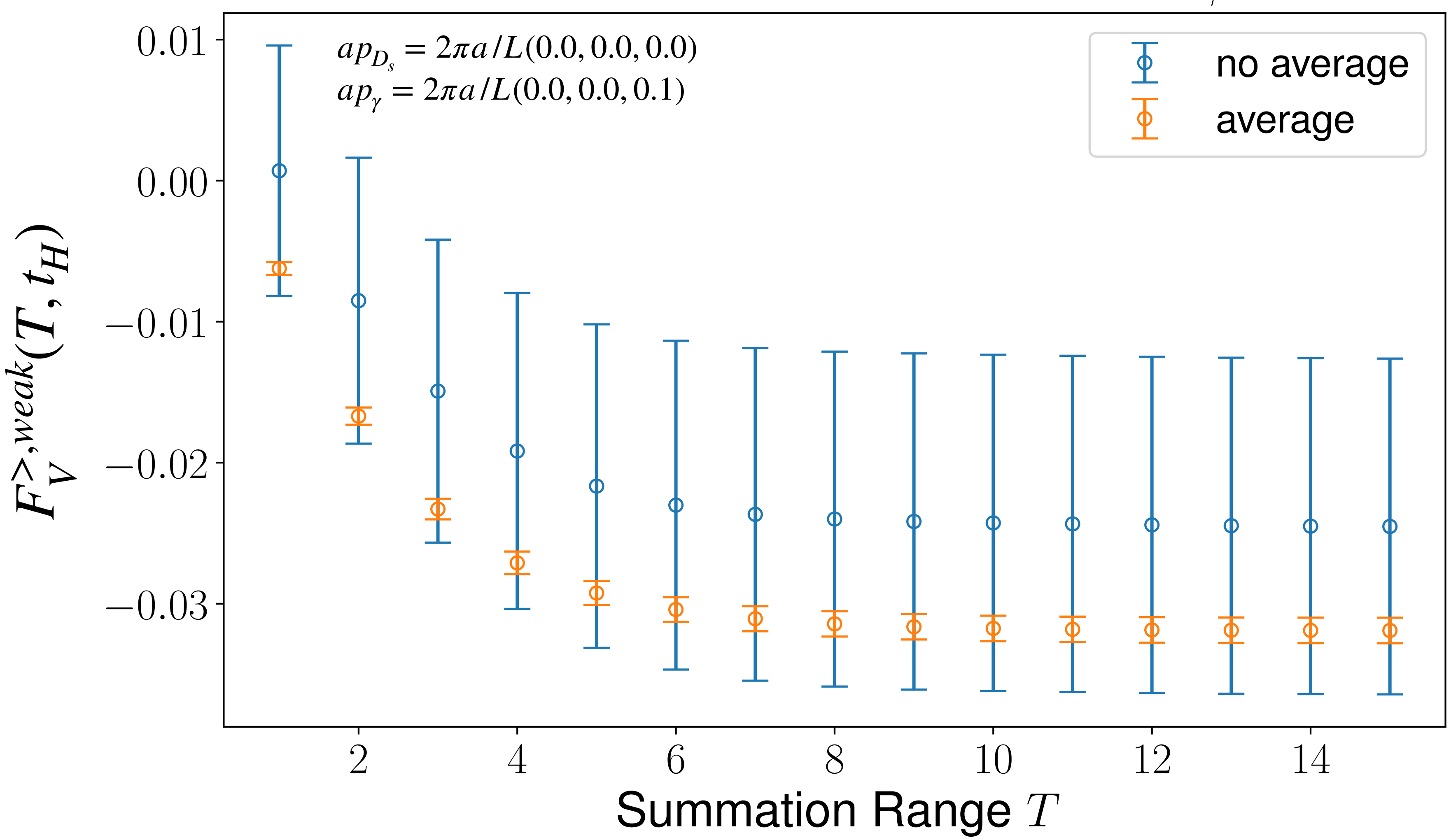
Improved form factors estimators



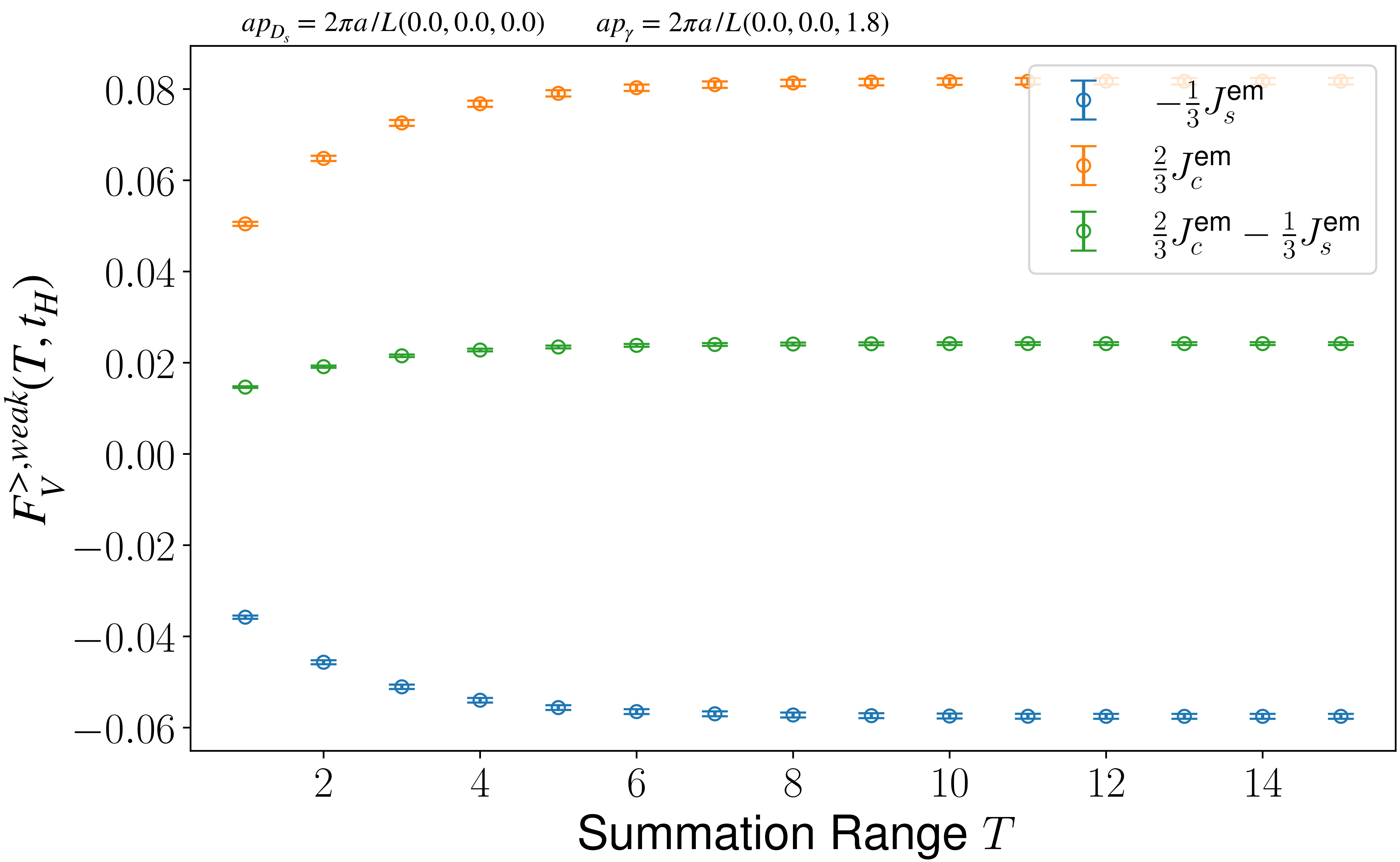
Improved form factors estimators [2]

$\pm\vec{p}_\gamma$ average





Cancellation between quark components



Fit form: 3d method

Include terms to fit

(1) unwanted exponential from first intermediate state

(2) first excited state

Fit form factors F_V and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

$\boxed{t_H < t_{em} < 0 \quad t_H < 0 < t_W}$

$$F^{weak}_{<}(t_H, T) = F^{<} + B^{<}_{F}\left(1+B^{<}_{F,exc}\overbrace{e^{\Delta E(T+t_H)}}\right)\overbrace{e^{-(E_\gamma-E_H+E^{<})T}}+C^{<}_{F}\overbrace{e^{\Delta E t_H}}$$

$$F^{em}_{>}(t_H, T) = F^{<} + B^{<}_{F}\left[1+B^{<}_{F,exc}\frac{E_\gamma+E^{<}-(\Delta E+E_H)}{E_\gamma+E^{<}-E_H}e^{\Delta E t_H}\right]e^{-(E_\gamma-E_H+E^{<})T}+\tilde{C}^{<}_{F}e^{\Delta E t_H}$$

$\boxed{t_H < 0 < t_{em} \quad t_H < t_W < 0}$

$$F^{weak}_{>}(t_H, T) = F^{>} + B^{>}_{F}\left(1+B^{>}_{F,exc}e^{\Delta E t_H}\right)e^{(E_\gamma-E^{>})T}+C^{>}_{F}e^{\Delta E t_H}$$

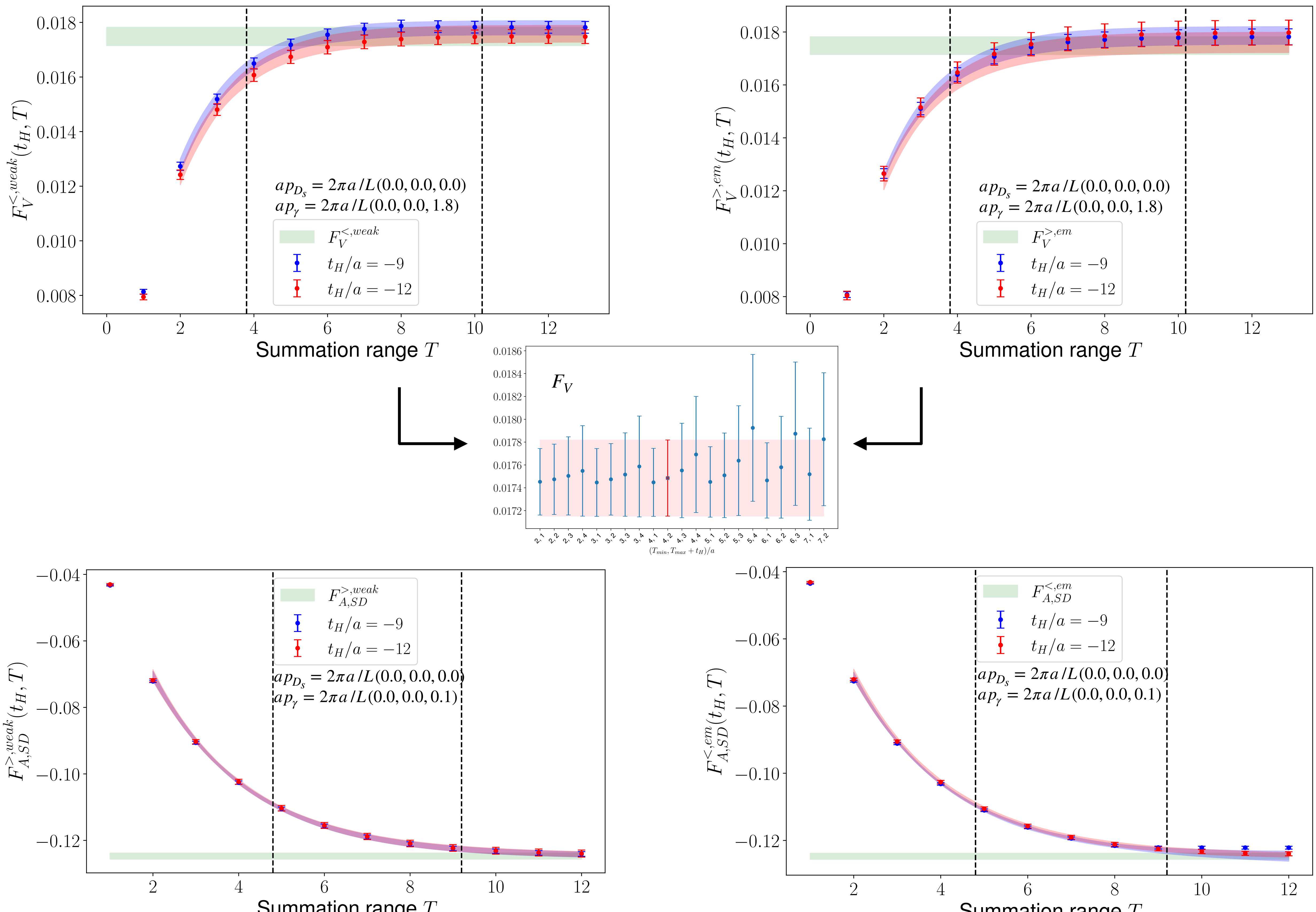
$$F^{em}_{<}(t_H, T) = F^{>} + B^{>}_{F}\left[1+B^{>}_{F,exc}\frac{E_\gamma-E^{>}}{E_\gamma-E^{>}+\Delta E}e^{\Delta E(T+t_H)}\right]e^{(E_\gamma-E^{>})T}+\tilde{C}^{>}_{F}e^{\Delta E t_H}$$

Only have two values of t_H, fitting multiple exponentials not possible

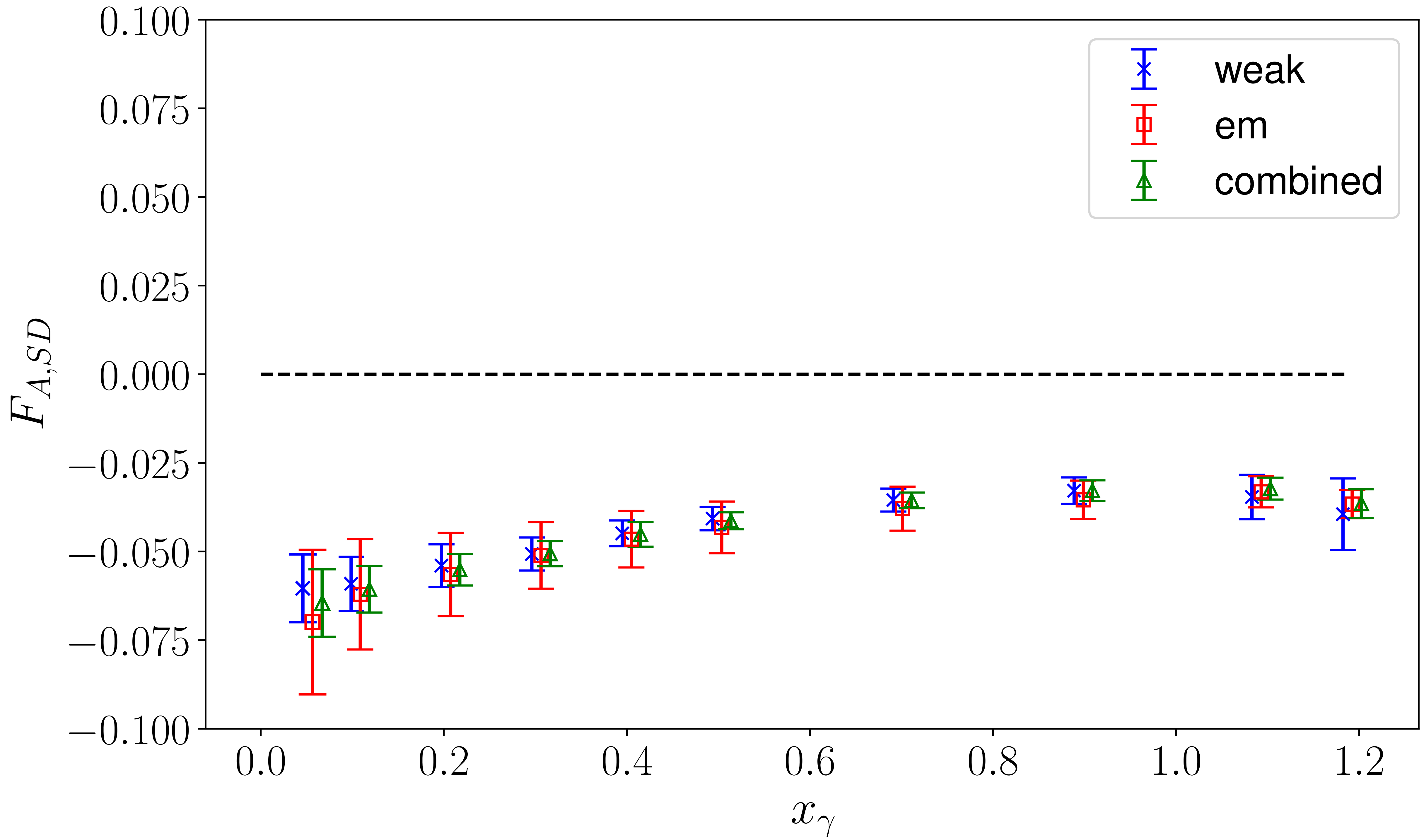
$\rightarrow$ Determine ΔE from the pseudoscalar two-point correlation function

