OHIO UNIVERSITY

ASCSN

BAND

Bayesian Analysis of Nuclear Dynamics

# *Bridging the gap:* a Bayesian model mixing approach to the dense matter equation of state


Alexandra C. Semposki

*in collaboration with:* C. Drischler, R. J. Furnstahl, J. A. Melendez, D. R. Phillips

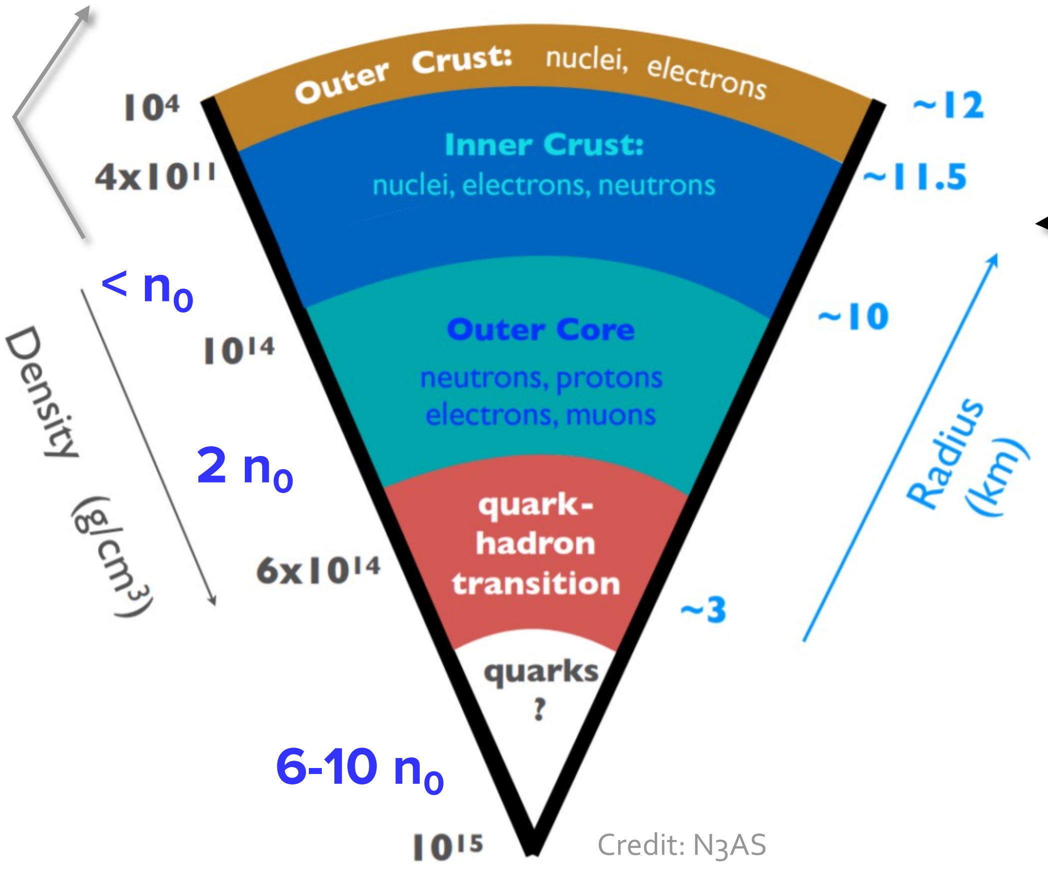

arXiv:2404.06323

# Motivation: the equation of state (EOS)

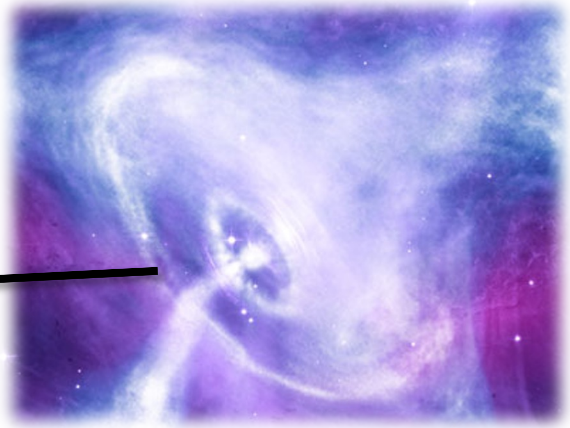
Credit: NASA

# Motivation: the equation of state (EOS)

Units of nuclear saturation density

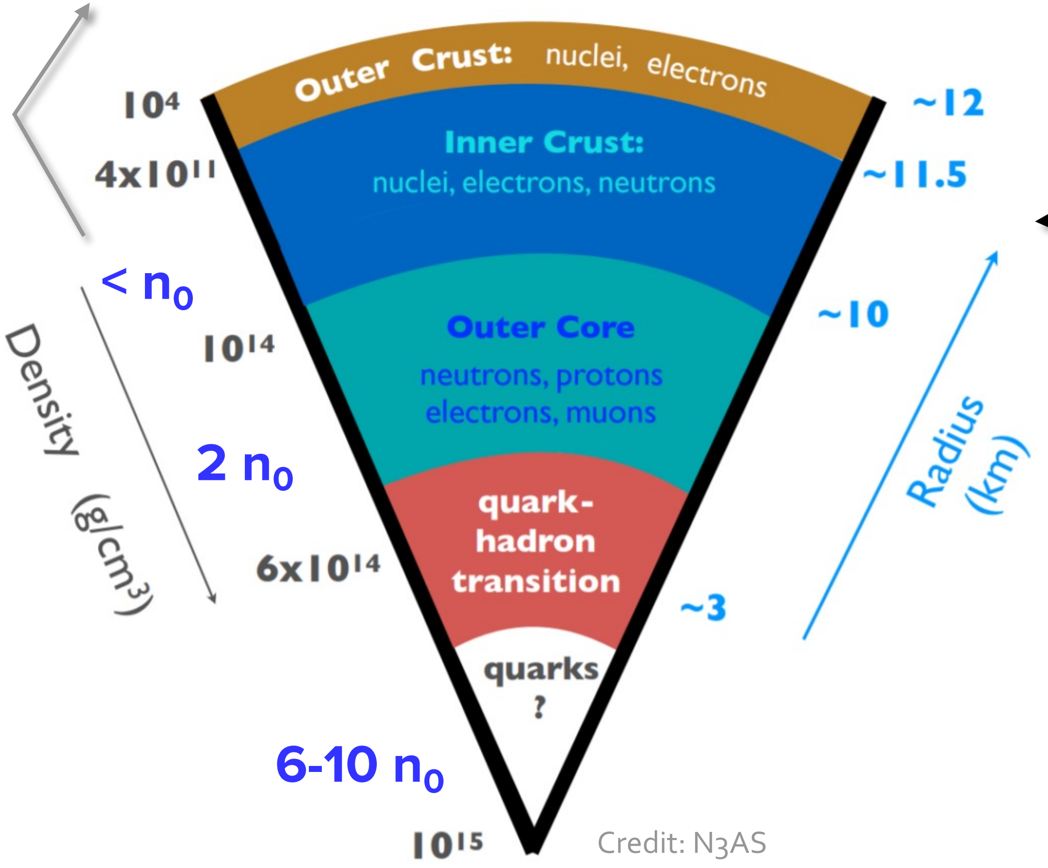



Credit: N3AS

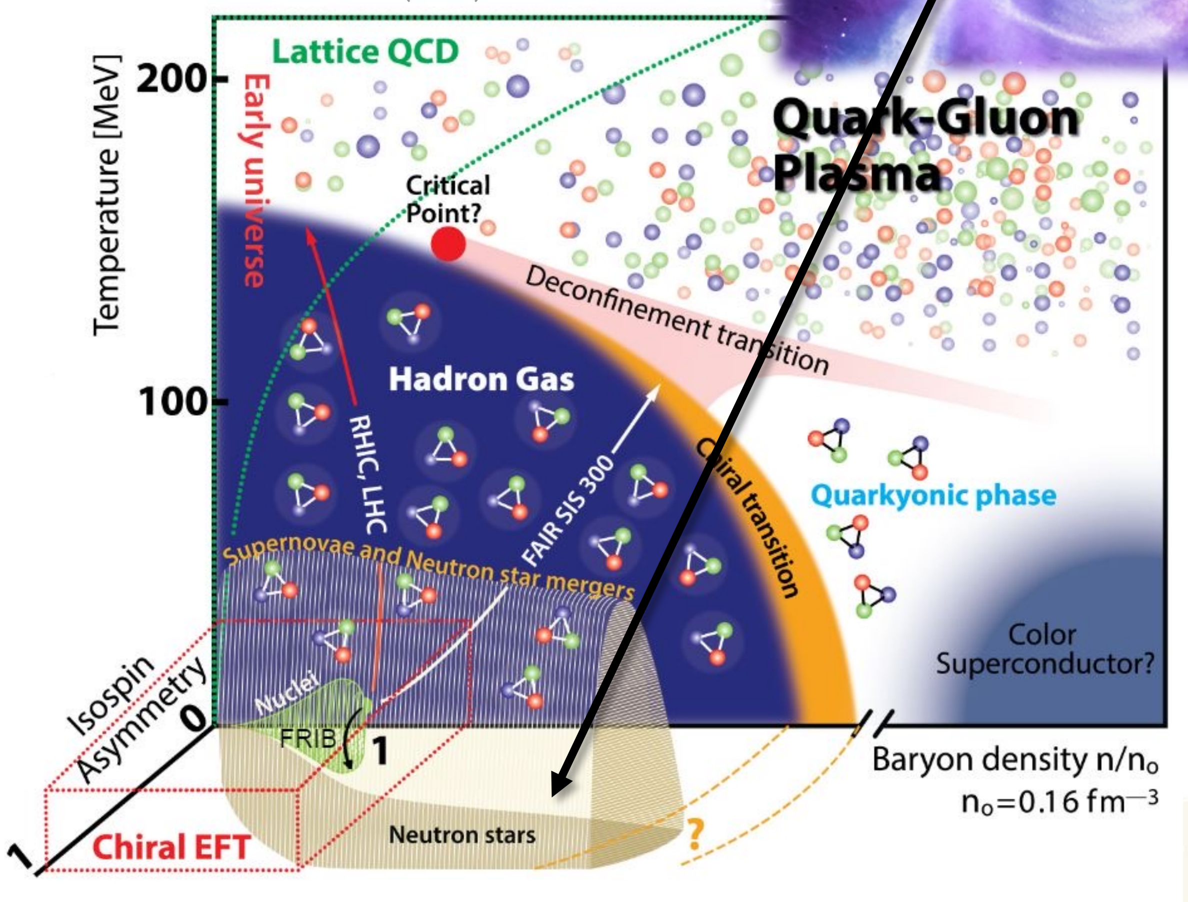


Credit: NASA

# Motivation: the equation of state (EOS)




Credit: N3AS

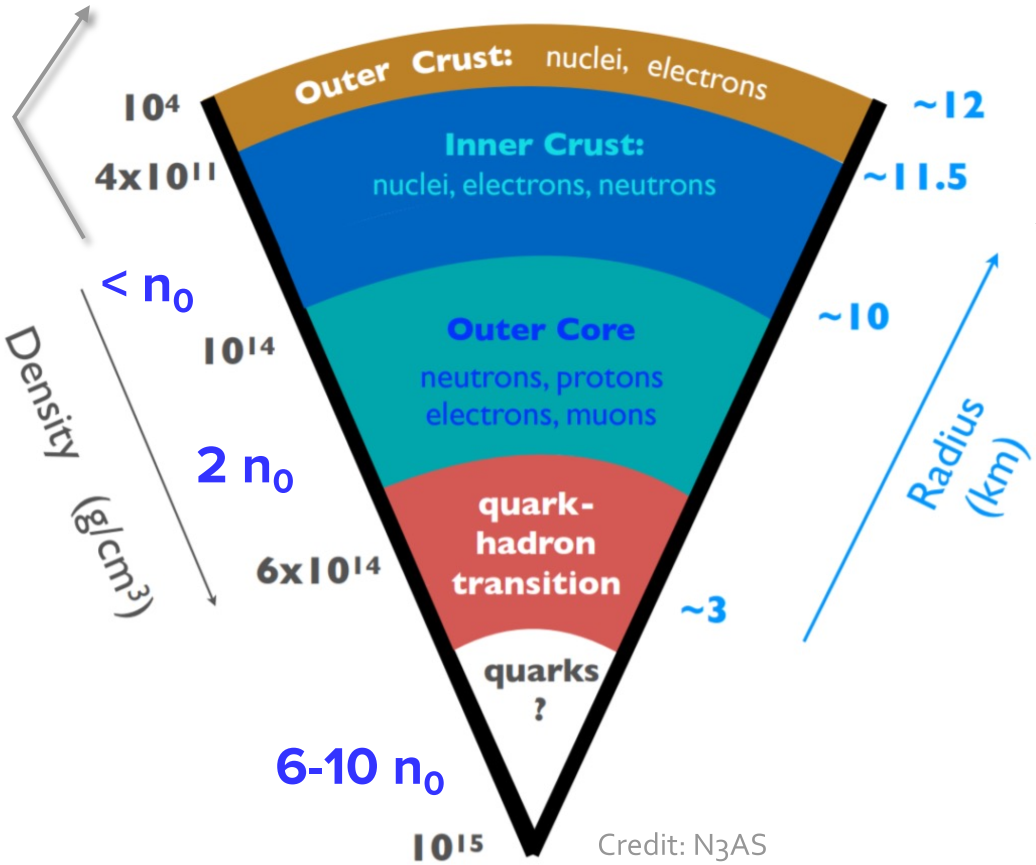



Credit: NASA

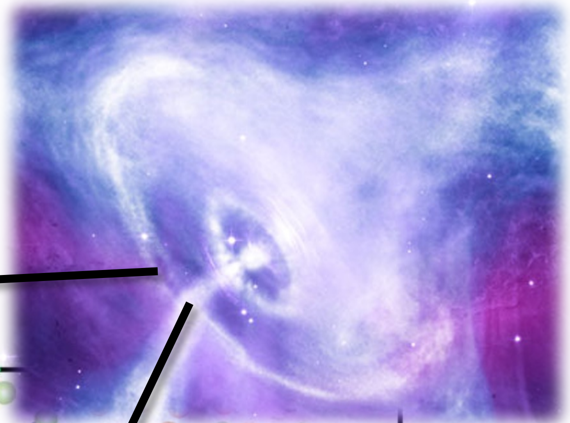
EOS describes the state of matter from nuclei to neutron stars to quarks and deconfinement

