Novel Aspects of Energy Correlators in N=4 super Yang-Mills Theory

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MITP Programme: Energy Correlators at Collider Frontier

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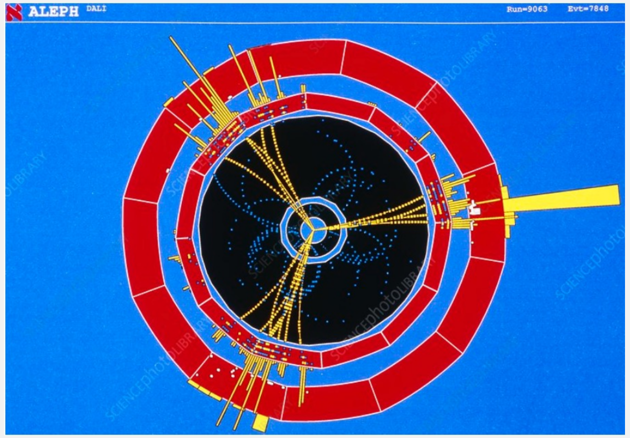
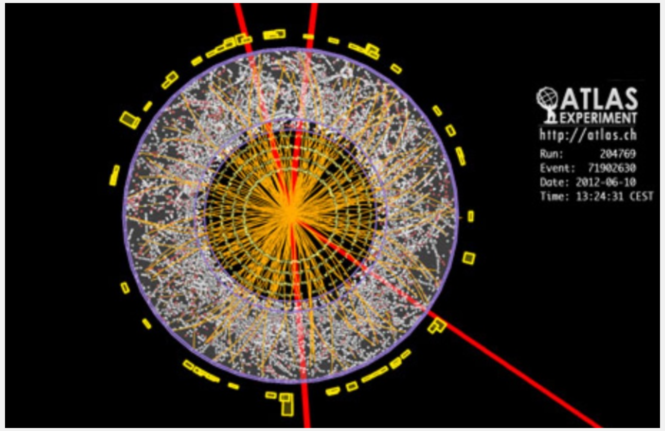
EVENT SHAPE

How do we describe the final states produced in particle scattering?

How is the total scattering energy divided in the hadronic final state?

Event shape:
measure the geometric distribution of energy flow, discriminate between jet-like vs. spherical events
e.g. Thrust, Sphericity, C-parameter, N-jettiness, jet finding algorithms,…





Explicit calculations will be crucial for uncovering hidden simplicity and structures

MULTI-POINT ENERGY FLOW CORRELATION

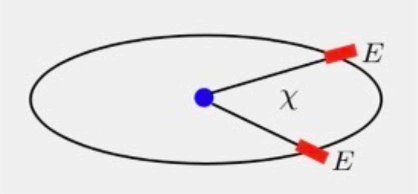
Final state is an ensemble of varying number of particles, characterized by energy flow

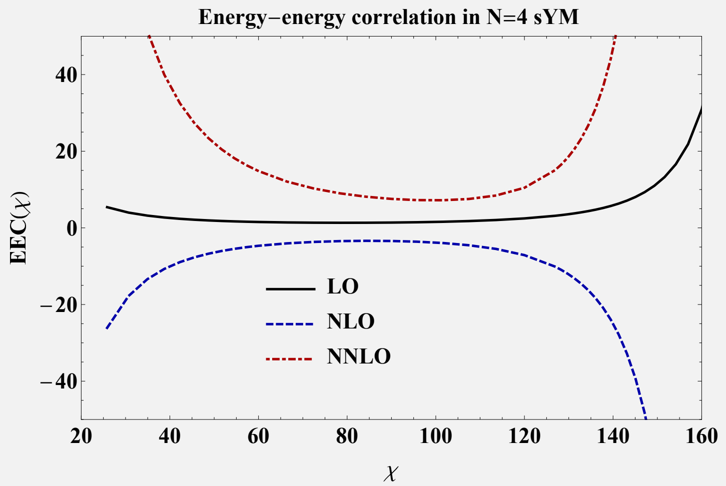
$$\mathrm{E}(n) := \sum_{i \in X} E_i \, \delta^2(n - \Omega_i)$$

- Simplest event shape: expectation value of energy flow in fixed direction: $\langle \mathrm{E}(n) \rangle$

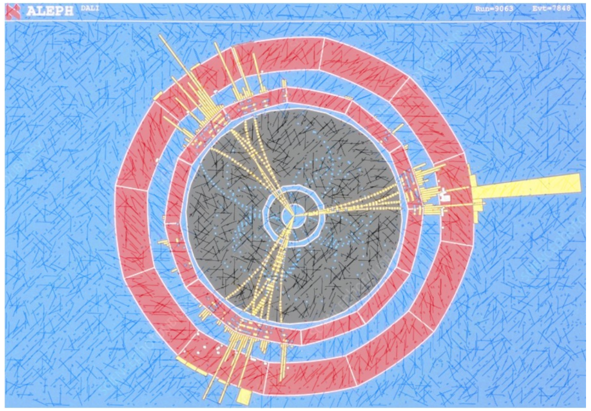
Energy flow distribution $\langle \mathrm{E}(n) \rangle = \frac{q^0}{4\pi}$ Evenly distributed for unpolarized source.

- Two-particle correlators $\langle \mathrm{E}(n_1)\mathrm{E}(n_2) \rangle$





Analytic structure for higher-point correlators?



(Muiti-point) Energy Correlators: correlation of energy deposited in detectors in different directions as function of the angles between them

Novel observables in collider physics measured at the LHC

N −point Energy Correlators in the multi-collinear limit:

$$\mathrm{E^N C} \overset{\text{coll.}}{=} \int_0^1 dx_1 \cdots dx_N \, \delta(1 - \sum_i x_i) \, (x_1 \cdots x_N)^2 \, \mathcal{P}^{(0)}_{1\to N}$$

x_i: energy carried by collinear particles

Parametric representation similar to Feynman loop integrals

Potential for developing amplitudes methods for the study of physical observables

Energy correlators:
correlation function of flow operators

$$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\ldots\mathrm{E}(n_N)\rangle = \int d^4x e^{iq\cdot x}\frac{\langle O(x)\mathcal{E}(n_1)\cdots\mathcal{E}(n_N)O(0)\rangle}{\langle O(x)O(0)\rangle}$$

$$\mathcal{E}(n) = \int_{-\infty}^{+\infty} du \lim_{r\to\infty} r^2 T_{0i}(t = u + r, r\,\vec{n})n^i$$

Exhibits an OPE in the (multi-) collinear limit

key for understanding the conformal light-ray OPE [Chang et al, 2202.04090][Chen et al, 2202.04085]

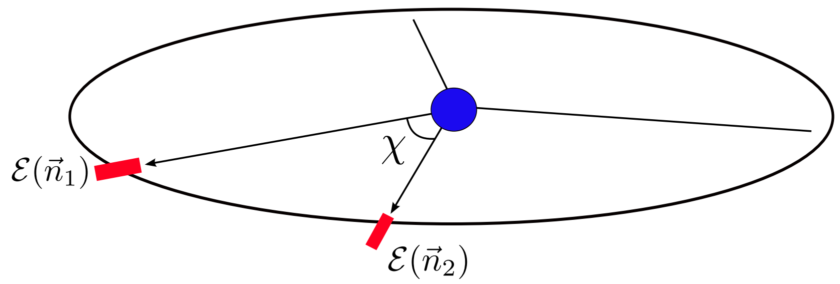
Multi-collinear limit: relevant in the studies of jet physics

new jet substructure calculations [Komiske et al, 2201.07800]

A potential playground for novel onshell methods in scattering amplitudes[Shounak De et al,2308.03753] [Arkani-Hamed,Yuan,1712.09991.] [Gong, Yuan, 2206.06507]

Offshell definition

$$EEC(\chi; q^2) = \int d^4x\, e^{i\,q\cdot x} \langle O(x)\,\mathrm{E}(n)\mathrm{E}(n')O(0)\rangle$$



O: half BPS operator (N=4)
electromagnetic current (QCD)

$$\sim \int dx_{2,-} dx_{3,-} \langle O\ T\ T\ O\rangle$$

$$\mathrm{E}(n) := \lim_{r\to\infty} r^2 \int d\,x_-\, n^j T_{0j}(t = x_- + r, r\,\vec{n})$$

$$\mathrm{EEC}(\zeta) \sim \int d^4x\, e^{iq\cdot x} \int_{-\infty}^{\infty} dx_{2-} dx_{3-} \lim_{x_{2+,3+}\to\infty} x_{2+}^2 x_{3+}^2 \langle 0| O^\dagger(x)\mathcal{O}(x_2)\mathcal{O}(x_3)O(0)|0\rangle$$

Detector time integration

Wightman correlation function

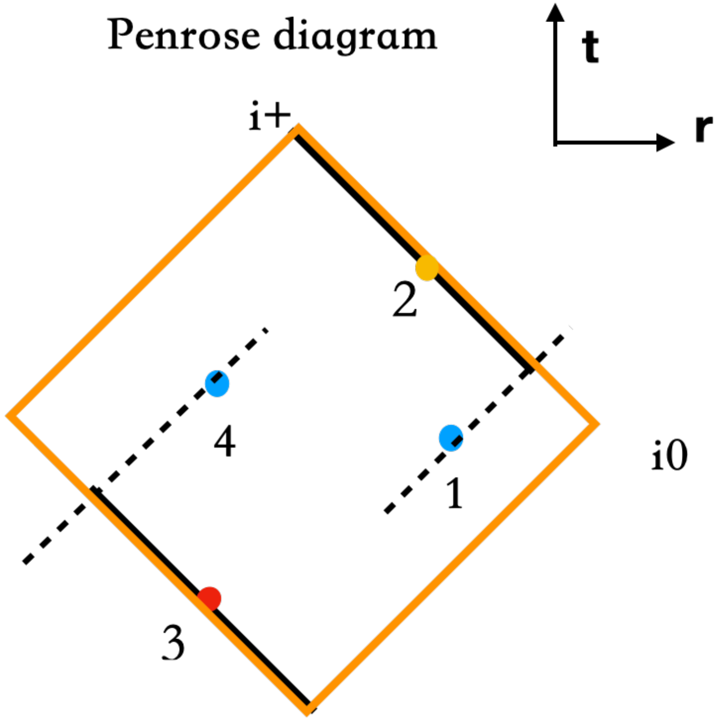
[Henn,Sokatchev,Yan,Zhiboedov,19‘]

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = \frac{\Phi\,(u,v)}{x_{12}^4 x_{34}^4} \qquad u = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, v = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}.$$

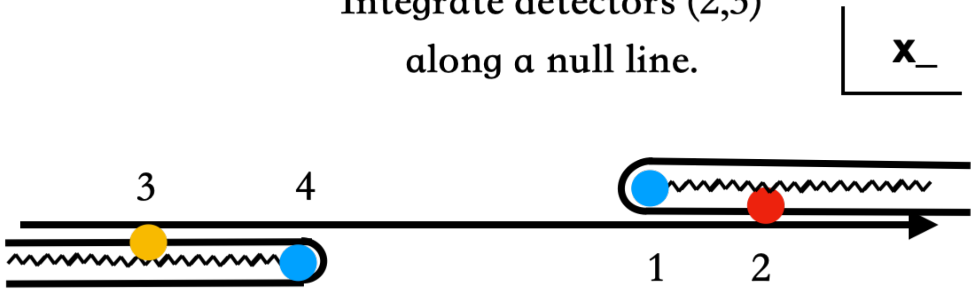
Analytically continue to Minkowski space: → G(u,v)

Double discontinuity formula:

$$\int_{C_2} dx_{2-} \int_{C_3} dx'_{3-} \boxed{\mathrm{disc}_{x_{2-}=x_-}\,\mathrm{disc}_{x'_{3-}=0}[\mathcal{G}(u,v)]}$$



Integrate detectors (2,3) along a null line.



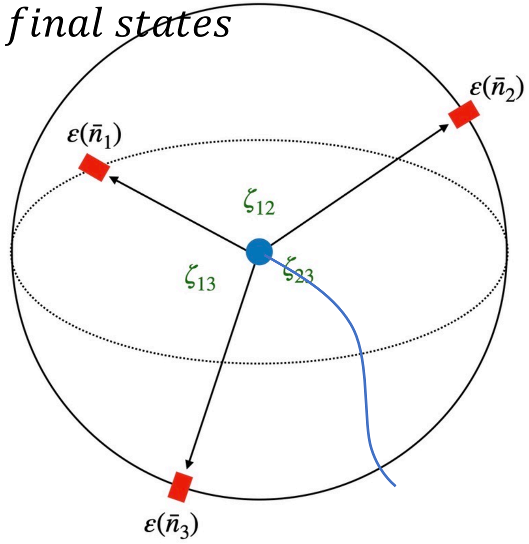
Works well for two-point correlator (in particular in N=4 SYM).

Difficult to generalize to higher point.