$\rightarrow$ use result as Gaussian prior in form factor fits

$D_s \to \ell \nu_\ell \gamma$: 3d method



$D_s \to \ell\nu_\ell\gamma$: 3d method

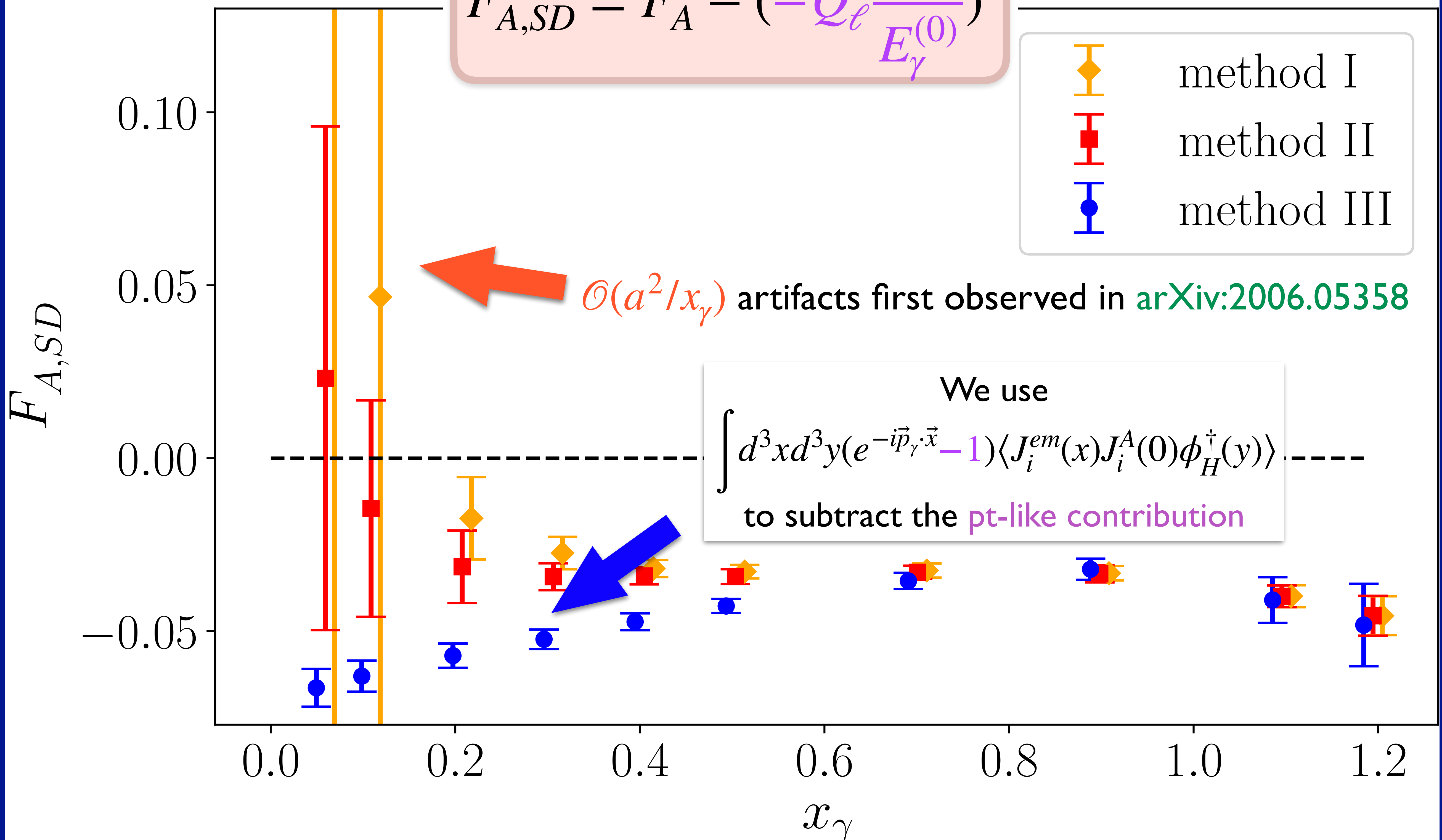


$$x_\gamma = \frac{2p_H \cdot p_\gamma}{m_H^2} \xrightarrow{\vec{p}_H = \mathbf{0}} x_\gamma = \frac{2E_\gamma^{(0)}}{m_H}$$

$$0 \le x_\gamma \le 1 - \frac{m_\ell^2}{m_H^2}$$

NP subtraction of IR-divergent discretization effects

$$F_{A,SD} = F_A - \left(-Q_\ell \frac{f_H}{E_\gamma^{(0)}}\right)$$



$\mathcal{O}(a^2/x_\gamma)$ artifacts first observed in arXiv:2006.05358

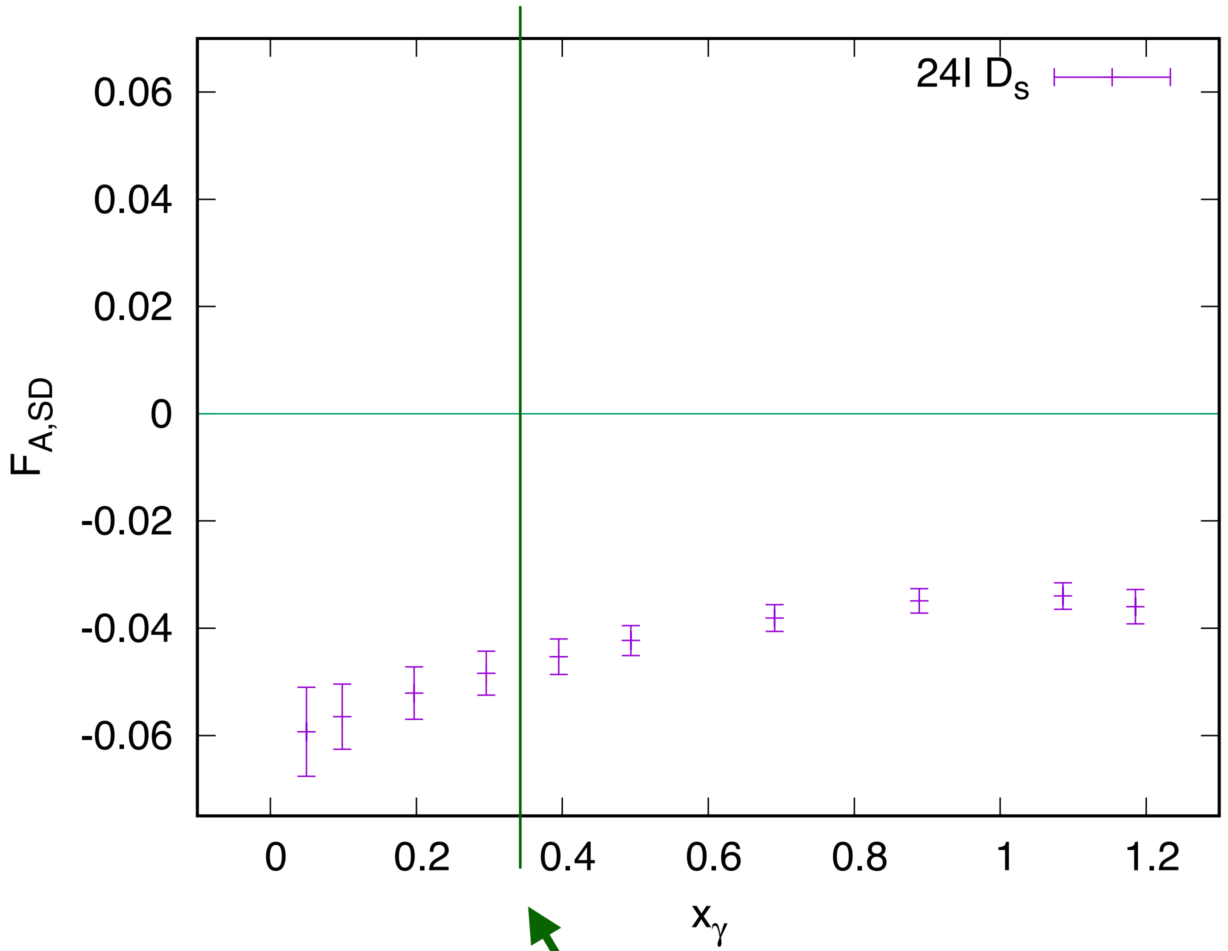
We use

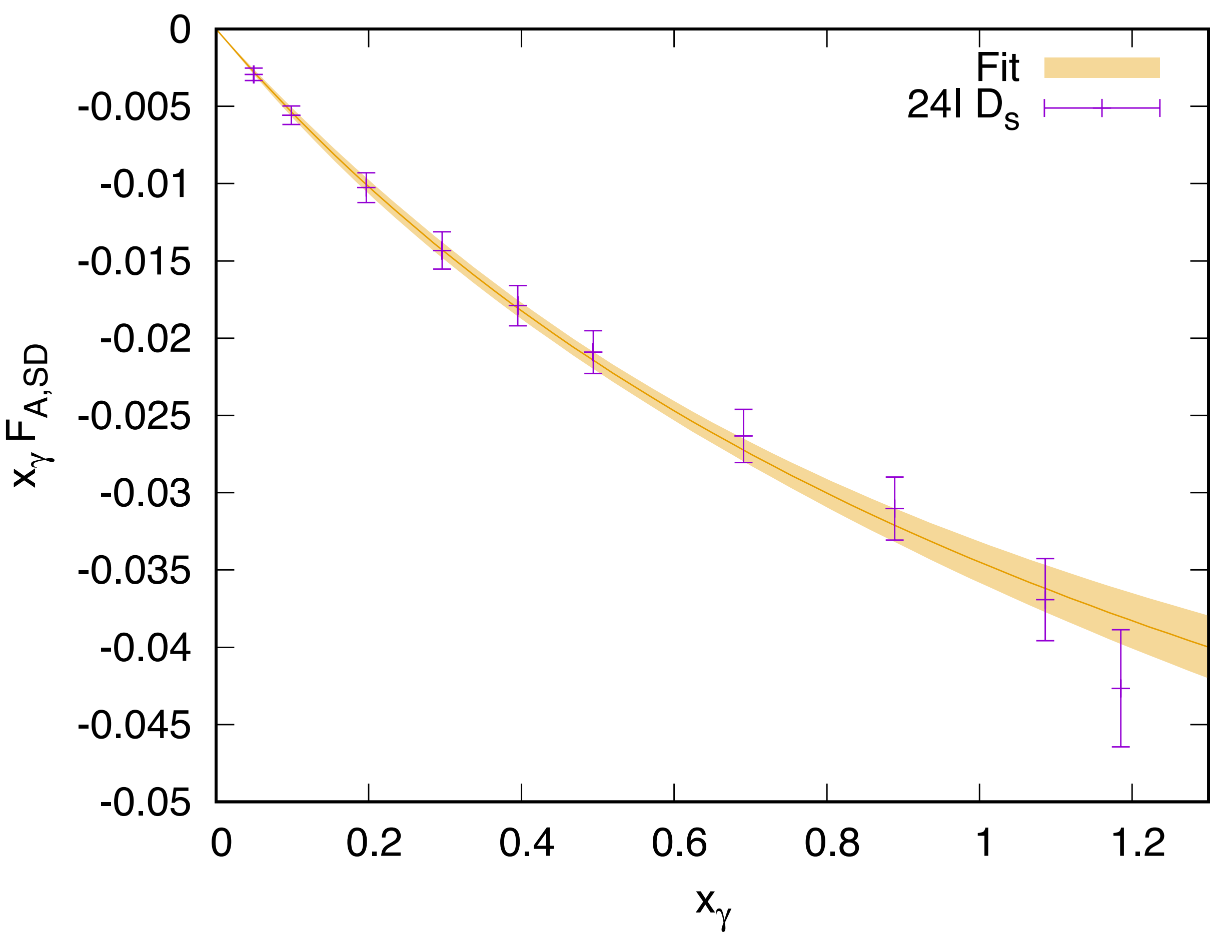
$$\int d^3x d^3y (e^{-i\vec{p}_\gamma \cdot \vec{x}} - 1) \langle J_i^{em}(x) J_i^A(0) \phi_H^\dagger(y) \rangle$$

to subtract the pt-like contribution

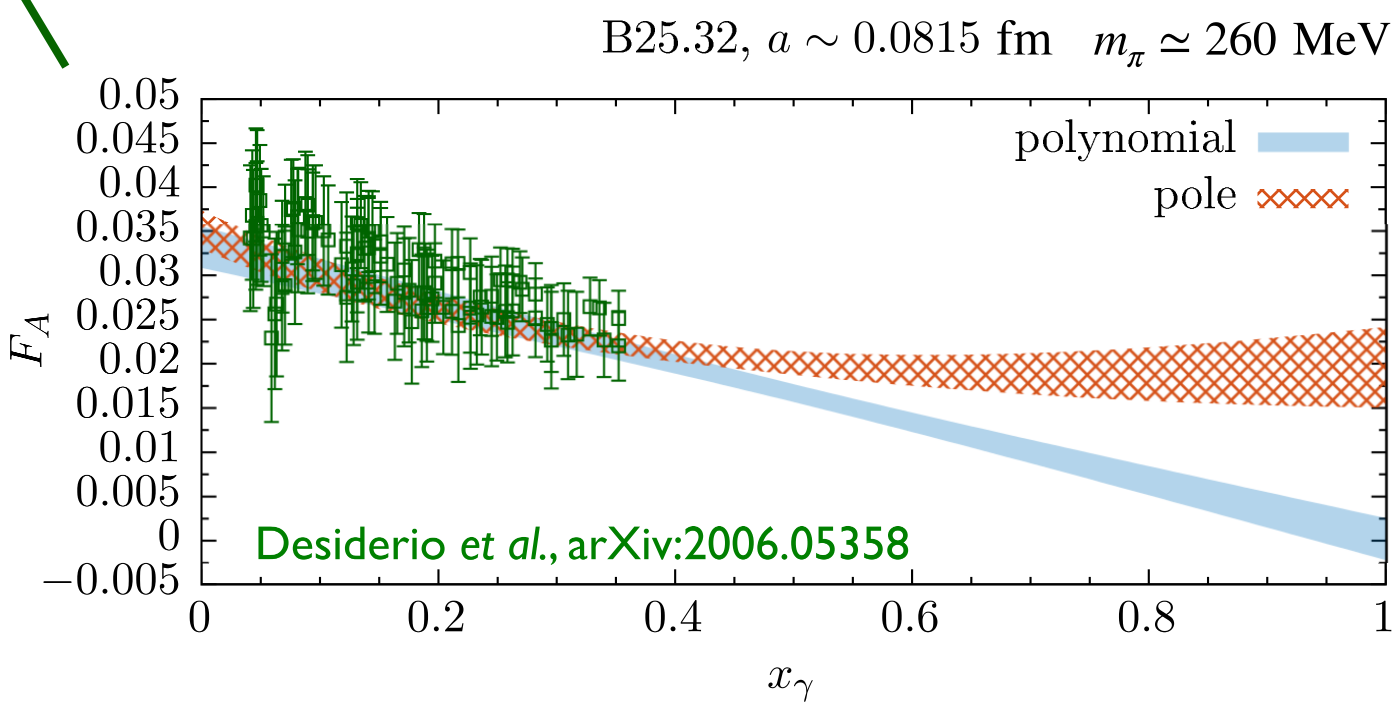
Blue data: improved subtraction of pt-like contribution