# Motivation: the equation of state (EOS)

Units of nuclear saturation density

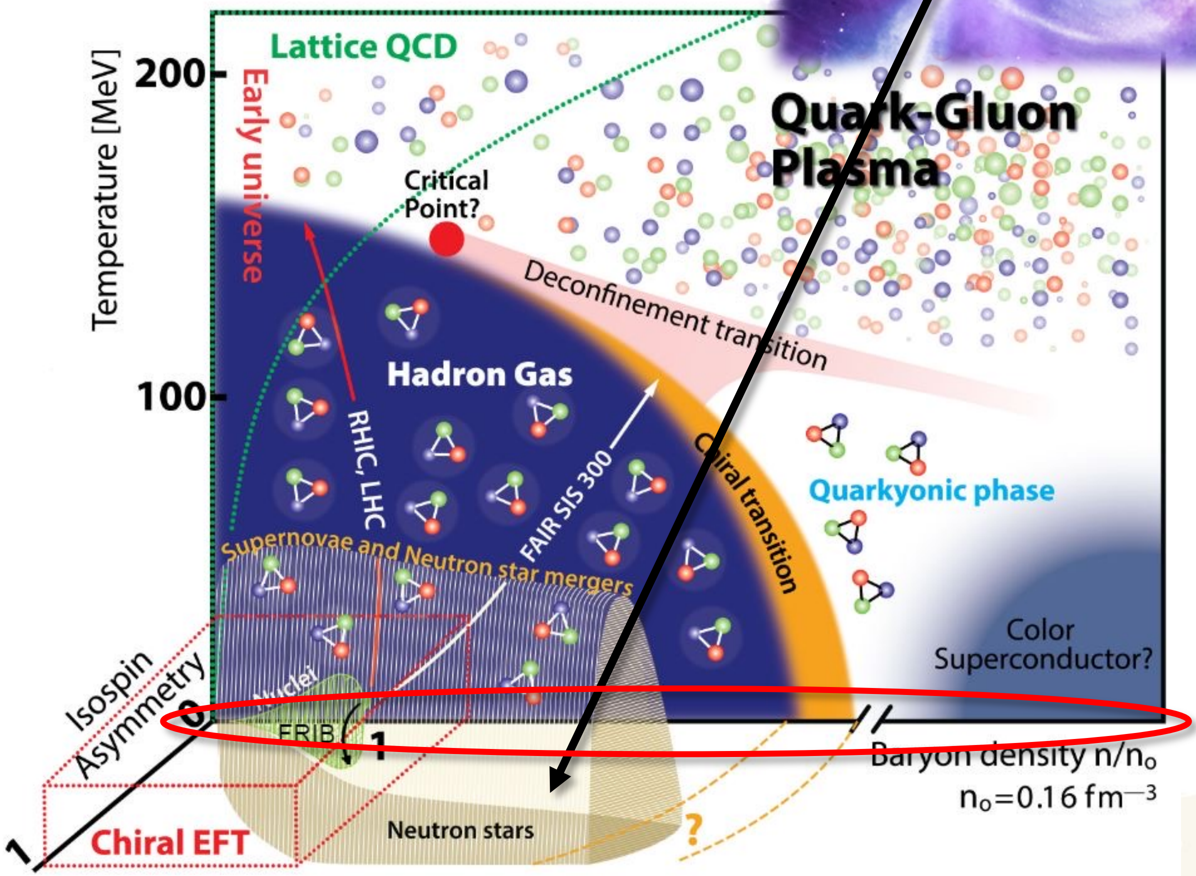



Credit: N3AS



Credit: NASA

Drischler et al. (2021)




We are here

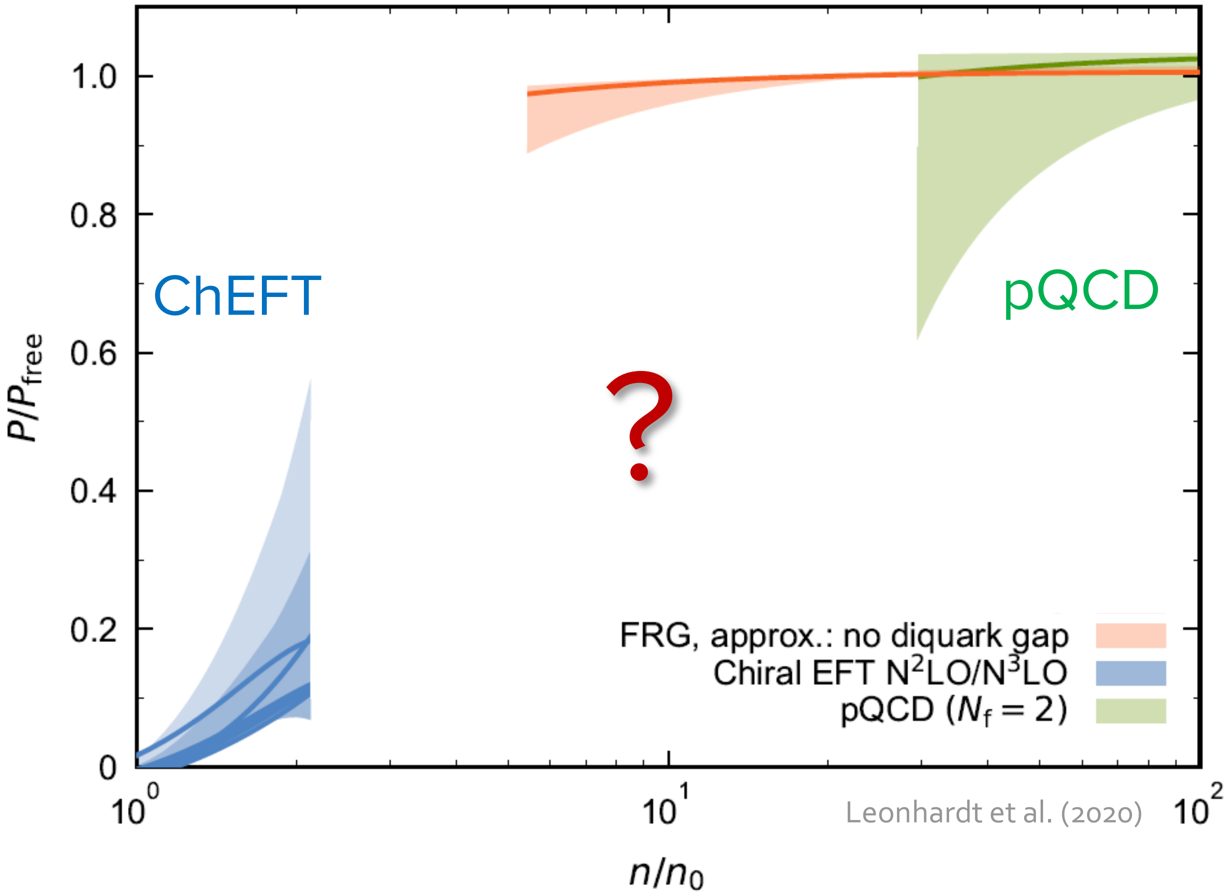
EOS describes the state of matter from nuclei to neutron stars to quarks and deconfinement

# Motivation: between two extremes

How to develop an overall model from several individual models for the EOS?

Symmetric Nuclear Matter
(50% n, 50% p)

Our goal: improve current estimates & constrain the gap using UQ



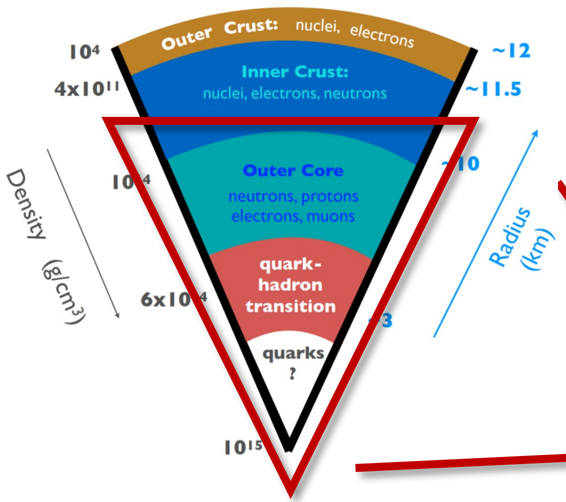
# Motivation: between two extremes

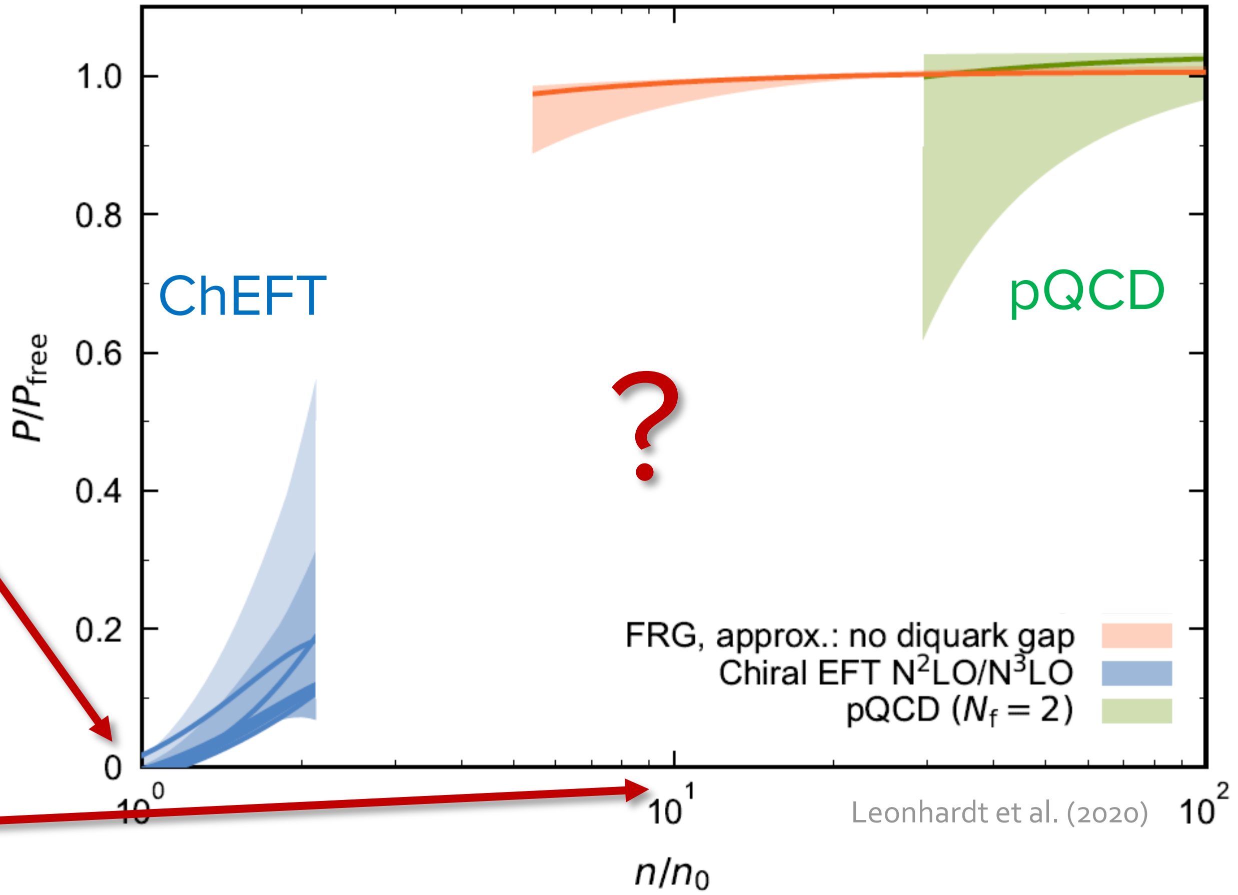
How to develop an overall model from several individual models for the EOS?