Onshell definition

$$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\ldots\mathrm{E}(n_N)\rangle = \sum_{k>N}\int d\,\Pi_k\,\delta^2(n_i-\Omega_i)\cdots\delta^2(n_N-\Omega_N)\frac{E_1\cdots E_N}{Q^N}FF_k(O)$$

$FF_k(O)$: $|\langle 0|O|1,2,\ldots,k\rangle|^2$ *summed over helicity, color and permutations on final states*



Analytic function of distances on the celestial sphere: $\zeta_{ij} = \frac{q^2(n_i\cdot n_j)}{2(n_i\cdot q)(n_j\cdot q)}$

$k = N+1$: Leading order. Manifestly finite integration over tree-level matrix element $FF_{N+1}^{(0)}$

N −point Energy correlators $@LO$

Master formula : energy integrations over onshell tree-level $(N+1)$-point squared Form Factor

$$E^N C = \mathcal{F}_N\left(\zeta_{ij}\right) + \; perms(1, 2 \cdots, N)$$

$$\mathcal{F}_N^{LO} := \int_0^1 d\,x_1 \;\ldots d\,x_N\, \delta(1 - Q_N)(x_1 \cdots x_N)^2 \left|F_{N+1}^{(0)}(O)\right|^2$$

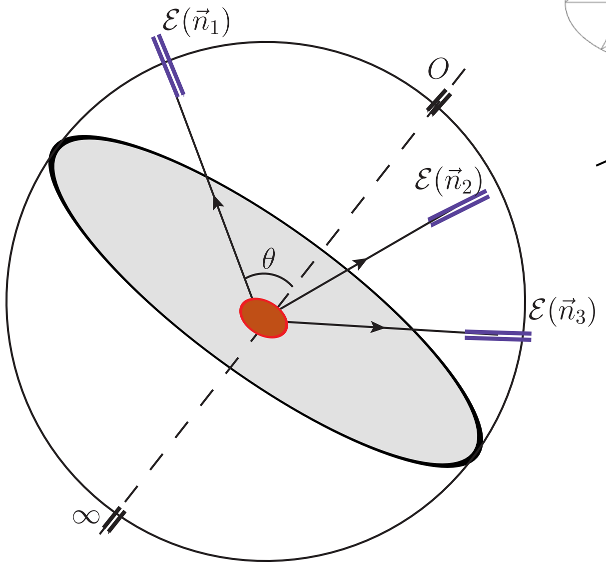
$$Q_N := x_1 + \cdots + x_N - \textstyle\sum_{i,j} x_i x_j\, \zeta_{ij}$$

$$x_i = \frac{2E_i}{Q} \qquad s_{ij} = x_i x_j \zeta_{ij} \qquad s_{ijk} = x_i x_j \zeta_{ij} + x_j x_k \zeta_{jk} + x_i x_k \zeta_{ik}, \text{ etc}$$

$$\left|F_{N+1}^{(0)}(O)\right|^2 = \left|F_{N+1}^{(0)}(O)\right|_{MHV} \times \text{(products of ratios between madelstam variables)}$$

Three-point Energy Correlator in N=4 sYM @LO

Zhang, Yan, Phys.Rev.Lett. 129 (2022) 2, 021602



detectors the directions of the unit vectors n1,n2 and n3

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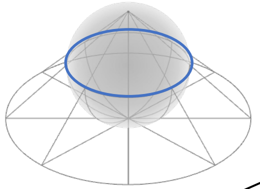
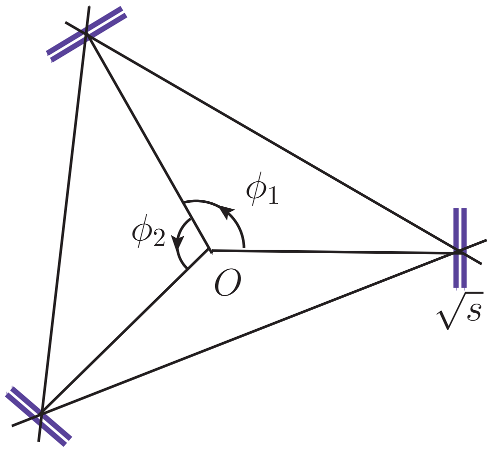
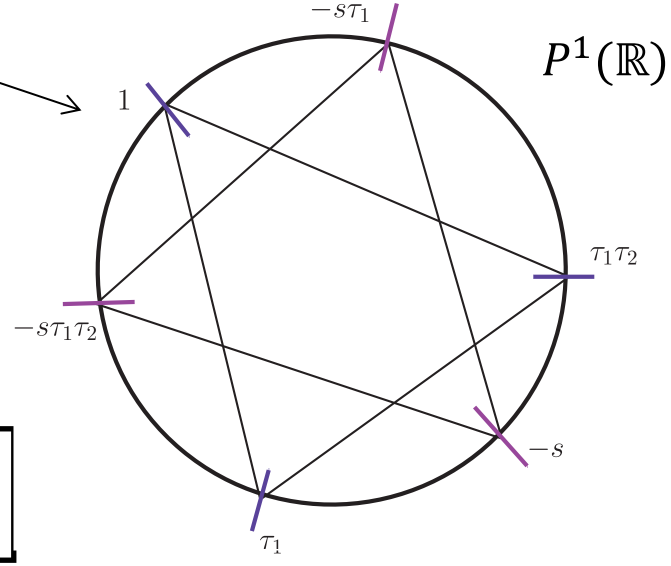




three points on the celestial sphere on a circle centered at the origin

$$s := \tan^2\frac{\theta}{2}, \tau_1 := e^{i\phi_{23}}, \tau_2 := e^{i\phi_{13}}$$



Kinematic data embedded in 6 points on unit circle

Function Space

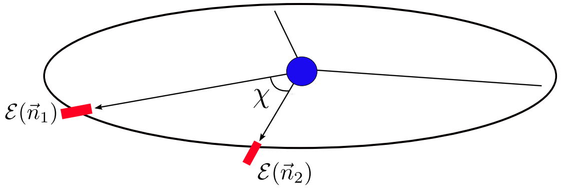
$Li_w(-S)$ *modulo* products of logarithms

$$S := \{\frac{\langle 14\rangle\langle I3\rangle}{\langle 13\rangle\langle 4I\rangle}, \frac{\langle 12\rangle\langle 54\rangle}{\langle 15\rangle\langle 24\rangle}, \frac{\langle 12\rangle\langle 43\rangle}{\langle 14\rangle\langle 23\rangle}, \frac{\langle 13\rangle\langle 54\rangle}{\langle 15\rangle\langle 43\rangle}, \frac{\langle 12\rangle\langle 34\rangle\langle 56\rangle}{\langle 23\rangle\langle 45\rangle\langle 61\rangle}, \frac{\langle 14\rangle\langle 23\rangle\langle 56\rangle}{\langle 34\rangle\langle 25\rangle\langle 61\rangle}$$

$\}$+ D_6 images

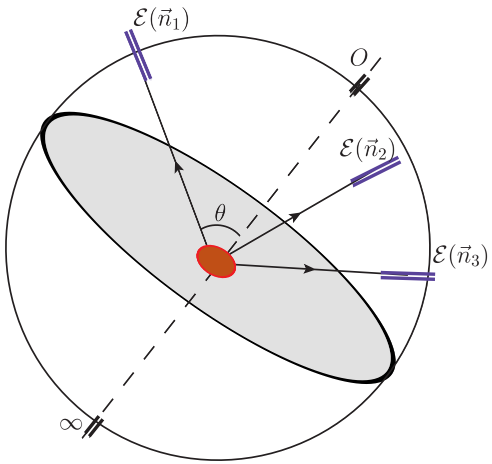
Perturbative results

$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\rangle$



	$O(\alpha)$	$O(\alpha^2)$	$O(\alpha^3)$
$\mathcal{F}\left(\zeta = \sin^2\frac{\chi}{2}\right)$	@LO	@NLO	@NNLO
	$\ln(1-\zeta)$	max. weight 3 polylogs $\mathcal{A}_{EE} = \{\zeta, 1-\zeta, \frac{1-\sqrt{\zeta}}{1+\sqrt{\zeta}}\}$	max. weight 5 HPL + Elliptic

$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\mathrm{E}(n_3)\rangle$



$O(\alpha^2)$

$\mathcal{F}(\zeta_{12}, \zeta_{13}, \zeta_{23})$ @LO

max. weight 2 polylogs

16 letters, 2 types of squre roots

Basis of classical polylogarithms
First entry conditions

$\langle \mathrm{E}(n_1)\mathrm{E}(n_2)\mathrm{E}(n_3)\mathrm{E}(n_4)\rangle$? $O(\alpha^3)$

N −point Energy correlators $@LO$

For arbitrary N, given by manifestly *finite N-fold energy integrals. Go to higher order in the coupling constants without encountering IR divergence.*

They admit parametric representation similar to Feynman loop integrals.

Built entirely from tree-level quantities, the leading-order energy correlators, with an increasing number of detectors, may provide insights on the structures for higher-loop amplitudes.

In the multi-collinear limit

$$p_i^{\mu} = (p_i^{+}, p_i^{-}, p_i^{\perp}) = \frac{x_i}{\sqrt{1+|z_i|^2}}(1, |z_i|^2, z_i) \to x_i(1, |z_i|^2, z_i) \qquad \zeta_{ij} \to |z_{ij}|^2 \sim 0$$

z_{ij}: small angular separations
x_i: energy fractions carried by collinear particles

$$E^N C_{coll.} = \mathcal{G}_N(z_i) + \; perms(1, 2 \cdots, N)$$

$$\mathcal{G}_N^{LO}(z_i) := \int_0^1 d\,x_1 \ldots d\,x_N \delta(1 - x_1 - \cdots - x_N)(x_1 \cdots x_N)^2 \left|Split_{1\to N}^{(0)}\right|^2$$

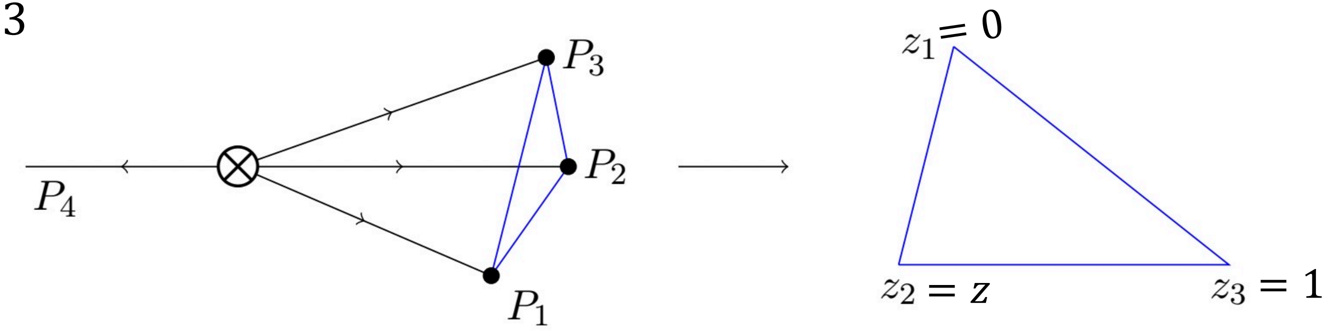
Splitting function contains poles either linear or bi-linear in the x_i −parameters

The N−point correlators define a class of manifestly *finite integrals* in $(N-1)$−dimensional projective space $[x_1 : \cdots : x_N] \in \mathbb{P}_{N-1}(\mathbb{R}_+)$

$E^N C@LO$ in the collinear limit:

Single-valued function of coordinates $(z_i, \bar{z}_i)$ on the celestial sphere

$N = 3$



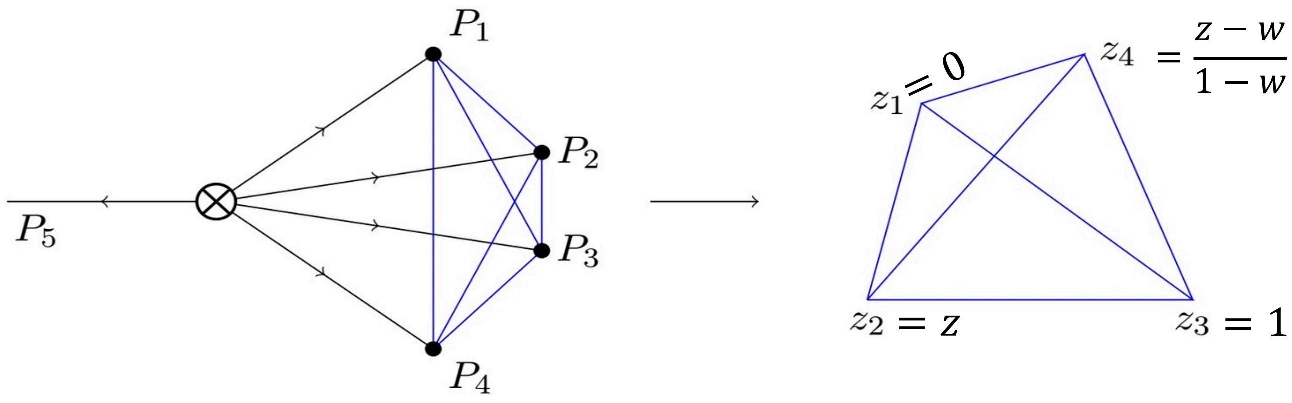
$G(z)$ @ LO

[H.Chen et al, 1912.11050],

$\ln u \quad \Phi_2(z)$

$Li_2(1-u) + \frac{1}{2} \ln u \, \ln v$

$N = 4$



$G(z,w)$ @ LO

$max.weight\ 3\ polylogarithms$

Chicherin, Sokatchev, Moult,Yan,Zhu

Onshell methods applied to physical cross sections

-Integrand:

The squared (N+1)-point super form factor for ½ BPS operator, with manifest dual conformal symmetry.

