MITP, July 2024

Isospin-breaking effects with $C^\star$ boundary conditions

RC*star collaboration

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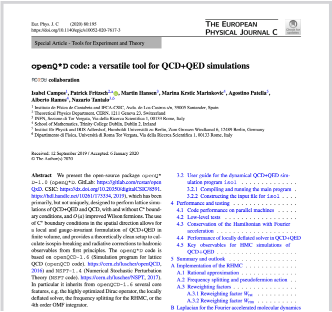

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Special Article - Tools for Experiment and Theory

openQ*D code: a versatile tool for QCD+QED simulations

RC*QED collaboration

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Abstract We present the open-source package openQ*D-1.0 (openQ*D. GitLab: https://gitlab.com/rcstar/openQxD. CSIC: https://dx.doi.org/10.20350/digitalCSIC/8591. https://hdl.handle.net/10261/173334, 2019), which has been primarily, but not uniquely, designed to perform lattice simulations of QCD+QED and QCD, with and without C* boundary conditions, and $O(a)$ improved Wilson fermions. The use of C* boundary conditions in the spatial direction allows for a local and gauge-invariant formulation of QCD+QED in finite volume, and provides a theoretically clean setup to calculate isospin-breaking and radiative corrections to hadronic observables from first principles. The openQ*D code is based on openQCD-1.6 (Simulation program for lattice QCD (openQCD code). https://cern.ch/luscher/openQCD, 2016) and NSPT-1.4 (Numerical Stochastic Perturbation Theory (NSPT code). https://cern.ch/luscher/NSPT, 2017). In particular it inherits from openQCD-1.6 several core features, e.g. the highly optimized Dirac operator, the locally deflated solver, the frequency splitting for the RHMC, or the 4th order OMF integrator.

- Isospin transformations (i.e. unitary transformations of the up/down doublet) are are approximated symmetries of Nature.
- Isospin symmetry is broken by $m_u \neq m_d$ and $q_u \neq q_d$.
- Isospin-breaking effects are typically of order 1% on hadronic observables.
- In order to calculate hadronic observables at the percent or subpercent precision level, one needs to consider QCD+QED.

Introduction

- Often, electroquenched approximation (i.e. valence quarks are charged, sea quarks are neutral)! We need to do better.
- Including the sea-quark effects is difficult. Full simulations or RM123 method?
- RC* collaboration: full QCD+QED simulations with C* boundary conditions.
- RC* collaboration: on-going detailed study of variance of sea-quark effects in RM123 method.

(Quick) theoretical intro

Two ways for QCD+QED on the lattice

1. RM123 method

G. M. de Divitiis *et al.* [RM123], "Leading isospin breaking...," Phys.Rev.D 87 (2013) 11, 114505.

$$\begin{aligned} S_{\mathrm{QCD+QED}} =& S_{\mathrm{QCD}} + S_\gamma + \frac{\delta\beta}{\beta} S_{gluon} + \sum_{xf} \delta m_f \, \bar{\psi}_f \psi_f(x) \\ &+ e \sum_{x\mu} A_\mu(x) \mathcal{J}_\mu(x) + e^2 \sum_{x\mu} A_\mu(x)^2 \mathcal{T}_\mu(x) + O(e^3) \end{aligned}$$

Pros:

- Calculate directly isospin-breaking and radiative correction to QCD (10% precision is enough).
- Reuse QCD configurations (careful with the finite-volume effects).
- Tuning is trivial: QED counterterms are calculated by solving linear equations.

Cons:

- Complicated observables, quark-disconnected pieces, expensive variance-reduction techniques.
- Correction-to-QCD noise ratio diverges with $V^{1/2}$ and a^{-2}. Bad scaling with V can be killed with coordinate-space techniques, bad scaling with a is irreducible.

Gluon and photon fields are treated on equal footing. Fully interacting $SU(3) \times U(1)$ configurations are generated. Used in:

S. Borsanyi *et al.* [BMW], "Ab initio calculation...," Science 347 (2015), 1452-1455.

R. Horsley *et al.* [QCD-SF], "QED effects...," JHEP 04 (2016), 093. R. Horsley *et al.* [QCD-SF], "Isospin splittings of meson and baryon masses...," J.Phys.G 43 (2016) 10, 10LT02.

A. Altherr *et al.* [RC*], "First results on QCD+QED with C* boundary conditions," JHEP 03 (2023), 012, 1452-1455.

Pros:

- Standard algorithms can be used.
- Simpler observables.
- The scaling of the noise in QCD+QED with V and a is like in QCD.

Cons:

- Expensive simulations.
- Observables need to be calculated at the permille precision level.
- Up and down quark masses need to be tuned independently.

Charged states

The Gauss's law forbids charged states with periodic boundary conditions:

$$Q = \int_0^L d^3x\, \rho(\boldsymbol{x}) = \int_0^L d^3x\, \nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = 0 \ .$$

Some popular solutions:

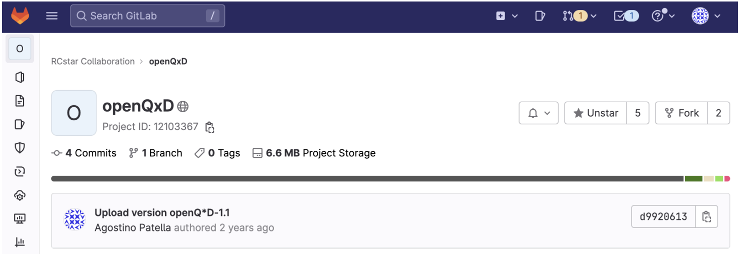
- $\mathrm{QED_L}$: non-local constraint $\int d^3x\, A_\mu(t, \boldsymbol{x}) = 0$.
- $\mathrm{QED_m}$: massive photon.
- $\mathrm{QED_\infty}$: (only with RM123) reconstruct infinite-volume QCD n-point functions and integrate them with infinite-volume photon propagators.
- $\mathrm{QED_C}$: C-periodic boundary conditions in space $\phi(t, \boldsymbol{x} + L\boldsymbol{e}_k) = \phi^C(t, \boldsymbol{x})$.