Symmetric Nuclear Matter (50% n, 50% p)

Our goal: improve current estimates & constrain the gap using UQ

Low densities can be described by $\chi$EFT, constraints from heavy-ion collisions




Leonhardt et al. (2020)

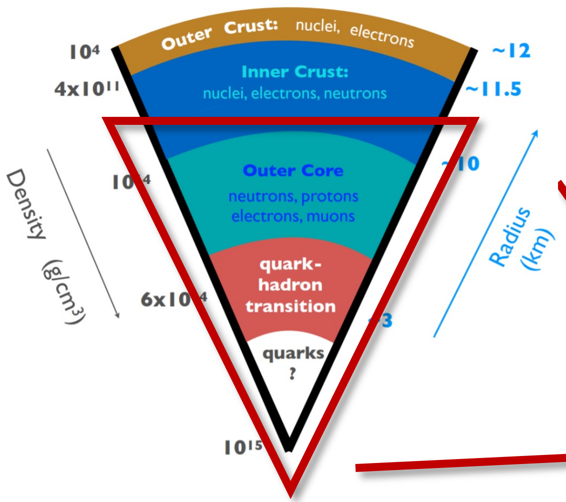
# Motivation: between two extremes

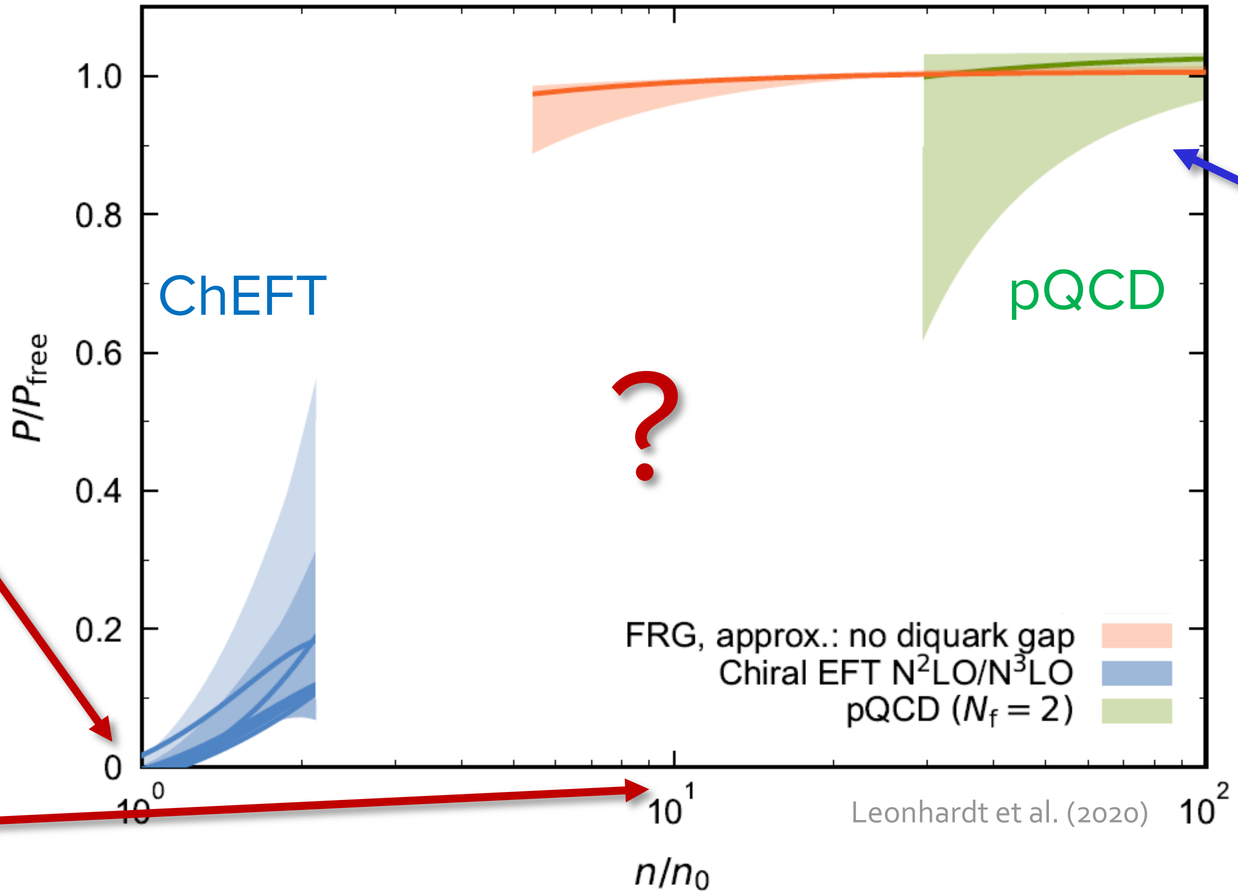
How to develop an overall model from several individual models for the EOS?

Symmetric Nuclear Matter (50% n, 50% p)

Our goal: improve current estimates & constrain the gap using UQ

Low densities can be described by $\chi$EFT, constraints from heavy-ion collisions




High densities:

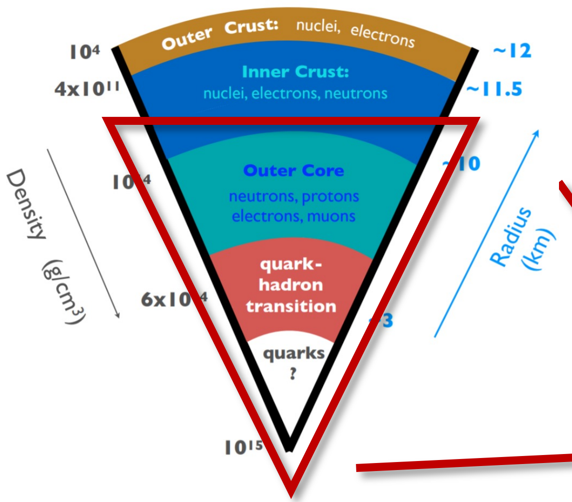
perturbative QCD describes quark matter in the densest regions

FRG EOS in the gap, but uncertainties are not well-quantified (yet!) …

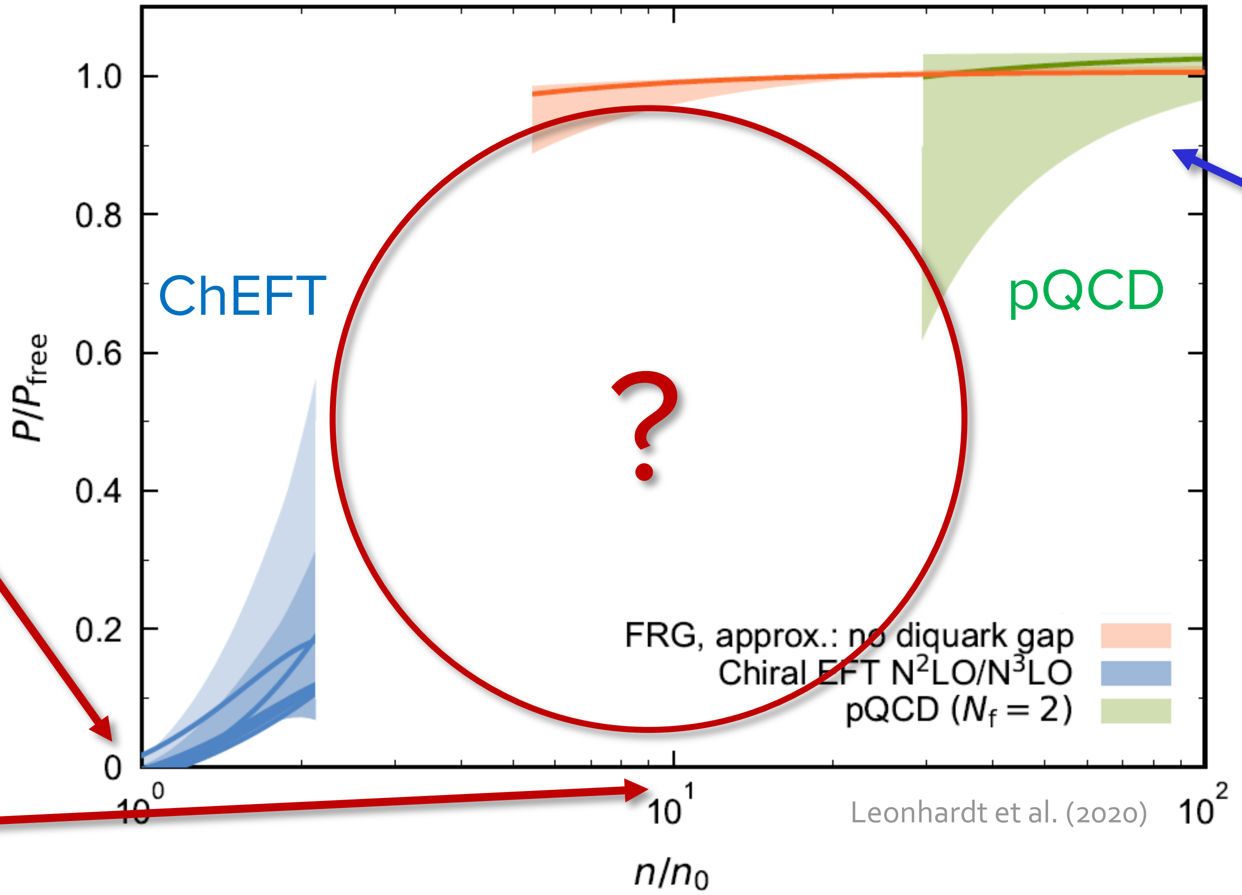
# Motivation: between two extremes

How to develop an overall model from several individual models for the EOS?

Low densities can be described by $\chi$EFT, constraints from heavy-ion collisions




Symmetric Nuclear Matter (50% n, 50% p)




Our goal: improve current estimates & constrain the gap using UQ

High densities:

perturbative QCD describes quark matter in the densest regions

FRG EOS in the gap, but uncertainties are not well-quantified (yet!) …

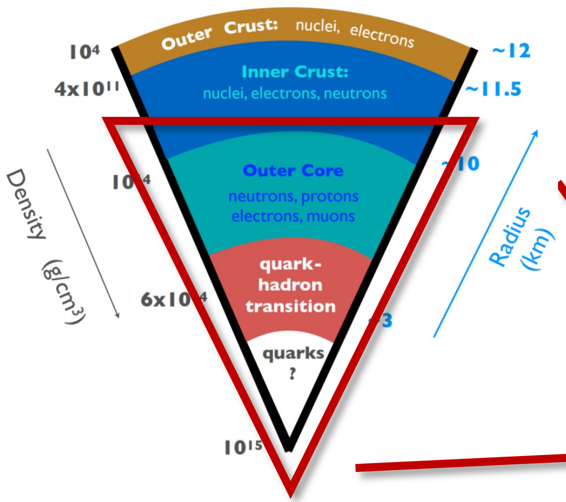
# Motivation: between two extremes

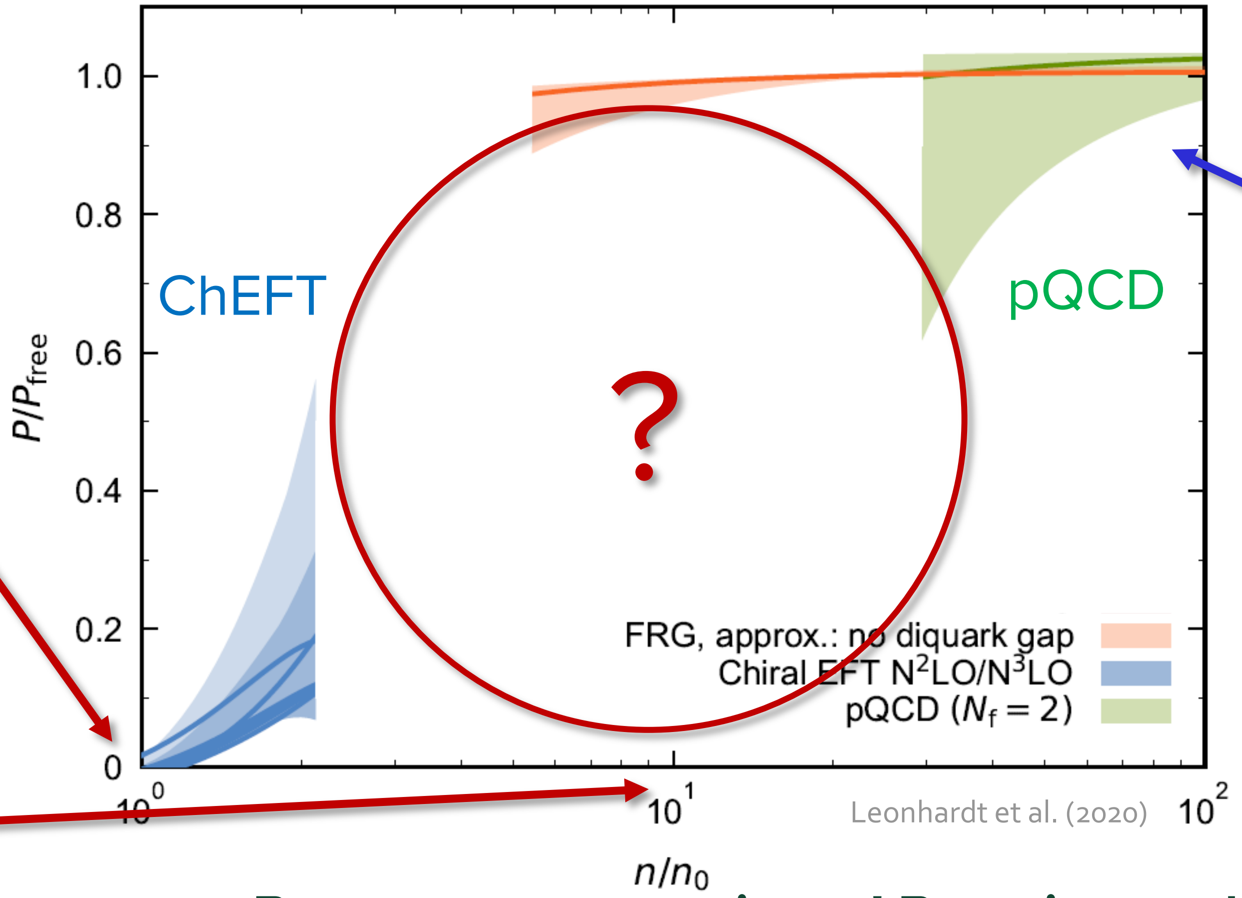
How to develop an overall model from several individual models for the EOS?