-Integration

- Integration-by-part algorithm operating directly on the (Feynman-) parameter space . Profit from simplicity that exist for *finite* integrals.

-Intersection theory method

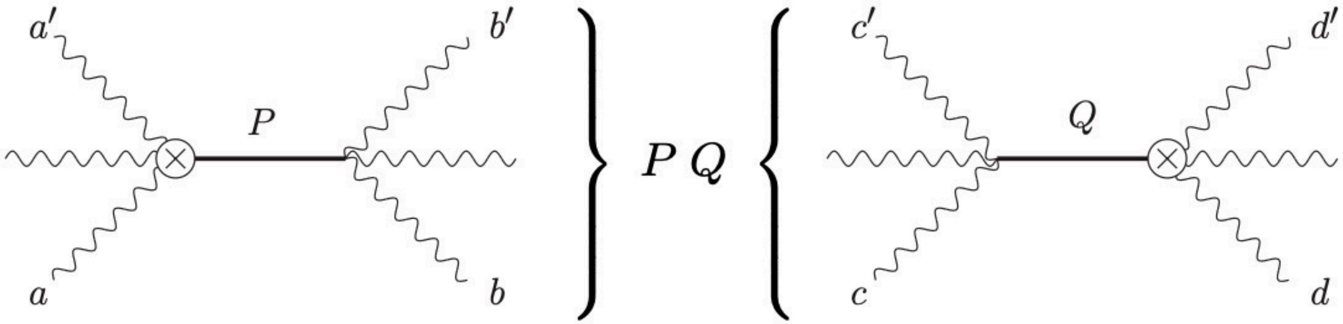
-Function

- Symbols, landau singularity analysis
- Physical constraints and asymptotic limits → Boostrap

Form factor $|F_{NMHV}(O'_{20})|^2$

The tree-level $1 \to N$ splitting function can be obtained from the squared $(N+1)$-point form factor where $p_1 \cdots p_N$ are collinear and p_{N+1} is anti-collinear.

$|F_{NMHV}(\mathcal{L})|^2$ given by product of chiral and anti-chiral diagrams describing the MHV rules.

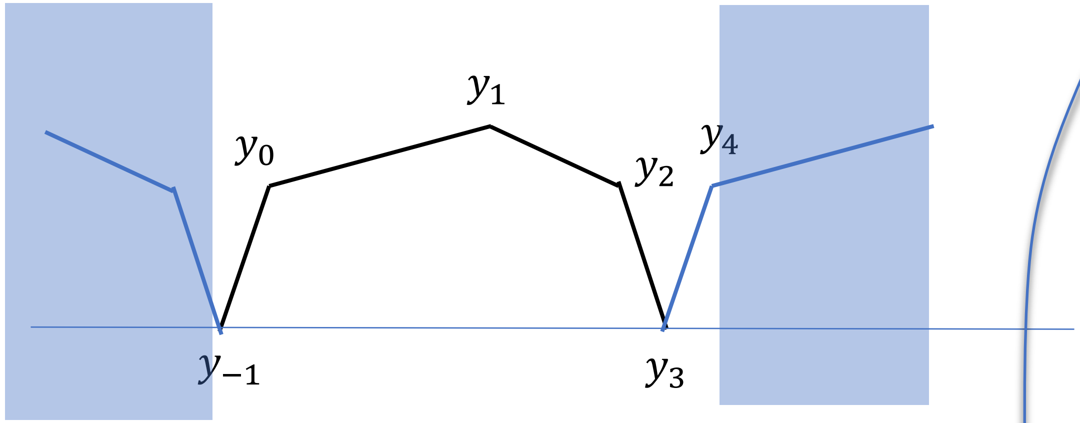


$$K_{ab} \equiv \sum_{i=a}^{b} z_{ai} x_i, \quad L_{ab} \equiv \sum_{i=a}^{b} z_{bi} x_i,$$

$$\lim_{1||2\ldots||n-1} \mathrm{FF}_{n,1} = \sum_{\{b,b'\};\{c,c'\}} \left(\sum_{i=b}^{b'} \sum_{j=c}^{c'} \sum_{k=e}^{e'} x_i x_j x_k z_{ik} \bar{z}_{kj} \right)^4 \times \frac{z_{b-1\,b} z_{b'\,b'+1}}{s_{bb'} K_{b-1,b'} K_{b,b'} L_{b,b'} L_{b,b'+1}} \frac{\bar{z}_{c-1\,c} \bar{z}_{c'\,c'+1}}{s_{cc'} \overline{K}_{c-1,c'} \overline{K}_{c,c'} \overline{L}_{c,c'} \overline{L}_{c,c'+1}}$$

In the collinear limit, $|F_{N+1}|^2$ can be expressed in terms of coordinates on a section in the periodic dual coordinate space including *one period* $+1$ *point*

$$[y_{-1},\dots,y_N]$$



$$p_i = y_i - y_{i-1},\ p_{i+4} \coloneqq p_i,\ s_{i+1,k} = y_{ik}^2$$

Compact form of the splitting function manifestly dual conformal invariant

$$(a,b,c,d) \equiv \frac{y_{ab}^2 y_{cd}^2}{y_{ac}^2 y_{bd}^2}$$

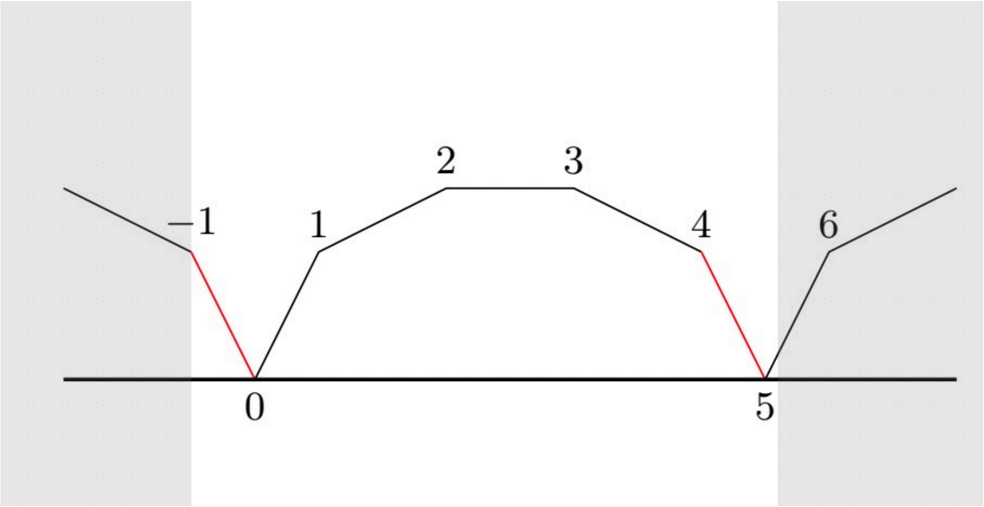
For N=3,

$$\lim_{1||2||3} \frac{|F_4^{\mathrm{NMHV}}|^2}{|F_4^{\mathrm{MHV}}|^2} = (-1,1,2,4) + (-1,3,2,0) + (3,1,0,4)$$

In the N −particle collinear limit,

$$y_{iN}^2 \to x_{i+1} + \cdots + x_{N-1}$$
$$y_{-1i}^2 \to x_1 + \cdots + x_i$$

In the quadruple collinear limit



$$p_i = y_i - y_{i-1},\ p_{i+5} \coloneqq p_i,\ s_{i+1,k} = y_{ik}^2$$

$$\lim_{1||2||3||4} \frac{|F_5|_{NMHV}^2}{|F_5|_{MHV}^2} = \begin{aligned} &-1 + (-1,1,2,5) + (-1,2,3,5) + (-1,4,3,0) + (4,1,0,5) + (-1,3,2,0) + (4,2,1,5) \\ &+ (0,4,3,1) + (0,4,3,1)(-1,1,3,5) + (-1,4,3,1)(3,1,0,5) \\ &+ (-1,4,2,0)(0,2,3,5) + (-1,1,2,4)(4,2,0,5) + (-1,3,2,0)(4,2,0,5) + (-1,4,2,0)(4,2,1,5) \\ &+ (-1,4,3,0)(-1,1,2,4) + (4,1,0,5)(0,2,3,5) + (-1,4,3,1)(-1,4,2,0) + (3,1,0,5)(4,2,0,5) \\ &+ (-1,4,3,1)(-1,1,2,5) + (3,1,0,5)(-1,2,3,5) + (-1,1,2,4)(-1,1,3,5) + (0,2,3,5)(-1,1,3,5) \end{aligned}$$

Integration-by-parts for finite integrals

We developed techniques suitable for the computation of the $E^N C$ for arbitrary N; based on methods for finite integrals [Caron Huot, Henn, 1404.2922][Henn, Ma, Yan,Zhang,2211.13967]

Advantages:
Bypass solving large linear systems of IBPs ;
DEs exhibit a ”grading” structure, visualizes the iterative structure of loop integrals.
may apply in more general setup.

Energy integration over splitting function

$$\mathrm{E^NC} \overset{\text{coll.}}{=} \frac{1}{|z_{12}\cdots z_{N-1N}|^2}\int \frac{d^N x}{\mathrm{GL}(1)}(x_1+\cdots+x_N)^{-N}\mathcal{G}_N + \text{perm}\,(z_1,\cdots,z_N)$$

$$G_N := \frac{|F|^2}{|F_{MHV}|^2}$$

$N = 3$: $\quad (3,1,0,4) = \frac{s_{23}x_{123}}{s_{123}x_{23}} \qquad (-1,3,2,0) = \frac{s_{12}x_{123}}{s_{123}x_{12}} \qquad (-1,1,2,4) = \frac{x_1 x_{123}}{x_{12}x_{23}}$

G_N contains only multi-particle poles, no two-particle pole

$$s_{I\dots J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2 \qquad x_{I,\dots,J} := x_I + \cdots + x_J$$

$$\mathrm{E^NC} \overset{\text{coll.}}{=} \frac{1}{|z_{12}\cdots z_{N-1N}|^2}\int \frac{d^N x}{\mathrm{GL}(1)}(x_1+\cdots+x_N)^{-N}\mathcal{G}_N + \mathrm{perm}(z_1,\cdots,z_N)$$

$$G_N := \frac{|F|^2}{|F_{MHV}|^2}$$

$N = 4$:

$$(-1,4,3,0) = \frac{s_{123}\, x_{1234}}{s_{1234}\, x_{123}} \qquad (-1,4,2,0)(0,2,3,5) = \frac{s_{12}^2\, x_{1234}x_4}{s_{1234}s_{123}\, x_{12}x_{34}}$$

$$(-1,4,3,0)(-1,1,2,4) = \frac{s_{123}s_{34}\, x_{1234}x_1}{s_{1234}s_{234}\, x_{12}x_{123}} \qquad (0,4,3,1)(-1,1,3,5) = \frac{s_{1234}s_{23}}{s_{123}s_{234}}\frac{x_1x_4}{x_{123}x_{234}}$$

$$s_{I\ldots J} := \sum_{(i,j)\in[I,J]} x_i x_j |z_{ij}|^2 \qquad x_{I,\ldots,J} := x_I + \cdots + x_J$$

$$\mathrm{E^N C} \overset{\text{coll.}}{=} \frac{1}{|z_{12} \cdots z_{N-1N}|^2} \int \frac{d^N x}{\mathrm{GL}(1)} (x_1 + \cdots + x_N)^{-N} \mathcal{G}_N + \text{perm}(z_1, \cdots, z_N)$$

Multi kinematic scales and high degree poles in the integrand poses great challenge to partial fractioning and multi-fold integration.

Goal:

-lower the degree of denominators in target integrals.

transform them to simpler, manageable integrals with simple or at most double pole

Integration-by-part method can be designed to achieve these goals.