Some properties of $\mathrm{QED_C}$:

- Continuum limit described by Symanzik effective theory (like $\mathrm{QED_m}$).
- Leading finite-volume effects dominated by low-energy states (like $\mathrm{QED_m}$).
- Power-like finite-volume effects to single-particle masses and matrix elements (like $\mathrm{QED_L}$).
- Incompatible with θ-periodic boundary conditions.
- Partially-broken flavour symmetry.

Full QCD+QED simulations

openQ*D code



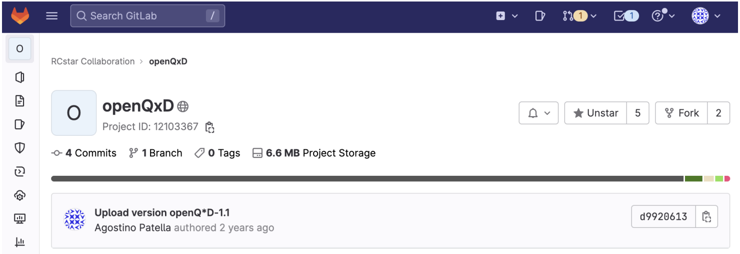
https://gitlab.com/rcstar/openQxD

- Extension of openQCD-1.6
- Simulation of QCD and QCD+QED
- $C^\star$ boundary conditions in space
- Compact photon action
- Wilson flow for photon field
- Fourier acceleration for photon field
- Multiple deflation subspaces
- U(1)-invariant quark propagators

See talk at Lattice2024: Roman Gruber, $O(a)$-improved QCD+QED Wilson Dirac operator on GPUs, Software Development and Machines, 30 July 2024 14:25

I. Campos *et al.*, "openQ*D code: a versatile tool for QCD+QED simulations," Eur.Phys.J.C 80 (2020) 3, 195.

openQ*D code



`https://gitlab.com/rcstar/openQxD`

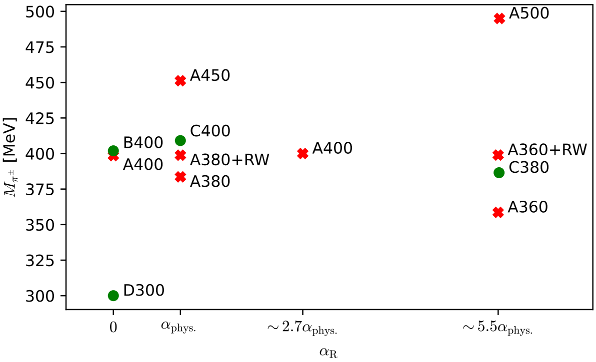
Coming up (testing phase):

- Sign of determinant/Pfaffian
- Mass reweighting
- Interface to QUDA solvers
- Various noise-reduction techniques for RM123 method (hopping expansion, frequency splitting, truncated solvers, expandable...)

See talk at Lattice2024: Roman Gruber, $O(a)$-improved QCD+QED Wilson Dirac operator on GPUs, Software Development and Machines, 30 July 2024 14:25

I. Campos *et al.*, "openQ*D code: a versatile tool for QCD+QED simulations," Eur.Phys.J.C 80 (2020) 3, 195.

Ensembles

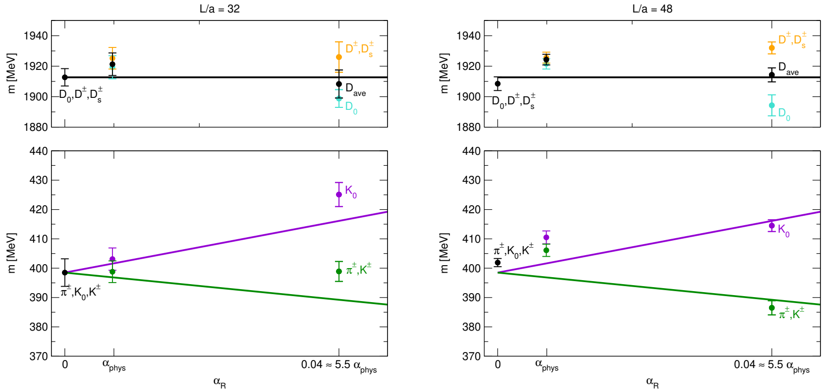


$a \simeq 0.05$ fm
u+d+s+c quarks

A $= 64 \times 32^3$
B $= 80 \times 48^3$
C $= 96 \times 48^3$
D $= 128 \times 64^3$

$m_\pi L \lesssim 4$
$m_\pi L \gtrsim 5$

Meson masses



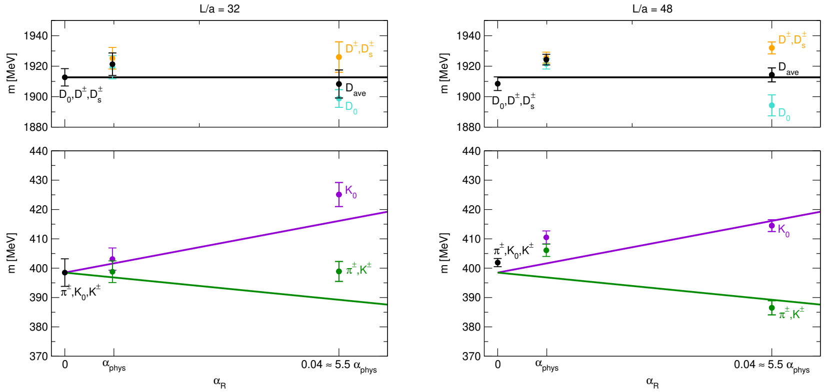
$$\phi_0 = 8t_0(M_{K^\pm}^2 - M_{\pi^\pm}^2) = 0$$

$$\phi_1 = 8t_0(M_{\pi^\pm}^2 + M_{K^\pm}^2 + M_{K_0}^2) \simeq \phi_1^{\text{phys}}$$

$$\phi_2 = 8t_0\alpha_R^{-1}(M_{K_0}^2 - M_{K^\pm}^2) \simeq \phi_2^{\text{phys}}$$

$$\phi_3 = \sqrt{8t_0}(M_{D_0} + M_{D^\pm} + M_{D_s^\pm}) \simeq \phi_3^{\text{phys}}$$