Symmetric Nuclear Matter (50% n, 50% p)

Our goal: improve current estimates & constrain the gap using UQ

Low densities can be described by $\chi$EFT, constraints from heavy-ion collisions




High densities:

perturbative QCD describes quark matter in the densest regions

FRG EOS in the gap, but uncertainties are not well-quantified (yet!) …

**Bayes your way to victory! Bayesian model mixing (BMM)!**

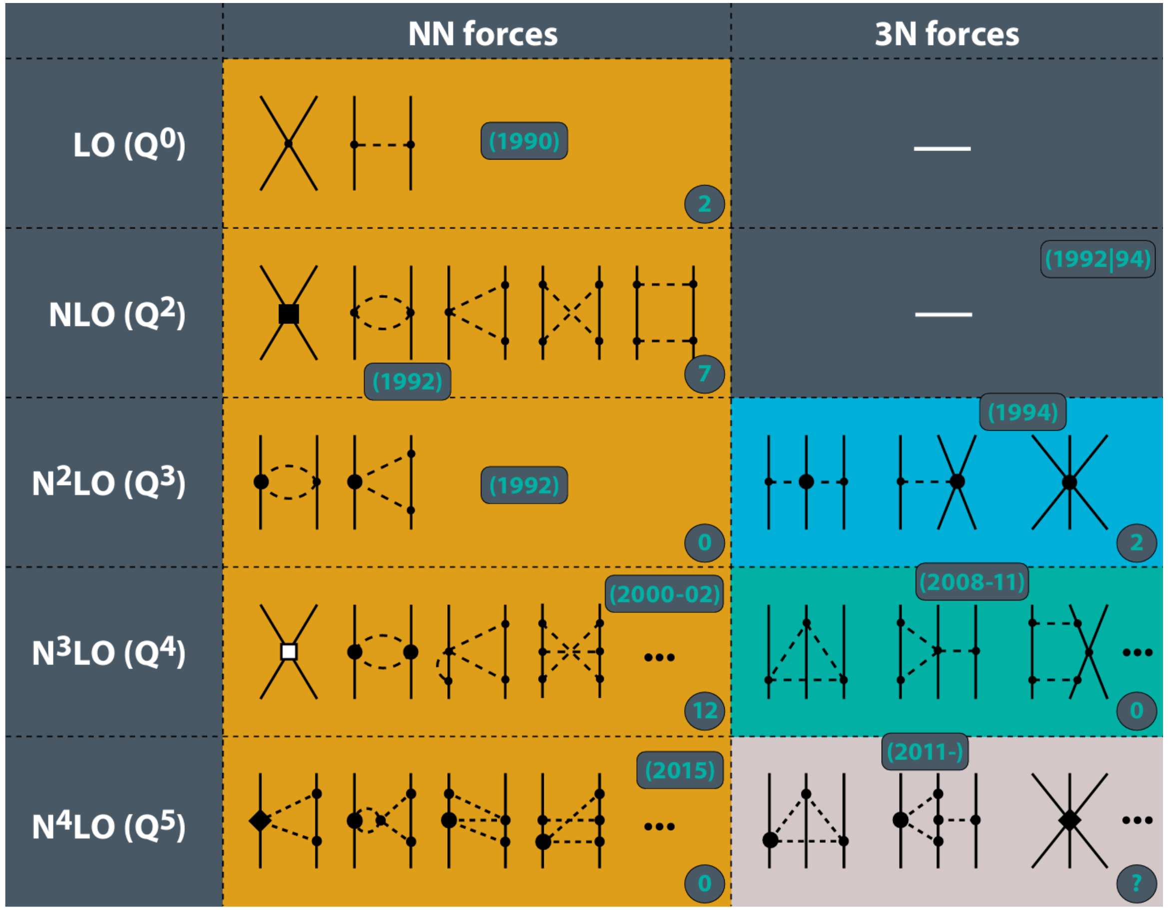
OHIO UNIVERSITY

# "Low" densities: EOS from chiral EFT

QCD non-perturbative at low energies, build *effective description* using nucleons, pions as degrees of freedom

$$Q = \max\left(\frac{p}{\Lambda_b}, \frac{m_\pi}{\Lambda_b}\right)$$

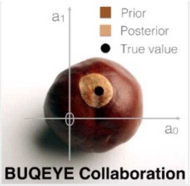
Quantifiable truncation error, obeys all symmetries of QCD




C. Drischler, S. Bogner (2021)

OHIO UNIVERSITY

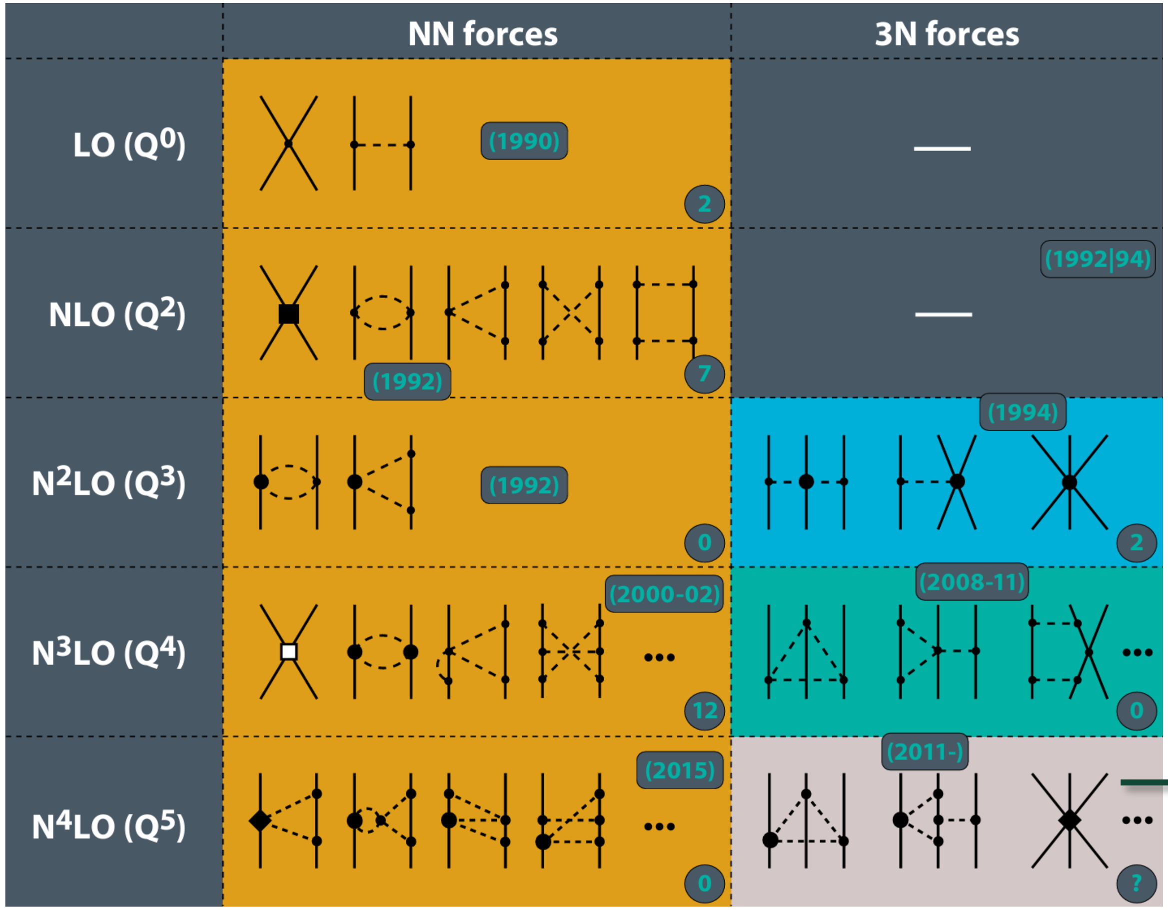
# “Low” densities: EOS from chiral EFT




QCD non-perturbative at low energies, build *effective description* using nucleons, pions as degrees of freedom

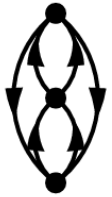
$$Q = \max\left(\frac{p}{\Lambda_b}, \frac{m_\pi}{\Lambda_b}\right)$$

Quantifiable truncation error, obeys all symmetries of QCD



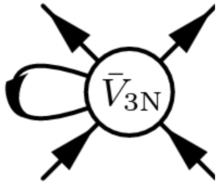

C. Drischler, S. Bogner (2021)

$\mathcal{E}_2(k_\mathrm{F}) =$  $= -\frac{1}{4}\sum_{ijab} V^{ij,ab} V^{ab,ij} f_{ij} \bar{f}_{ab} \frac{1}{D_{ab,ij}}$

$\mathcal{E}_{3,pp}(k_\mathrm{F}) =$ 

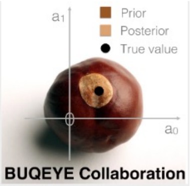
Use **MBPT*** for energy per particle (E/A) calculations

*A choice, not a necessity---BMM framework is model-independent

$\left\langle \mathbf{2'3'} \middle| V_{\mathrm{NN}}^{\mathrm{med}} \middle| \mathbf{23} \right\rangle =$ 

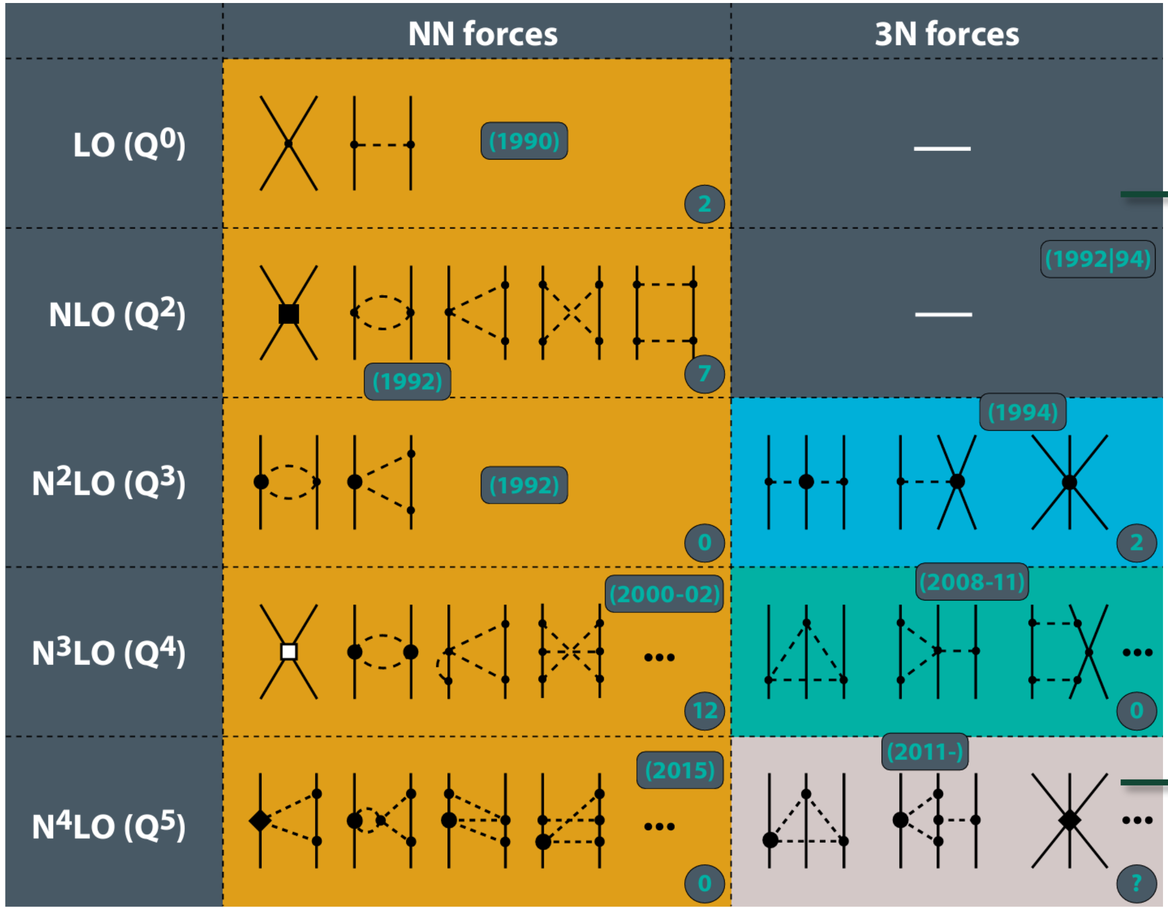

C. Drischler, J. Holt, C. Wellenhofer (2021)

Original calculations and ideas: C. Drischler, K. Hebeler, A. Schwenk, Phys. Rev. Lett. **122**, 042501 (2019)