$D_s \to \ell\nu_\ell\gamma$: results (3d method)

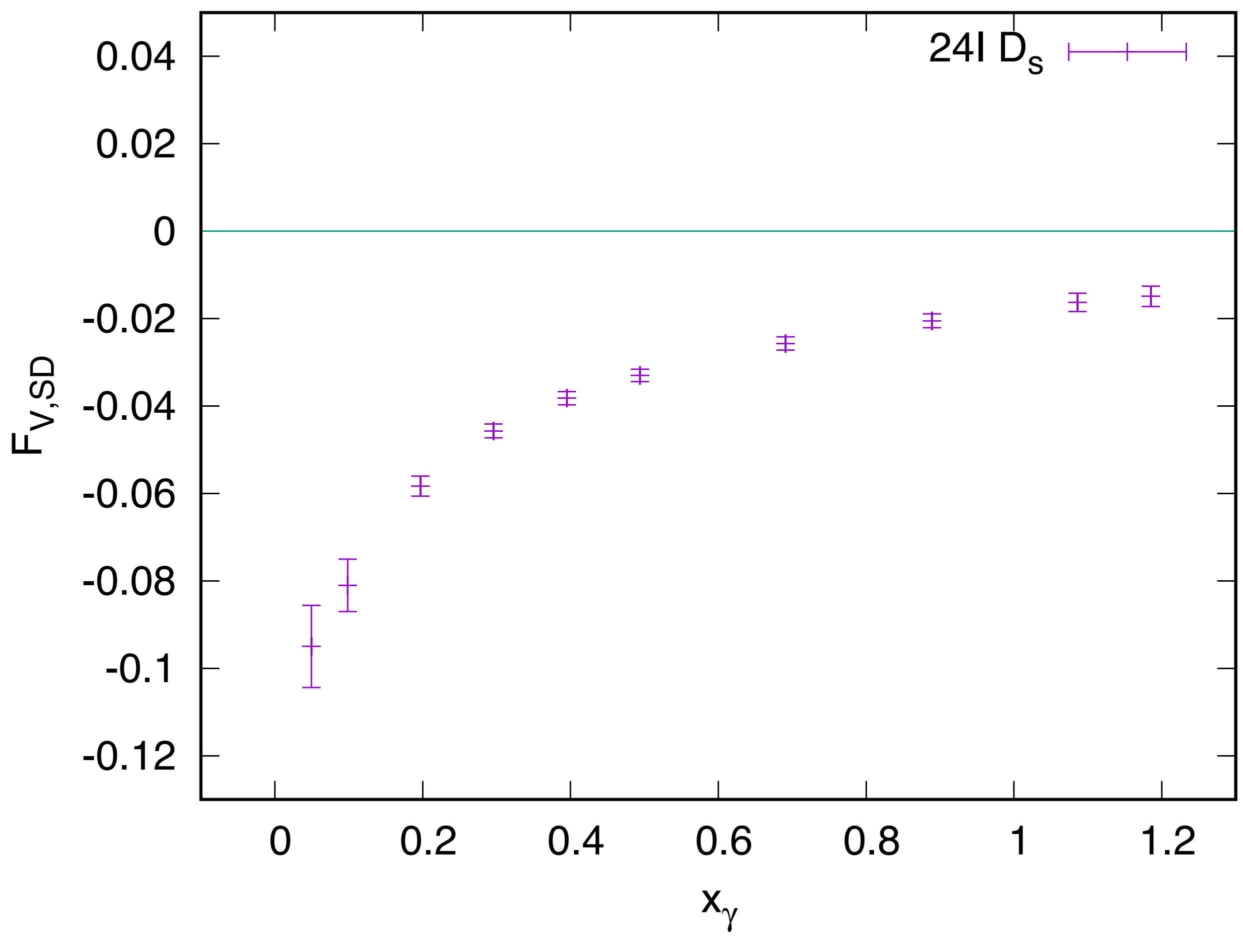


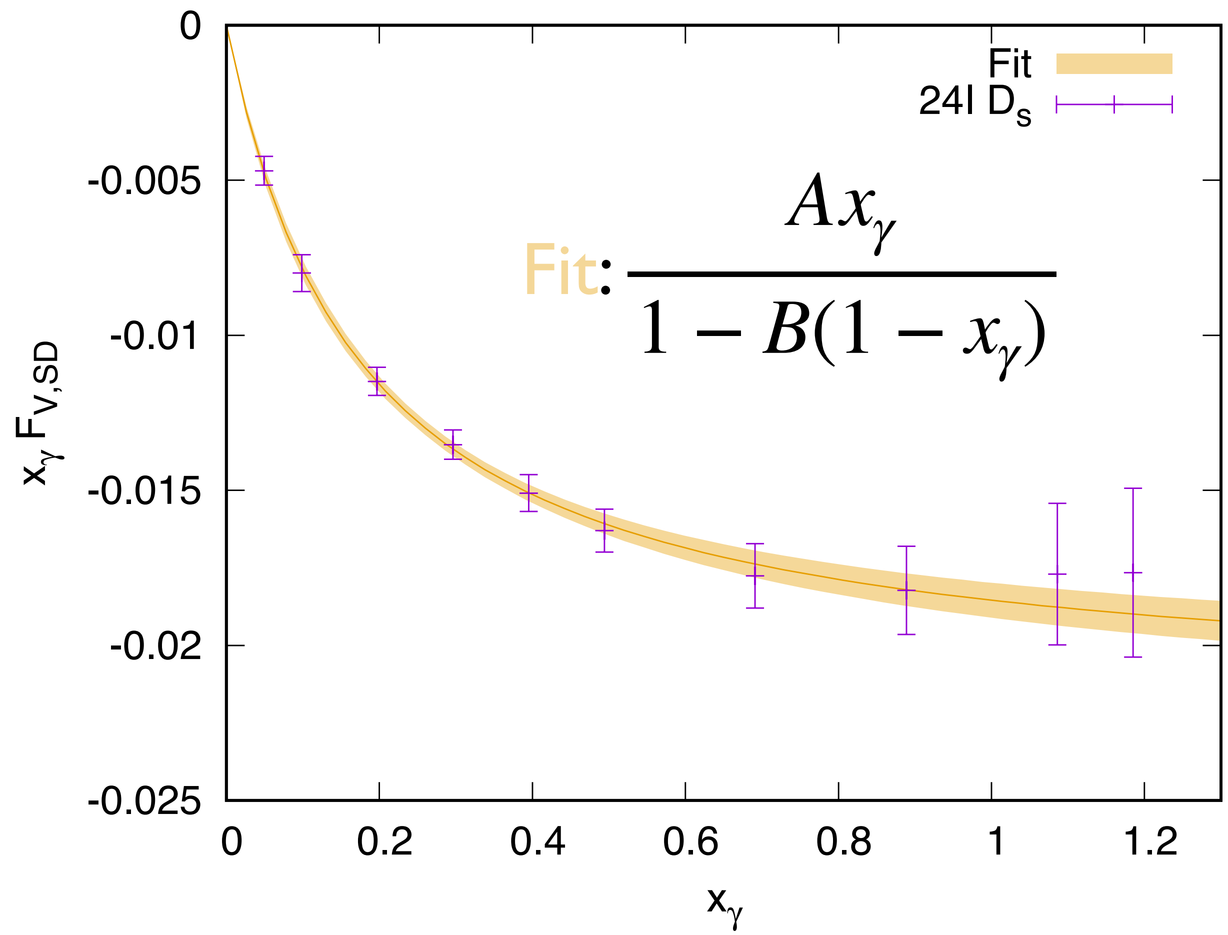


sign: different FFs parameterization



$D_s \to \ell\nu_\ell\gamma$: results (3d method) [2]



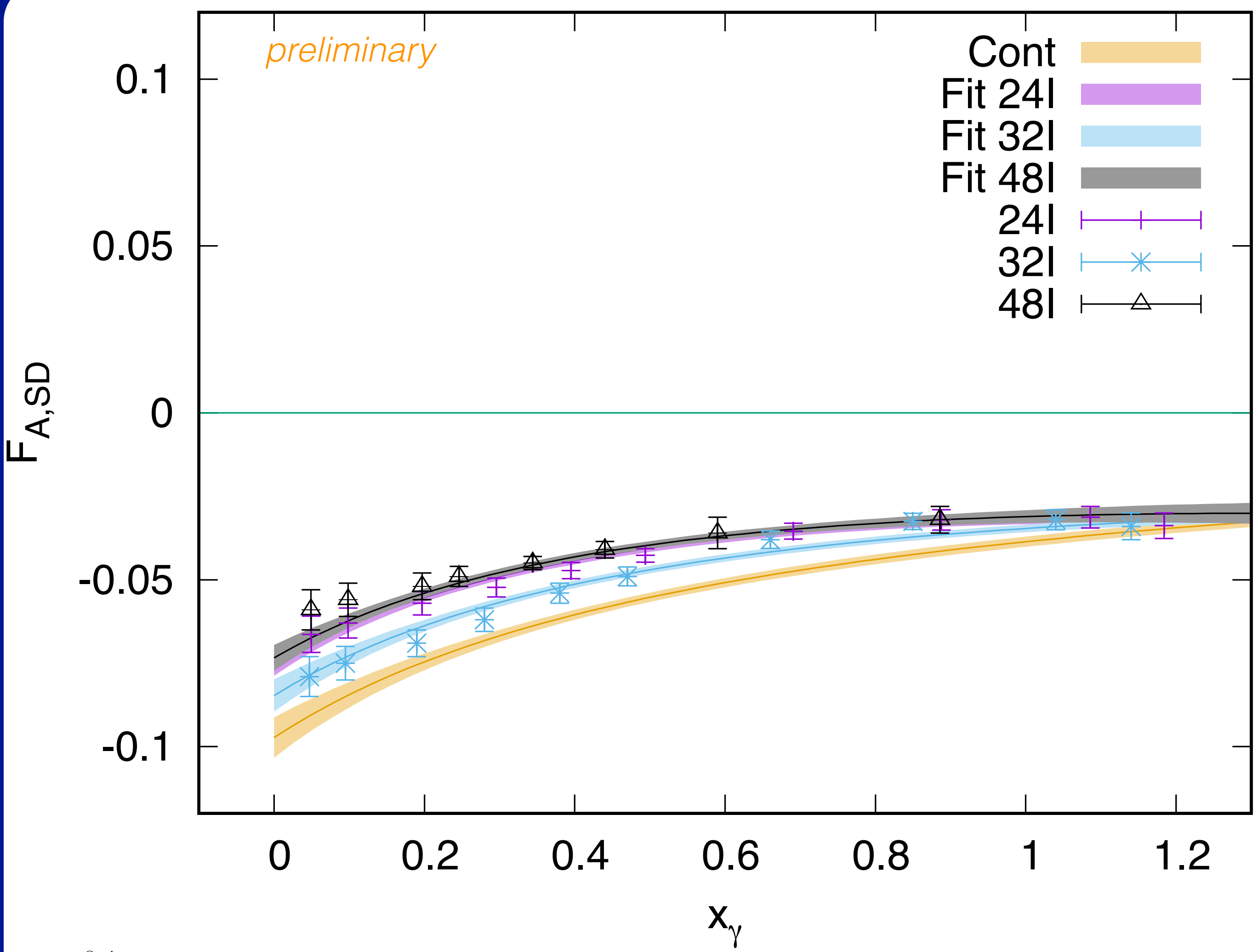


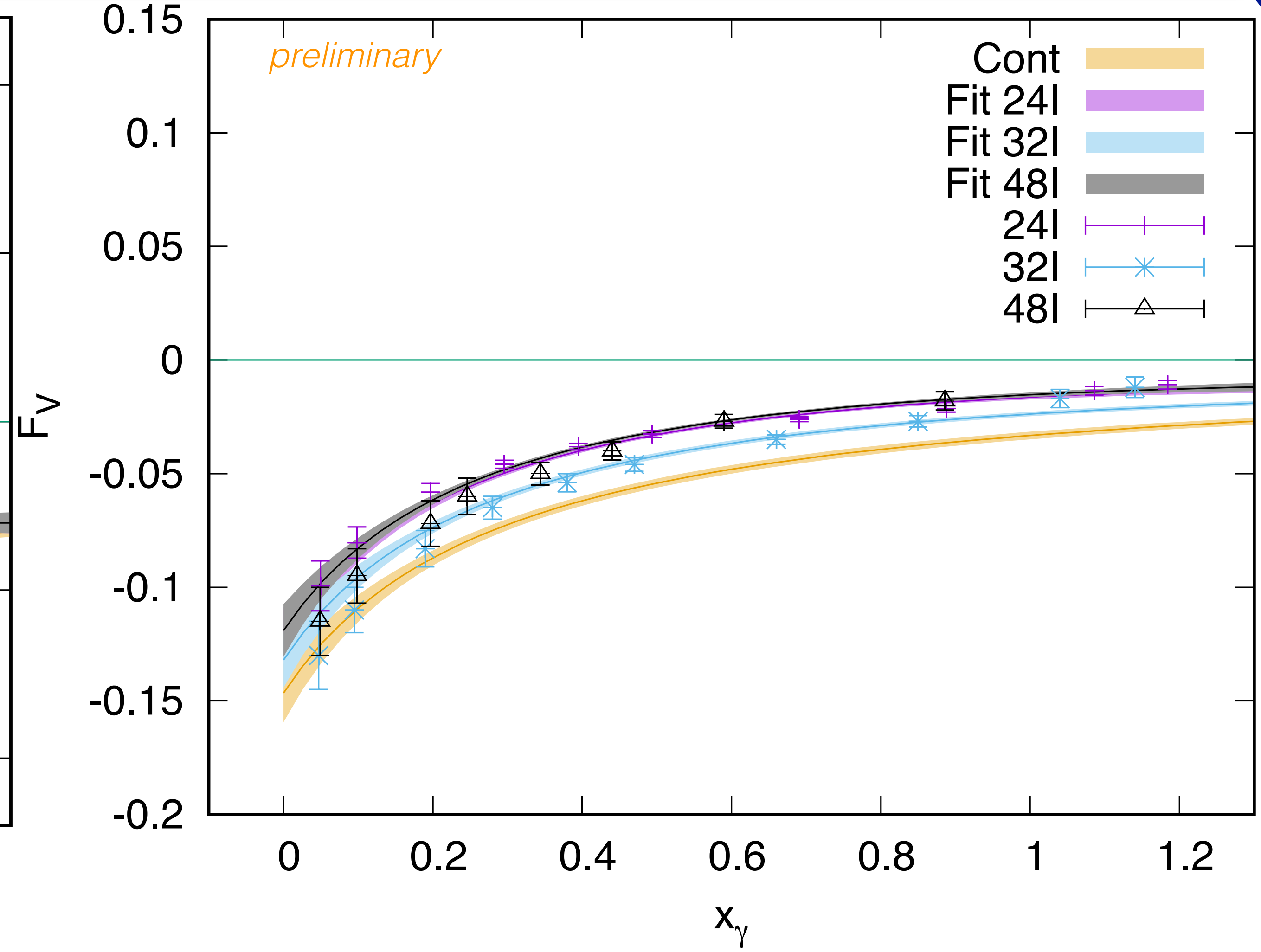
- $D_s^+ \to e^+\nu\gamma$: $\mathcal{B}(E_\gamma > 10\text{ MeV}) < 1.3 \times 10^{-4}$ SM: $\mathcal{O}(10^{-4})$