Meson masses



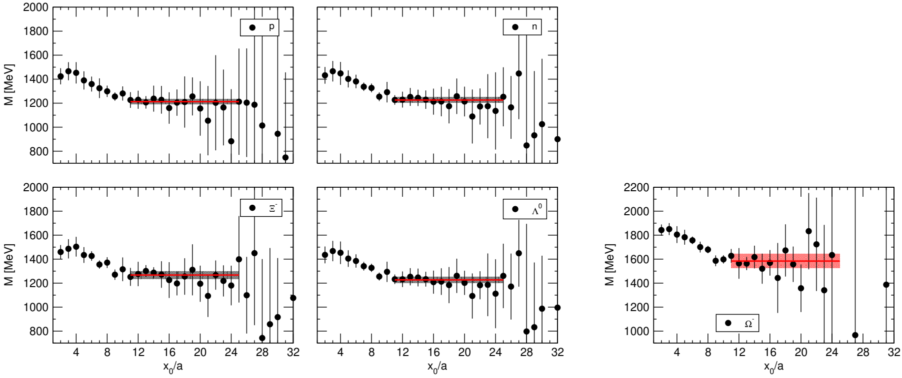
$$M(L) = M(\infty) - \frac{\alpha_R q^2 c_1}{2L} - \frac{\alpha_R q^2 c_2}{2ML^2} + O\left(\frac{1}{L^4}\right)$$

Universal FV correction for $K^\pm$ at $\alpha_R \simeq 5.6\alpha_{\text{phys}}$

$L/a = 32$: $1.09(1)\% + 0.308(8)\%$

$L/a = 48$: $0.751(4)\% + 0.145(2)\%$

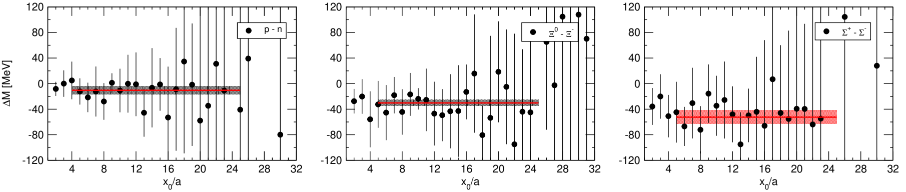
Baryon masses



$a \simeq 0.05\text{fm}$ $\alpha_R \simeq 0.04$ $M_{\pi^\pm} \simeq 400\text{MeV}$ $M_{\pi^\pm} L \simeq 4$

2000 configurations, 4 stochastic sources per configuration, smearing

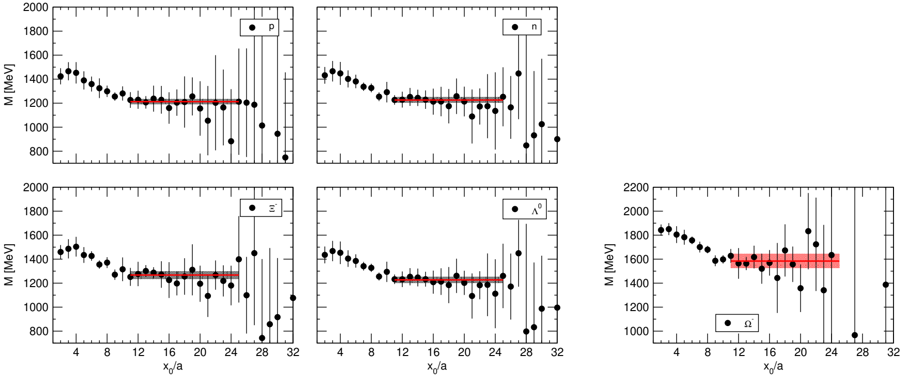
Baryon masses



$a \simeq 0.05\text{fm} \qquad \alpha_R \simeq 0.04 \qquad M_{\pi^\pm} \simeq 400\text{MeV} \qquad M_{\pi^\pm} L \simeq 4$

2000 configurations, 4 stochastic sources per configuration, smearing

Baryon masses

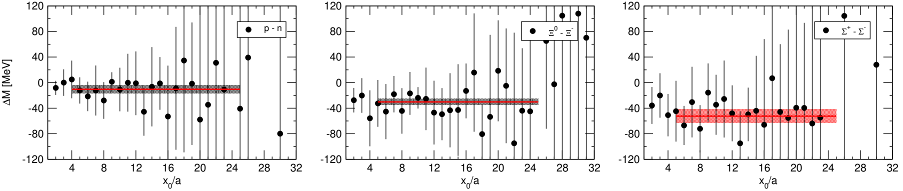


$a \simeq 0.05\text{fm}$ $\quad$ $\alpha_R \simeq 0.04$ $\quad$ $M_{\pi^\pm} \simeq 400\text{MeV}$ $\quad$ $M_{\pi^\pm} L \simeq 4$

2000 configurations, 4 stochastic sources per configuration, smearing

See poster at Lattice2024: Sara Rosso, Partially connected contributions to baryon masses in QCD+QED, 30 July 2024 18:15

Baryon masses



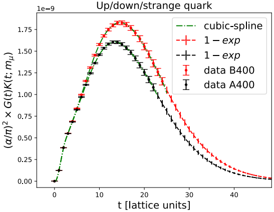
$a \simeq 0.05\text{fm}$ $\alpha_R \simeq 0.04$ $M_{\pi^\pm} \simeq 400\text{MeV}$ $M_{\pi^\pm} L \simeq 4$

2000 configurations, 4 stochastic sources per configuration, smearing

See poster at Lattice2024: Sara Rosso, Partially connected contributions to baryon masses in QCD+QED, 30 July 2024 18:15

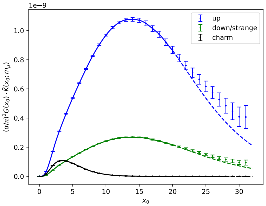
Isospin-breaking corrections to HVP

QCD only ensembles



type	$a_\mu^{u/d/s} \times 10^{-10}$	$am_V^{u/d/s}$	$a_\mu^c \times 10^{-10}$	am_V^c
Ensemble A400a00b324				
ll	338(8)	0.2644(50)	7.83(8)	0.8463(5)
cl	334(9)	0.2652(55)	6.18(7)	0.8462(5)
Ensemble B400a00b324				
ll	402(9)	0.2522(33)	7.81(9)	0.8458(9)
cl	397(9)	0.2530(32)	6.16(7)	0.8454(8)

QCD+QED ensembles



type	$a_\mu^u \times 10^{10}$	am_V^u	$a_\mu^{d/s} \times 10^{10}$	$am_V^{d/s}$	$a_\mu^c \times 10^{10}$
Ensemble A380a07b324, $\alpha_R = 0.007$					
ll	–	–	–	–	–
cl	331(7)	0.266(4)	83(2)	0.265(6)	9.78(10)
cc	–	–	–	–	–
Ensemble A360a50b324, $\alpha_R = 0.040633(80)$					
cl	309(11)	0.267(8)	77(2)	0.262(7)	10.62(11)

In all cases, tails are fitted to a single exponential after ~ 1.2 fm.