# “Low” densities: EOS from chiral EFT

QCD non-perturbative at low energies, build *effective description* using nucleons, pions as degrees of freedom




C. Drischler, S. Bogner (2021)

$$Q = \max\left(\frac{p}{\Lambda_b}, \frac{m_\pi}{\Lambda_b}\right)$$

Quantifiable truncation error, obeys all symmetries of QCD

Using NN potential from **Entem, Machleidt, Nosyk** (2017) + **3N forces**

Fit **low-energy constants** (LECs) to experimental data, empirical saturation point of SNM

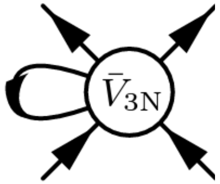
Employ momentum **cutoff** of 500 MeV

$\mathcal{E}_2(k_{\mathrm{F}}) =$  $= -\frac{1}{4}\sum_{ijab} V^{ij,ab} V^{ab,ij} f_{ij} \bar{f}_{ab} \frac{1}{D_{ab,ij}}$

$\mathcal{E}_{3,pp}(k_{\mathrm{F}}) =$ 

Use **MBPT*** for energy per particle (E/A) calculations

*A choice, not a necessity--- BMM framework is model-independent

$\left\langle \mathbf{2'3'} \middle| V_{\mathrm{NN}}^{\mathrm{med}} \middle| \mathbf{23} \right\rangle =$ 

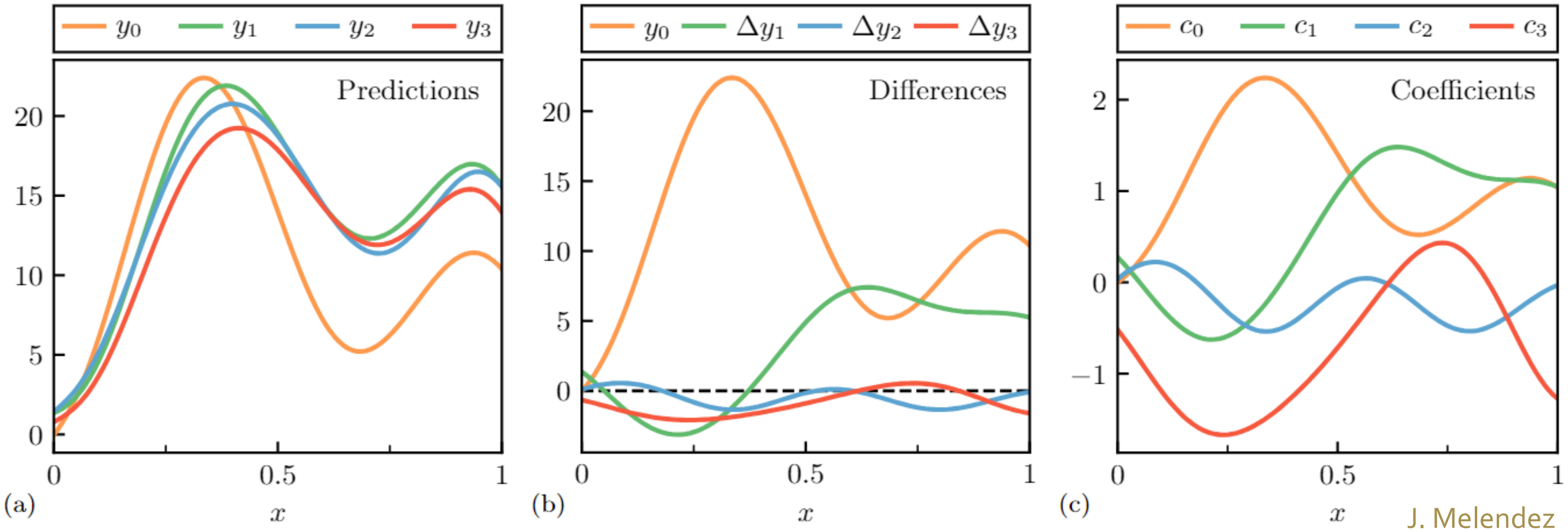

C. Drischler, J. Holt, C. Wellenhofer (2021)

# Uncertainty quantification for the EOS

Employ the BUQEYE truncation error model (**gsum**)

$$y_k(x) = y_{\mathrm{ref}}(x) \sum_{n=0}^{k} c_n(x) Q^n(x)$$

Known orders of expansion




J. Melendez et al. (2019)



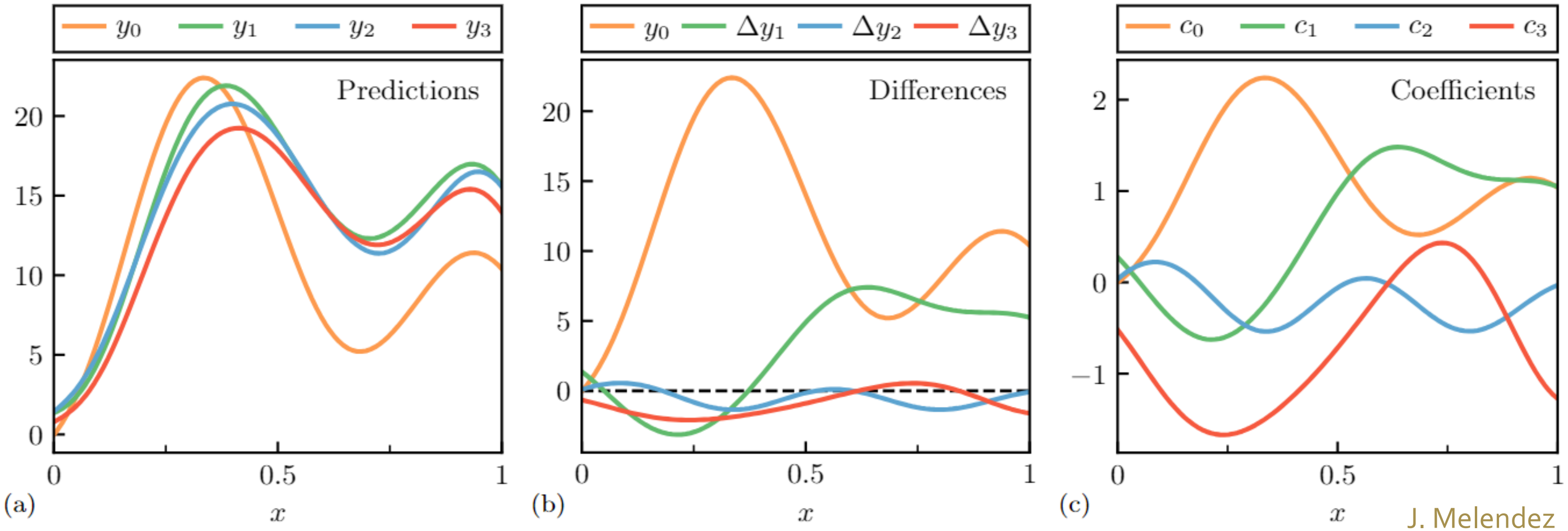
Original work: Melendez, Wesolowski, Furnstahl, Phillips, Pratola, PRC (2019)

# Uncertainty quantification for the EOS

Employ the BUQEYE truncation error model (**gsum**)

$$y_k(x) = y_{\mathrm{ref}}(x) \sum_{n=0}^{k} c_n(x) Q^n(x)$$

Known orders of expansion




J. Melendez et al. (2019)

Estimations for unknown orders

$$\delta y_k(x) = y_{\mathrm{ref}}(x) \sum_{n=k+1}^{\infty} c_n(x) Q^n(x)$$

Dimensional analysis

Natural i.i.d. curves

Expansion parameter (suppresses higher orders)



Original work: Melendez, Wesolowski, Furnstahl, Phillips, Pratola, PRC (2019)

# Uncertainty quantification for the EOS

BUQEYE Collaboration

Employ the BUQEYE truncation error model (**gsum**)

$$y_k(x) = y_{\mathrm{ref}}(x) \sum_{n=0}^{k} c_n(x) Q^n(x)$$

Known orders of expansion

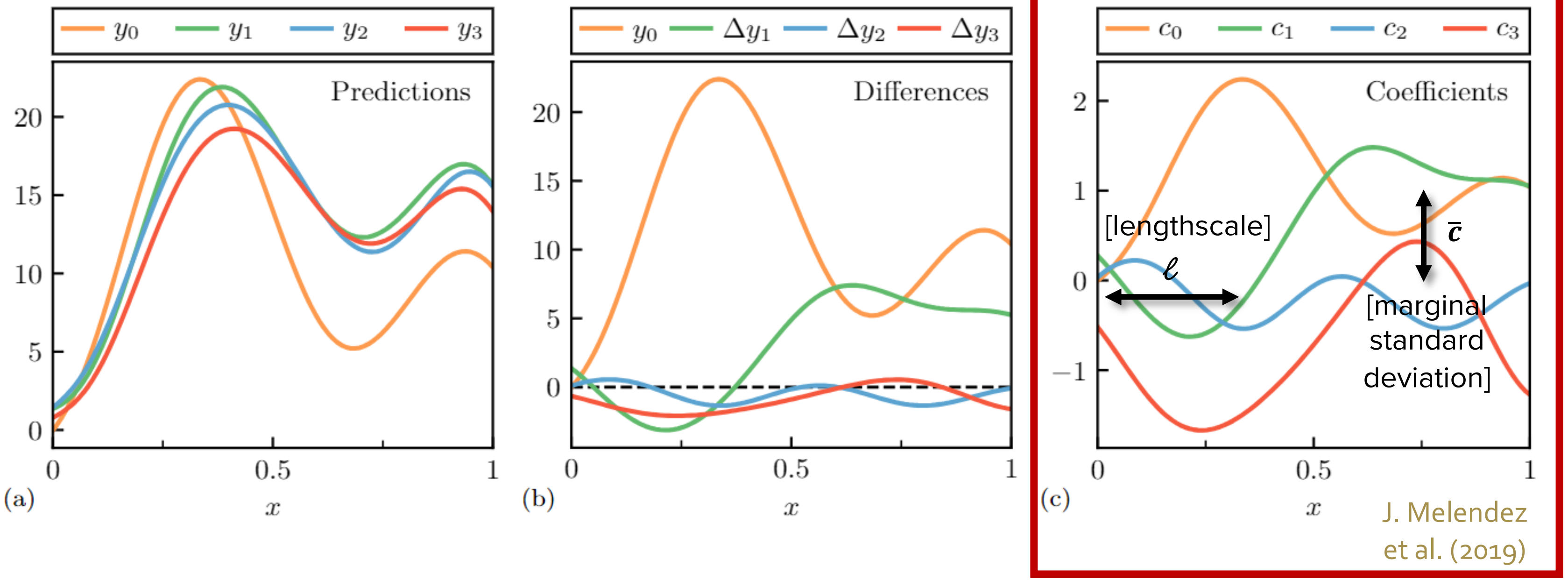
Estimations for unknown orders

$$\delta y_k(x) = y_{\mathrm{ref}}(x) \sum_{n=k+1}^{\infty} c_n(x) Q^n(x)$$

Dimensional analysis

Natural i.i.d. curves

Expansion parameter (suppresses higher orders)




**Gaussian Processes**: learn coefficient distributions

Incorporates correlations in y(x), continuous model, priors constrain parameters of GP

Easily test with *diagnostics* (Mahalanobis distance, pivoted Cholesky decomposition,...)