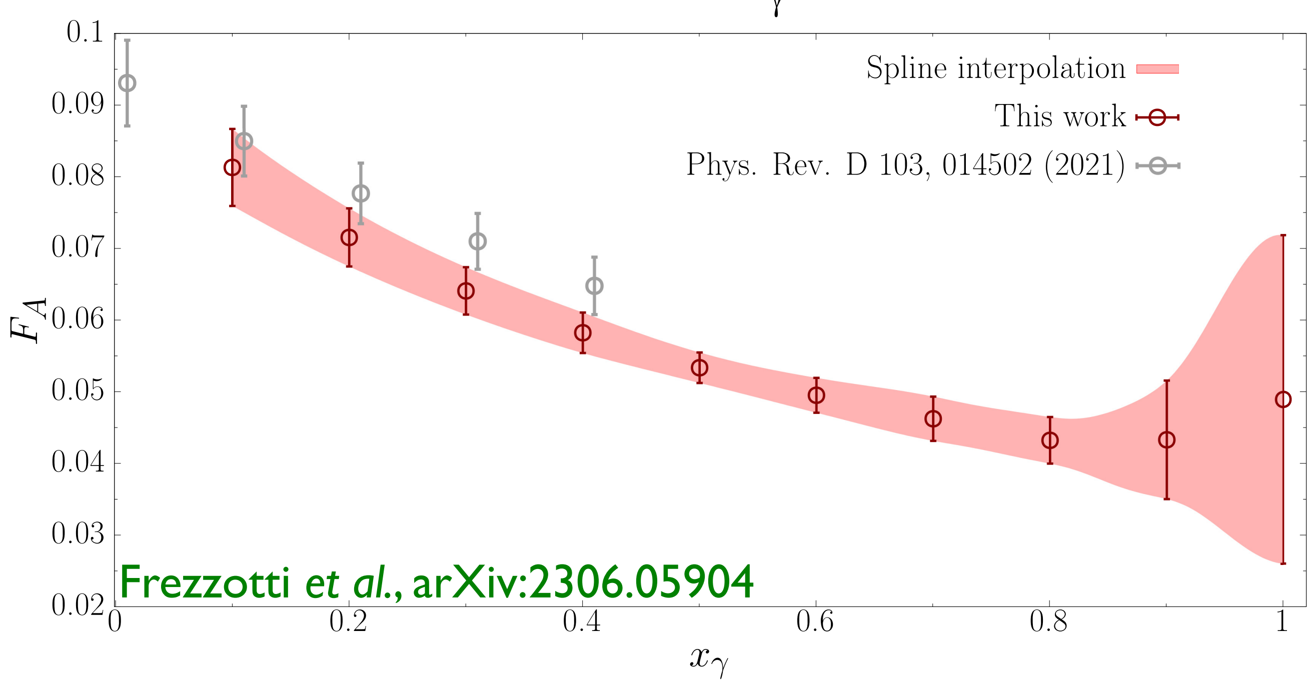
  [BESIII Collaboration, arXiv:1902.03351]

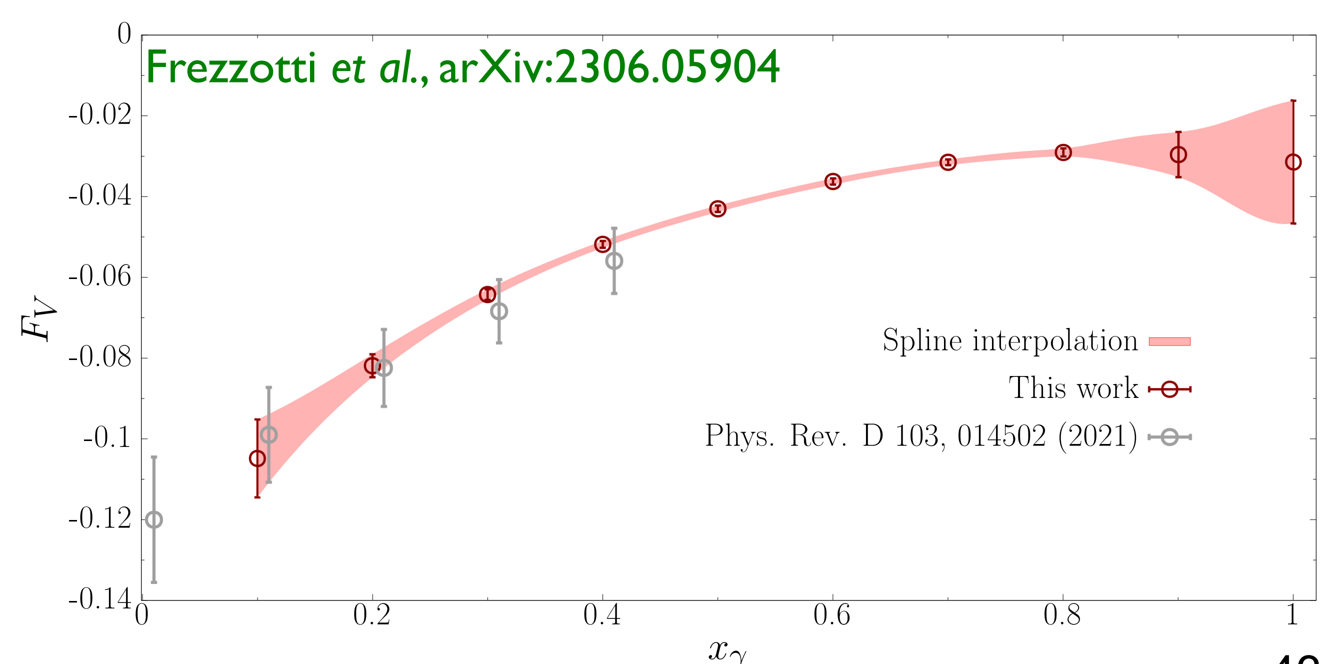
- Fit Ansatz inspired by the phenomenological analysis of arXiv:0907.1845