See talk at Lattice2024: Letizia Parato, Update on the isospin breaking corrections to the HVP with C-periodic boundary conditions, Hadronic and nuclear spectrum and interactions, 1 August 2024 12:10

Isospin-breaking corrections to HVP

1. Corrections from Z

$$\delta Z_V a_\mu^{\mathrm{HVP},u} = (-510(32)\Delta m_u - 22(3)e^2) \times 10^{-10}$$

$$\delta Z_V a_\mu^{\mathrm{HVP},d/s} = (-128(8)\Delta m_d - 1.4(2)e^2) \times 10^{-10}$$

$$\delta Z_V a_\mu^{\mathrm{HVP},c} = (-6.37(2)\Delta m_c - 0.578(2)e^2) \times 10^{-10}$$

2. Corrections from G
   - $G^{ll}(t)$ – local-local case

$$\delta_G a_\mu^{HVP,u} = (-4364(266)\Delta m_u - 216(14)e^2) \times 10^{-10}$$

$$\delta_G a_\mu^{HVP,d/s} = (-1091(67)\Delta m_d - 13.5(9)e^2) \times 10^{-10}$$

$$\delta_G a_\mu^{HVP,c} = (-59.2(3)\Delta m_c - 3.119(13)e^2) \times 10^{-10}$$

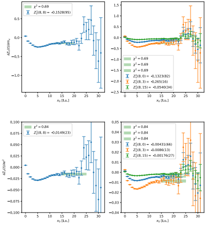
   - $G^{cl}(t)$ – conserved-local case

$$\delta_G a_\mu^{HVP,u} = (-4591(288)\Delta m_u - 227(15)e^2) \times 10^{-10}$$

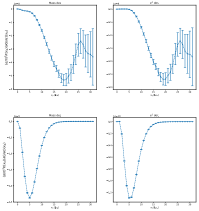
$$\delta_G a_\mu^{HVP,d/s} = (-1148(72)\Delta m_d - 14.2(1.0)e^2) \times 10^{-10}$$

$$\delta_G a_\mu^{HVP,c} = (-57.2(2)\Delta m_c - 3.295(14)e^2) \times 10^{-10}$$

δa_μ from G larger than those from Z



Derivatives of $Zv(8,8)$ (left) and mixed components $Zv(8,0)$, $Zv(8,3)$, $Zv(8,15)$ (right) over m_u (top) or α (bottom)



Derivatives of G^{up} (top) and G^{charm} (bottom) over lass (left) and α (right)

RM123 method

$$
\begin{aligned}
D(e) = D(e=0) &+ \delta m \\
&+ i\frac{eq}{2}\sum_\mu (A_\mu H_\mu - \bar{H}_\mu A_\mu) \\
&- \frac{e^2 q^2}{4}\sum_\mu (A_\mu^2 H_\mu + \bar{H}_\mu A_\mu^2) \\
&- \frac{eq c_{\mathrm{sw}}^{\mathrm{U(1)}}}{4}\sum_{\mu\nu} \sigma_{\mu\nu} F_{\mu\nu}^{\mathrm{sym}} - \frac{\delta c_{\mathrm{sw}}^{\mathrm{SU(3)}}}{4}\sum_{\mu\nu}\sigma_{\mu\nu}\widehat{G}_{\mu\nu} \\
&+ O(e_0^3)
\end{aligned}
$$

Hopping operators:

$$
H_\mu \chi(x) = (1-\gamma_\mu) U(x,\mu)\chi(x+\hat{\mu}) \qquad \bar{H}_\mu \chi(x) = (1+\gamma_\mu) U(x-\hat{\mu},\mu)^\dagger \chi(x-\hat{\mu})
$$

$$
\begin{aligned}
D(e) = D(e=0) &+ \boxed{\delta m} \\
&\boxed{+\, i\frac{eq}{2}\sum_{\mu}(A_\mu H_\mu - \bar{H}_\mu A_\mu)} \\
&\boxed{-\, \frac{e^2 q^2}{4}\sum_{\mu}(A_\mu^2 H_\mu + \bar{H}_\mu A_\mu^2)} \\
&\boxed{-\, \frac{eq c_{\mathrm{sw}}^{\mathrm{U(1)}}}{4}\sum_{\mu\nu}\sigma_{\mu\nu}F_{\mu\nu}^{\mathrm{sym}} - \frac{\delta c_{\mathrm{sw}}^{\mathrm{SU(3)}}}{4}\sum_{\mu\nu}\sigma_{\mu\nu}\widehat{G}_{\mu\nu}} \\
&+ O(e_0^3)
\end{aligned}
$$

mass shift 

current 

tadpole 

improvement 

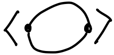
in practice $\delta c_{\mathrm{sw}}^{\mathrm{SU(3)}} = 0$, $c_{\mathrm{sw}}^{\mathrm{U(1)}} = 1$

Hopping operators:

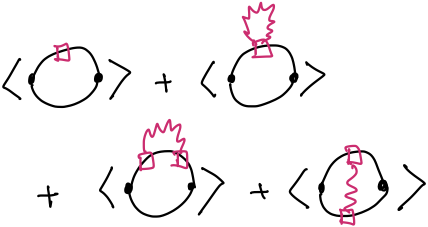
$$
H_\mu \chi(x) = (1-\gamma_\mu)U(x,\mu)\chi(x+\hat{\mu}) \qquad \bar{H}_\mu \chi(x) = (1+\gamma_\mu)U(x-\hat{\mu},\mu)^\dagger \chi(x-\hat{\mu})
$$