N-point Integral Family

$N = 3:$

$$B_{a_1,\cdots,a_6} \equiv \int \frac{d^3x}{\mathrm{GL}(1)} \frac{s_{12}^{-a_5}\, s_{23}^{-a_6}}{s_{123}{}^{a_1}\, x_{12}^{a_2}\, x_{23}^{a_3}\, x_{123}^{a_4}}$$

$N = 4:$

$$A_{a_1,\cdots,a_{10}} \equiv \int \frac{d^4x}{\mathrm{GL}(1)} \frac{1}{\prod_i D_i^{a_i}}$$

$$D_1 = s_{1234}, \quad D_2 = s_{123}, \quad D_3 = s_{234},$$
$$D_4 = x_{1234}, \quad D_5 = x_{234}, \quad D_6 = x_{123}, \quad D_7 = x_{34},$$
$$D_8 = s_{12}, \quad D_9 = s_{23}, \quad D_{10} = s_{34}\,.$$

D_{1-7} are multi-particle poles corresponding to physical singularities

N-point Integral Family

1. Homogeneity: integrand has the overall scaling dimension $-N$

as $\{x_1, x_2, x_3, x_4\} \to \kappa\{x_1, x_2, x_3, x_4\}$

2. Finiteness: free from IR divergences as any subset of energy variables go to zero.

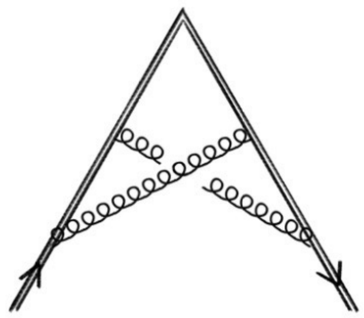
Given *condition 1,*
condition 2 is equivalent to the following UV power-counting behaviour:

$$\prod_i D_i^{-a_i} \sim O(\kappa^{-1-|\breve{S}|}) \quad \text{as} \quad S \to \kappa S\,, \kappa \to \infty\,, \forall S \subset \{x_1, \cdots, x_N\}$$

An analog: Wilson-line web functions

$$\mathrm{T2}[a_1,\cdots,a_7] = \int \frac{d^Dk_1}{i\pi^{D/2}}\frac{d^Dk_2}{i\pi^{D/2}}\frac{D_7^{-a_7}}{D_1^{a_1}\cdots D_6^{a_6}}$$

$$D_1 = -2k_1\cdot v_1+\delta,\quad D_2 = -2(k_1+k_2)\cdot v_1+\delta,\quad D_3 = -2(k_1+k_2)\cdot v_2+\delta,$$
$$D_4 = -2k_2\cdot v_2+\delta,\quad D_5 = -k_1^2,\quad D_6 = -k_2^2,\quad D_7 = k_1\cdot k_2\,.$$



The $E^N C$ fulfill the same criterions for the so-called 'admissible integrals' (the absence of sub-divergences) in *[Henn, Ma, Yan, Zhang 2211.13967]*,

No regulators are needed for the "leading" divergences.
Four-dimensional IBP and DE methods apply.

Integrals in families $A_{\vec{a}}$ and $B_{\vec{a}}$ are defined in integer dimension

1. There are partial fractioning identities among the integrals carrying different propagator indices.

E.g. for $\left\{\frac{x_1 x_{12}}{s_{123}^2 x_{123}}, \frac{x_1^2}{s_{123}^2 x_{123}}, \frac{x_{12}^2}{s_{123}^2 x_{123}}, \frac{x_1}{s_{123}^2}, \frac{x_{12}}{s_{123}^2}, \frac{1}{s_{123} x_{123}}\right\}$, there is relation $|z_{12}|^2 \frac{x_1(x_{12}-x_1)}{s_{123}^2 x_{123}} +$

$|z_{23}|^2 \frac{(x_{12}-x_1)(x_{123}-x_{12})}{s_{123}^2 x_{123}} + |z_{13}|^2 \frac{x_1(x_{123}-x_{12})}{s_{123}^2 x_{123}} - \frac{1}{s_{123} x_{123}} = 0$

2. A finite integral may appear as linear combination of divergent ones

$\left\{\frac{x_1 x_{12}}{s_{123}^2 x_{123}}, \frac{x_1^2}{s_{123}^2 x_{123}}\right\}$ divergent $\qquad \frac{x_1(x_{12}-x_1)}{s_{123}^2 x_{123}}$ finite

Solutions:

Integrand reduction performed together with seeding and IBP reduction

Setup integrand in a way that only allow x-monomials in the numerator
Search for basis of ”single finite integrals ”

$$A_{a_1,\cdots,a_7;\,q_1,\cdots,q_4} \equiv \int \frac{d^4x}{\mathrm{GL}(1)} \frac{x_1^{-q_1} x_2^{-q_2} x_3^{-q_3} x_4^{-q_4}}{D_1^{a_1} D_2^{a_2} \cdots D_7^{a_7}}$$

Here we demand $a_i \geq 0$, $q_k \leq 0$ D_{1-7} are the physical, multi-particle poles

IBP identities in projective space

$$O_i = \frac{\partial}{\partial x_i} v \,, \; i = 1, \cdots, N,$$

Differential operators acting on projective coordinates [x_1,…,x_N]

$$\int \frac{d^N x}{\mathrm{GL}(1)} O_i \circ f = -\int \frac{d^{N-1} x}{\mathrm{GL}(1)} v \circ f \Big|_{x_i=0}$$

$$v = \prod_k x_k^{-q_k} \,, \quad f = \frac{1}{\prod_j D_j^{a_j}} \,.$$

$O_i \circ f$* must satisfy the power-counting *condition 1* and *condition 2

$$O_i \to \kappa^{\beta_S} O_i \,, \; \forall\, S \subset \{x_1, x_2, x_3, x_4\}$$

$$\kappa_S = -\sum_{k \in S} q_k - d_i^S \qquad d_i^S \equiv 1 \text{ if } i \in S \text{ and } 0 \text{ otherwise}$$

IBP identities in projective space

$$\int \frac{d^N x}{\mathrm{GL}(1)} O_i \circ f = -\int \frac{d^{N-1}x}{\mathrm{GL}(1)} v \circ f \Big|_{x_i=0}$$

$$v = \prod_k x_k^{-q_k}\,, \quad f = \frac{1}{\prod_j D_j^{a_j}}\,.$$

Expanded over $A_{a,\ldots a_7;q_1..q_4}$

Each term is finite

boundary terms are generated on the surface of integration domain

$$B^{[j,k,l]}_{a_1,a_2,a_3,a_4;\,q_1,q_2,q_3} \equiv \int \frac{d^3x}{\mathrm{GL}(1)} \frac{x_j^{-q_1} x_k^{-q_2} x_l^{-q_3}}{s_{jkl}^{a_1}\, x_{jk}^{a_2}\, x_{kl}^{a_3}\, x_{jkl}^{a_4}}$$

Boundary integral family: lower-point integrals defined in $[x_1, .. \widehat{x_i}, .., x_N]$

Seeding
(with power counting)

In sector $A_{0,1,0,1,1,0,0}$ consider

$$O_1 = \frac{\partial}{\partial x_1} x_2^{-q_2} x_3^{-q_3} \qquad f = \frac{1}{s_{123}^{a_2} x_{234}^{a_5} x_{1234}^{a_4}} \qquad a_2, a_5, a_4 > 0$$

Imposing power counting *condition 1, condition 2* on $O_1 \circ f$

$$2a_2 + a_4 + a_5 + q_2 + q_3 + 1 = 4$$ *Overall scaling =0*

Seeding
(with power counting)

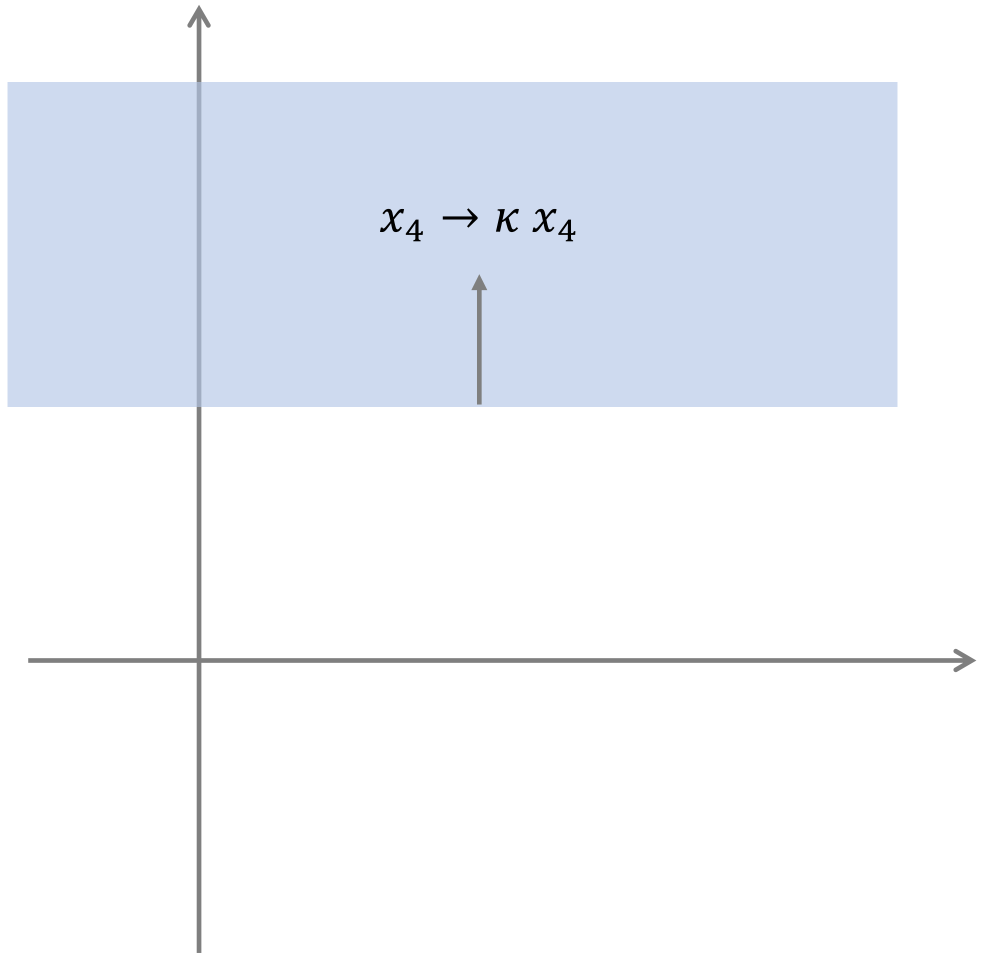
In sector $A_{0,1,0,1,1,0,0}$ consider

$$O_1 = \frac{\partial}{\partial x_1} x_2^{-q_2} x_3^{-q_3} \qquad f = \frac{1}{s_{123}^{a_2} x_{234}^{a_5} x_{1234}^{a_4}}$$

Imposing power counting *condition 1, condition 2* on $O_1 \circ f$

$$2a_2 + a_4 + a_5 + q_2 + q_3 + 1 = 4$$

$$a_4 + a_5 > 1 \qquad\qquad x_4 \to \infty$$



Seeding
(with power counting)

In sector $A_{0,1,0,1,1,0,0}$ consider

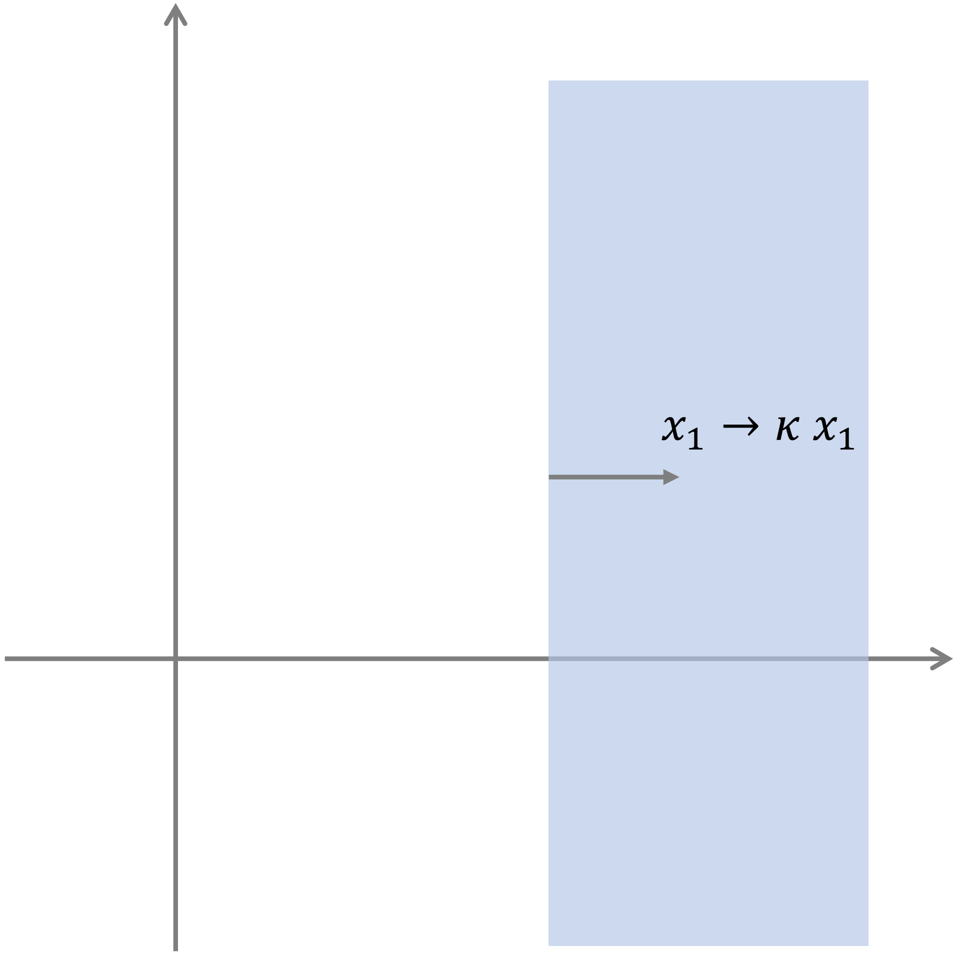
$$O_1 = \frac{\partial}{\partial x_1} x_2^{-q_2} x_3^{-q_3} \qquad f = \frac{1}{s_{123}^{a_2} x_{234}^{a_5} x_{1234}^{a_4}}$$

