Electromagnetic finite-size effects in the hadronic vacuum polarisation

Nils Hermansson-Truedsson

Higgs Centre for Theoretical Physics, *University of Edinburgh*

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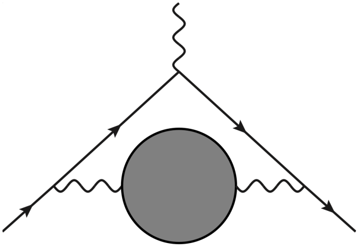
MAINZ

Introduction



- Hadronic vacuum polarisation (HVP) in lattice QCD+QED
- Finite-volume effects (FVEs)
- QED long-range force: FVEs potentially much larger than QCD
- Analytically in [Bijnens, Harrision, Hermansson-Truedsson, Janowski, Jüttner, Portelli 19]
- Today: Review what is known and extensions

I. Some preliminaries about FVEs

QED in a finite volume

- What is difficult about QED?
- Gauss' law: Difficult to define charged states in finite volume with periodic boundary conditions

$$Q = \int_V d^3\mathbf{x}\, \nabla \cdot \mathbf{E}(t, \mathbf{x}) = \int_{\partial V} d\mathbf{S} \cdot \mathbf{E} = 0$$

- Related to the photon propagator:

$$D_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{k^2}$$

- Problem: Photon zero-momentum modes and absence of mass gap
- Need to define QED in a finite volume

QED in a finite volume

- Several prescriptions

1. $\mathrm{QED_M}$: Photon mass m_γ
   [Endres, Shindler, Tiburzi, Walker-Loud 2016; Bussone, Della Morte, Janowski 2018]
2. QED_∞: Do the QED part in infinite volume
   [Feng, Jin 2018; Christ, Feng, Jin, Sachrajda, Wang 2023]
3. $\mathrm{QED_C}$: Charge-conjugated boundary conditions
   [Kronfeld, Wiese 1991–1993; RC* 2019]
4. $\mathrm{QED_L^{IR}}$: Exclude/redistribute photon zero-mode
   [Davoudi, Harrison, Jüttner, Portelli, Savage 2019]
   - $\mathrm{QED_L}$: Exclude photon zero-mode [Hayakawa, Uno 2008]
   - $\mathrm{QED_r}$: Redistribute photon zero-mode [Di Carlo, Hansen, NHT, Portelli in prep.]

- Each has advantages/challenges

Finite-size effects

- Massless photon + no zero-mode ($\mathrm{QED_L}$, $\mathrm{QED_r}$, $\mathrm{QED_C}$)
- $V = \mathbb{R} \times L^3$: $\implies$ Finite-size effects in observable $\mathcal{O}(L)$:

$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}(\infty) = \kappa_0 + \kappa_{\log} \log m_P L + \kappa_1 \frac{1}{m_P L} + \kappa_2 \frac{1}{(m_P L)^2} + \ldots$$

- QCD only: $e^{-m_P L}$ [Lüscher 86] $\implies$ QED potentially bigger FVEs
- Derive using EFT methods
- Scaling in L is observable-dependent: e.g. HVP $\kappa_0 = \kappa_{\log} = 0$
- Coefficients depend on: masses, charges, structure
- NB: Coefficients are prescription dependent!

[Davoudi, Savage 14; BMW 15; RM-123/Soton 17; Davoudi, Harrison, Jüttner, Portelli, Savage 19; Bijnens, Harrison, NHT, Janowski, Jüttner, Portelli 19; Di Carlo, Hansen, NHT, Portelli 22]

QED prescriptions

- Photon momentum $k = (k_0, \mathbf{k})$
- 3-momentum discretised by finite volume extent $\mathbf{k} = \frac{2\pi\mathbf{n}}{L} \neq \mathbf{0}$
- $\mathrm{QED_L}$: Propagator (Feynman gauge)

$$D_{\mu\nu}(k) = \delta_{\mu\nu}\,\frac{1}{k^2}\,(1-\delta_{\mathbf{k},\mathbf{0}})$$

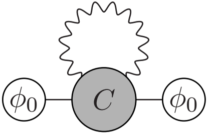
- $\mathrm{QED_L^{IR}}$: Weights $w_{|\mathbf{n}|^2}$ to tame FVEs [Davoudi, Harrison, Jüttner, Portelli, Savage 19]

$$D_{\mu\nu}(k) = \delta_{\mu\nu}\,\frac{1+w_{|\mathbf{n}|^2}}{k^2}\,(1-\delta_{\mathbf{k},\mathbf{0}})$$

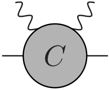
- $\mathrm{QED_r}$ $w_{|\mathbf{n}|^2} = \delta_{|\mathbf{n}|,1}/6$ [Di Carlo, Hansen, NHT, Portelli In prep.]
- $\mathrm{QED_L}$ $w_{|\mathbf{n}|^2} = 0$
- We focus on $\mathrm{QED_L}$ and make comments about other formulations