Expansion of pion correlator

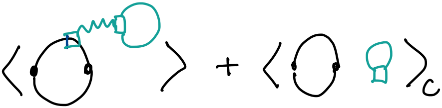
LO

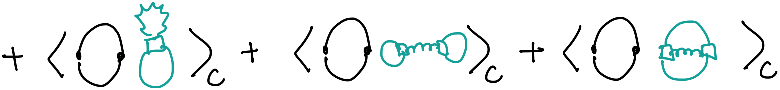


Electroquenched contributions



Sea contributions





- Consider the quantity at positive flowtime

$$A(t,x) = \frac{t^2}{2} \mathrm{tr}\, \hat{F}_{\mu\nu} \hat{F}_{\mu\nu}(t,x)$$

- In QCD simulations, both $A(t,x)$ and its variance are finite in the $V \to +\infty$ and $a \to 0$ limits.

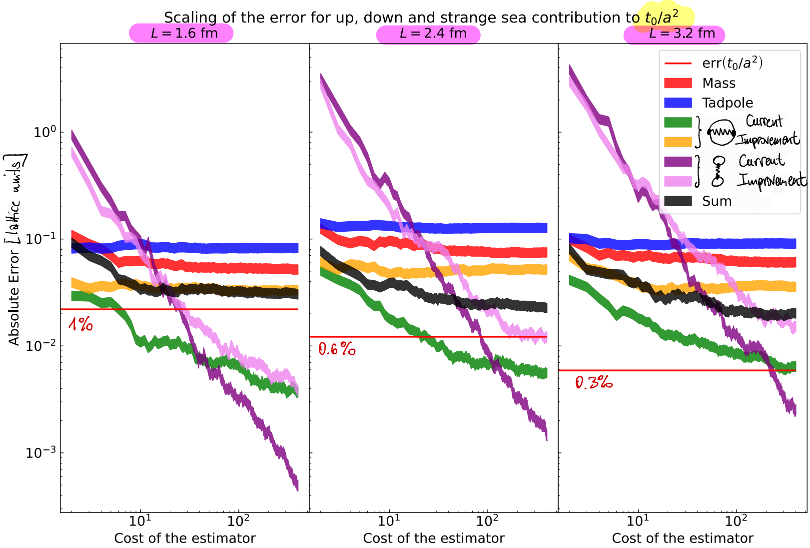
- In full QCD+QED simulations, both $A(t,x)$ and its variance are finite in the $V \to +\infty$ and $a \to 0$ limits.

- If R is one of the see diagrams of the RM123 method, and α is the configuration index, then the estimator for the RM123 insertion

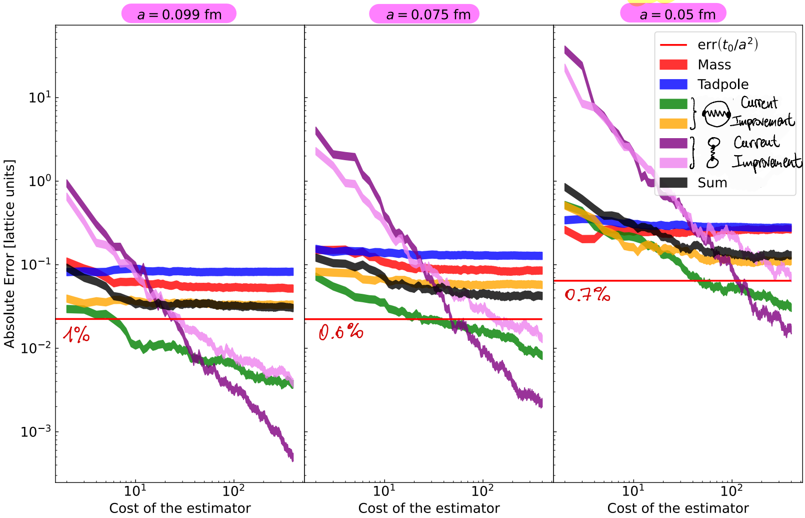
$$\frac{1}{N_g} \sum_{\alpha} A_\alpha(t,x) R_\alpha - \frac{1}{N_g(N_g - 1)} \sum_{\alpha \neq \beta} A_\alpha(t,x) R_\beta$$

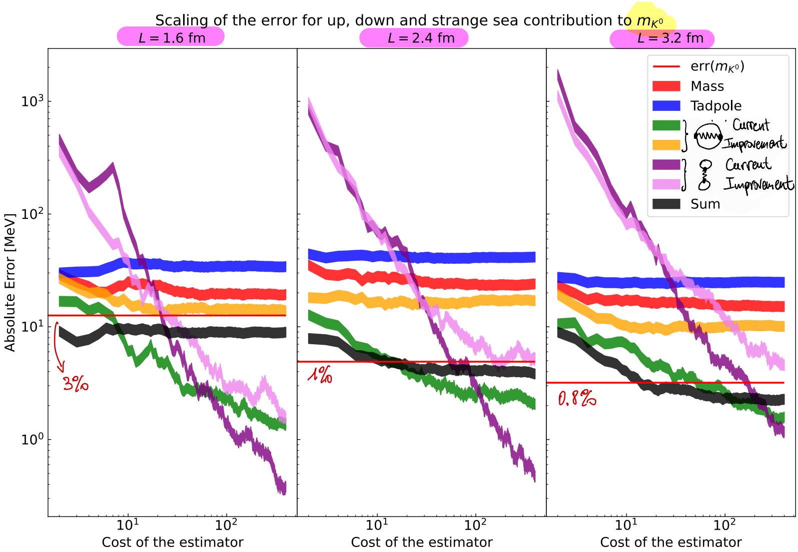
  has a variance that diverges like $V^{1/2}$ in the $V \to \infty$ limit and a^{-2} in the $a \to 0$ limit.