$D_s \to \ell \nu_\ell \gamma$: preliminary results on more ensembles

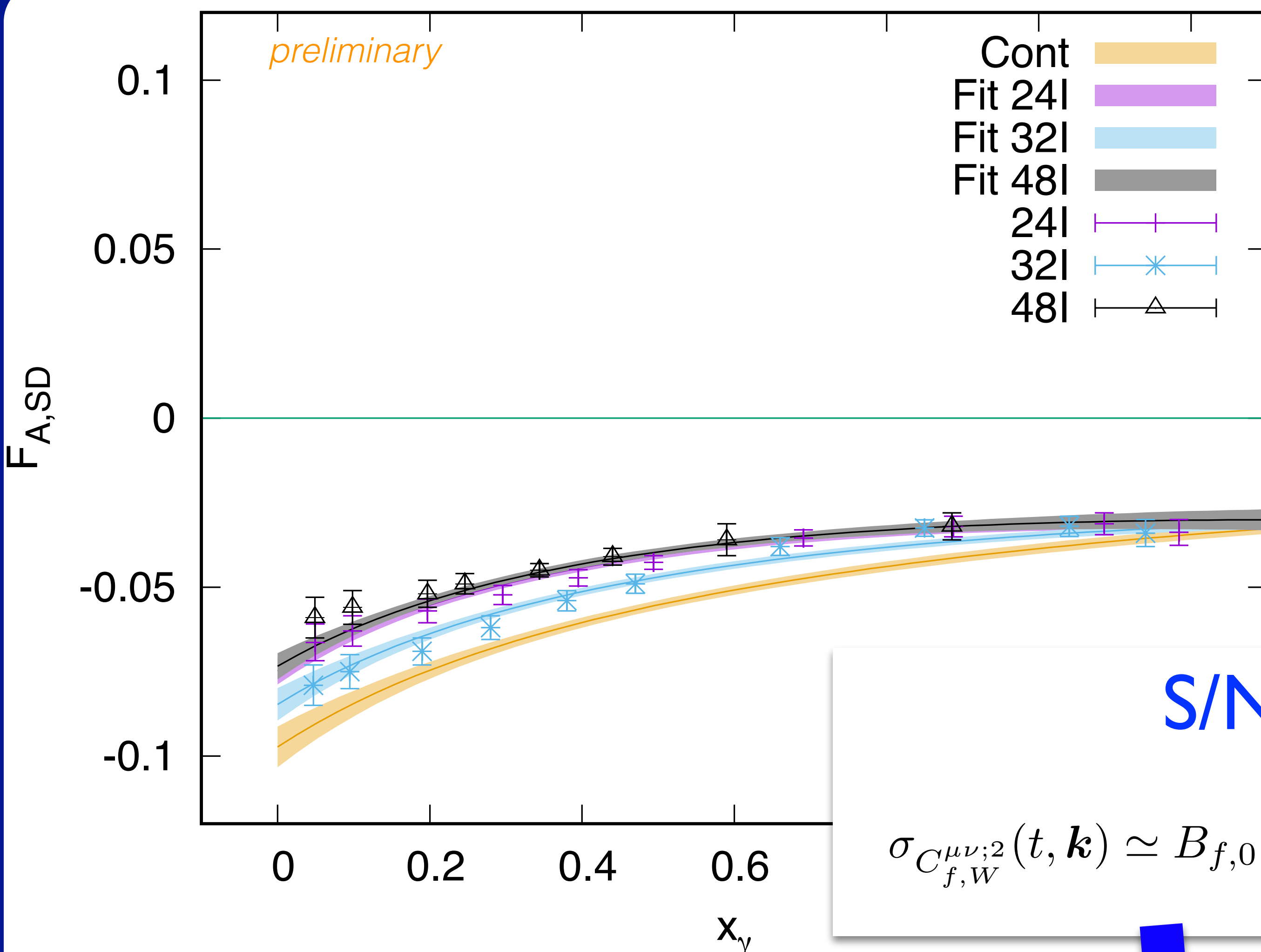


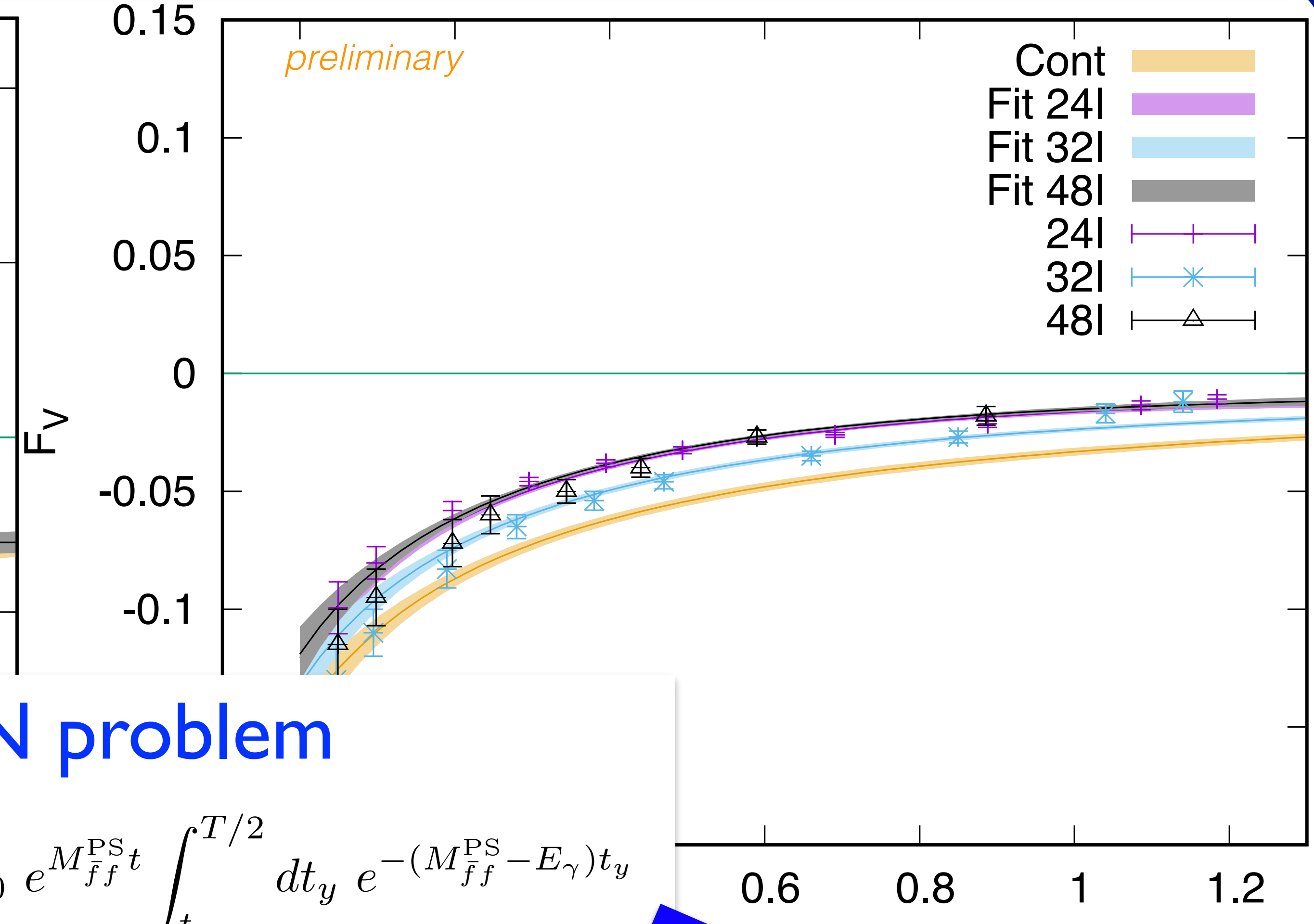






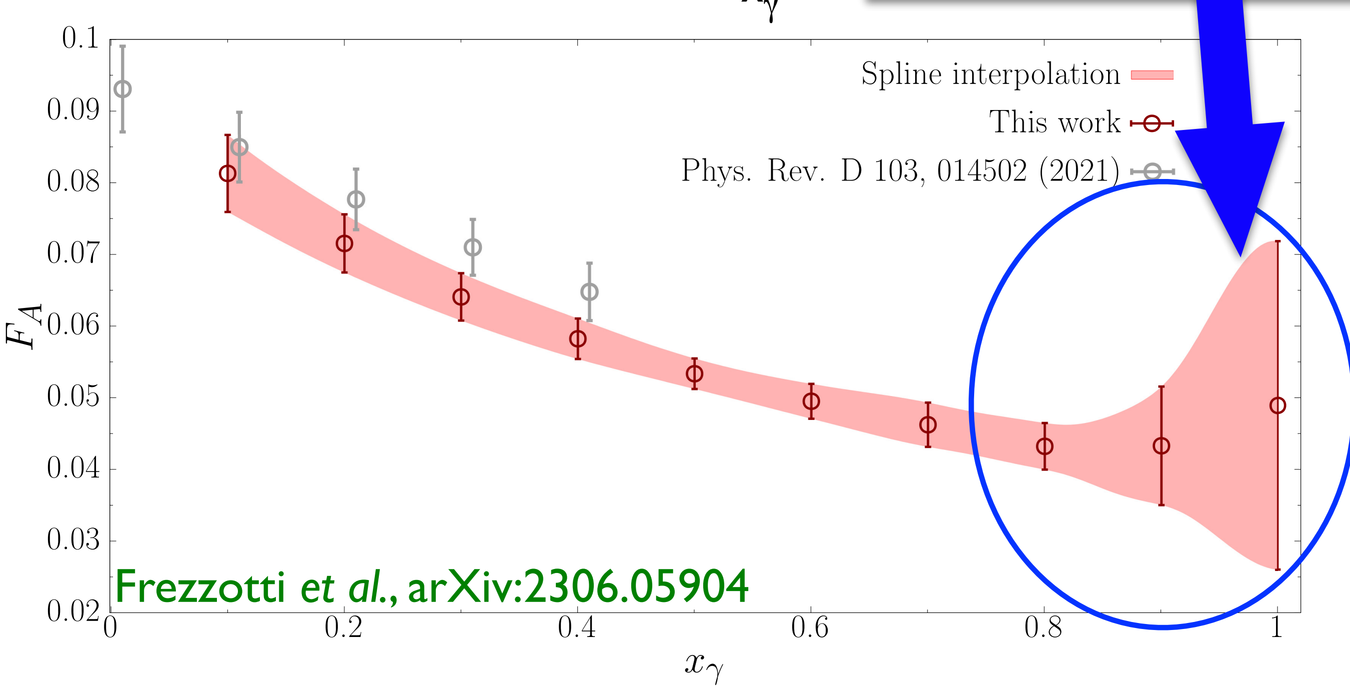
$D_s \to \ell\nu_\ell\gamma$: preliminary results on more ensembles

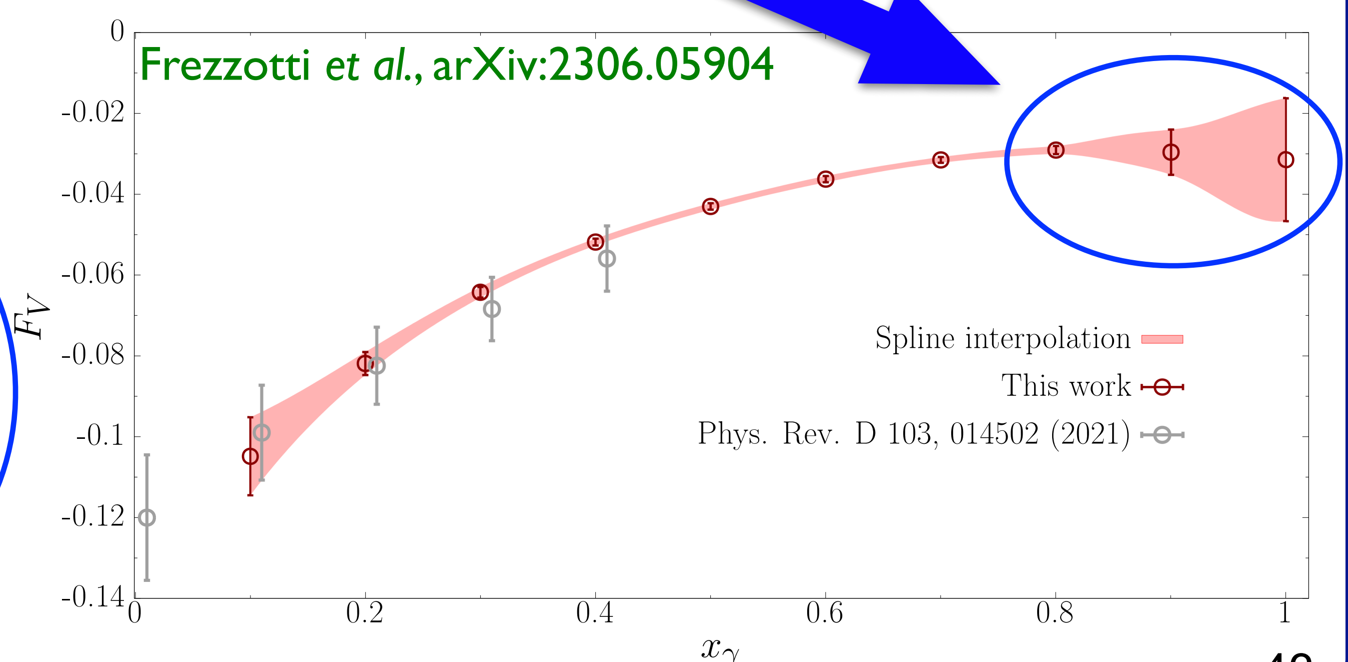




S/N problem

$$\sigma_{C^{\mu\nu;2}_{f,W}}(t,\boldsymbol{k}) \simeq B_{f,0}\; e^{M^{\rm PS}_{\bar f f} t} \int_t^{T/2} dt_y\; e^{-(M^{\rm PS}_{\bar f f} - E_\gamma) t_y}$$



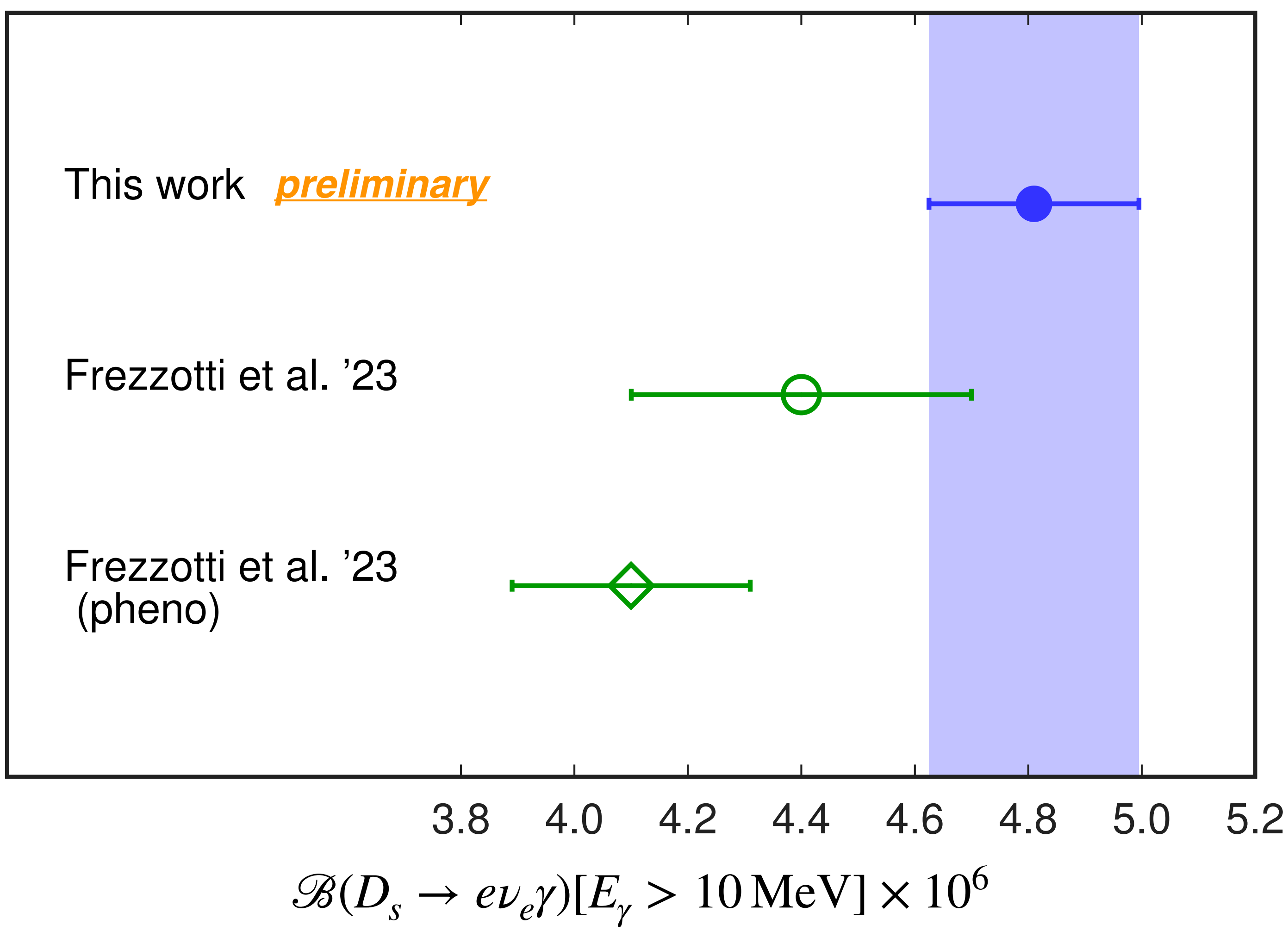


$\mathscr{B}(D_s \to e\nu_e\gamma)$

$$\frac{dR^{\rm pt}}{dx_\gamma} = -\frac{2}{(1-r_\ell^2)^2}\frac{1}{x_\gamma}\left\{\left[\frac{(2-x_\gamma)^2}{1-x_\gamma} - 4r_\ell^2\right](1-x_\gamma-r_\ell^2) - \left[2(1-r_\ell^2)(1+r_\ell^2-x_\gamma)+x_\gamma^2\right]\log\left(\frac{1-x_\gamma}{r_\ell^2}\right)\right\},$$

$$\frac{dR^{\rm int}}{dx_\gamma} = -\frac{2M_{D_s}}{f_{D_s}(1-r_\ell^2)^2}\left\{F_A\, x_\gamma\left[\frac{r_\ell^4}{1-x_\gamma} - 1 + x_\gamma + 2r_\ell^2\log\left(\frac{1-x_\gamma}{r_\ell^2}\right)\right] + (F_V-F_A)\, x_\gamma^2\left[\frac{r_\ell^2}{1-x_\gamma} - 1 + \log\left(\frac{1-x_\gamma}{r_\ell^2}\right)\right]\right\},$$

$$\frac{dR^{\rm SD}}{dx_\gamma} = \frac{M_{D_s}^2}{f_{D_s}^2}\left(F_V^2+F_A^2\right)\frac{x_\gamma^3}{r_\ell^2(1-r_\ell^2)^2}\frac{(2+r_\ell^2-2x_\gamma)(1-x_\gamma-r_\ell^2)^2}{6(1-x_\gamma)^2}\,.$$



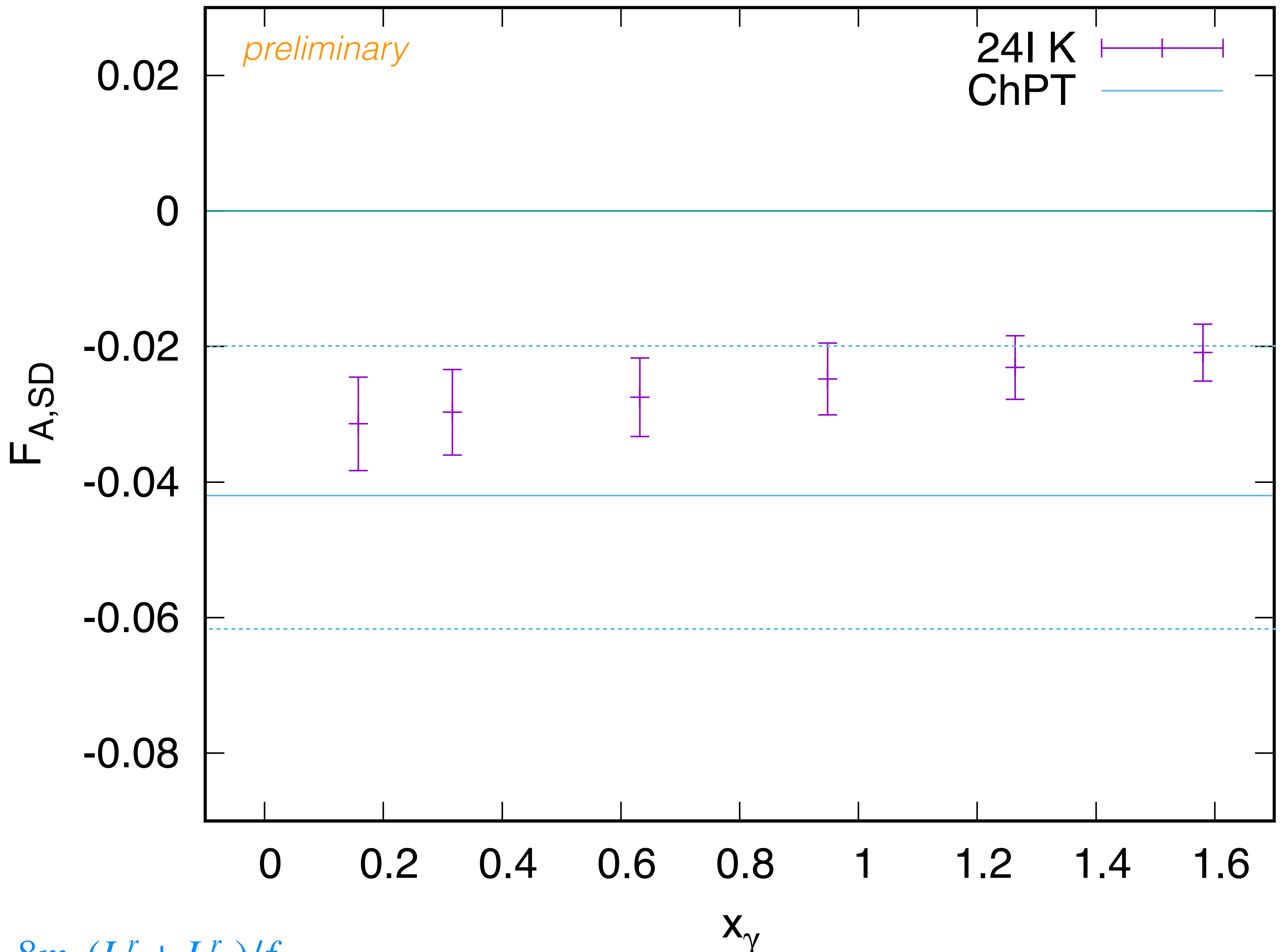
Experiment

BESIII Collaboration upper bound

$$\mathscr{B}(D_s^+ \to e^+\nu_e\gamma)[E_\gamma > 10\,\mathrm{MeV}] < 1.3\times 10^{-4}$$