# "Low" densities: EOS from chiral EFT

Obtain pressure as a function of number density, P(n), for model mixing calculations

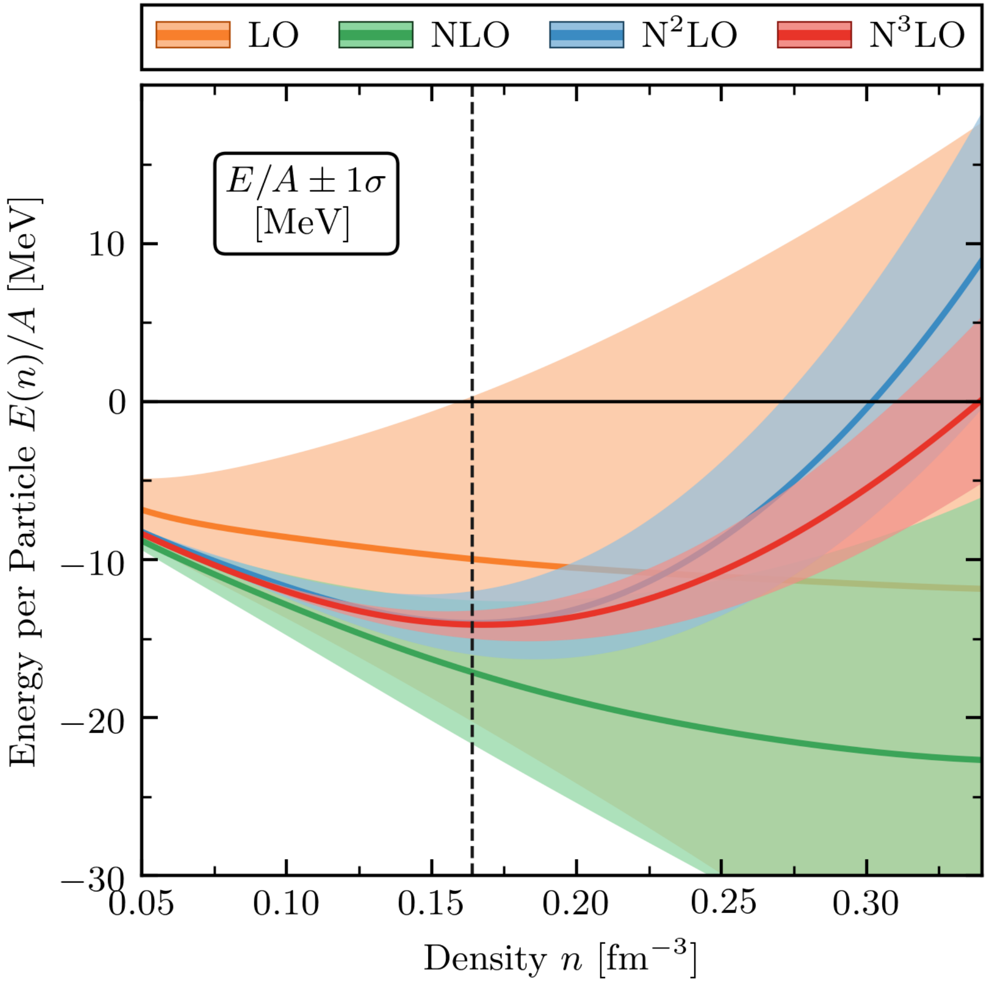
$$P(n) = n^2 \frac{\mathrm{d}}{\mathrm{d}n} \frac{E}{A}(n)$$

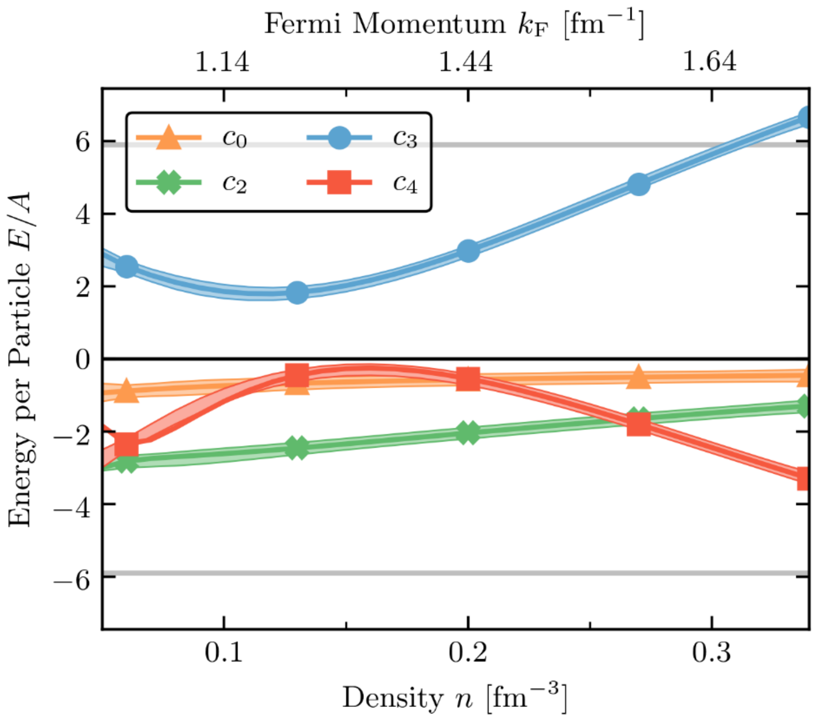
Coefficient extraction for truncation error estimation done via **gsum**

$$Q(k_\mathrm{F}) = \frac{k_\mathrm{F}}{\Lambda_b}$$ ≈ 600 MeV

$$y_\mathrm{ref}(k_\mathrm{F}) = 16\,\mathrm{MeV} \times \left(\frac{k_\mathrm{F}}{k_{\mathrm{F},0}}\right)^2$$

*Symmetric nuclear matter*



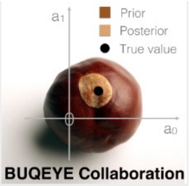




Truncation error scheme yields natural-sized curves as expected



Truncation error analysis: C. Drischler, J. A. Melendez, R. J. Furnstahl, D. R. Phillips, Phys. Rev. C **102**, 054315 (2020)

# "Low" densities: EOS from chiral EFT




Obtain pressure as a function of number density, P(n), for model mixing calculations

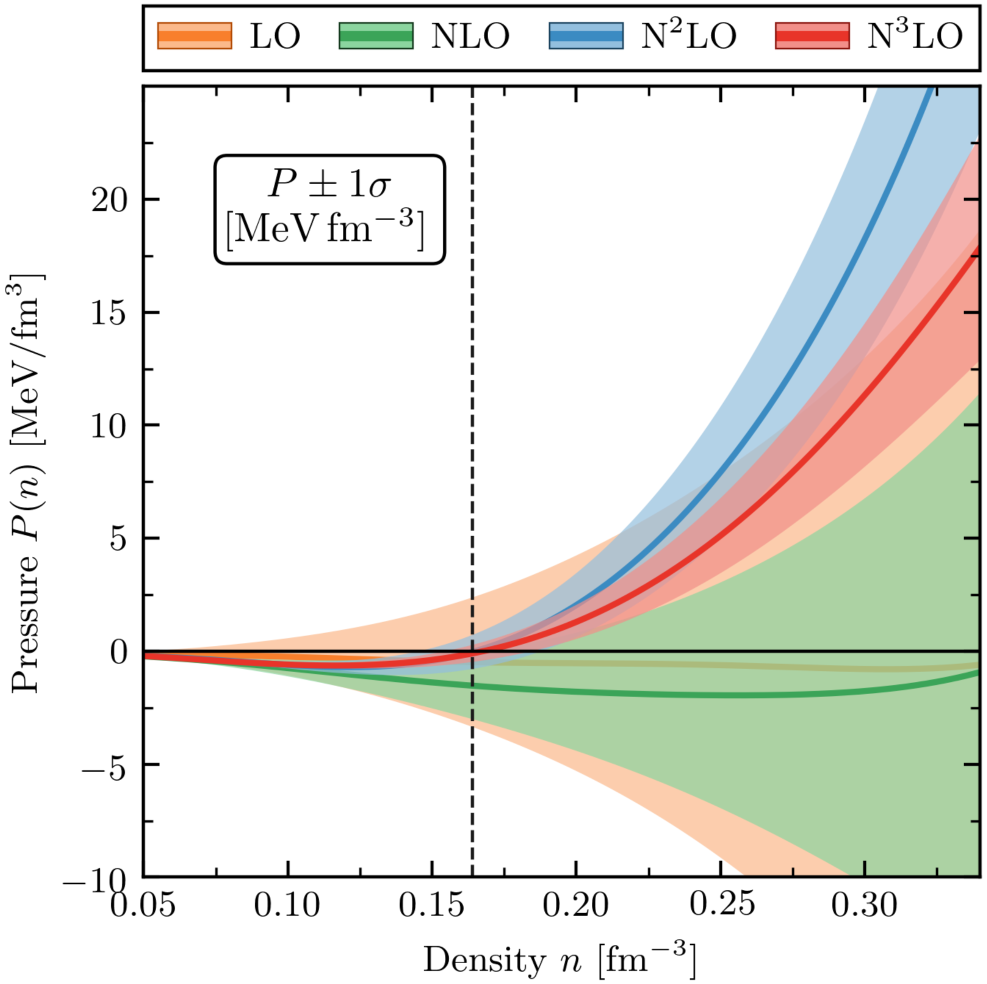
$$P(n) = n^2 \frac{\mathrm{d}}{\mathrm{d}n}\frac{E}{A}(n)$$

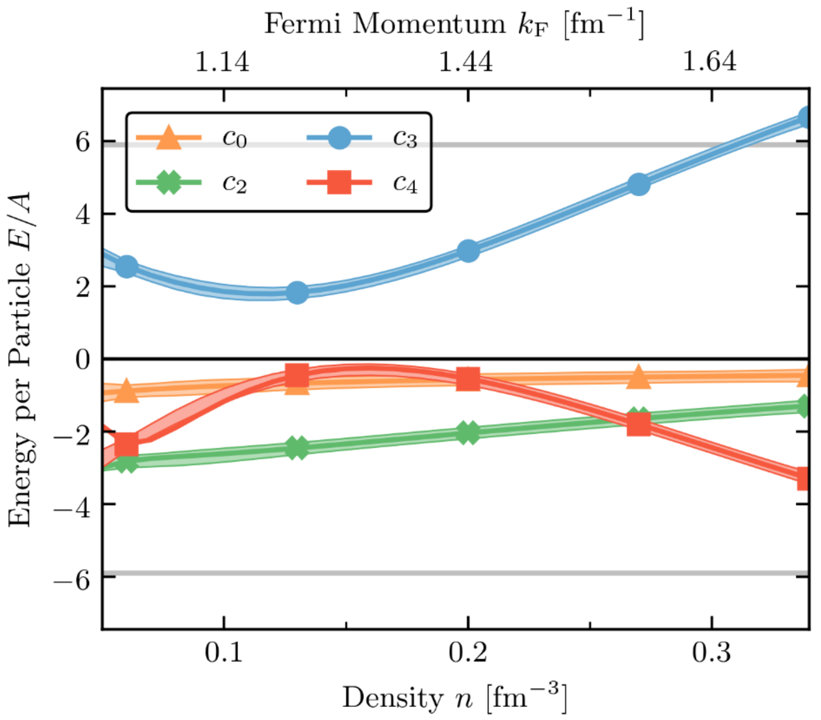
Coefficient extraction for truncation error estimation done via **gsum**

$$Q(k_F) = \frac{k_F}{\Lambda_b}$$ ≈ 600 MeV

$$y_{\mathrm{ref}}(k_F) = 16\,\mathrm{MeV} \times \left(\frac{k_F}{k_{F,0}}\right)^2$$

*Symmetric nuclear matter*




Truncation error scheme yields natural-sized curves as expected



Truncation error analysis: C. Drischler, J. A. Melendez, R. J. Furnstahl, D. R. Phillips, Phys. Rev. C **102**, 054315 (2020)