See talk at Lattice2024: Alessandro Cotellucci, Error Scaling of Sea Quark Isospin-Breaking Effects, Hadronic and nuclear spectrum and interactions, 1 August 2024 11:50

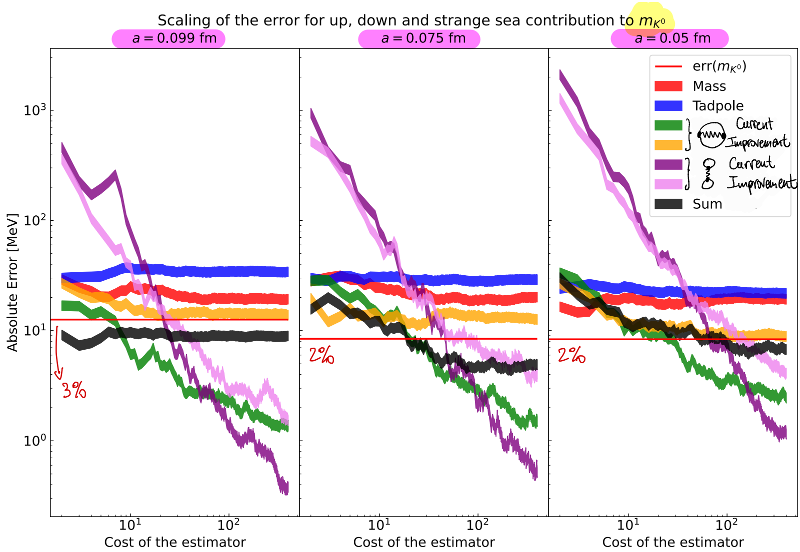

Scaling of the error for up, down and strange sea contribution to t_0/a^2
$L = 1.6$ fm
$L = 2.4$ fm
$L = 3.2$ fm
Absolute Error [lattice units]
Cost of the estimator
Cost of the estimator
Cost of the estimator
1%
0.6%
0.3%
err(t_0/a^2)
Mass
Tadpole
Current
Improvement
Current
Improvement
Sum

Scaling of the error for up, down and strange sea contribution to t_0/a^2

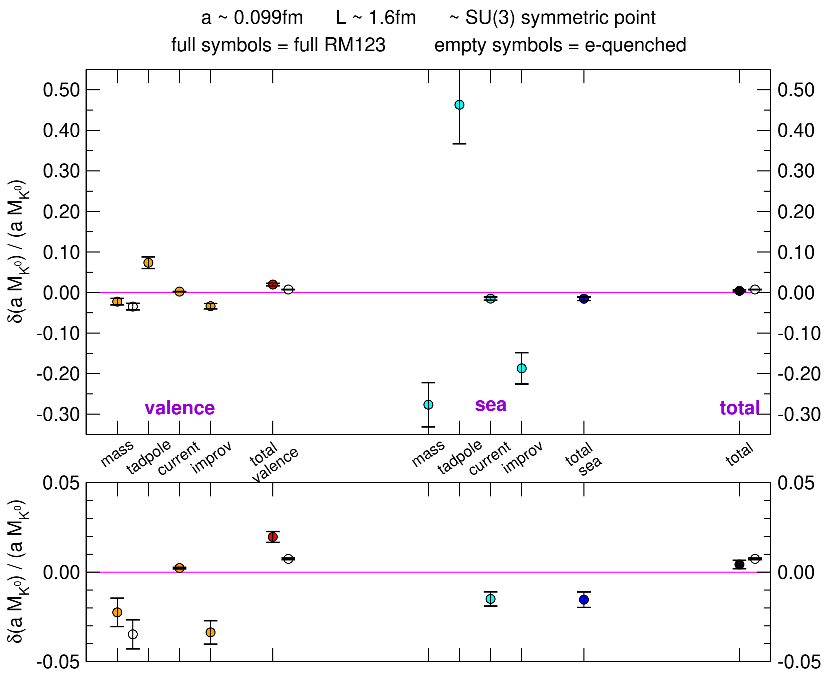




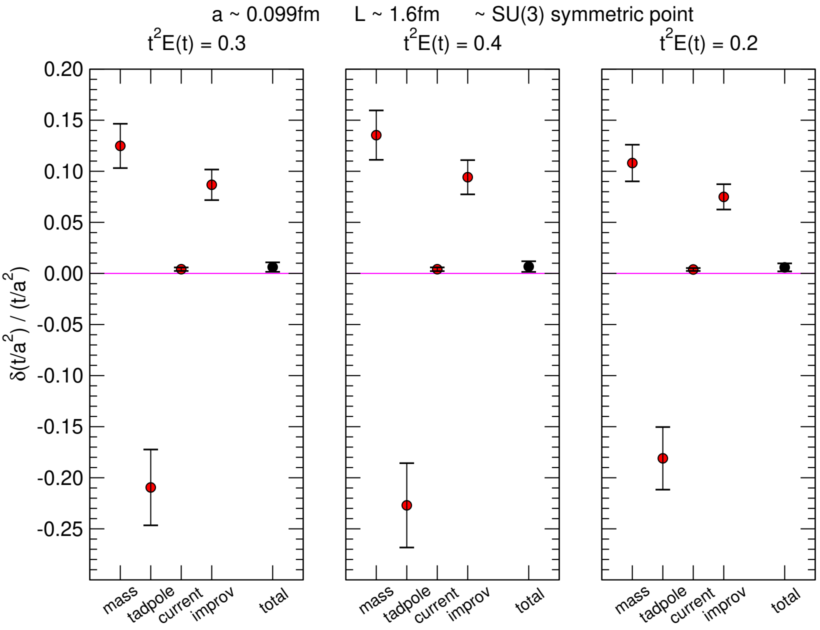
Scaling of the error for up, down and strange sea contribution to m_{K^0}
L = 1.6 fm
L = 2.4 fm
L = 3.2 fm
Absolute Error [MeV]
Cost of the estimator
err(m_{K^0})
Mass
Tadpole
Current
Improvement
Current
Improvement
Sum
3%
1%
0.8%