arXiv:1902.03351

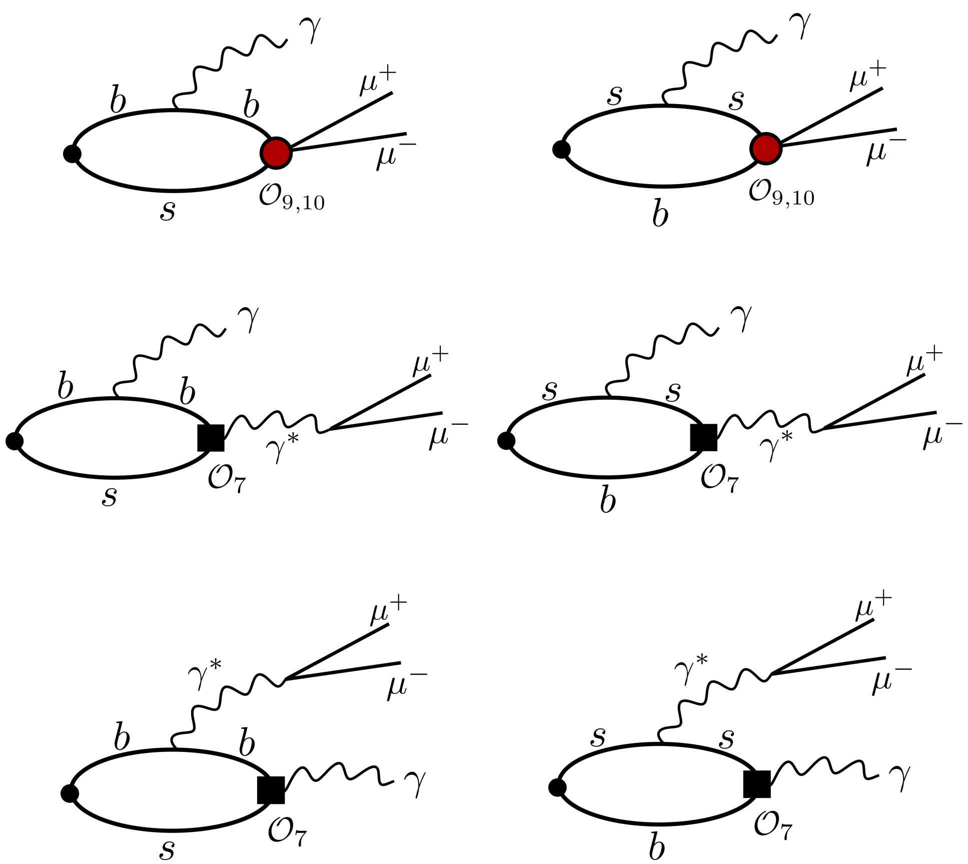
$K \to \ell\nu_\ell\gamma$: results



ChPT : $8m_K(L_9^r + L_{10}^r)/f_K$

J. Bijnens *et al.*, 1993

$B_s \to \mu^+\mu^-\gamma$

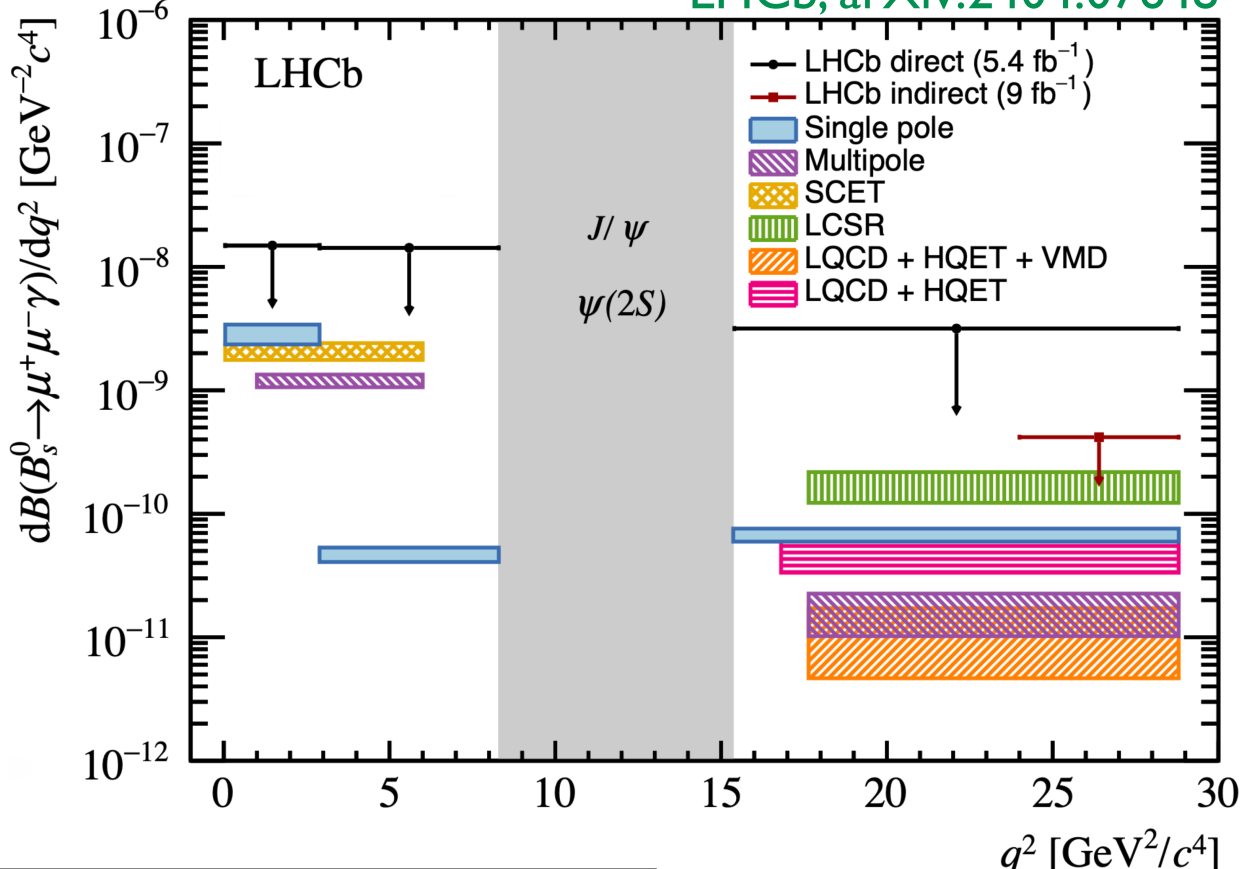


F_A, F_V

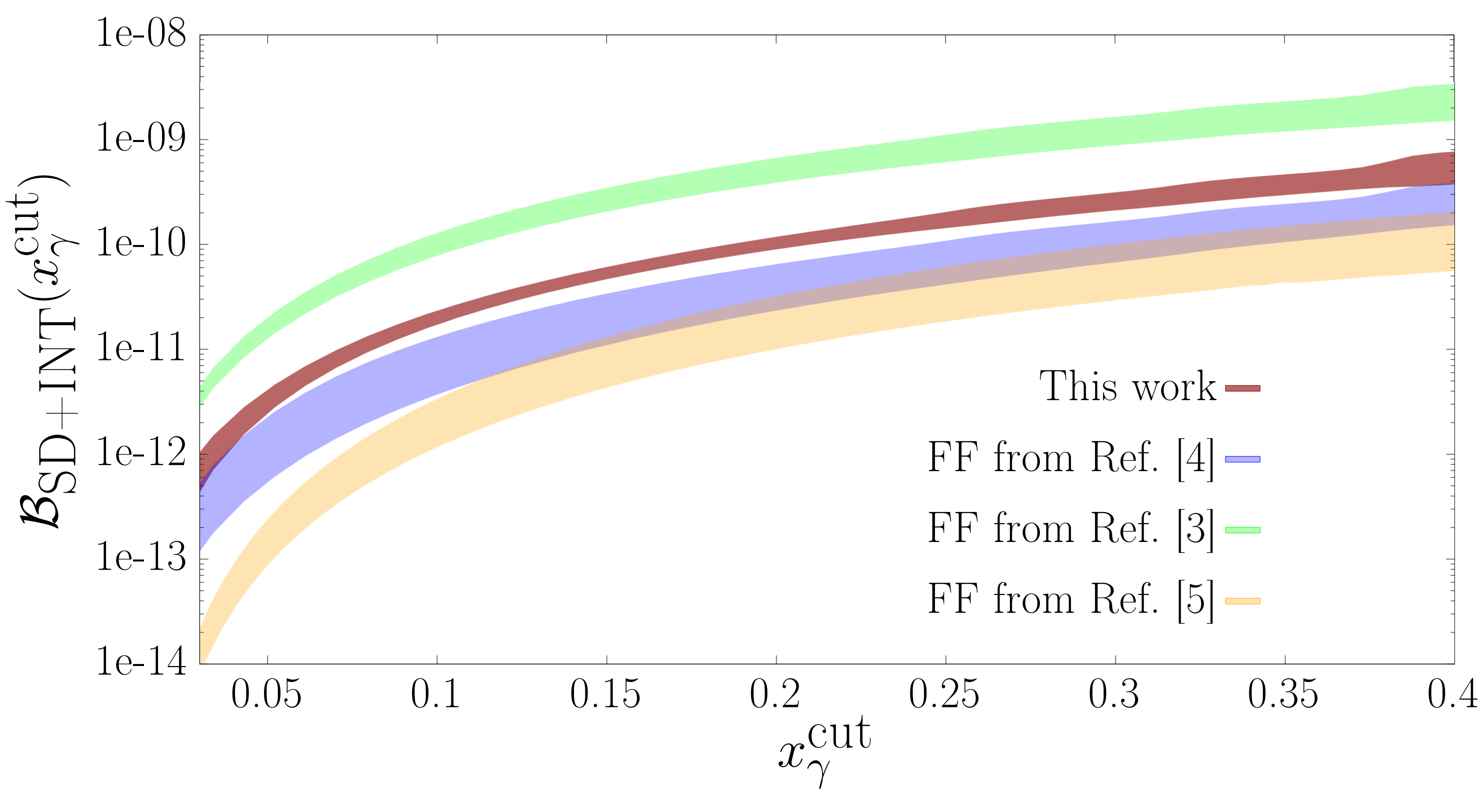
F_{TA}, F_{TV}

$\overline{F}_T$

LHCb, arXiv:2404.07648



Frezzotti *et al.*
arXiv:2402.03262



Conclusions and future perspectives

- The form factors for real emissions are accessible from Euclidean correlators

- We compared analysis methods using 3d and 4d data. 3d method results in smallest statistical uncertainties and allows to tame S/N problems at large photon energies.

- With moderate statistics we are able to provide rather precise, first-principles results for the form factors in the full kinematical (photon-energy) range

- Lattice calculations of radiative leptonic heavy-meson decays at high photon energy could provide useful information to better understand the internal structure of hadrons

- To extend the study to B-meson decays we will take advantage of new RBC/UKQCD ensembles at $a^{-1} \approx (3.5, 4.5)$ GeV

	48I	64I	96I
$L^3 \cdot T/a^4$	$48^3 \cdot 96$	$64^3 \cdot 128$	$96^3 \cdot 192$
β	2.13	2.25	2.31
am_l	0.00078	0.000678	0.0054
am_h	0.0362	0.02661	0.02132
α	2.0	2.0	2.0
a^{-1} (GeV)	1.730(4)	2.359(7)	≈ 2.8
a (fm)	0.1141(3)	0.0837(3)	≈ 0.071
L (fm)	5.476(12)	5.354(16)	≈ 6.8
L_s/a	24	12	12
m_π (MeV)	139.2(4)	139.2(5)	≈ 135
$m_\pi L$	3.863(6)	3.778(8)	≈ 4.7
N_{conf}	120	160	20

PRACE

Supplementary slides

A strategy for Lattice QCD:

The isospin-breaking part of the Lagrangian is treated as a perturbation

Expand in: $m_d - m_u$ + α_{em}

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Isospin breaking effects due to the up-down mass difference in lattice QCD

RM123 collaboration

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PHYSICAL REVIEW D **87,** 114505 (2013)

Leading isospin breaking effects on the lattice

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arXiv:1303.4896

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RM123 Collaboration

① The (md-mu) expansion

- Identify the isospin-breaking term in the QCD action

$$S_m = \sum_x \left[m_u \bar{u}u + m_d \bar{d}d \right] = \sum_x \left[\frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d) - \frac{1}{2}(m_d - m_u)(\bar{u}u - \bar{d}d) \right] =$$

$$= \sum_x \left[m_{ud} (\bar{u}u + \bar{d}d) - \Delta m (\bar{u}u - \bar{d}d) \right] = S_0 - \Delta m \hat{S} \longleftarrow \boxed{\hat{S} = \Sigma_x(\bar{u}u - \bar{d}d)}$$

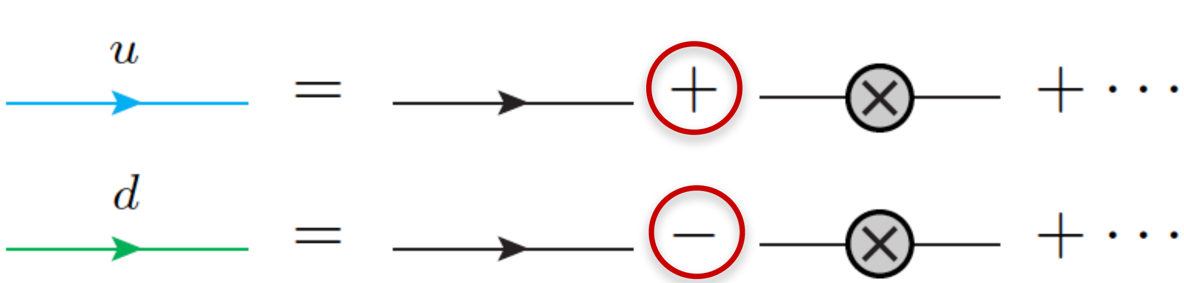
- Expand the functional integral in powers of Δm

Advantage: factorized out

$$\langle O \rangle = \frac{\int D\phi\, O e^{-S_0 + \Delta m \hat{S}}}{\int D\phi\, e^{-S_0 + \Delta m \hat{S}}} \overset{1st}{\simeq} \frac{\int D\phi\, O e^{-S_0} (1 + \Delta m \hat{S})}{\int D\phi\, e^{-S_0} (1 + \Delta m \hat{S})} \simeq \frac{\langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0}{1 + \cancel{\Delta m \langle \hat{S} \rangle_0}} = \langle O \rangle_0 + \Delta m \langle O \hat{S} \rangle_0$$

for isospin symmetry

- At leading order in Δm the corrections only appear in the valence-quark propagators:



(disconnected contractions of ūu and d̄d vanish due to isospin symmetry)

② The QED expansion

- Non-compact QED: the dynamical variable is the gauge potential $A_\mu(x)$ in a fixed covariant gauge ($\nabla^-_\mu A_\mu(x) = 0$)

$$\boxed{S_{QED} = \frac{1}{2}\sum_{x;\mu\nu} A_\nu(x)\left(-\nabla^-_\mu \nabla^+_\mu\right) A_\nu(x) \overset{(p.b.c.)}{=} \frac{1}{2}\sum_{k;\mu\nu} \tilde{A}^*_\nu(k)\left(2\sin(k_\mu/2)\right)^2 \tilde{A}_\nu(k)}$$