Determining the volume effects

- Once a QED prescription is fixed, we look at the correlation function
- Example: Pion self-energy



- Cottingham formula: Related to Compton tensor $C_{\mu\nu}(p, k, q)$

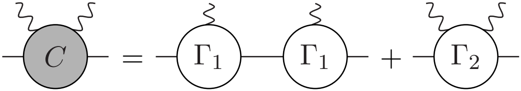


$$= C_{\mu\nu}(p, k, q) = e^2 \int \mathrm{d}^4 x \, e^{-ik\cdot x} \, \langle \pi, \mathbf{p} | \, T\left\{ J_\mu(x) J_\nu(0) \right\} | \pi, \mathbf{p} \rangle$$

- This is the object that has to be understood (also for HVP)

Skeleton expansion

- Can insert complete set of states and decompose



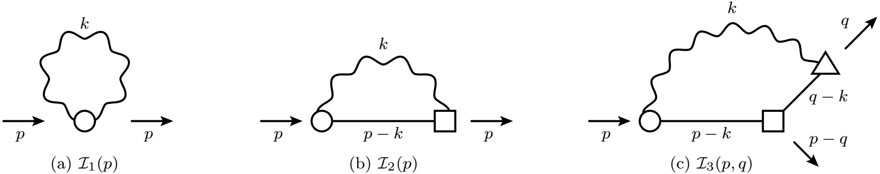
- Irreducible vertex functions $\Gamma_1 = \Gamma_\mu(p, k)$ and $\Gamma_2 = \Gamma_{\mu\nu}(p, k, q)$
- Contain structure dependence through form factors, e.g. [Fearing, Scherer 94]

$$\Gamma_\mu(p, k) = (2p + k)_\mu F(k^2) + k_\mu G(k^2, (p + k)^2, p^2)$$
$$F'(0) = \frac{\langle r_\pi^2 \rangle}{6}$$

- Vertex form governed by Ward identities
- FVEs: Structure for mass and leptonic decays [Di Carlo, Hansen, NHT, Portelli 22]

Feynman diagrams

- Consider observable $\mathcal{O}$ at order α, with photon momentum $k = (k_0, \mathbf{k})$
- Example diagrams



(a) $\mathcal{I}_1(p)$ (b) $\mathcal{I}_2(p)$ (c) $\mathcal{I}_3(p,q)$

- Finite-size effects in $\mathcal{O}(L)$ given by:

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\right) \int \frac{dk_0}{2\pi} \frac{f_{\mathcal{O}}\left(k_0, |\mathbf{k}|, \mathbf{p}\cdot\mathbf{k}\right)}{[(p-k)^2 + m^2]} \frac{1}{k^2}$$

- Do k_0 integral + expand in L ($\mathbf{k} = 2\pi\mathbf{n}/L$)

$$\Delta\mathcal{O}(L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) g_{\mathcal{O}}(|\mathbf{k}|, \mathbf{p}\cdot\mathbf{k})$$

- Example: moving frame mass [Di Carlo, Hansen, NHT, Portelli In prep.]

$$\Delta m_P^2(L) = e^2 m_P^2 \Bigg\{ \frac{1}{\gamma(|\mathbf{v}|)} \frac{c_2(\mathbf{v})}{4\pi^2 m_P L} + \frac{c_1}{2\pi(m_P L)^2} + \frac{c_0}{(m_P L)^3} \left[\frac{(\gamma(|\mathbf{v}|)^2 - 1)\left(1 - \frac{2\langle r_P^2\rangle m_P^2 \gamma(|\mathbf{v}|)^2}{3}\right)^2}{2\gamma(|\mathbf{v}|)^3} - \frac{2\langle r_P^2\rangle m_P^2}{3}\gamma(|\mathbf{v}|) + \mathcal{C} \right] \Bigg\}$$

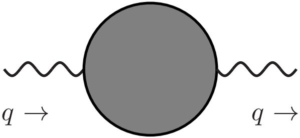
- Structure dependence: $\langle r_P^2\rangle$ known, $\mathcal{C}$ unknown (cut)
- Finite-volume coefficients: Prescription dependent

$$c_j(\mathbf{v}) = \left(\sum_{\mathbf{n}\neq\mathbf{0}} - \int d^3\mathbf{n} \right) \frac{1}{|\mathbf{n}|^j (1 - \mathbf{v}\cdot\hat{\mathbf{n}})}$$

II. Hadronic vacuum polarisation

Hadronic vacuum polarisation

- Recall central object is vector 2-point function



$$\Pi_{\mu\nu}(q) = \int d^4x \, e^{iq\cdot x} \langle 0| \, T\left[J_\mu(x) J_\nu^\dagger(0)\right] |0\rangle$$