Imposing power counting *condition 1, condition 2* on $O_1 \circ f$

$$2a_2 + a_4 + a_5 + q_2 + q_3 + 1 = 4$$

$$a_4 + a_5 > 1$$

$$a_2 + a_4 - 1 > 1 \qquad x_1 \to \infty$$



Seeding (with power counting)

In sector $A_{0,1,0,1,1,0,0}$ consider

$$O_1 = \frac{\partial}{\partial x_1} x_2^{-q_2} x_3^{-q_3} \qquad f = \frac{1}{s_{123}^{a_2} x_{234}^{a_5} x_{1234}^{a_4}}$$

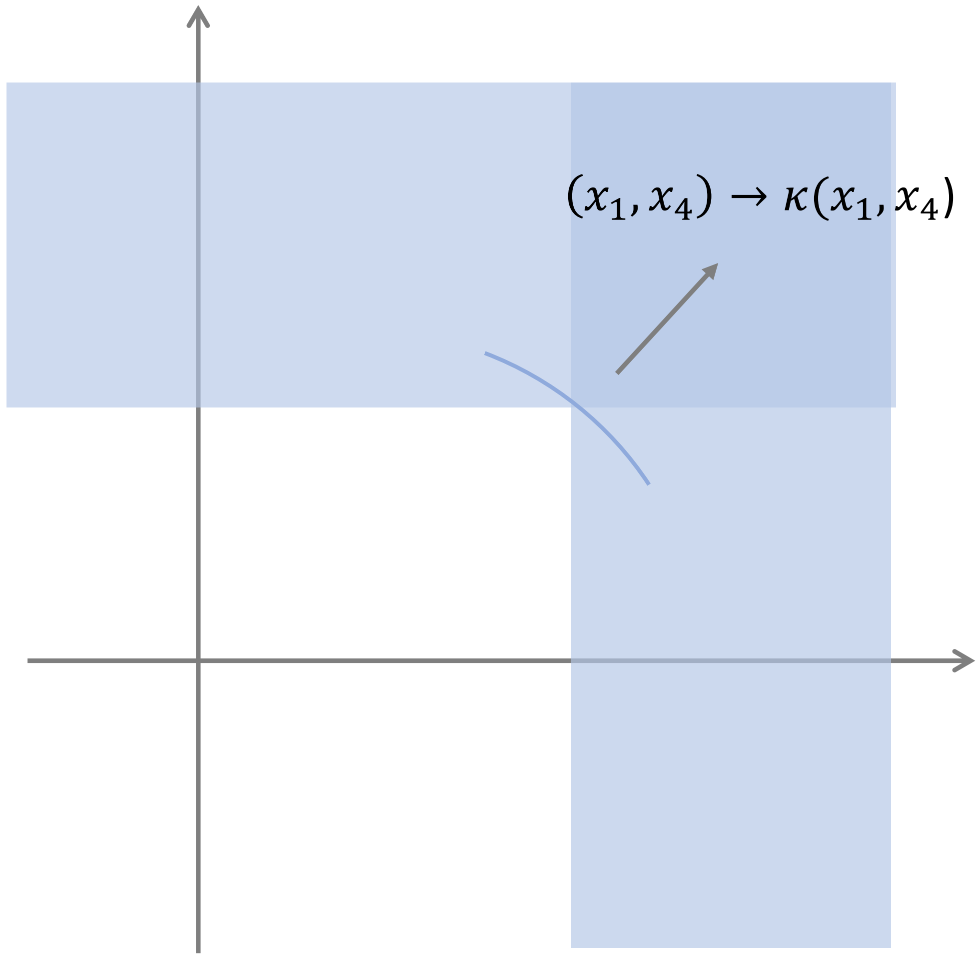
Imposing power counting *condition 1, condition 2* on $O_1 \circ f$

$$2a_2 + a_4 + a_5 + q_2 + q_3 + 1 = 4$$

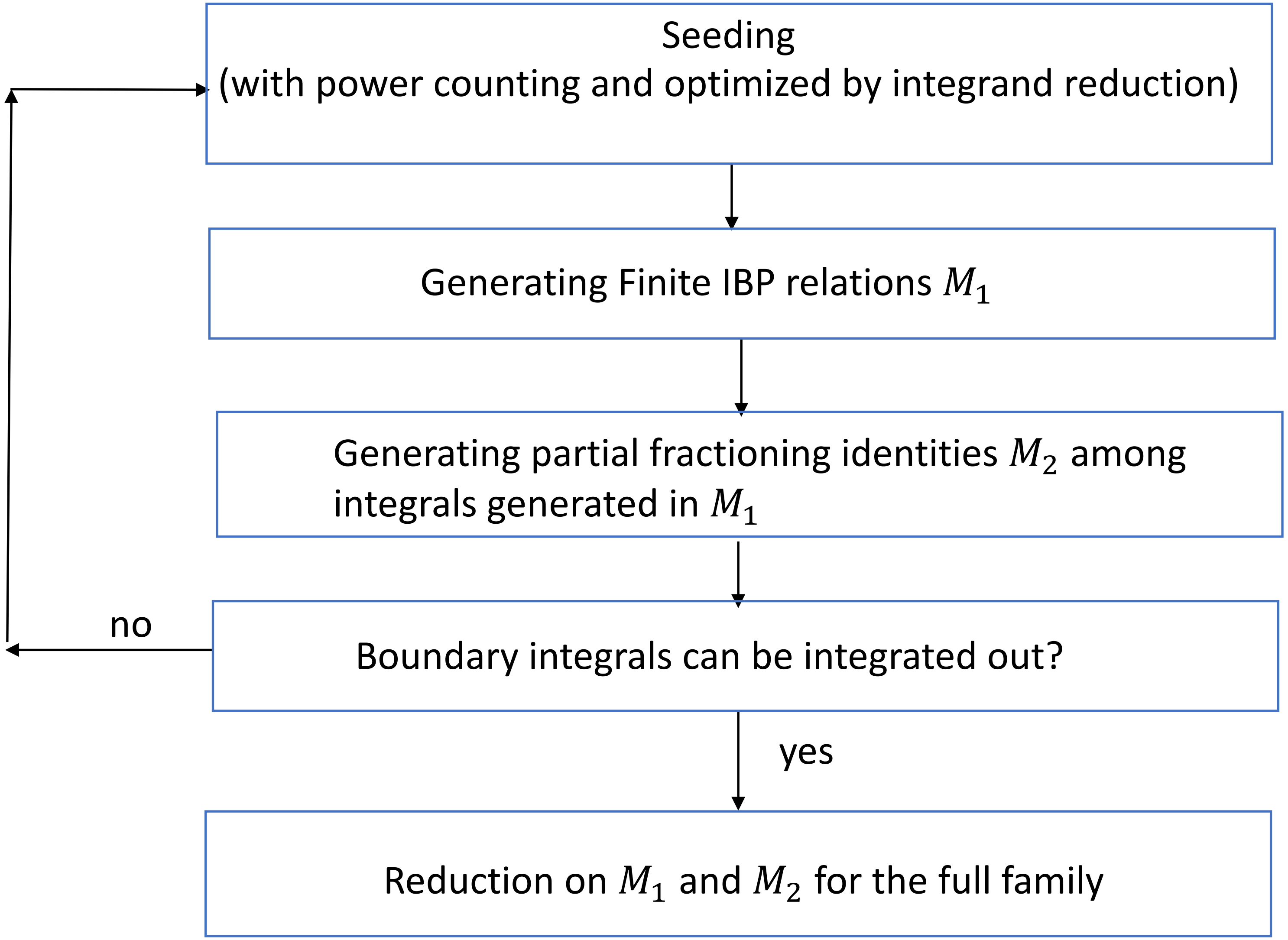
$$a_4 + a_5 > 1$$

$$a_2 + a_4 - 1 > 1$$

$$a_2 + a_4 + a_5 - 1 > 2 \qquad (x_1, x_4) \to \infty$$



Work flow



Seeding
(with power counting and optimized by integrand reduction)
Generating Finite IBP relations M_1
Generating partial fractioning identities M_2 among integrals generated in M_1
Boundary integrals can be integrated out?
no
yes
Reduction on M_1 and M_2 for the full family

4-point energy integrals mapped onto 6 sub-topologies plus their images under a *reflection* symmetry which flips the detector orientation : 1 ↔ 4, 2 ↔ 3.
The 6 topologies are further divided into three categories:

$$\begin{aligned}&\text{Type-I (one 3−particle cut)}: \quad (2,4,5,6) \quad (2,4,6,7)\\&\text{Type-II (4− and 3−particle cut)}: \quad (1,2,4,5) \quad (1,2,4,7)\\&\text{Type-III (two 3−particle cuts)}: \quad (2,3,4,5) \quad (2,3,5,6)\end{aligned}$$

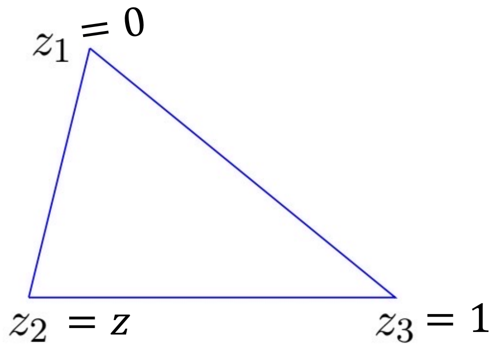
5 distinct boundary integral topologies related by S4- symmetry:

$$B^{[j,k,l]}_{a_1,a_2,a_3,a_4;\, q_1,q_2,q_3} \equiv \int \frac{d^3x}{\mathrm{GL}(1)} \frac{x_j^{-q_1} x_k^{-q_2} x_l^{-q_3}}{s_{jkl}^{a_1} x_{jk}^{a_2} x_{kl}^{a_3} x_{jkl}^{a_4}}$$

$$B^{[1,2,3]}_{1,0,1,1} \qquad B^{[1,2,4]}_{1,0,1,1} \qquad B^{[1,3,4]}_{1,0,1,1} \qquad B^{[2,3,4]}_{1,0,1,1} \qquad B^{[2,3,4]}_{1,1,0,1}$$

A total number of 28 four-point master integrals , 14 three-point boundary integrals and and 1 constant function.

EECC master integrals (triple collinear)



$$B_{a_1,0,a_3,a_4,q_1,q_2,q_3} = \int \frac{d^3x}{GL(1)} \frac{x_1^{-q_1} x_2^{-q_2} x_3^{-q_3}}{s_{123}^{a_1} x_{23}^{a_3} x_{123}^{a_4}}$$

$$B_1: \frac{x_2}{s_{123} x_{23} x_{123}}$$

$$B_2: \frac{1}{s_{123} x_{123}} \qquad B_3: \frac{x_2}{s_{123} x_{123}^2} \qquad B_4: \frac{x_3}{s_{123} x_{123}^2}$$

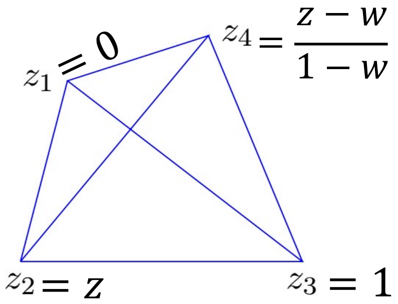
$$C_1: \boxed{1}$$

Boundary terms are kinematic independent integrals which integrates to rational numbers.

$$d\,\vec{g} = dA\,\vec{g}$$

$$\begin{bmatrix} dlog\left(\frac{1-z}{1-\bar{z}}\right) & dlog\left(\frac{z}{\bar{z}}\right) & & \\ dlog\left(\frac{|1-z|^2}{1-|z|^2}\right) & dlog|z|^2 & & \\ & & dlog|z|^2 & \\ & & dlog|1-z|^2 & \end{bmatrix}$$

EEEEC Master Integrals



$$\frac{x_1 x_2}{s_{1234} s_{123} x_{34} x_{1234}}, \frac{x_2^2}{s_{1234} s_{123} x_{234} x_{1234}}$$

$$\frac{x_1}{s_{1234} s_{123} x_{1234}}, \frac{x_2}{s_{1234} s_{123} x_{1234}}, \frac{x_3}{s_{1234} s_{123} x_{1234}},$$

$$\frac{x_3}{s_{1234} s_{123} x_{34}}, \frac{x_2}{s_{1234} s_{123} x_{234}}, \frac{x_3}{s_{1234} s_{123} x_{234}},$$

$$\frac{1}{s_{1234} x_{34} x_{1234}}, \frac{1}{s_{1234} x_{234} x_{1234}}, \frac{1}{s_{123} x_{34} x_{1234}}, \frac{x_3}{s_{123} x_{34} x_{1234}^2}, \frac{1}{s_{123} x_{234} x_{1234}}$$

$$\frac{1}{s_{1234}^2 x_{34}}$$

$$\frac{x_2 x_3}{s_{123} s_{234} x_{234} x_{1234}}, \frac{x_2^2}{s_{123} s_{234} x_{234} x_{1234}}$$

$$\frac{x_2}{s_{123} s_{234} x_{1234}}, \frac{x_3}{s_{123} s_{234} x_{1234}},$$

$$\frac{x_1 x_2}{s_{123} s_{234} x_{1234}^2}, \frac{x_1 x_3}{s_{123} s_{234} x_{1234}^2}, \frac{x_1 x_3}{s_{123} s_{234} x_{1234}^2}$$

$$B^{[1,2,3]}_{1,0,1,1} \quad B^{[1,2,4]}_{1,0,1,1} \quad B^{[1,3,4]}_{1,0,1,1} \quad B^{[2,3,4]}_{1,0,1,1}$$