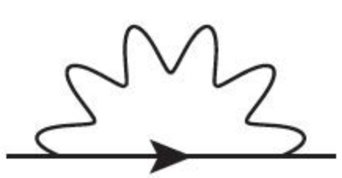
- The photon propagator is IR divergent → subtract the zero momentum mode

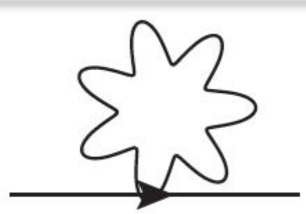
- Full covariant derivatives are defined introducing QED and QCD links:

$$\boxed{A_\mu(x) \to E_\mu(x) = e^{-iaeA_\mu(x)}} \qquad \boxed{D^+_\mu q_f(x) = \left[E_\mu(x)\right]^{e_f} U_\mu(x)\, q_f(x+\hat{\mu}) - q_f(x)}$$

QED QCD

- Since $E_\mu(x) = e^{-ieA_\mu(x)} = 1 - ieA_\mu(x) - 1/2\, e^2 A^2_\mu(x) + \ldots$ the expansion leads to:

$(e_f e)^2$ 

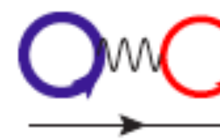
$(e_f e)^2$ 

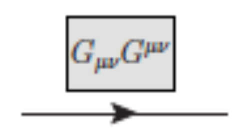
+ counterterms

The QED expansion for the quark propagator

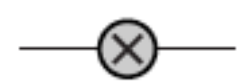
Δ  $^{\pm} =$

$(e_f e)^2$  $+ (e_f e)^2$  $- [m_f - m_f^0]$  $\mp [m_f^{cr} - m_0^{cr}]$ 

$-e^2 e_f \sum_{f_1} e_{f_1}$  $- e^2 \sum_{f_1} e_{f_1}^2$  $- e^2 \sum_{f_1} e_{f_1}^2$  $+ e^2 \sum_{f_1 f_2} e_{f_1} e_{f_2}$ 

$+ \sum_{f_1} \pm [m_{f_1}^{cr} - m_0^{cr}]$  $+ \sum_{f_1} [m_{f_1} - m_{f_1}^0]$  $+ [g_s^2 - (g_s^0)^2]$  .

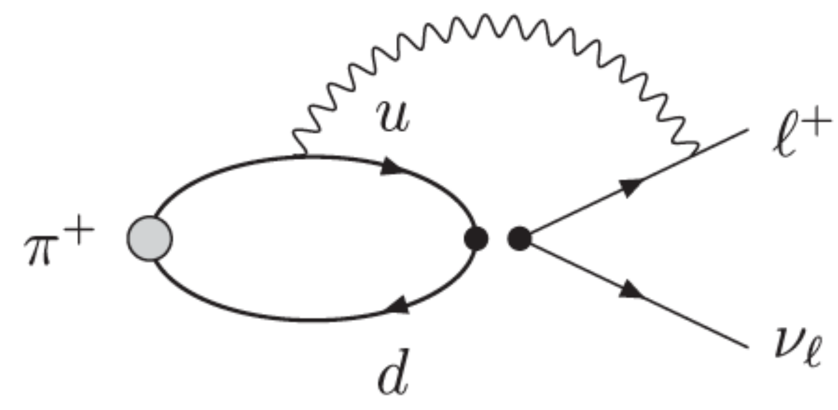
In the electro-quenched approximation:

Δ  $^{\pm} = (e_f e)^2 \Big[$  $+$  $\Big] - [m_f - m_f^0]$  $\mp [m_f^{cr} - m_0^{cr}]$  .

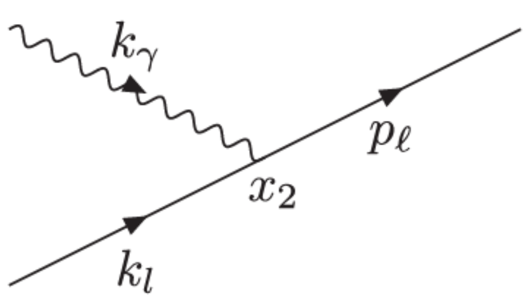
Lattice calculation of Γ_0 at $O(\alpha)$

- A technical but important point:

$$\delta C^{(q\ell)}(t)_{\alpha\beta} = -\int d^3\vec{x}\, d^4x_1\, d^4x_2 \langle 0|T\left\{J_W^{\nu}(0)\, j_\mu(x_1)\, \phi^\dagger(\vec{x},-t)\right\}|0\rangle \times \Delta(x_1,x_2)\left(\gamma_\nu(1-\gamma^5)S(0,x_2)\gamma_\mu\right)_{\alpha\beta} e^{E_\ell t_2 - i\,\vec{p}_\ell\cdot\vec{x}_2}$$



We need to ensure that the t_2 integration converges as $t_2 \to \infty$. The large t_2 behavior is given by the factor $\exp\left[\left(E_\ell - \omega_\ell - \omega_\gamma\right)t_2\right]$



$$E_\ell = \sqrt{\vec{p}_\ell^{\,2} + m_\ell^2} \qquad \omega_\ell = \sqrt{\vec{k}_\ell^{\,2} + m_\ell^2} \qquad \omega_\gamma = \sqrt{\vec{k}_\gamma^{\,2} + m_\gamma^2} \qquad \vec{k}_\ell + \vec{k}_\gamma = \vec{p}_\ell$$

$$\Rightarrow \quad \left(\omega_\ell + \omega_\gamma\right)_{\min} = \sqrt{\left(m_\ell^2 + m_\gamma^2\right) + \vec{p}_\ell^{\,2}} > E_\ell$$

The integral is convergent and the continuation from Minkowski to Euclidean space can be performed (same if we set $m_\gamma=0$ but remove the photon zero mode in FV).

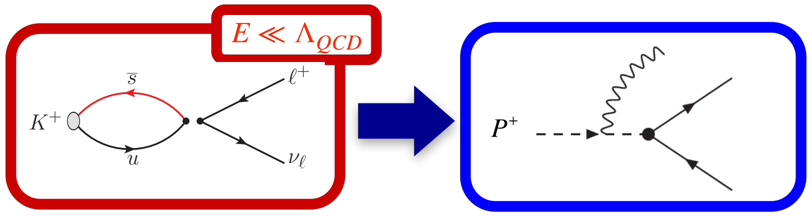
CONDITIONS: - mass gap between the decaying particle and the intermediate states
- absence of lighter intermediate states

The strategy

$$\Gamma[P_{\ell 2}] = (\Gamma_0 - \Gamma_0^{pt}) + (\Gamma_0^{pt} + \Gamma_1^{pt}(E))$$

- The contributions from soft virtual photon to Γ_0 and Γ_0^{pt} in the first term are exactly the same and the IR divergence cancels in the difference $\Gamma_0 - \Gamma_0^{\mathrm{pt}}$.

- The sum $\Gamma_0^{pt} + \Gamma_1^{pt}(E)$ in the second term is also IR finite since it is a physically well defined quantity. This term can be thus calculated in perturbation theory with a different IR cutoff.

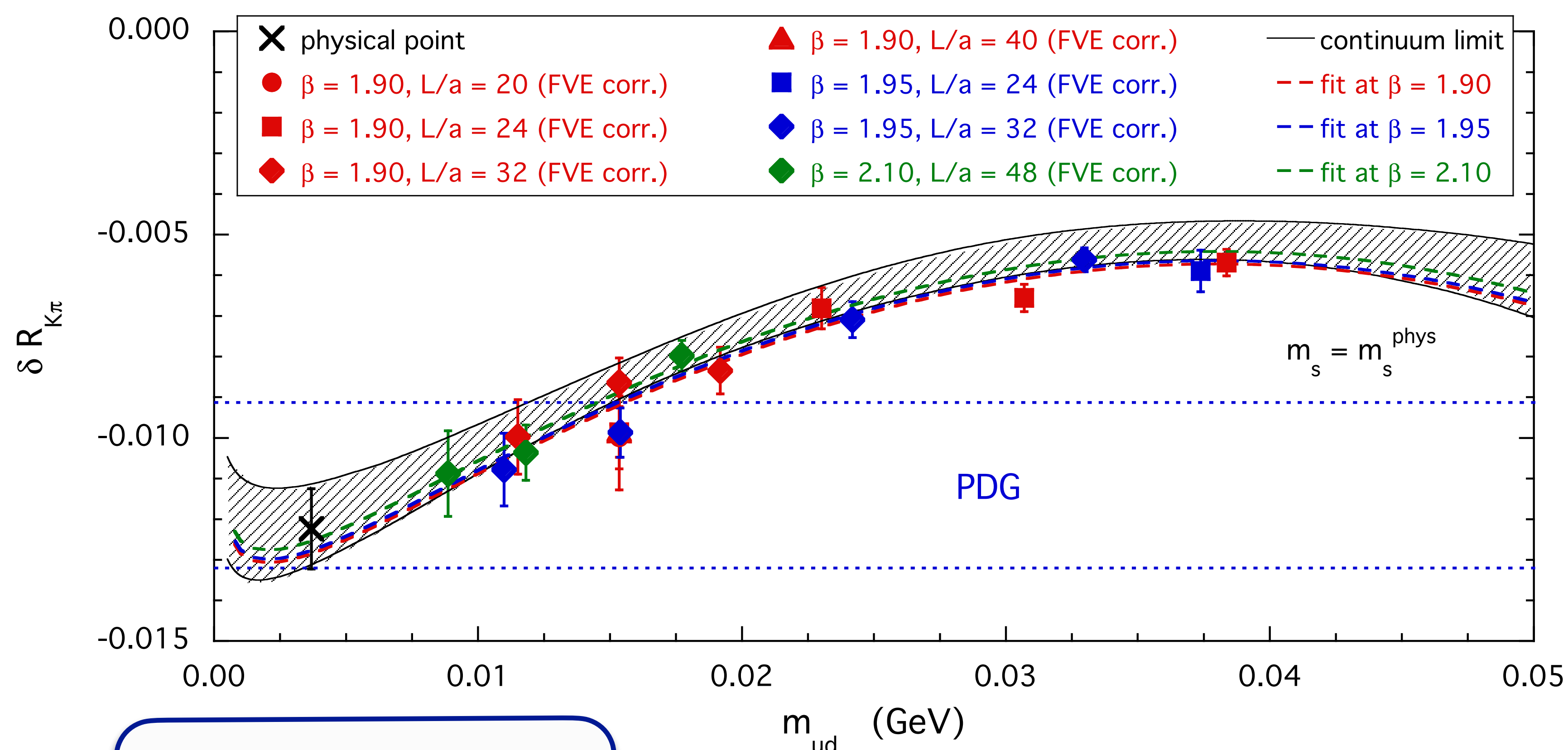


- The two terms are also separately gauge invariant.

$$\Delta\Gamma_0(L) = \Gamma_0(L) - \Gamma_0^{pt}(L)$$

$$\Gamma^{pt}(E) = \lim_{m_\gamma \to 0} \left[\Gamma_0^{pt}(m_\gamma) + \Gamma_1^{pt}(E, m_\gamma)\right]$$

Leptonic decays at O(α): RESULTS



RM123 & Soton Coll., 2017

$$\delta R_{K\pi} = C_0 + C_\chi \log(m_{ud}) + C_1 m_{ud} + C_2 m_{ud}^2 + D a^2 + \frac{K_2}{L^2}\left[\frac{1}{M_K^2} - \frac{1}{M_\pi^2}\right] + \frac{K_2^\mu}{L^2}\left[\frac{1}{(E_\mu^K)^2} - \frac{1}{(E_\mu^\pi)^2}\right] + \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max,K}\right) - \delta\Gamma^{pt}\left(\Delta E_\gamma^{\max,\pi}\right)$$

LATTICE RESULT

$$\delta R_{K\pi} = -0.0126(10)_{stat}(2)_{input}(5)_{chir}(5)_{FVE}(4)_{disc}(6)_{qQED} = -0.0126(14)$$

ChPT

$$\delta R_K - \delta R_\pi = -0.0112(21)$$

V.Cirigliano and H.Neufeld, PLB 700 (2011) 7

$$\left|\frac{V_{us}}{V_{ud}}\right| \frac{f_K^{(0)}}{f_\pi^{(0)}} = 0.27683(29)_{exp}(20)_{th}$$

$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23135(24)_{exp}(39)_{th}$$

FLAG(2019) N_f=2+1+1 $\frac{f_K^{(0)}}{f_\pi^{(0)}} = 1.1966(18)$

$|V_{ud}|$ from

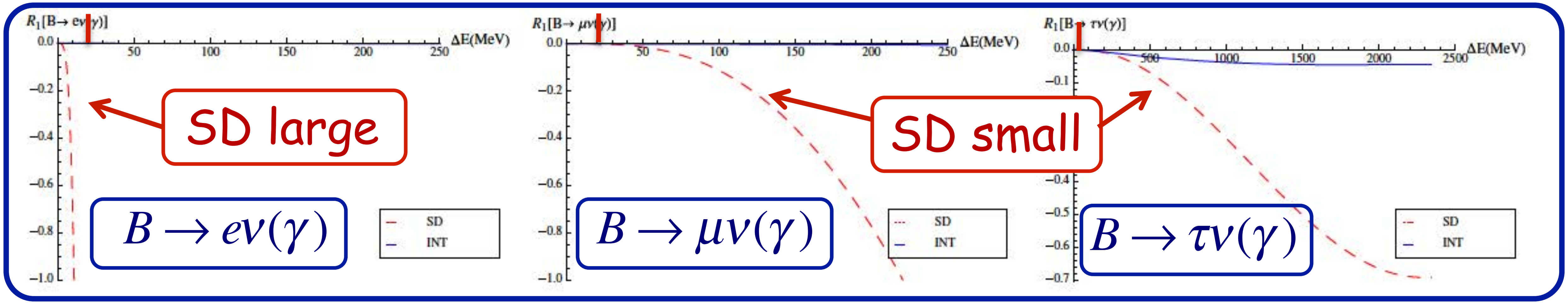
$|V_{us}| = 0.22538(46)$ Hardy and Towner, 2016

$|V_{us}| = 0.22526(46)$ Seng *et al.*, 2018

Structure dependent contributions to decays of D and B mesons

- For the studies of D and B mesons decays we cannot apply ChPT
- For B mesons in particular we have another small scale, $m_{B^*} - m_B \simeq 45\ \mathrm{MeV}$
  ➡ the radiation of a soft photon may still induce sizeable SD effects
- A phenomenological analysis based on a simple pole model for F_V and F_A confirms this picture