# High densities: pQCD EOS

## QCD perturbative at high energies

Expansion in the strong coupling constant $\alpha_s$

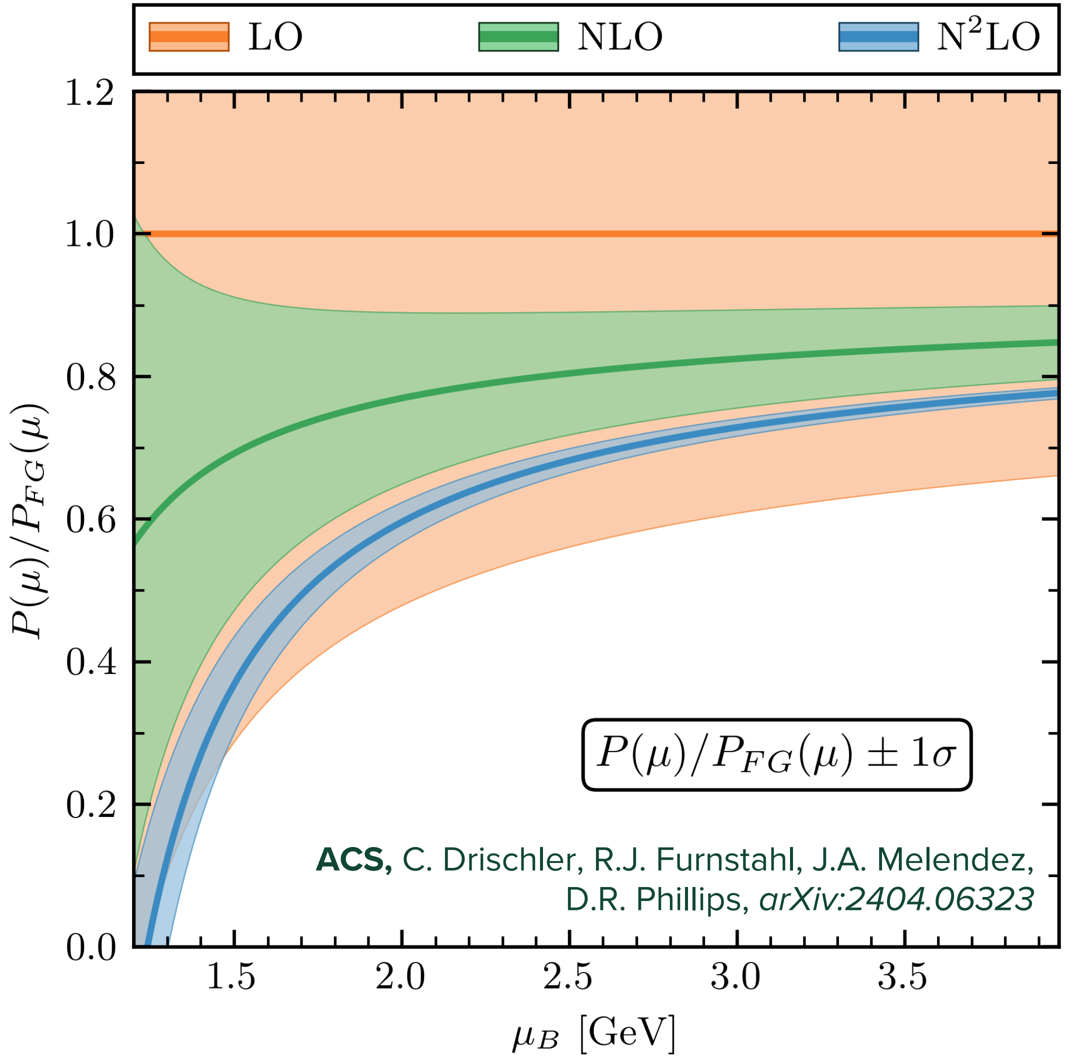
$$\frac{P(\mu)}{P_{FG}(\mu)} \simeq 1 + a_{1,1}\left(\frac{\alpha_s(\mu)}{\pi}\right) + N_f\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[a_{2,1}\ln\left(\frac{N_f\alpha_s(\mu)}{\pi}\right) + a_{2,2}\ln\frac{\bar{\Lambda}}{2\mu} + a_{2,3}\right] + \mathcal{O}(\alpha_s^3),$$

Two-loop running:

$$\alpha_s(\bar{\Lambda}) = \frac{4\pi}{\beta_0 L}\left[1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln L}{L}\right] \quad \begin{cases} L = \ln(\bar{\Lambda}^2/\Lambda_{\overline{MS}}^2), \\ \bar{\Lambda} = 2X\mu \end{cases}$$

Degrees of freedom: quarks and gluons

Massless u, d quarks with equal $\mu$






Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# High densities: pQCD EOS

## QCD perturbative at high energies

Expansion in the strong coupling constant $\alpha_s$

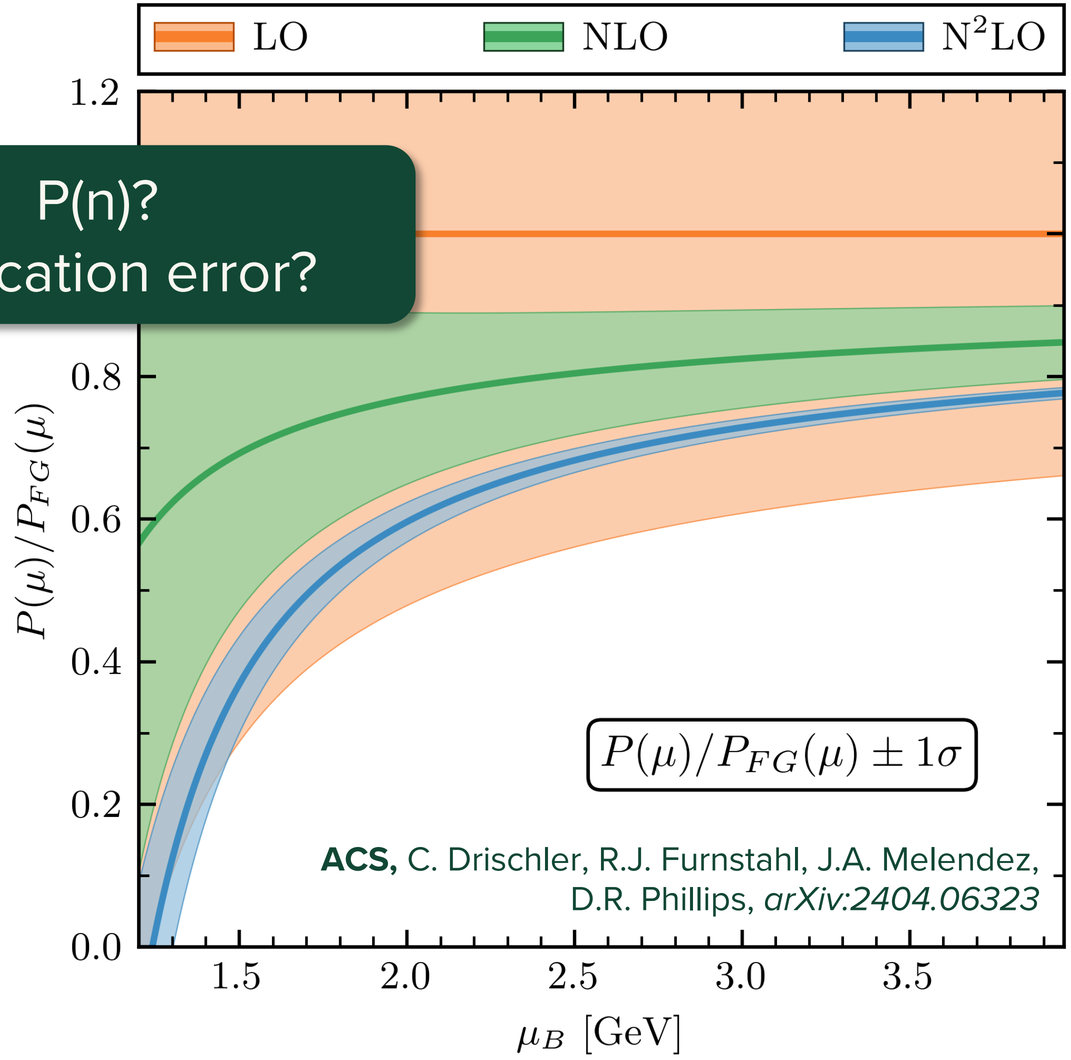
$$\frac{P(\mu)}{P_{FG}(\mu)} \simeq 1 + a_{1,1}\left(\frac{\alpha_s(\mu)}{\pi}\right) + N_f\left(\frac{\alpha_s(\mu)}{\pi}\right)^2\left[a_{2,1}\ln\left(\frac{N_f\alpha_s(\mu)}{\pi}\right) + a_{2,2}\ln\frac{\bar{\Lambda}}{2\mu} + a_{2,3}\right] + \mathcal{O}(\alpha_s^3),$$

Two-loop running:

$$\alpha_s(\bar{\Lambda}) = \frac{4\pi}{\beta_0 L}\left[1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln L}{L}\right] \quad \begin{cases} L = \ln(\bar{\Lambda}^2/\Lambda_{\overline{MS}}^2), \\ \bar{\Lambda} = 2X\mu \end{cases}$$

Degrees of freedom: quarks and gluons

Massless u, d quarks with equal $\mu$



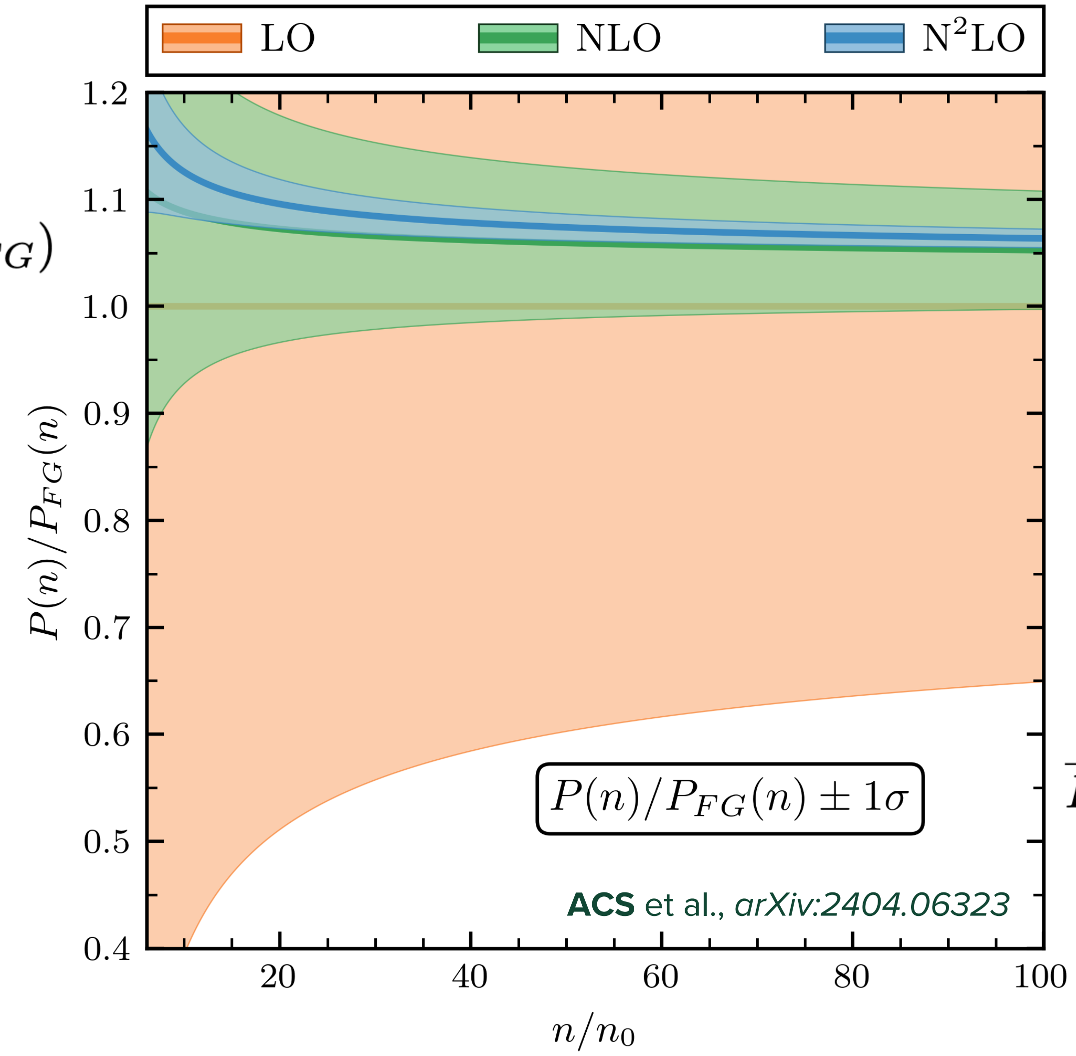

**ACS,** C. Drischler, R.J. Furnstahl, J.A. Melendez, D.R. Phillips, *arXiv:2404.06323*

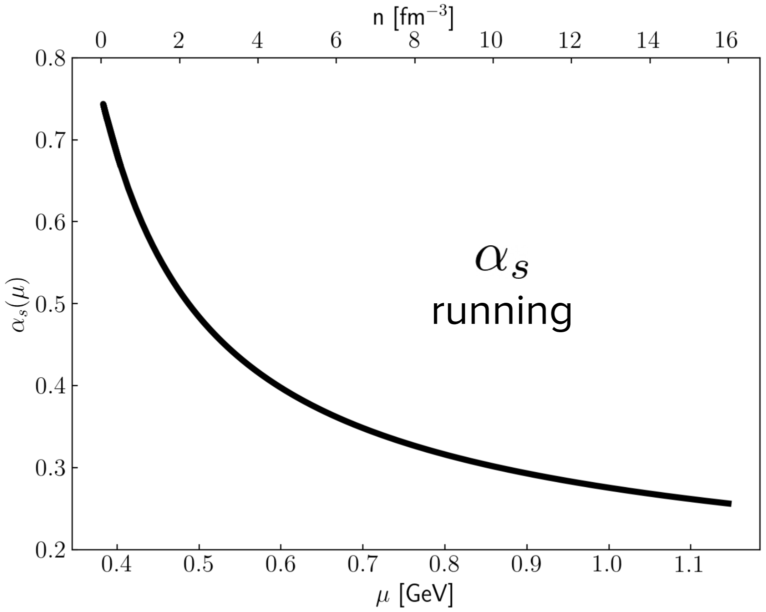
P(n)?
Truncation error?