1

Evaluate to up to weight-3 polylogarithms

All except two integrals involve only rational letters

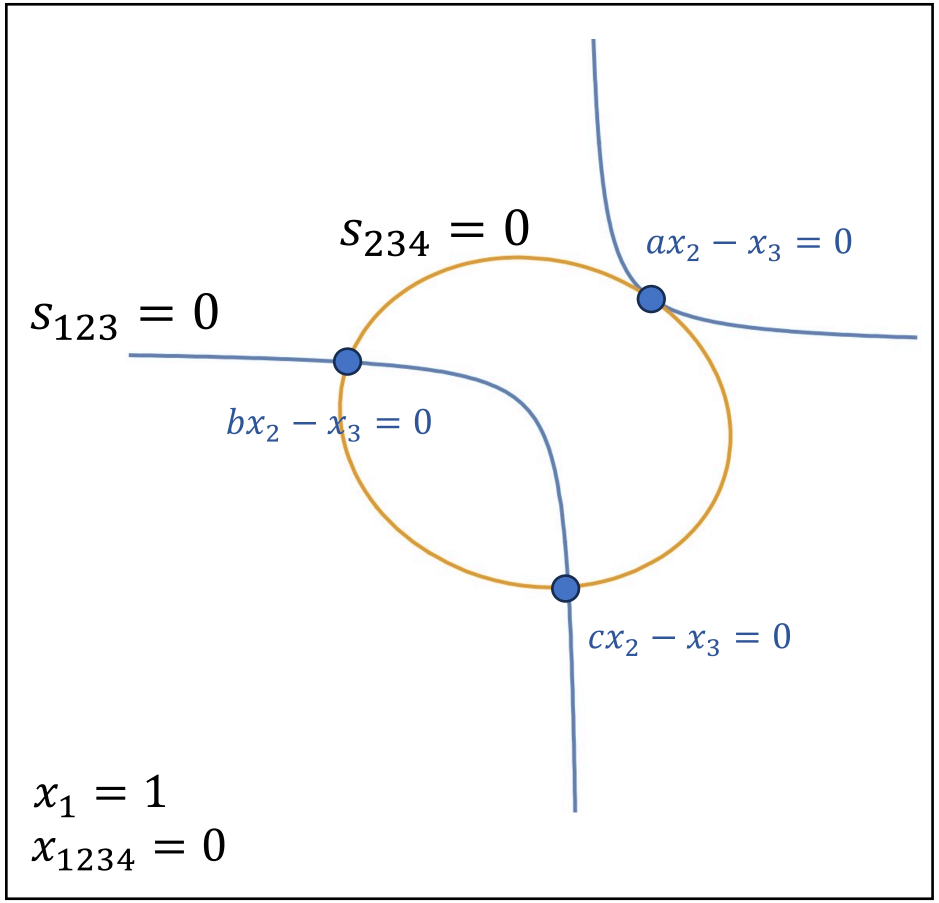
Analytic properties

$$f_1 := \int \frac{d^4x}{GL(1)} \frac{x_2}{s_{123}s_{234}x_{1234}},$$
$$f_2 := \int \frac{d^4x}{GL(1)} \frac{x_3}{s_{123}s_{234}x_{1234}}.$$

Cutting three propagators

$s_{123} = s_{234} = x_{1234} = 0$ defines a cubic curve

$$x_2^3|z_{12}|^2|z_{24}|^2 + x_3^3|z_{13}|^2|z_{34}|^2 + x_2^2x_3((|z_{13}|^2 - |z_{23}|^2)|z_{24}|^2 + |z_{12}|^2(-|z_{23}|^2 + |z_{24}|^2 + |z_{34}|^2)) + x_3^2x_2((|z_{13}|^2 - |z_{23}|^2)|z_{24}|^2 + |z_{12}|^2(-|z_{23}|^2 + |z_{24}|^2 + |z_{34}|^2)$$
$$= 0 := -|z_{13}|^2|z_{34}|^2(a\, x_2 - x_3)(b\, x_2 - x_3)(c\, x_2 - x_3)$$



$$f_1 \coloneqq \int \frac{d^4x}{GL(1)} \frac{x_2}{s_{123}s_{234}x_{1234}},$$
$$f_2 \coloneqq \int \frac{d^4x}{GL(1)} \frac{x_3}{s_{123}s_{234}x_{1234}}.$$

$$f_1 = \frac{1}{(c-a)(a-b)} \int d\ln x_3 \wedge \ln \frac{a\,x_2 - x_3}{b\,x_2 - x_3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_4} - \frac{1}{(b-c)(c-a)} \int d\ln x_3 \wedge \ln \frac{b\,x_2 - x_3}{c\,x_2 - x_3} \wedge d\ln \frac{D_2}{D_4} \wedge d\ln \frac{D_3}{D_4}$$

dlog basis:

$$g_1 \coloneqq g_a - g_b\,.$$

$$g_2 \coloneqq g_b - g_c\,.$$

$$|z_{13}|^2|z_{34}|^2 f_1 \coloneqq \frac{g_a}{(c-a)(a-b)} + \frac{g_b}{(a-b)(b-c)} + \frac{g_c}{(b-c)(c-a)},$$

$$|z_{12}|^2|z_{24}|^2 f_2 \coloneqq -\frac{g_a}{\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{a}-\frac{1}{b}\right)} - \frac{g_b}{\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{b}-\frac{1}{c}\right)} - \frac{g_c}{\left(\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{c}-\frac{1}{a}\right)}.$$

f_1, f_2 are totally symmetric when shuffling the three cubic roots

g_a, g_b, g_c: pure functions related by cyclic permutation

Under proper parametrization, e.g. using $(a, b, |z|^2, |w|^2)$, $g_a . g_{b,} g_c$ can evaluated in *HyperInt.*

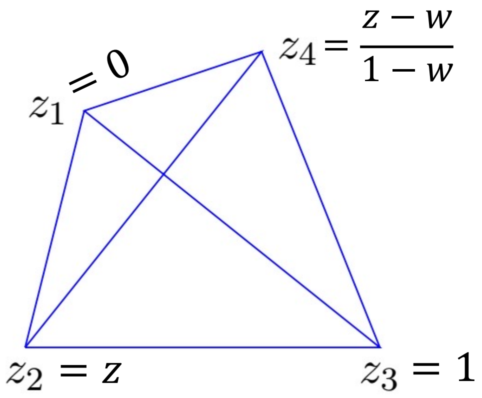
g_1, g_2 contain 10 more letters involving the cubic roots, which only appear in the last entry

$$g_1 \coloneqq g_a - g_b \,.$$

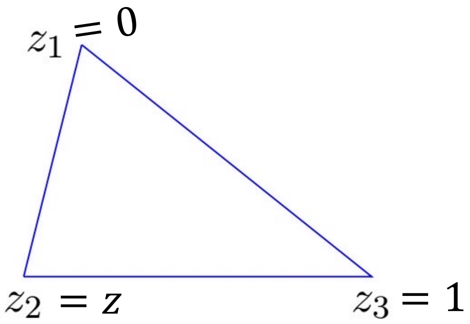
$$g_2 \coloneqq g_b - g_c \,.$$

$$\mathcal{A}_{cubic} \coloneqq \left\{ \frac{a}{b}, \frac{a+|z|^2}{b+|z|^2}, \frac{a+|w|^2}{b+|w|^2}, \frac{a+z}{a+\bar{z}} \frac{b+\bar{z}}{b+z}, \frac{a+w}{a+\bar{w}} \frac{b+\bar{w}}{b+w} \right\} \cup (a \to b, b \to c)$$

Despite the cubic-root dependence, the master integrals are single-valued function whose branch cuts cancel on the Euclidean sheet.
They all satisfy a first-entry condition: the first entry of the symbol must be $|z_{ij}|^2$.



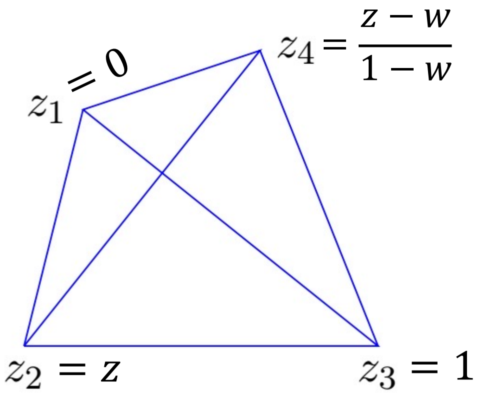
Symbol alphabets



Three-point correlator :

$$\mathcal{A}_3 := \{z, \bar{z}, 1-z, 1-\bar{z}, 1-|z|^2, 1-|1-z|^2, |z|^2-|1-z|^2\}$$

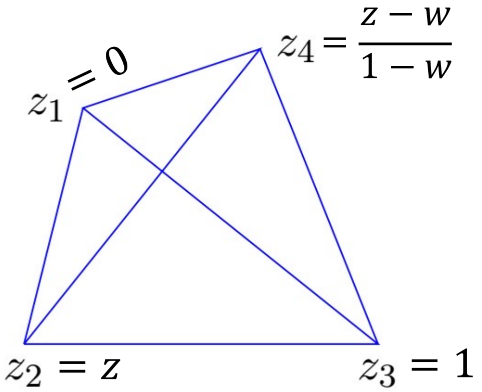
Four-point correlator : $define\ \overline{\mathcal{A}_3} \coloneqq \mathcal{A}_3 \cup \{z-\bar{z}\}$



$$\mathcal{A}_4 = \overline{\mathcal{A}_3}(1,2,3) \cup \overline{\mathcal{A}_3}(2,3,4) \cup \overline{\mathcal{A}_3}(1,2,4) \cup \overline{\mathcal{A}_3}(1,3,4) \cup \mathcal{A}_{1234} \cup \mathcal{A}_{cubic}$$

$$\mathcal{A}_{1234}: = \{\bar{w}z - w, w\bar{z} - \bar{w}, 1 - w - \bar{w} + \bar{w}z, 1 - w - \bar{w} + w\bar{z}, w - |z|^2, \bar{w} - |z|^2, \bar{w}z - \bar{z}w, |z|^2 - |w|^2\} \cup (w \leftrightarrow \frac{1}{z}, \bar{w} \leftrightarrow \frac{1}{\bar{z}})$$

Symbol alphabets



$$\left|\frac{z_{12}z_{34}}{z_{13}z_{24}}\right|^2 = \frac{|z|^2}{|w|^2}$$

$\mathcal{S}(E^4C)$

$1st\ entry$:

$$\{|z|^2, |1-z|^2, |w|^2, |1-w|^2, |z-w|^2\}$$

$2nd\ entry$:

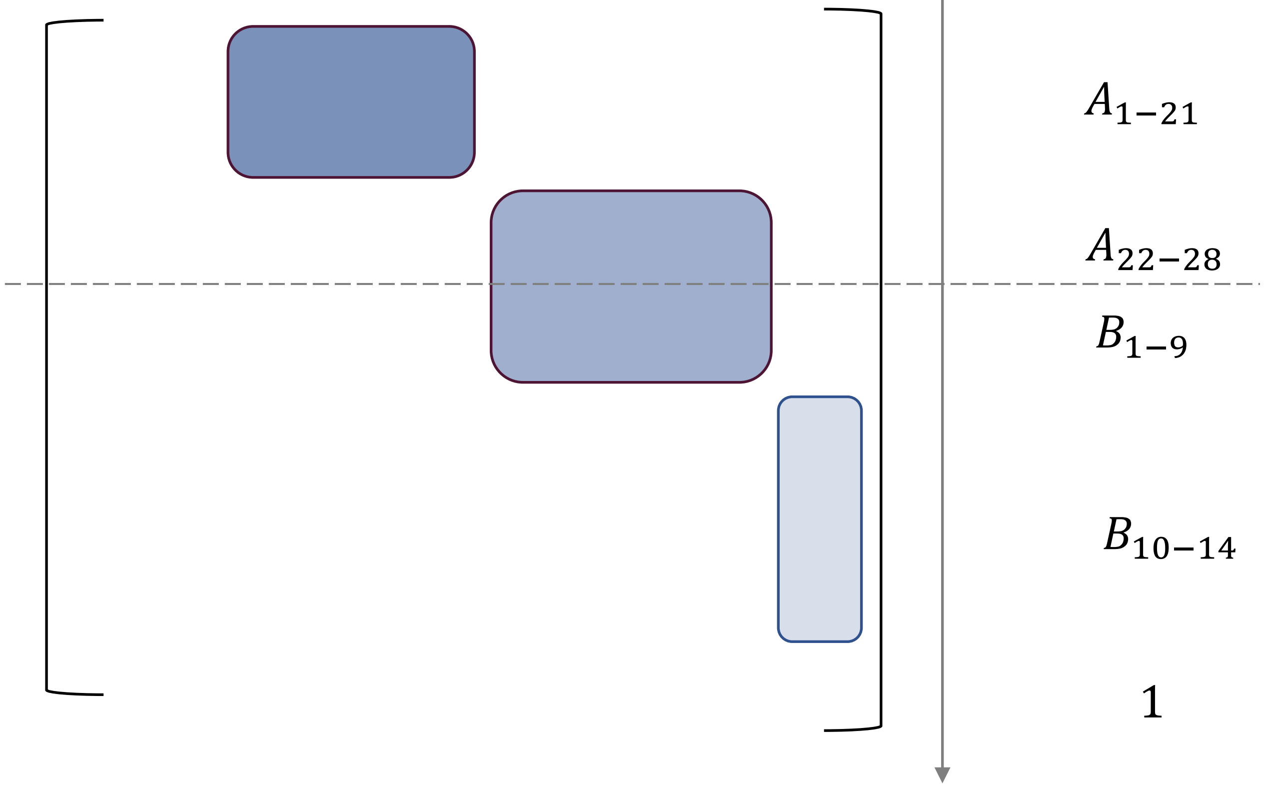
$$\mathcal{A}_3(1,2,3) \cup \mathcal{A}_3(2,3,4) \cup \mathcal{A}_3(1,2,4) \cup \mathcal{A}_3(1,3,4) \cup \{|z|^2 - |w|^2\}$$