- Virtuality: $q^2 > 0$: $q = (q_0, \mathbf{0})$
- Ward identity $q_\mu \Pi_{\mu\nu} = 0$

$$\Pi_{\mu\nu}(q^2) = \left(q_\mu q_\nu - q^2 \delta_{\mu\nu}\right) \Pi(q^2)$$

Hadronic vacuum polarisation

- We want the subtracted quantity

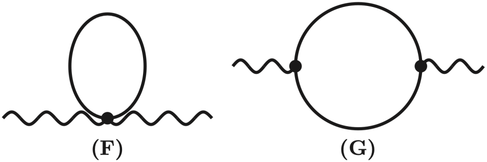
$$\hat{\Pi}(q^2) = \Pi(q^2) - \Pi(0) = \frac{1}{3q_0^2}\sum_{j=1}^{3}\Big(\Pi_{jj}(q^2) - \Pi_{jj}(0)\Big)$$

- Finite-volume effects:

$$\Delta\hat{\Pi}(q^2, L) = \hat{\Pi}(q^2, L) - \hat{\Pi}(q^2, \infty)$$

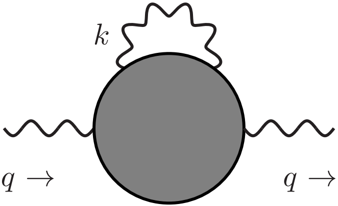
HVP in QCD

- In QCD only:



- FVEs studied by [Aubin et al. 16/20; Bijnens, Relefors 17; Hansen, Patella 20]
- Exponentially suppressed $e^{-m_\pi L}$
- Related to Compton amplitude

HVP including QED

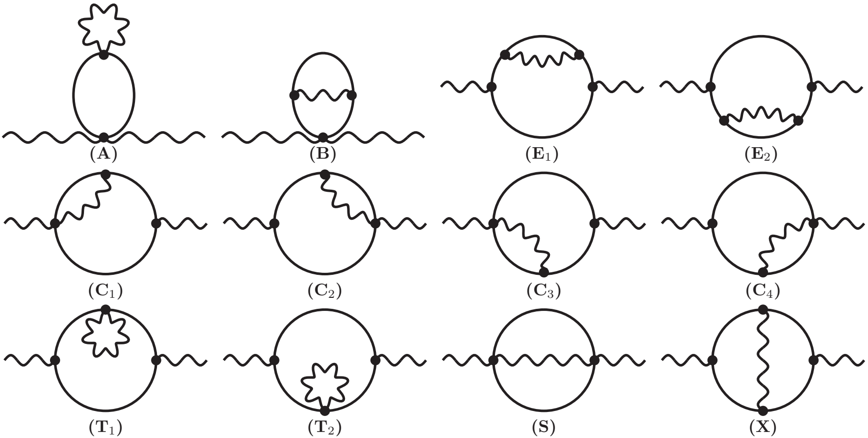


- This is now our object, related to HLbL tensor

$$\Pi^{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2, q_3, q_4) \longrightarrow \frac{\Pi^{\mu\mu\nu\nu}(q, q, k, -k)}{k^2}$$

- Can calculate QED corrections to HVP this way [Biloshytskyi et al. 23]
  $\rightarrow$ Can avoid powerlike FVEs
- In $\mathrm{QED_L}$ and $\mathrm{QED_C}$ there will be powerlike FVEs
- We are interested in $\mathrm{QED_L}$: Saturate bubble with $\pi^{\pm}$, γ

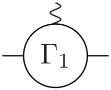
Diagrams including QED

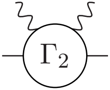


- 7 independent diagrams: A, B, E, C, T, S, X
- 2-loop diagrams: 2 momenta in finite volume

Pointlike scalar QED

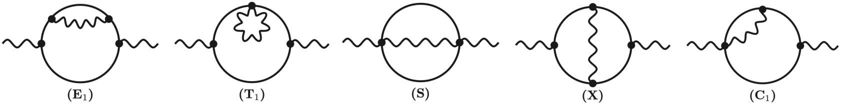
- Same 2 vertices appearing from before:



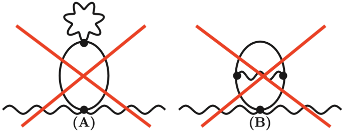
$= \Gamma_\mu(p,k)\,,$ 

$= \Gamma_{\mu\nu}(p,k,q)$

- Pointlike scalar QED: $\Gamma_\mu(p,k) = (2p+k)_\mu$, $\Gamma_{\mu\nu}(p,k,q) = -2\delta_{\mu\nu}$



- A and B independent of q: $\hat{\Pi}_{A,B}(q^2) = \Pi_{A,B}(q^2) - \Pi_{A,B}(0) = 0$