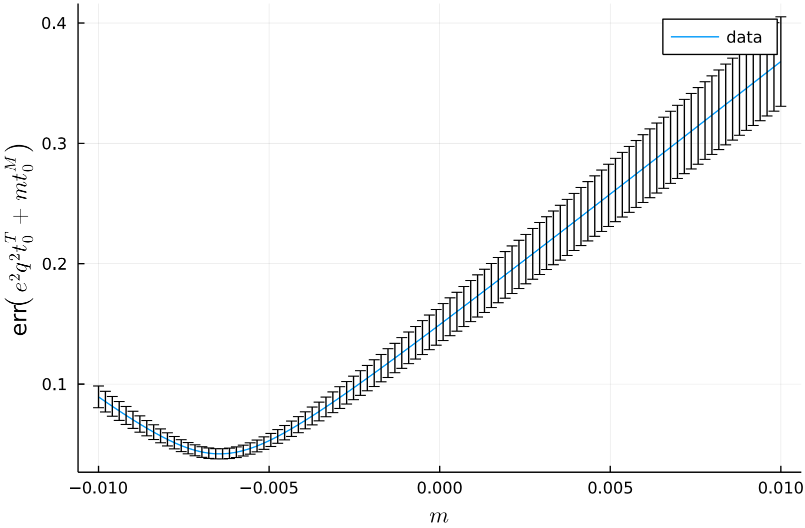
Scaling of the error for up, down and strange sea contribution to m_{K^0}
$a = 0.099$ fm
$a = 0.075$ fm
$a = 0.05$ fm
Absolute Error [MeV]
Cost of the estimator
err(m_{K^0})
Mass
Tadpole
Current
Improvement
Current
Improvement
Sum
3%
2%
2%



a ~ 0.099fm L ~ 1.6fm ~ SU(3) symmetric point
full symbols = full RM123 empty symbols = e-quenched
$\delta(a M_{K^0}) / (a M_{K^0})$
valence
sea
total
mass
tadpole
current
improv
total valence
mass
tadpole
current
improv
total sea
total
0.50
0.40
0.30
0.20
0.10
0.00
-0.10
-0.20
-0.30
0.05
0.00
-0.05



a ~ 0.099fm L ~ 1.6fm ~ SU(3) symmetric point
$t^2E(t) = 0.3$
$t^2E(t) = 0.4$
$t^2E(t) = 0.2$
$\delta(t/a^2) / (t/a^2)$
0.20
0.15
0.10
0.05
0.00
-0.05
-0.10
-0.15
-0.20
-0.25
mass
tadpole
current
improv
total



0.4
0.3
0.2
0.1
−0.010
−0.005
0.000
0.005
0.010
m
$\text{err}\left(e^2q^2t_0^T + mt_0^M\right)$
data

Summary

Full QCD+QED simulations:

- Simulations run as well/bad as QCD ones. More expensive because of $C^\star$ boundary conditions and RHMC for all quarks.
- Tuning of quark masses is difficult but not hopeless. Which precision do we need?
- Meson effective masses are obtained with a statistical precision similar to QCD. Finite-volume effects need to be quantified better.
- We calculated p, n, Ξ^-, Λ_0, Ω^- masses. Too noisy for now. We are neglecting extra Wick contractions due to $C^\star$ boundary conditions. We are working on it
- We are calculating isospin-breaking corrections to HVP contribution to muon $g-2$ on QCD+QED configurations.

RM123 method:

- Reaching the gauge noise for the sea-quark insertions is painful but can be done.
- Error of sea-quark insertions diverges asymptotically as $V^{1/2}$ and a^{-2}.
- In the considered range of parameters we see the $V^{1/2}$ quite well, while the lattice spacing behaviour seems better than the asymptotic one. More statistics is needed...
- Good news: large cancellation of errors among various constributions. Needs more investigation...

Backup

IB corrections - Last updates from Zürich

RC* collaboration

July 21, 2024

Comparing two methods for calculating Isospin Breaking Effects

Goal: Cross-validate and compare costs and challenges of two approaches to compute IB effects at fixed lattice spacing and volume:

1. Direct QCD+QED with dynamical U(1) and $m_u \neq m_d$
2. IsoQCD + RM123: perturbative expansion in $m_d - m_u$ and α_{QED}, including all sea effects

Setup: 2 ensembles with Wilson fermions, $O(a)$ improved action with coeff. $c_{\text{sw}}^{SU(3)} = 2.18859$ and $c_{\text{sw}}^{U(1)} = 1$, *same volume and* β, but *different* κ_q *and* α:

ensemble	lattice	β	α	κ_u	$\kappa_d = \kappa_s$	κ_c
A400a00b324	64×32^3	3.24	0	0.13440733	0.13440733	0.12784
A380a07b324	64×32^3	3.24	0.007299	0.13457969	0.13443525	0.12806355
$\delta m_q = m_q^{\text{A380}} - m_q^{\text{A400}}$				-0.00476435	-0.000772590	-0.00682735

Target observable: HVP contribution to $(g-2)_\mu$.

Key steps:

- Compute all relevant observables at LO
- Correlator derivatives: $\partial G/\partial m_f$, $\partial G/\partial e^2$ and derivatives to Z_V (see next slide)
- Combine $\delta\vec{\epsilon} \equiv (\delta\beta, \delta\alpha, \delta m_u, \delta m_{d/s}, \delta m_c)$ and derivatives to get IB effects to a_μ^{HVP}

IB corrections to the HVP

Using the local-local implementation[1] for the correlator, $G^{\ell\ell}(t)$,

$$a_\mu^{HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, Z_V^2 G^{\ell\ell}(t) \tilde{K}(t; m_\mu)\,.$$

a_μ^{HVP} receives two types of IB corrections:

1. Corrections to the correlator:

$$\delta a_{\mu,(1)}^{HVP} = \left(\tfrac{\alpha}{\pi}\right)^2 \int dt\, (Z_V^{(0)})^2 \delta G^{\ell\ell}(t) \tilde{K}(t; m_\mu)$$

$$G^{\ell\ell}(t) = G^{\ell\ell}(t)^{(0)} + \delta G^{\ell\ell}(t) = G^{\ell\ell}(t)^{(0)} + \sum_f \delta m_f \frac{\partial G^{\ell\ell}(t)}{\partial m_f}\bigg|_{(0)} + \frac{e^2}{2} \frac{\partial^2 G^{\ell\ell}(t)}{\partial e^2}\bigg|_{(0)}$$

2. Corrections to renormalization constants:

$$\delta a_{\mu,(2)}^{HVP} = \left(\tfrac{\alpha}{\pi}\right)^2 \int dt\, 2 Z_V^{(0)} \delta Z_V G^{\ell\ell}(t)^{(0)} \tilde{K}(t; m_\mu)$$

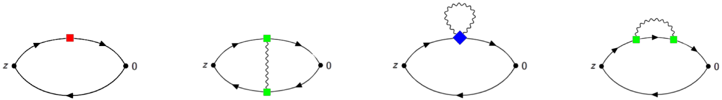
$$Z_V = Z_V^{(0)} + \delta Z_V = Z_V^{(0)} + \sum_f \delta m_f \frac{\partial Z_V}{\partial m_f}\bigg|_{(0)} + \frac{1}{2} e^2 \frac{\partial^2 Z_V}{\partial e^2}\bigg|_{(0)}$$

Alternatively, we also use conserved-local correlator $G^{c\ell}(t)$.