$\mathcal{A}_{1234} \backslash \{|z|^2 - |w|^2\}, \mathcal{A}_{cubic}$ only appear in last entry

43× 43 system of differential equations $\vec{g} = dB\,\vec{g}$

Master integrals graded by transcendental weight

$dB =$



Iterative structures for (the symbol of) N −point correlators / adjacency relations?

EEEEC in N=4 SYM in the quadruple collinear limit :

$\mathcal{G}\ (z,w)$: the sum of integrals in four sub topologies

$A_{1,1,0,1,1,0,1}$ $A_{0,1,1,1,1,1,0}$

$A_{0,1,0,1,1,1,1}$ $A_{0,0,1,1,1,1,1}$

$$EEEEC_{N=4\ SYM}\Big|_{coll.} = \frac{1}{|z_{12}|^2|z_{23}|^2|z_{34}|^2}[\mathcal{G}\ (z,w)] + \text{perms}(1,2,3,4)$$

$$\mathcal{G}\ (z,w) = [R_i A_i + r_j\ B_j + r_0]$$

R_i, r_j : Algebraic functions

A_i : 28 pure master integrals 4-pt integral family

21 weight-3 + 7 weight-2

B_j: 14 pure master integrals in the boundary integral family

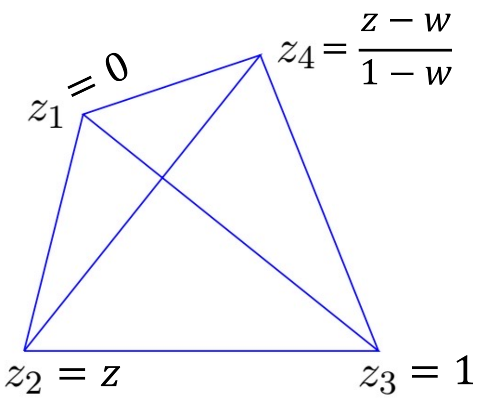
9 weight-2 + 5 weight-1

The expressions for R_i, r_j are complicated, contain high degree poles (up to six degree), most are spurious.

Factorization Limits

(1,2,3) triple collinear : $(w, \overline{w}) \to 1$

$$EEEEC_{N=4\ SYM}\Big|_{coll.} \propto \frac{1}{|z_{4i}|^2} EEEC_{N=4SYM}\Big|_{coll.}$$



(1,2) (3,4) double collinear : $\left(z, \bar{z}, \frac{1}{w}, \frac{1}{\overline{w}}\right) \to 0$

$$EEEEC_{N=4\ SYM}\Big|_{coll.} \propto \frac{1}{|z_{23}|^2} EEC_{N=4SYM}\Big|_{coll} \times EEC_{N=4SYM}\Big|_{coll}$$

Bootstrapping the Energy Correlators

Are there ways to bypass heavy IBPs and determine the E^NC via bootstrap?

Challenges: mixed weight, complication in the rational coefficients

Opportunities:
Imposing physical constraints,
lower-weight terms could be fixed from the higher-weight functions

Probing the structure of the symbol

-Differential equations

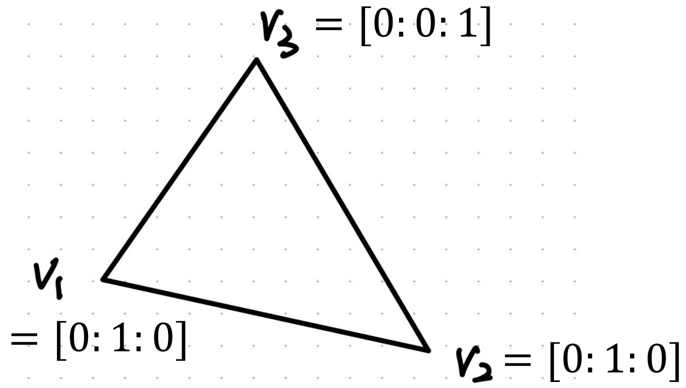
intersection theory method as a short cut for building the system of DEs

-Discontinuity

based on the method of projective geometry developed in 1712.09991 (also refering to 2206.06507)

Ongoing collaborations with Hofie Hannesdottir, Andrzej Pokraka, Xiaoyuan Zhang, Ellis Ye Yuan, Jianyu Gong

E^NC as simplex contour integral



Δ: Canonical simplex

$$I_N = \int_\Delta \frac{T[X^m]\,\langle X d^{N-1} X\rangle}{\prod_I (XQ_IX)^{a_I}\,\prod_J (H_JX)^{b_J}}$$

(n-1)-Simplex is uniquely determined by its 0-faces. In R^{n-1},

$$\sum_{i=1}^{n} x_i V_i, \qquad \sum_{i=1}^{n} x_i = 1 \text{ and } (\forall i)\ x_i \geq 0$$

We believe energy correlators can be analytically continued, so they are functions of complex variables.

In CP^{n-1},
domain of x_i are promoted to complex field. $\overline{V_i V_j}$ can be deformed within the CP^1subspace it belongs to.

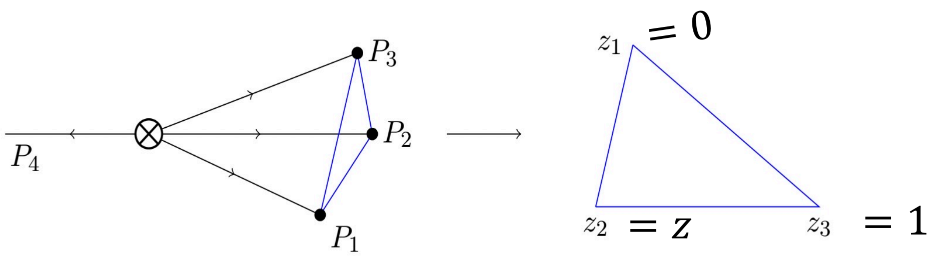
$E^N C$ from projective geometry

The ENC integrals are projective in CP^{n-1}

$$h(\boldsymbol{\omega})) \to \mathrm{d}^3\omega/\mathrm{GL}(1)/(\omega_1+\omega_2+\omega_3)^{\mathbb{N}}$$

$$F_1 = \int \mathrm{d}^3\omega \, \frac{\omega_2\omega_3 \; \delta(1-h(\boldsymbol{\omega}))}{\omega_1\omega_2 + |z|^2\omega_2\omega_3 + |1-z|^2\omega_1\omega_3} \qquad = 3\int_{CP^3} \frac{\langle \boldsymbol{\omega} \; \mathrm{d}^3\boldsymbol{\omega}\rangle \omega_2\omega_3}{(\boldsymbol{\omega} Q \boldsymbol{\omega})^4}$$

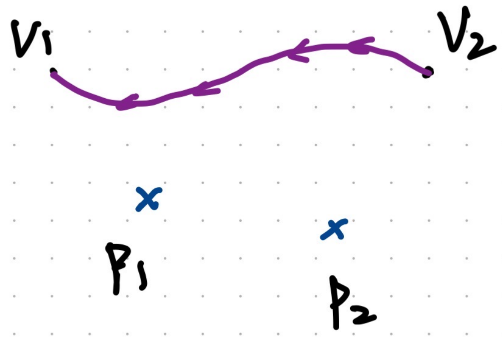
$$Q = \frac{1}{2}\begin{pmatrix} 0 & 1 & |z-1|^2 & 1 \\ 1 & 0 & |z|^2 & 1 \\ |z-1|^2 & |z|^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$



Spherical contour approach:

Taking deformed integration contour to compute discontinuity associated to the branch cut, (i.e. symbol entry), which can be read off from matrix Q.

Warm up: 1-simplex



$$Q = \frac{1}{2}\begin{pmatrix} 2 & -r_1 - r_2 \\ -r_1 - r_2 & 2 \end{pmatrix}$$

$$I = \int_{\Delta} \frac{\sqrt{\det Q}\,\langle X\, d\, X\rangle}{X\, Q\, X} = \int_{[1:0]}^{[0:1]} \frac{(r_1 - r_2)(x_1 dx_2 - x_2 dx_1)}{(x_1 - r_1 x_2)(x_1 - r_2 x_2)}$$

Type equation here.

$$S[I] = \otimes \frac{\langle P_1 V_1\rangle\langle P_2 V_1\rangle}{\langle P_1 V_2\rangle\langle P_1 V_2\rangle} = \otimes\, r(Q^{-1}) \qquad P_{1,2} = [r_{1,2}\colon 1]$$

$$r(M) \equiv \frac{M_{12} + \sqrt{M_{12}^2 - M_{11}\, M_{22}}}{M_{12} - \sqrt{M_{12}^2 - M_{11} M_{22}}} = \frac{r_1}{r_2}$$

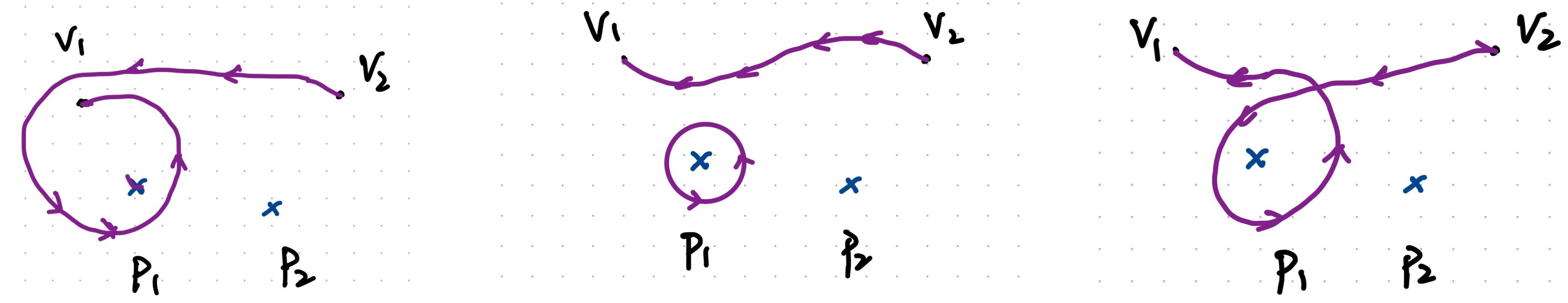
(The first entry of) the symbol emerge where the integrand singularities hits the contour boundary.

$$\langle P_i V_j\rangle = 0$$

Residue contour

Compute discontinuity:

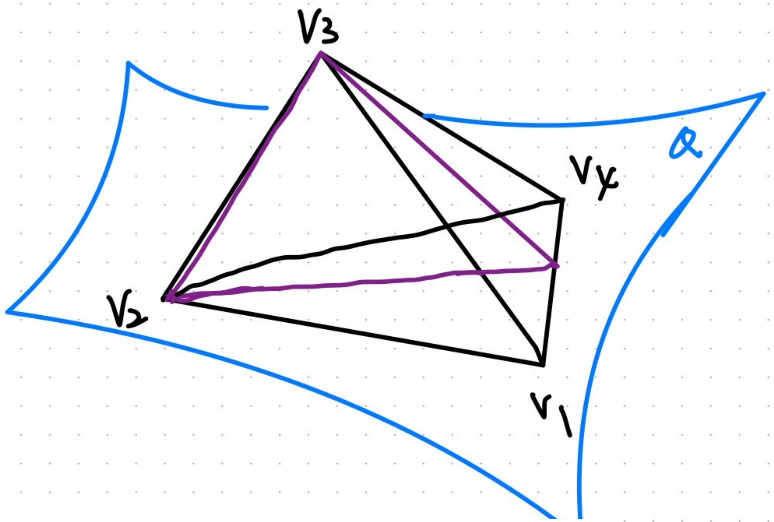
Pick up $(V_{i,}P_j)$ and analytically continue their bracket $\langle V_i P_j\rangle$ around zero, or equivalently, letting V_i to deform around P_j .