Diagrams

- 2-loop sum-integral differences for each diagram

$$\hat{\Pi} = 2\,\hat{\Pi}_E + 2\,\hat{\Pi}_T + \hat{\Pi}_S + \hat{\Pi}_X + 4\,\hat{\Pi}_C$$

$$\Delta\hat{\Pi}_U = \left(\frac{1}{L^3} \sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{L^3} \sum_{\boldsymbol{\ell}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{d\ell_0}{2\pi}\, \hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- Integrand $\hat{\pi}_U\left(q_0^2, k, \ell\right)$ contains propagator poles
- We let pions be in IV

$$\text{Poisson:} \quad \sum_{\boldsymbol{\ell}} = \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} + \mathcal{O}\left(e^{-m_\pi L}\right)$$

$$\Delta\hat{\Pi}_U = \left(\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right)\int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3}\int \frac{dk_0}{2\pi}\frac{d\ell_0}{2\pi}\,\hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- Do energy integrals analytically

$$\hat{\rho}_U(\mathbf{k}, \boldsymbol{\ell}, q_0) = \int \frac{dk_0}{2\pi}\frac{d\ell_0}{2\pi}\,\hat{\pi}_U\left(q_0^2, k, \ell\right)$$

- Finally gives

$$\Delta\hat{\Pi}_U = \left(\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right)\int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3}\,\hat{\rho}_U(\mathbf{k}, \boldsymbol{\ell}, q_0)$$

- Photon FVEs give rise to coefficients c_j as before
- Pion integrals:

$$\Omega_{i,j}(z = q_0^2/m^2) = \int dx\, x^2 \frac{1}{(1+x^2)^{i/2}[z + 4(x^2+1)]^j}$$

Finite volume corrections

- Diagram by diagram:

$$\Delta\hat{\Pi}_E(z) = \frac{c_1}{\pi(mL)^2}\left(-\frac{4}{3}\,\Omega_{-1,3} + \frac{1}{2}\,\Omega_{1,2} + \frac{4}{3}\,\Omega_{1,3} - \frac{1}{4}\,\Omega_{3,1}\right)$$
$$- \frac{c_0}{(mL)^3}\left(-\frac{8}{3}\,\Omega_{0,3} + \frac{32}{3}\,\Omega_{0,4} + \frac{1}{16}\,\Omega_{2,2} + \frac{10}{3}\,\Omega_{2,3} - \frac{32}{3}\,\Omega_{2,4} - \frac{23}{128}\,\Omega_{4,1} + \frac{5}{16}\,\Omega_{4,2} - \frac{2}{3}\,\Omega_{4,3}\right)$$

$$\Delta\hat{\Pi}_C(z) = \frac{c_1}{\pi(mL)^2}\frac{1}{8}\,\Omega_{3,1} - \frac{c_0}{(mL)^3}\left(\frac{8}{3}\,\Omega_{0,3} + \frac{1}{6}\,\Omega_{2,2} - \frac{8}{3}\,\Omega_{2,3} + \frac{1}{8}\,\Omega_{4,1} - \frac{1}{6}\,\Omega_{4,2}\right)$$

$$\Delta\hat{\Pi}_T(z) = \frac{c_1}{\pi(mL)^2}\frac{1}{4}\,\Omega_{3,1}$$

Finite volume corrections

$$\Delta\hat{\Pi}_S(z) = -\frac{c_1}{\pi(mL)^2}\frac{1}{4}\,\Omega_{3,1} + \frac{c_0}{(mL)^3}\left(2\,\Omega_{2,2} + \frac{1}{4}\,\Omega_{4,1}\right)$$

$$\begin{aligned}\Delta\hat{\Pi}_X(z) = &\frac{c_1}{\pi(mL)^2}\left(\frac{8}{3}\,\Omega_{-1,3} - \Omega_{1,2} - \frac{8}{3}\,\Omega_{1,3} - \frac{1}{4}\,\Omega_{3,1}\right)\\ &- \frac{c_0}{(mL)^3}\left(-\frac{128}{3}\,\Omega_{-2,4} - \frac{16}{3}\,\Omega_{0,3} + 64\,\Omega_{0,4} - \frac{11}{24}\,\Omega_{2,2} + \frac{20}{3}\,\Omega_{2,3}\right.\\ &\left. - \frac{64}{3}\,\Omega_{2,4} - \frac{17}{64}\,\Omega_{4,1} + \frac{29}{24}\,\Omega_{4,2} - \frac{4}{3}\,\Omega_{4,3}\right)\end{aligned}$$

- We thus see that each diagram contributes starting from $1/L^2$
- Compare pion mass FVEs: $1/L$

Finite volume corrections

- However, in total:

$$\Delta\hat{\Pi}(q^2) = \frac{c_0}{(mL)^3}\left(-\frac{16}{3}\Omega_{0,3} - \frac{5}{3}\Omega_{2,2} + \frac{40}{9}\Omega_{2,3} - \frac{3}{8}\Omega_{4,1} + \frac{7}{6}\Omega_{4,2} + \frac{8}{9}\Omega_{4,3} \right) + \mathcal{O}\left(\frac{1}{L^4}, e^{-m_\pi L}\right)$$

- Suppression: Leading order is $1/L^3$
- Physics: Neutral current and photon far away sees no charge
- Also argued by [ETMC 17]
- Check: We find for charged currents it starts at order $1/L^2$