D. Becirevic *et al.*, PLB 681 (2009) 257



$$F_V \simeq \frac{\tilde{C}_V}{1-(p_B-k)^2/m_{B^*}^2}$$

$$F_A \simeq \frac{\tilde{C}_A}{1-(p_B-k)^2/m_{B_1}^2}$$

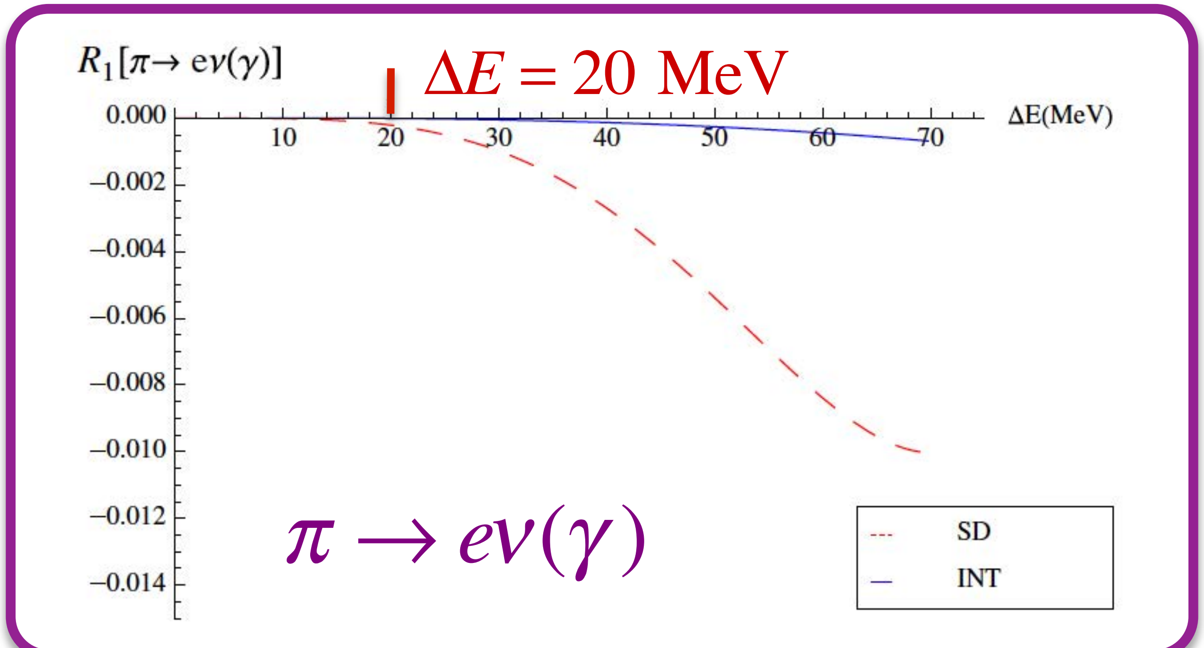
Under this assumption the SD contributions to $B \to e\nu(\gamma)$ for $E_\gamma \simeq 20$ MeV can be very large, but are small for $B \to \mu\nu(\gamma)$ and $B \to \tau\nu(\gamma)$

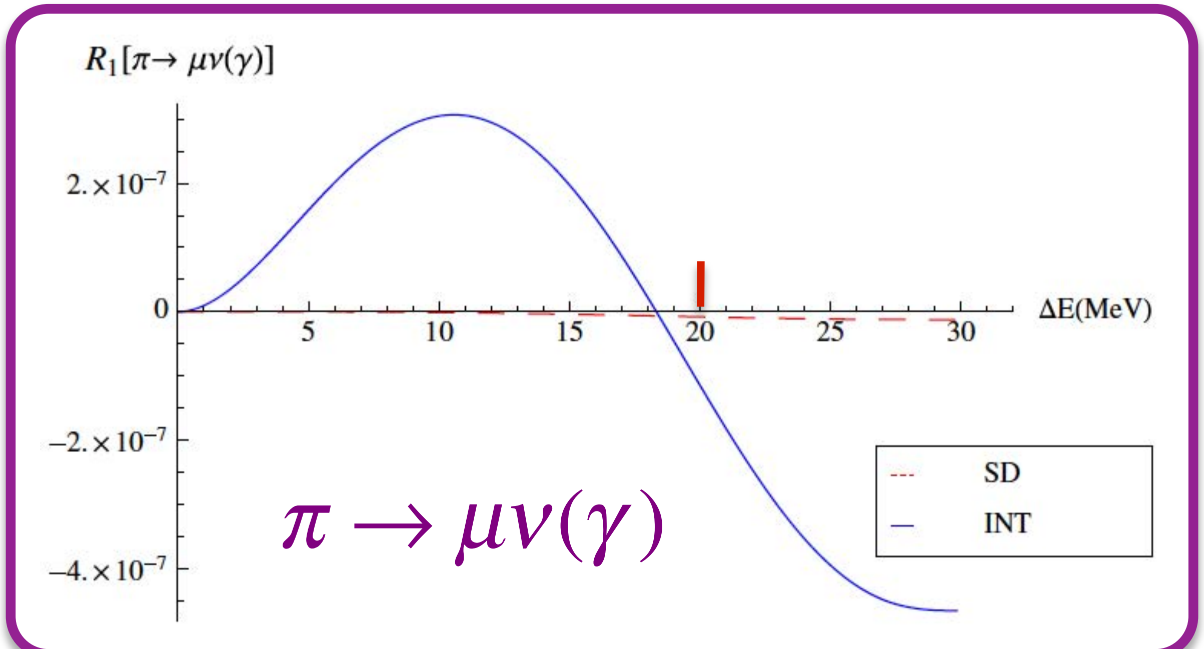
A lattice calculation of F_V and F_A would be very useful

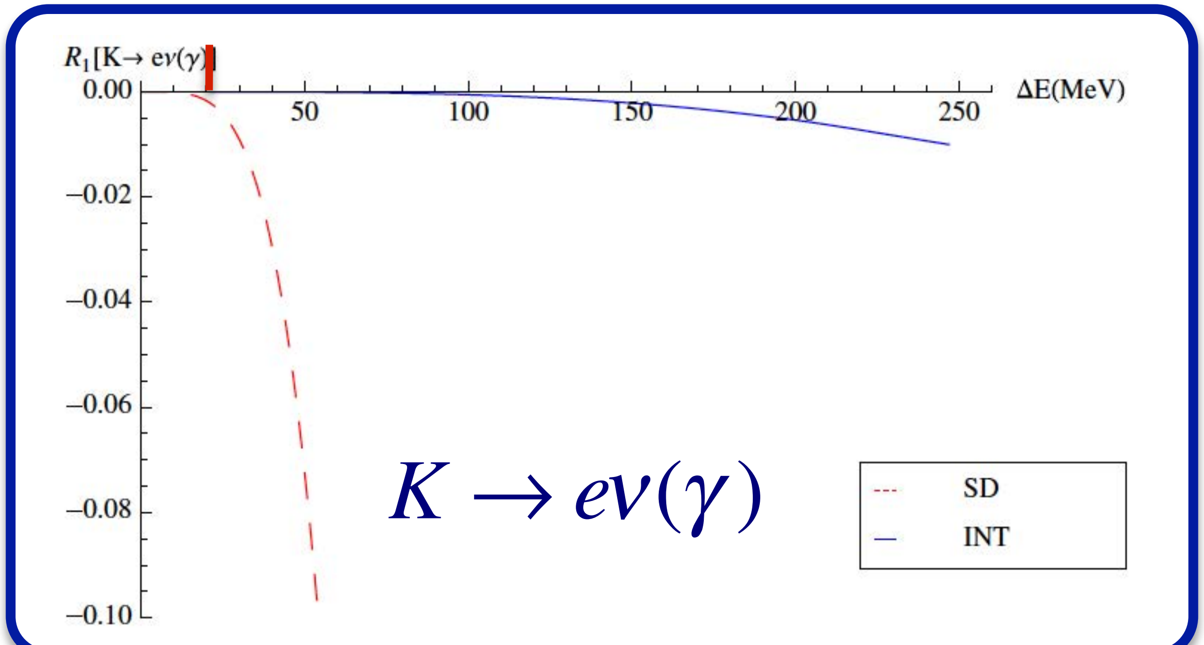
$$R_1^A(\Delta E) = \frac{\Gamma_1^A(\Delta E)}{\Gamma_0^{\alpha,pt} + \Gamma_1^{pt}(\Delta E)} \quad , \quad A = \{SD, INT\}$$

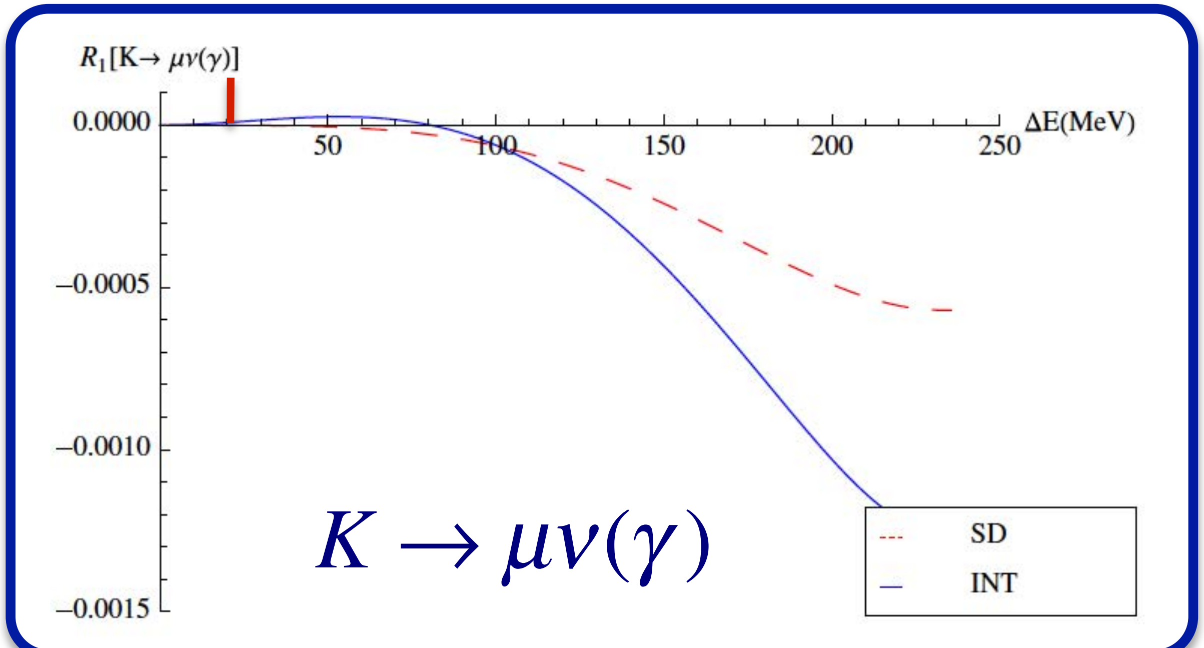
SD = structure dependent

INT = interference









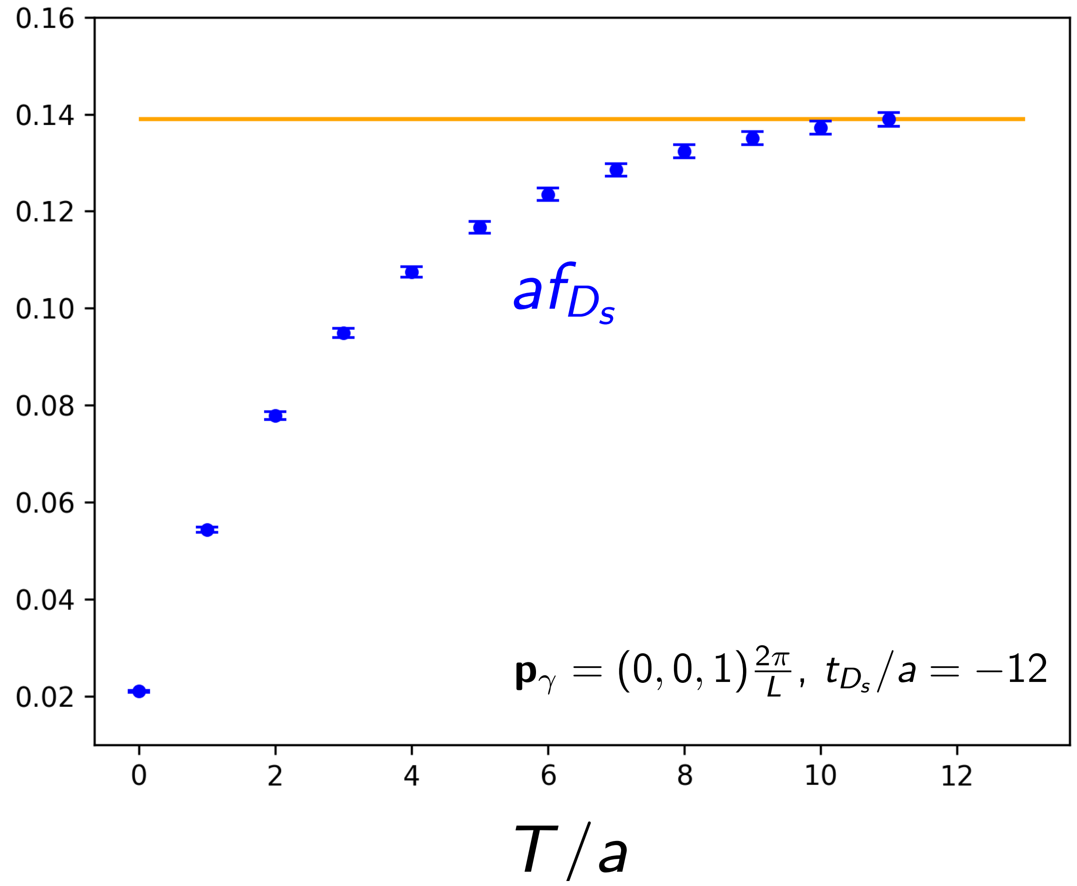
- Interference contributions are negligible in all the decays
- Structure-dependent contributions can be sizable for $K \to e\nu(\gamma)$ but they are negligible for $\Delta E < 20$ MeV (which is experimentally accessible)

Cross-checks

Recall

$$T_{\mu\nu} = \epsilon_{\mu\nu\tau\rho} p_\gamma^\tau v^\rho F_V + i[-g_{\mu\nu}(p_\gamma \cdot v) + v_\mu (p_\gamma)_\nu] F_A - i \frac{v_\mu v_\nu}{p_\gamma \cdot v} m_{D_s} f_{D_s}$$
$$+ (p_\gamma)_\mu\text{-terms}$$

$\longrightarrow$ also extract f_{D_s} as a cross-check



Yellow line = FLAG 2021 average

Infinite-volume approximation

We assume there exist $c, d, \Lambda, \Lambda' \in \mathbb{R}^+$ and $L_0 \in \mathbb{N}$ for which

$$\tilde{C}^L(q) \equiv \sum_{x=-L/2}^{L/2-1} C^L(x) e^{iqx}$$

for all x with $-L/2 \leq x \leq L/2$ and $L \geq L_0$ and

$$|C^\infty(x)| \leq d e^{-\Lambda'|x|}$$

for all x with $|x| > L/2$. We now define

$$|C^\infty(x) - C^L(x)| \leq c e^{-\Lambda L} \qquad \text{and} \qquad \tilde{C}^\infty(q) \equiv \sum_{x=-\infty}^{\infty} C^\infty(x) e^{iqx}\,.$$

Under the above assumptions, it then follows that there is a $\tilde{c} \in \mathbb{R}^+$ for which

$$|\tilde{C}^\infty(q) - \tilde{C}^L(q)| \leq \tilde{c} e^{-\Lambda_0 L}$$

for all $q \in [-\pi, \pi]$ and all $L \geq L_0$, with $\Lambda_0 \equiv \min(\Lambda, \Lambda'/2)$.