Shifts of bare parameters $(e^2, \delta m_u, \ldots, \delta m_c)$ fixed to $\delta\epsilon_i = \underbrace{\epsilon_i^{\text{A380a07}}}_{QCD+QED} - \underbrace{\epsilon_i^{\text{A400a00}}}_{IsoQCD}$

RM123: Diagrams for leading IB effects (connected valence only here)

Derivatives from our action $S^{\text{QCD+QED+SW}} = S_f(e, m_f) + S_{\text{SW}}(e) + \delta Sb$



mass   exchange   tadpole   self-energy

- ■ $\propto \sum_x \bar{\psi}(x)\psi(x)$ — from δ_m of mass term $(4 + m)\bar{\psi}\psi$ in S_f,
- ♦ $\propto \sum_{x,\mu} T_\mu(x) A^2_\mu(x)$ — from δ^2_e of kinetic term $T(x)$ in S_f,
- ■ $\propto \sum_{x,\mu} V^c_\mu(x) A_\mu(x)$ — from δ_e of kinetic term $T(x)$ in S_f, *or*
- ■ $\propto c^{\text{U(1)}}_{\text{SW}} \sum_{x,\mu} \delta_e D_{\text{SW}}(x)$ — from $S_{\text{SW}} = S_{\text{SW}}\big|_{e=0} + e \cdot \delta_e S_{SW} + O(e^3)$.

With $G^{c\ell}(t)$, if conserved current $V^c_\mu(x)$ defined at the sink, no additional propagators needed, but two additional diagrams appear:

- ■ $\propto \sum_{x,\mu} V^c_\mu(x) A^2_\mu(x)$ — from $\delta^2_e V^c_\mu$
- ♦ $\propto \sum_{x,\mu} T_\mu(x) A_\mu(x)$ — from $\delta_e V^c_\mu$

Renormalization constants Z_V and their IB corrections

Renormalization condition in the adjoint basis of the identity λ_0 plus SU(4) generators $\lambda_3, \lambda_8, \lambda_{15}$:

$$Z_{V_R V_l} = \lim_{x_0 \to \infty} G^{cl}(x_0)(G^{\ell\ell}(x_0))^{-1} \to \begin{pmatrix} 0.6578(9) & 0.0(0) & 0.0(0) & 0.0220(6) \\ 0.0(0) & 0.6766(12) & 0.0(0) & 0.0(0) \\ 0.0(0) & 0.0(0) & 0.6766(12) & 0.0(0) \\ 0.0439(12) & 0.0(0) & 0.0(0) & 0.6224(11) \end{pmatrix}$$

Taking derivatives:

$$\frac{\partial Z_{V_R V_l}}{\partial \varepsilon_i} = \lim_{x_0 \to \infty} \left[\frac{\partial G^{cl}}{\partial \varepsilon_i}(x_0) - G^{cl}(x_0) \left(G^{\ell\ell}(x_0)\right)^{-1} \frac{\partial G^{\ell\ell}}{\partial \varepsilon_i}(x_0) \right] \cdot \left(G^{\ell\ell}(x_0)\right)^{-1}$$

Total correction:

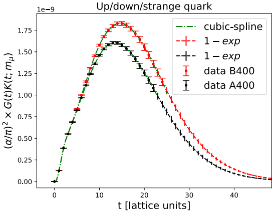
$$\delta Z_{V_R V_l} = \sum_f \Delta m_f \frac{\partial Z_{V_R V_l}}{\partial m_f} + e^2 \frac{\partial Z_{V_R V_l}}{\partial e^2} + \ldots$$

Using mass shifts:

$$\delta Z_{V_R V_l} = \begin{pmatrix} -0.00002(19) & 0.000260(95) & 0.00027(11) & 0.000147(76) \\ 0.000230(93) & -0.00008(16) & 0.00027(11) & 0.000094(38) \\ 0.000133(54) & 0.00027(11) & -0.00005(19) & 0.000054(22) \\ 0.00030(15) & 0.00038(15) & 0.000217(87) & -0.000259(62) \end{pmatrix}$$

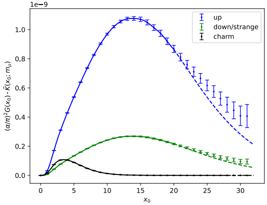
Results for a_μ^{HVP} from LO connected correlators

QCD only ensembles



type	$a_\mu^{u/d/s} \times 10^{-10}$	$am_V^{u/d/s}$	$a_\mu^c \times 10^{-10}$	am_V^c
Ensemble A400a00b324				
ll	338(8)	0.2644(50)	7.83(8)	0.8463(5)
cl	334(9)	0.2652(55)	6.18(7)	0.8462(5)
Ensemble B400a00b324				
ll	402(9)	0.2522(33)	7.81(9)	0.8458(9)
cl	397(9)	0.2530(32)	6.16(7)	0.8454(8)

QCD+QED ensembles



type	$a_\mu^u \times 10^{10}$	am_V^u	$a_\mu^{d/s} \times 10^{10}$	$am_V^{d/s}$	$a_\mu^c \times 10^{10}$
Ensemble A380a07b324, $\alpha_R = 0.007$					
ll	–	–	–	–	–
cl	331(7)	0.266(4)	83(2)	0.265(6)	9.78(10)
cc	–	–	–	–	–
Ensemble A360a50b324, $\alpha_R = 0.040633(80)$					
cl	309(11)	0.267(8)	77(2)	0.262(7)	10.62(11)

In all cases, tails are fitted to a single exponential after ~ 1.2 fm.