Pointlike vs. structure

- This was obtained in a pointlike approximation
- Universality: arguing from HLbL tensor the $1/L^2$ also cancels
- Recall QED effects to HVP $\approx$ HLbL in forward kinematics

$$\hat{\Pi}(q^2) = \int \frac{d^4k}{(2\pi)^4} \frac{\hat{\Pi}_{\mu\mu\nu\nu}(q,q,k,-k)}{k^2}$$

- From [Colangelo et al. 15] we know

$$\Pi^{\mu_1\mu_2\mu_3\mu_4}(q_1,q_2,q_3,q_4) = \sum_{i=1}^{54} T_i^{\mu_1\mu_2\mu_3\mu_4}\,\Pi_i$$

- Structure-dependence in form factors Π_i
- $1/L^2$ cancels regardless of structure

Other QED prescriptions

- $\mathrm{QED_L}$: Starts at $1/L^3$ with structure dependence

$$\Delta\hat{\Pi}(q^2) \overset{\mathrm{QED_L}}{\propto} \frac{c_0}{(m_\pi L)^3}, \qquad c_0 = -1$$

- $\mathrm{QED_C}$:

$$\Delta\hat{\Pi}(q^2) \overset{\mathrm{QED_C}}{\propto} \frac{c_0^\star}{(m_\pi L)^3}, \qquad c_0^\star = 0$$

- So $\mathrm{QED_C}$ expected to have smaller FVEs than $\mathrm{QED_L}$ here
- $\mathrm{QED_r}$ [Di Carlo, Hansen, NHT, Portelli In prep.]: Designed to remove c_0

$$\Delta\hat{\Pi}(q^2) \overset{\mathrm{QED_r}}{\propto} \frac{\bar{c}_0}{(m_\pi L)^3}, \qquad \bar{c}_0 = 0$$

- Leading scaling the same in $\mathrm{QED_r}$ as $\mathrm{QED_C}$
- Approach of [Biloshytskyi et al. 23]: Exponentially suppressed

IIa. Numerical validation in scalar QED

Numerical validation

Lattice scalar QED simulations:

- Eight different volumes with L/a between 16 and 64;
  Time extent $T/a = 128$
- $z = q^2/m^2 = 0.964$, $am = 0.2$

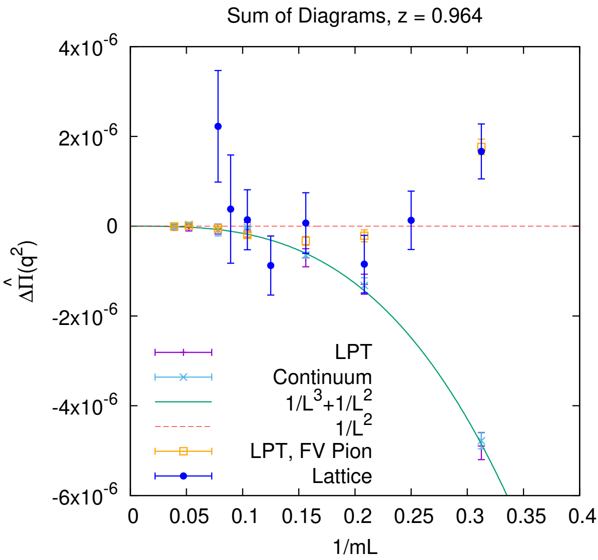
Lattice perturbation theory (LPT): $k_\mu,\, \ell_\mu \in (-\pi/a, \pi/a)$

- `Cuba VEGAS` Monte Carlo integration for both FV and IV pions

$$\left(\frac{1}{L^3} \sum_{\mathbf{k} \neq \mathbf{0}} \frac{1}{L^3} \sum_{\boldsymbol{\ell}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \right) \int \frac{dk_0}{2\pi} \frac{d\ell_0}{2\pi}$$

- We do this for several lattice spacings and then make a continuum extrapolation

Numerical validation

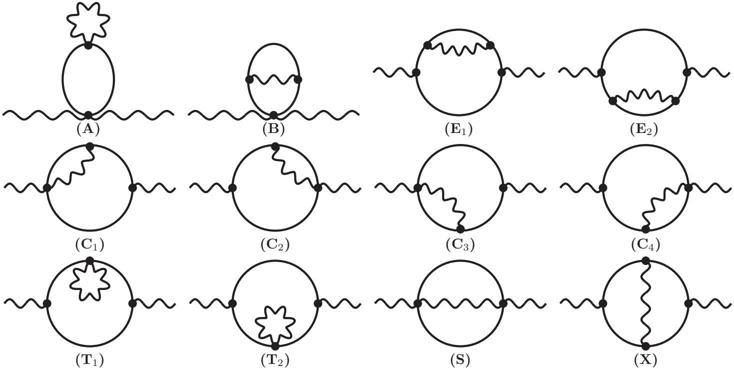


Numerical validation

- For $m_\pi L \gtrsim 4$ agreement with our scalar QED expectation: $1/L^3$
- One has $1/4^3 \sim 1.6\%$ and $e^{-4} \sim 1.8\%$, so naively similar size
- $1/(m_\pi L)^3 \lesssim 1.6\%$ on $\mathcal{O}(\alpha) \sim 1\%$ corrections to the HVP
- This was for pointlike scalar QED
- In $\mathrm{QCD{+}QED_L}$: [RBC/UKQCD 18; ETMC 19; BMW 21; Mainz 22]
- More work expected in future in also non-$\mathrm{QED_L}$ [Biloshytskyi et al.; RC∗]
- What about determining the leading structure dependence?