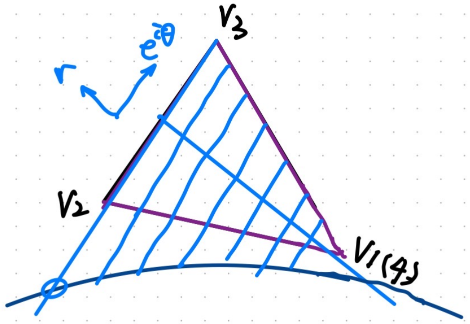


$$Disc_{V_1,P_1} I = \int_{|\langle X P_1\rangle|=\epsilon} \frac{\sqrt{\det Q}\,\langle X\ d\ X\rangle}{X\ Q\ X} = 2\pi i \qquad S[I] = \otimes\,\langle V_1 P_1\rangle - \otimes\,\langle V_2 P_1\rangle$$

Spherical contour

Fibration of CP^{n-1} over CP^2





$CP^2 \to CP^1 \times S^1$

Choose a partition, e.g {(2,3),(1,4)}

Taking (2,3) spherical contour: $(\omega_2, \omega_3) \to (\omega, \overline{\omega})$

$$\mathrm{Disc}_{23}[F_1] = 3\int_{\Delta^{(23)}} \frac{\mathrm{d}^2\boldsymbol{\omega}_{\widehat{23}}}{\mathrm{GL}(1)} \frac{1}{(z\bar{z})^4} \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \frac{2ir\ P_{23}}{\left(r^2 + \boldsymbol{\omega}_{\widehat{23}} Q_{\widehat{23}} \boldsymbol{\omega}_{\widehat{23}}\right)^4}$$

$$\boldsymbol{\omega}_{\widehat{23}} = (\omega_1, \omega_4) \qquad Q_{\widehat{23}} = -\begin{pmatrix} \frac{(1-z)(1-\bar{z})}{z\bar{z}} & \frac{2-z-\bar{z}}{2z\bar{z}} \\ \frac{2-z-\bar{z}}{2z\bar{z}} & \frac{1}{z\bar{z}} \end{pmatrix}$$

The discontinuity is a CP^1- integral over $[\omega_1 \colon \omega_4]$:

$$\mathrm{Disc}_{23}[F_1] = -\frac{\pi i}{(z\bar{z})^3} \int_{\Delta^{(23)}} \frac{\mathrm{d}^2\boldsymbol{\omega}_{\widehat{23}}}{\mathrm{GL}(1)} \frac{T_{23}}{\left(\boldsymbol{\omega}_{\widehat{23}} Q_{\widehat{23}} \boldsymbol{\omega}_{\widehat{23}}\right)^3}$$

Symbol construction

$$Disc_{14}[Disc_{23}[\Lambda]] = \frac{1}{z-\bar{z}} \quad \longrightarrow \quad S[Disc_{23}[\Lambda]] = \frac{1}{z-\bar{z}} \times \left(\otimes \frac{1-z}{1-\bar{z}}\right)$$

combine with $S[Disc_{13}[\Lambda]]$

$$\mathcal{S}[\Lambda] = \frac{1}{z-\bar{z}} \times \left(|z|^2 \otimes \frac{1-z}{1-\bar{z}} + |1-z|^2 \otimes \frac{z}{\bar{z}}\right)$$

Λ → $Disc_{ij}[\Lambda]$ → $Disc_{\widehat{ij}}[Disc_{ij}[\Lambda]]$

first entry: $r\left(Q^{-1}_{\{2,3\}\{2,3\}}\right) = z\bar{z}$

second entry: $r\left(Q^{-1}_{\widehat{23}}\right) = \frac{1-z}{1-\bar{z}}$

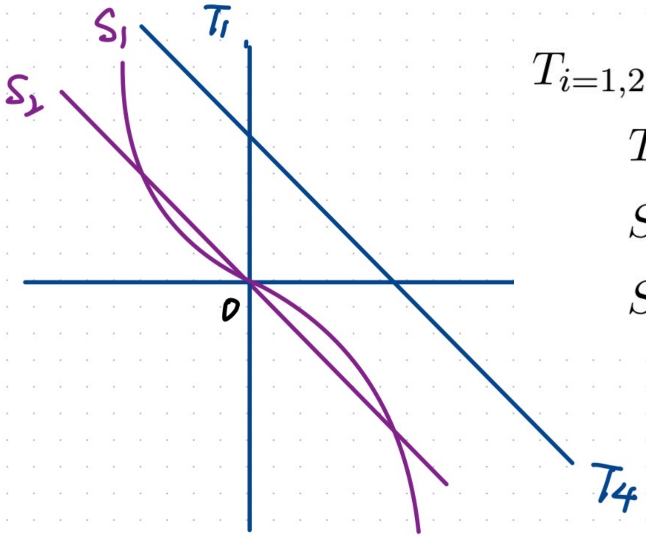
Leading singularity

$E^N C$ from intersection theory

The E^NC (in D=4-2e) defines a differential form which belong to a twisted cohomology

$$I_{\boldsymbol{\mu},\boldsymbol{\nu}} := \int \frac{\mathrm{d}^3\boldsymbol{\omega}}{\mathrm{GL}(1)} \frac{u}{\mathbf{T}^{\boldsymbol{\mu}}\mathbf{S}^{\boldsymbol{\nu}}} := \int u \, \varphi_{\boldsymbol{\mu},\boldsymbol{\nu}} \qquad \boldsymbol{\mu} \in \mathbb{Z}^4,\, \boldsymbol{\nu} \in \mathbb{Z}^2,\, u = \mathbf{T}^{(-\varepsilon,-\varepsilon,-\varepsilon,3\varepsilon)}$$

$$\varphi_{\boldsymbol{\mu},\boldsymbol{\nu}} \in H^3\left(X; \nabla_\omega\right), \qquad \nabla_\omega = \mathrm{d} + \omega\wedge\,, \qquad \omega = \mathrm{d}\log u \qquad X = \mathbb{CP}^3 \setminus \mathcal{TS}$$



$$T_{i=1,2,3} = \omega_i$$
$$T_4 = \omega_1 + \omega_2 + \omega_3$$
$$S_1 = \omega_1\omega_2|z_{12}|^2 + \omega_2\omega_3|z_{23}|^2 + \omega_3\omega_1|z_{31}|^2$$
$$S_2 = \omega_1 + \omega_2$$

(Potential) IR divergences are regulated at the twisted boundary

All differential forms are regular at the relative boundary

T_i: *twisted*

S_i: *unwisted*

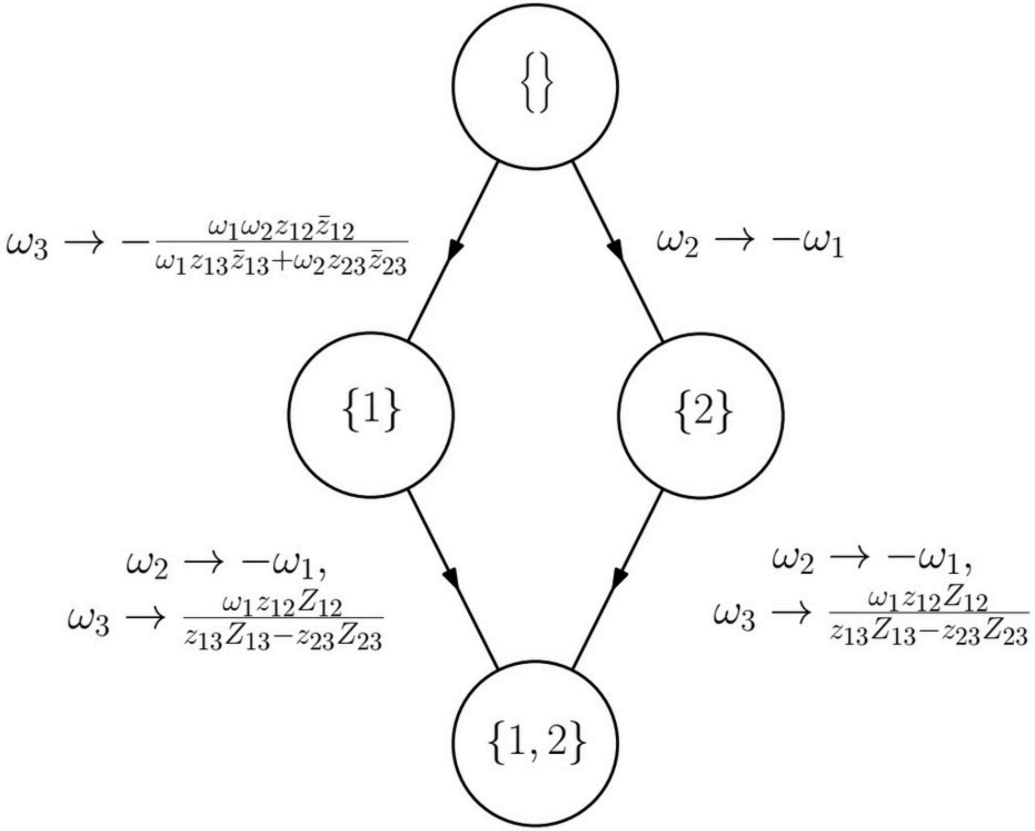
Define a dual relative twisted cohomology: $\mathbb{1} = \sum_{a,b} |\varphi_a\rangle \, C_{ab}^{-1} \, \langle\check{\varphi}_b|$ $\qquad C_{ab} = \langle\check{\varphi}_a|\varphi_b\rangle$

$$H^3\left(X^\vee, \mathcal{S}; \nabla_{-\omega}\right) = H^3\left(X^\vee; \nabla_{-\omega}\right) \bigoplus_{i=1,2} H^2\left(X^\vee \cap \mathcal{S}_i; \nabla_{-\omega}\right) \bigoplus H^1\left(X^\vee \cap \mathcal{S}_{12}; \nabla_{-\omega}\right)$$

$$X^\vee = \mathbb{CP}^3 \setminus \mathcal{T}\,, \quad \text{and} \quad \mathcal{S}_J = \bigcap_{i \in J} \mathcal{S}_i$$

It is more convenient to build the DEs for the dual forms.

$$\check{\varphi} = \sum_J \delta_J\left(\check{\phi}_J\right)$$



Boundary stratification of the relative twisted cohomology

A basis of dual forms :

$$\varphi_1^\vee = \delta_{\{\}}\left(\frac{\varepsilon^2}{\omega_1\omega_2\omega_3}\frac{\mathrm{d}\omega_1\wedge\mathrm{d}\omega_2\wedge\mathrm{d}\omega_3}{\mathrm{GL}(1)}\right)$$

$$\varphi_2^\vee = \delta_1\left(\frac{\varepsilon}{\omega_1\omega_2}\frac{\mathrm{d}\omega_1\mathrm{d}\omega_2}{\mathrm{GL}(1)}\right)$$

$$\varphi_3^\vee = \delta_1\left(-\frac{\varepsilon}{\omega_2(\omega_1|z_{13}|^2+\omega_2|z_{23}|^2)}\frac{\mathrm{d}\omega_1\mathrm{d}\omega_2}{\mathrm{GL}(1)}\right)$$

$$\varphi_4^\vee = \delta_1\left(\frac{\sqrt{(|z_{12}|^2)^2+(|z_{13}|^2)^2+(|z_{23}|^2)^2-2|z_{12}|^2|z_{13}|^2-2|z_{12}|^2|z_{23}|^2-2|z_{13}|^2|z_{23}|^2}}{\omega_1^2|z_{13}|^2+\omega_2^2|z_{23}|^2+(|z_{13}|^2+|z_{23}|^2-|z_{12}|^2)\omega_1\omega_2}\frac{\mathrm{d}\omega_1\mathrm{d}\omega_2}{\mathrm{GL}(1)}\right)$$

$$\varphi_5^\vee = \delta_{12}\left(\frac{1}{\omega_1}\frac{\mathrm{d}\omega_1}{\mathrm{GL}(1)}\right)$$

Number of dual basis on each boundary matches the number of master integrals sector by sector

C_1: 1

B_2: $\frac{1}{s_{123}x_{123}}$ B_3: $\frac{x_2}{s_{123}x_{123}^2}$

B_4: $\frac{x_3}{s_{123}x_{123}^2}$

B_1: $\frac{x_2}{s_{123}x_{23}x_{123}}$

The intersection method applies to higher-loop order, where phase-space integrals are IR divergent.

Summary

Further development of phase-space integration algorithms

-NLO:
Promoting to d=4-2e dimension, incorporating ideas from intersection theory methods.

-$E^N C$ at generic angle, away from collinear limit.

Algorithm for bootstrapping the $E^N C$

- What do we learn about the function space/rational structure?

-How to impose physical constraints, e.g. from various OPE limits of light-ray operators

THANK YOU FOR YOUR ATTENTION !