IIb. Leading structure dependence

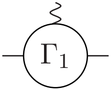
Diagrams including QED



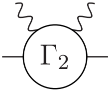
- What about determining the leading structure dependence in $\mathrm{QED_L}$?
- Should see $1/L^2$ cancellation and get estimate of $1/L^3$ coefficient
- NB: Preliminary

Structure dependence

- Structure-dependent vertex functions:



$= \Gamma_\mu(p,k)\,,$



$= \Gamma_{\mu\nu}(p,k,q)$

- Now we need form factor decompositions [Fearing, Scherer 94/96]:

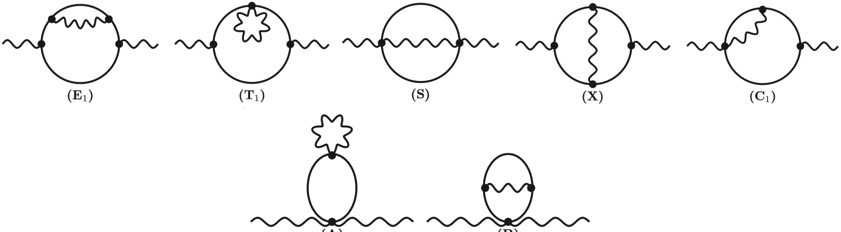
$$\Gamma_\mu(p,k) = (2p+k)_\mu\, F(k^2) + k_\mu \frac{(p+k)^2 - p^2}{k^2}\,[1 - F(k^2)]$$

$$\Gamma_{\mu\nu}(p,k,q) = 2\delta_{\mu\nu}[1 - F(k^2) - F(q^2)] - 2\,k_\mu k_\nu\,\frac{1-F(k^2)}{k^2} - 2\,q_\mu q_\nu\,\frac{1-F(q^2)}{q^2} + \Gamma^{\mathrm{T}}_{\mu\nu}(p,k,q)$$

- $\Gamma^{\mathrm{T}}_{\mu\nu}(p,k,q)$ depends on 5 form-factors: $\widetilde{C}_1$, $\widetilde{D}$, $\widetilde{E}$, $\bar{F}$, $\bar{G}$
- Physically: EM polarisabilities α, β + ...

Structure dependence

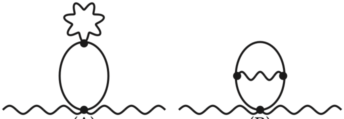
- Now we have 7 diagram topologies:



- A and B depend on q: $\hat{\Pi}_{A,B}(q^2) = \Pi_{A,B}(q^2) - \Pi_{A,B}(0) \neq 0$

$$\Gamma_{\mu\nu}(p,k,q) = 2\delta_{\mu\nu}[1 - F(k^2) - F(q^2)] - 2\,k_\mu k_\nu \frac{1-F(k^2)}{k^2} - 2\,q_\mu q_\nu \frac{1-F(q^2)}{q^2} + \Gamma^{\mathrm{T}}_{\mu\nu}(p,k,q)$$

Structure dependence



$$\Delta\hat{\Pi}_B = \left(\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right)\int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3}\int \frac{dk_0}{2\pi}\frac{d\ell_0}{2\pi}\,\hat{\pi}_B\left(q_0^2, k, \ell\right)$$

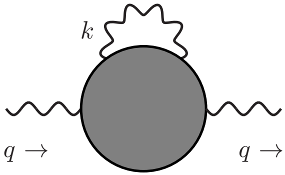
- Example: Diagram B

$$\hat{\pi}_B\left(q_0^2, k, \ell\right) = \frac{\Gamma_\mu(\ell, k)\Gamma_\mu(k+\ell, -\ell)}{3q_0^2\, k^2\,[\ell^2+m^2]^2\,[(k+\ell)^2+m^2]}\,\left[\Gamma_{jj}(\ell, q, q) - \Gamma_{jj}(\ell, 0, 0)\right]$$

- Not only poles from propagators in k_0, ℓ_0: Cuts in form factors

Cuts

- Let us go back to the HLbL tensor



$$\Delta\hat{\Pi}(q_0^2) = \left(\frac{1}{L^3}\sum_{\mathbf{k}\neq\mathbf{0}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right)\int \frac{dk_0}{2\pi}\frac{\hat{\Pi}^{\mu\mu\nu\nu}(q,q,k,-k)}{k^2}$$

- Analytical structure: Dispersion relations [Colangelo et al. 2015; Biloshytskyi et al. 23]
- Relation between 2π intermediate state and Feynman diagrams
- Need to figure out cut contributions from dispersive representation

Strategy

- Structure dependent vertex functions
- Separate

$$\Delta\hat{\Pi}(q_0^2) = \Delta_{\mathrm{poles}}\hat{\Pi}(q_0^2) + \Delta_{\mathrm{cuts}}\hat{\Pi}(q_0^2)$$

- Should analytically see universality of $1/L^2$ cancellation
- Should estimate $1/L^3$
- For now focus on $\Delta_{\mathrm{poles}}\hat{\Pi}(q_0^2)$

Strategy

- Old integrals:

$$\Omega_{i,j}(z = q_0^2/m^2) = \int dx\, x^2 \frac{1}{(1+x^2)^{i/2}[z+4(x^2+1)]^j}$$

- New integrals for $1/L^3$:

$$\bar{\Omega}_{i,j}(z = q_0^2/m^2) = \int dx\, x^2 \frac{A_1\left(\frac{x}{\sqrt{1+x^2}}\right)}{(1+x^2)^{i/2}[z+4(x^2+1)]^j}$$

$$A_1(x) = \frac{\mathrm{arctanh}(x)}{x}$$

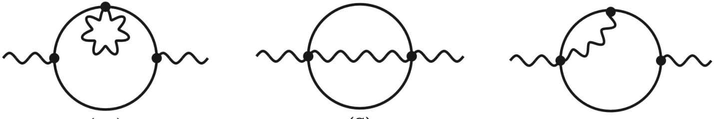
- Can do whole calculation now: Preliminary

Preliminary results

$$\Delta_{\text{poles}}\hat{\Pi}_{AB}(q_0^2) = \frac{c_1}{16\pi z\,(mL)^2}\Bigg\{8\,\widetilde{C}_1\,m^4 z\,(2\,\Omega_{3,0} - \Omega_{5,0}) + 12\,\Omega_{5,0}\,F\left(m^2 z\right) + 3\,\Omega_{5,0}\left(\bar{G}_0\,m^2 z - 4\right)\Bigg\} + \mathcal{O}\left[1/(mL)^3, e^{-mL}\right]$$

- Form factors
- Pointlike limit gives zero again

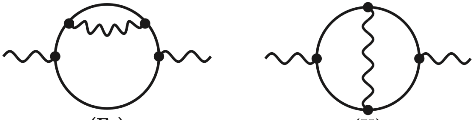
Preliminary results



$$\Delta_{\rm poles}\hat{\Pi}_{TSC}(q_0^2) = \frac{c_1}{12\pi z\,(mL)^2}\Big\{3\Big[8(3z-8)\,\Omega_{1,2} + z(3z-32)\,\Omega_{3,2} - 4z^2\Omega_{5,2} + 48\Omega_{-1,2}\Big] - 4\Big[(z-48)\Omega_{1,2} - 4z\,\Omega_{3,2} + 36\,\Omega_{-1,2}\Big] F\left(m^2 z\right)^2\Big\} + \mathcal{O}\Big[1/(mL)^3, e^{-mL}\Big]$$

- Form factors
- Pointlike limit gives back old result

Preliminary results



$$\Delta_{\text{poles}}\hat{\Pi}_{EX}(q_0^2) = \frac{c_1}{24\pi z\,(mL)^2}\Bigg\{4\Big[4z^2\,\Omega_{1,3} - 7z^2\,\Omega_{3,3} + 3z^2\,\Omega_{5,3} - 96z\,\Omega_{1,3}$$
$$+ 40z\,\Omega_{3,3} + 8(7z-66)\,\Omega_{-1,3} + 288\Omega_{-3,3} + 240\,\Omega_{1,3}\Big]\,F\left(m^2 z\right)^2$$
$$- 3\Big[6z^3\,\Omega_{3,3} - 11z^3\,\Omega_{5,3} + 5z^3\,\Omega_{7,3} + 72z^2\,\Omega_{1,3} - 132z^2\,\Omega_{3,3}$$
$$+ 60z^2\,\Omega_{5,3} - 528z\,\Omega_{1,3} + 240z\,\Omega_{3,3} + 32(9z-22)\,\Omega_{-1,3}$$
$$+ 384\,\Omega_{-3,3} + 320\,\Omega_{1,3}\Big]\Bigg\} + \mathcal{O}\Big[1/(mL)^3,\, e^{-mL}\Big]$$

- Form factors
- Pointlike limit gives back old result
- $1/L^3$ depends on charge radius $\langle r_\pi^2 \rangle$ as well, and $\bar{\Omega}_{i,j}$

Preliminary results

- Have done pole part $\Delta_{\text{poles}}\hat{\Pi}(q_0^2)$
- Recall

$$\Delta\hat{\Pi}(q_0^2) = \Delta_{\text{poles}}\hat{\Pi}(q_0^2) + \Delta_{\text{cuts}}\hat{\Pi}(q_0^2)$$

- Can be cancellations between $\Delta_{\text{poles}}\hat{\Pi}(q_0^2)$ and $\Delta_{\text{cuts}}\hat{\Pi}(q_0^2)$
- Very intricate cancellations between diagrams in $\Delta_{\text{poles}}\hat{\Pi}(q_0^2)$

Conclusions and outlook

- $\mathrm{QED_L}$: HVP starts at order $c_0/(mL)^3$
- Proven in [Bijnens, Harrision, Hermansson-Truedsson, Janowski, Jüttner, Portelli 19]
- Physics: Photon far away does not see charge of $\pi\pi$ pair
- New steps towards evaluating structure-dependent $c_0/(mL)^3$
- Should see cancellation and estimate $1/L^3$
- Finite-volume scaling depends on QED formulation
- $\mathrm{QED_r}/\mathrm{QED_C}$: Starts at order $1/(mL)^4$

Backup slides

Numerical validation

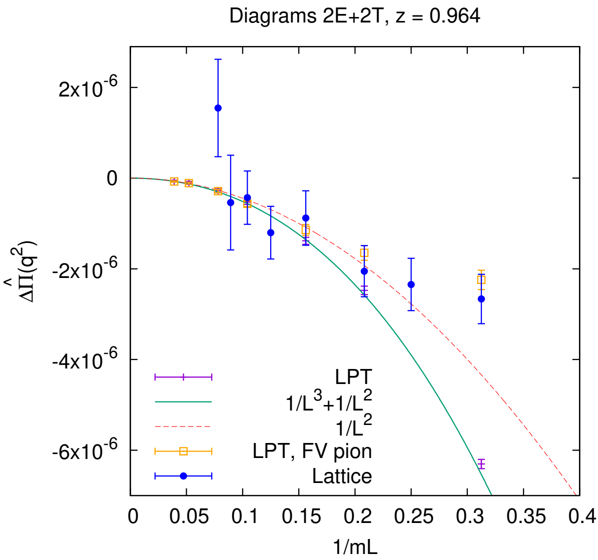
- Action $S[\phi, A] = S_\phi[\phi, A] + S_A[A]$

$$S_\phi\left[\phi, A\right] = \frac{a^4}{2}\sum_x \phi^*(x)\Delta\phi(x),$$

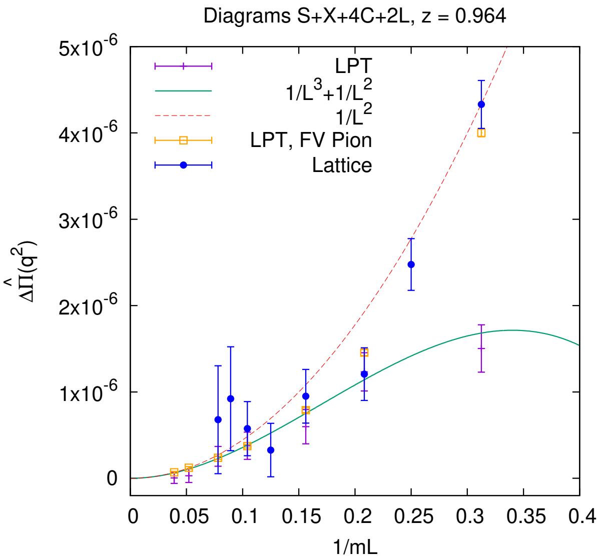
$$S_A[A] = -\frac{a^4}{2}\sum_{x,\mu} A_\mu(x)\delta^2 A_\mu(x)\,,$$

$$\Delta = m^2 - \sum_\mu D^*_\mu D_\mu$$

Numerical validation



Numerical validation



Charged currents

$$\Delta\hat{\Pi}_{\text{charged}}\left(q^2\right) = \frac{1}{m^3L^3}\left(-\frac{13}{24}\Omega_{2,2} + \frac{20}{9}\Omega_{2,3} - \frac{15}{64}\Omega_{4,1} + \frac{7}{24}\Omega_{4,2} + \frac{4}{9}\Omega_{4,3}\right) + \frac{c_1}{m^2L^2\pi}\left(-\frac{8}{3}\Omega_{-1,3} + \Omega_{1,2} + \frac{8}{3}\Omega_{1,3} + \frac{1}{8}\Omega_{3,1}\right) + \mathcal{O}\left(\frac{1}{L^4}, e^{-mL}\right)$$

Minimal choice: $\mathrm{QED_r}$

- Want $c_0^w = 0$

$$c_0^w = \underbrace{c_0}_{=-1} + \sum_{|\mathbf{n}|} w_{|\mathbf{n}|^2}$$

- Solution: $w_{|\mathbf{n}|^2} = \delta_{|\mathbf{n}|,1}/6$

$$\bar{c}_0 = 0$$

- Why it works: $|\mathbf{k}| \propto 1/L$
- Generally:

$$\bar{c}_j(\mathbf{v}) = \underbrace{c_j(\mathbf{v})}_{\mathrm{QED_L}} + \frac{1}{6} \sum_{|\mathbf{n}|=1} \frac{1}{(1 - \mathbf{v} \cdot \hat{\mathbf{n}})}$$