# High densities: pQCD EOS

Invoke the Kohn-Luttinger-Ward inversion theorem

$$P(n) = P_{FG}(n)\Big[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n) Q^2(\bar{\Lambda}_{FG}) + \ldots\Big]$$




P(n) scaled by FG pressure:

$$\frac{P(n)}{P_{FG}(n)} = 1 + \frac{2}{3\pi}\alpha_s(\bar{\Lambda}_{FG}) + \frac{8}{9\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{\beta_0}{3\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{N_f^2}{3\pi^2}c_2(\mu_{FG})\alpha_s^2(\bar{\Lambda}_{FG})$$



Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# High densities: pQCD EOS

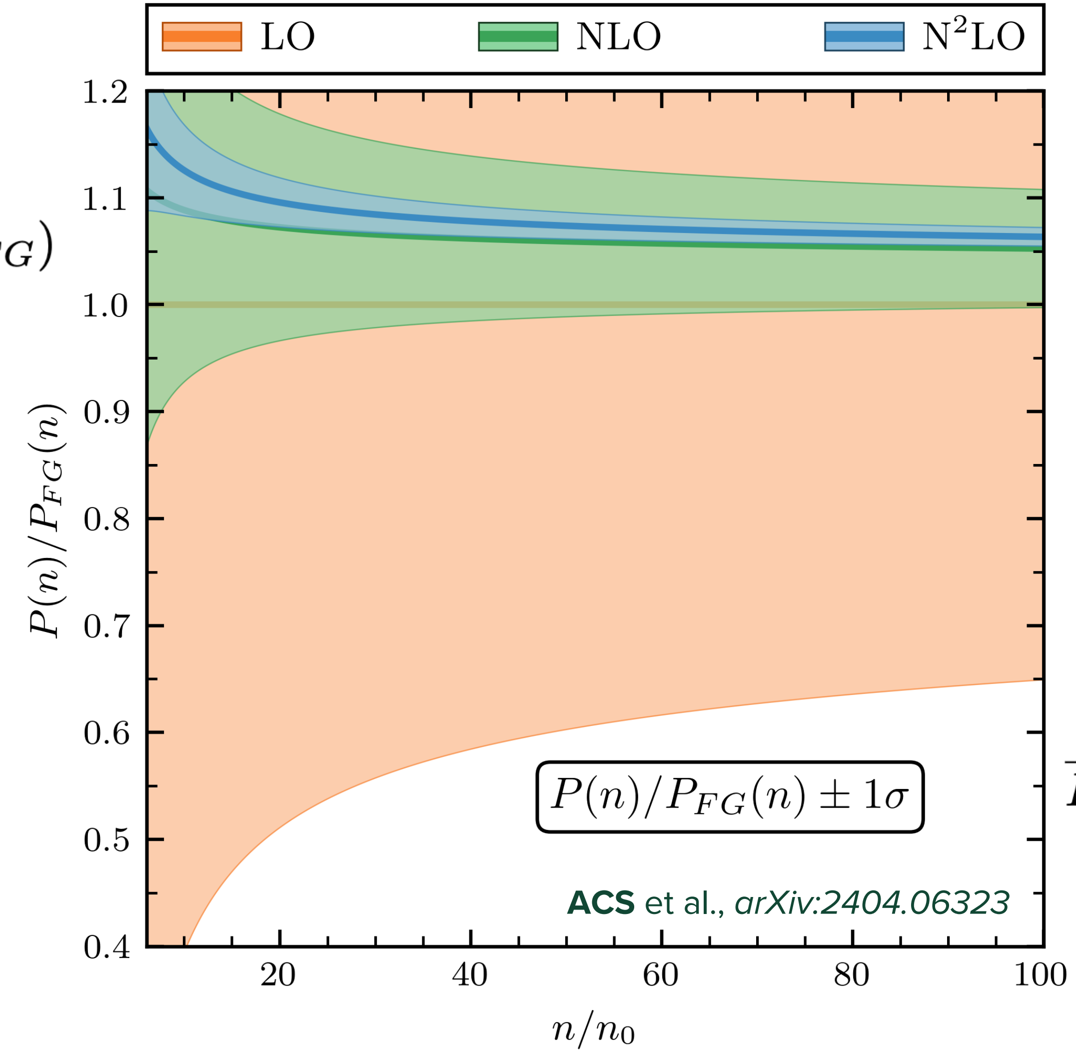
Invoke the Kohn-Luttinger-Ward inversion theorem

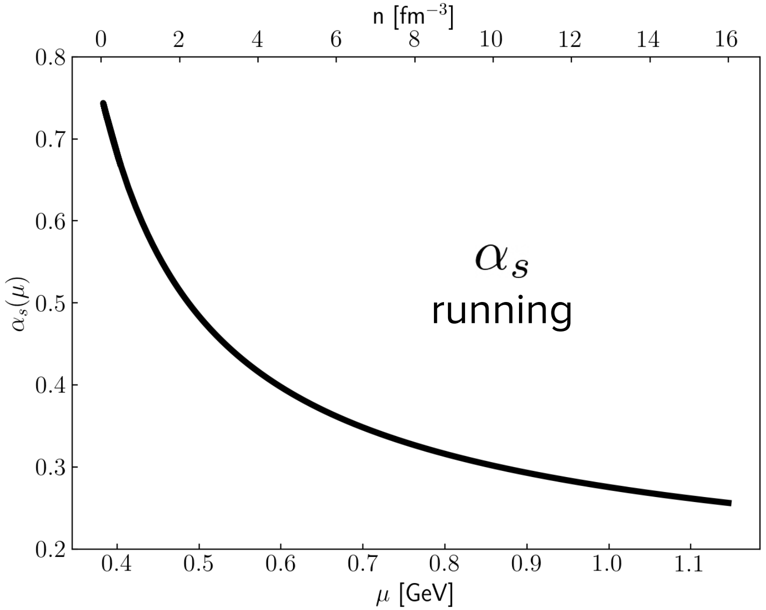
$$P(n) = P_{FG}(n)\left[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n) Q^2(\bar{\Lambda}_{FG}) + \ldots\right]$$

Apply **gsum** to **P(n)**

$$Q = \frac{N_f}{\pi}\alpha_s(\bar{\Lambda})$$

$$y_{ref} = P_{FG}(n)$$




P(n) scaled by FG pressure:

$$\frac{P(n)}{P_{FG}(n)} = 1 + \frac{2}{3\pi}\alpha_s(\bar{\Lambda}_{FG}) + \frac{8}{9\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{\beta_0}{3\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{N_f^2}{3\pi^2}c_2(\mu_{FG})\alpha_s^2(\bar{\Lambda}_{FG})$$

Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# High densities: pQCD EOS

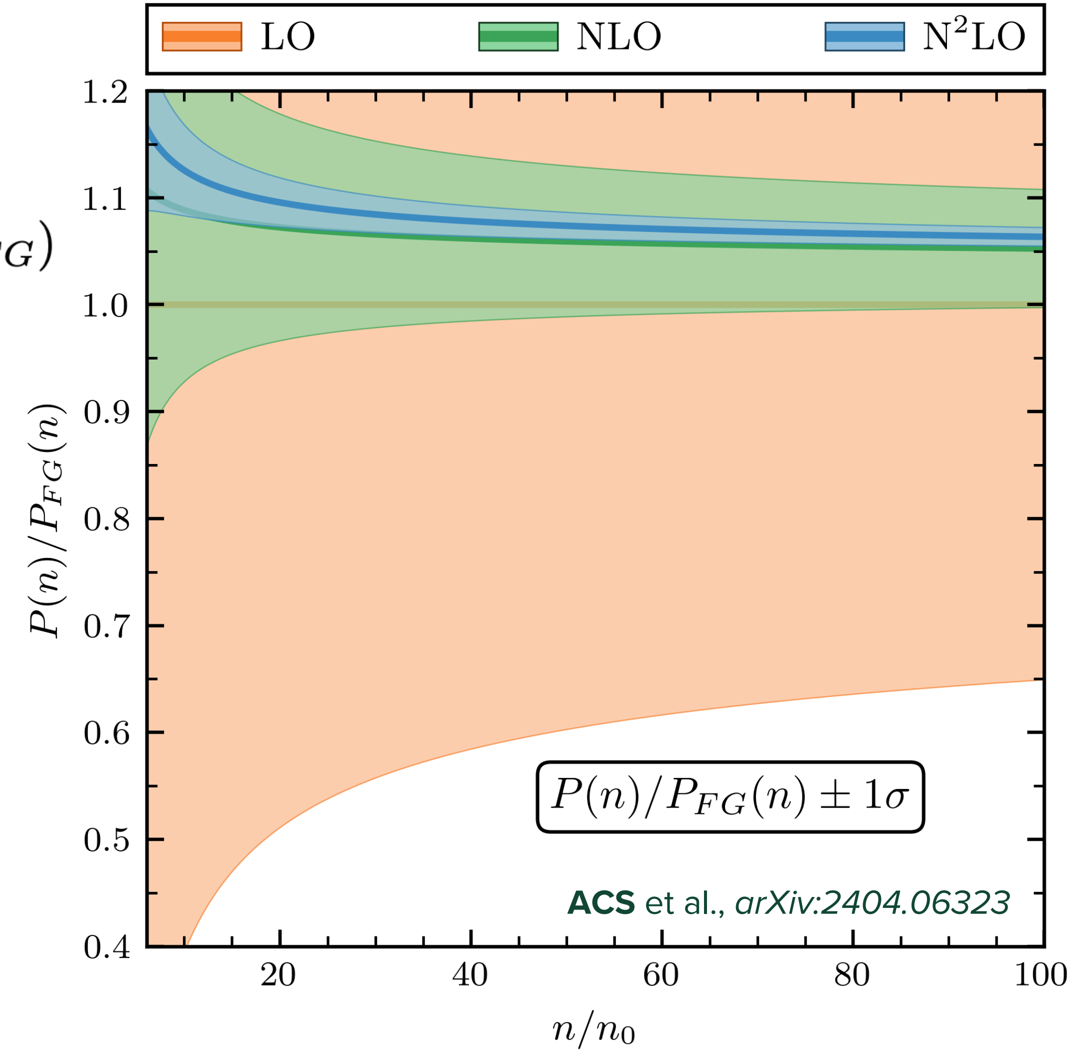
Invoke the Kohn-Luttinger-Ward inversion theorem

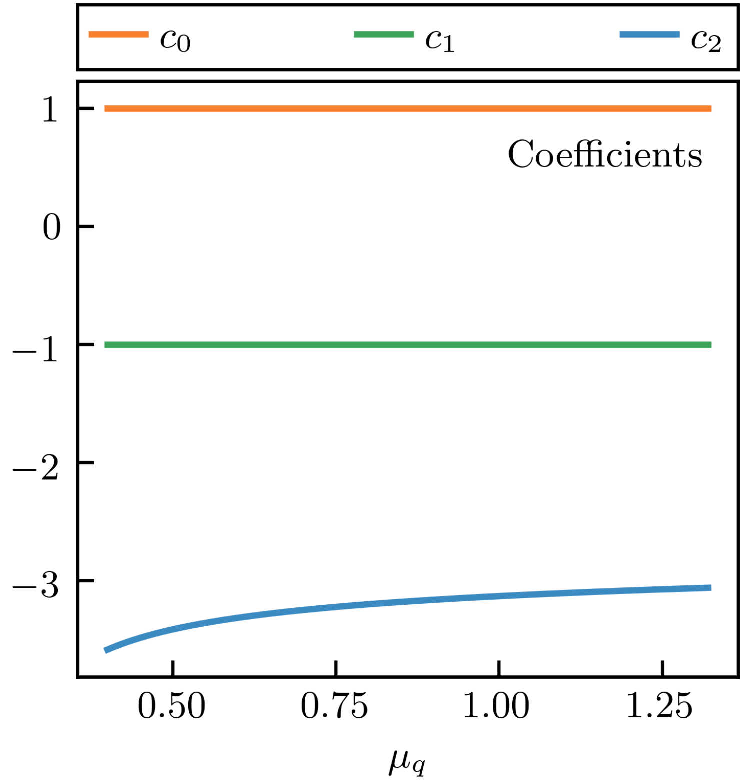
$$P(n) = P_{FG}(n)\left[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n)Q^2(\bar{\Lambda}_{FG}) + \ldots\right]$$

Apply **gsum** to **P(n)**

Q = $\frac{N_f}{\pi}\alpha_s(\bar{\Lambda})$

$y_{ref}$ = $P_{FG}(n)$




**Truncation error** often assessed by varying renormalization scale to obtain band---not statistically rigorous

* See newer works by Gorda et al. (2022, 2023), MiHO model for higher orders

# High densities: pQCD EOS

Invoke the Kohn-Luttinger-Ward inversion theorem

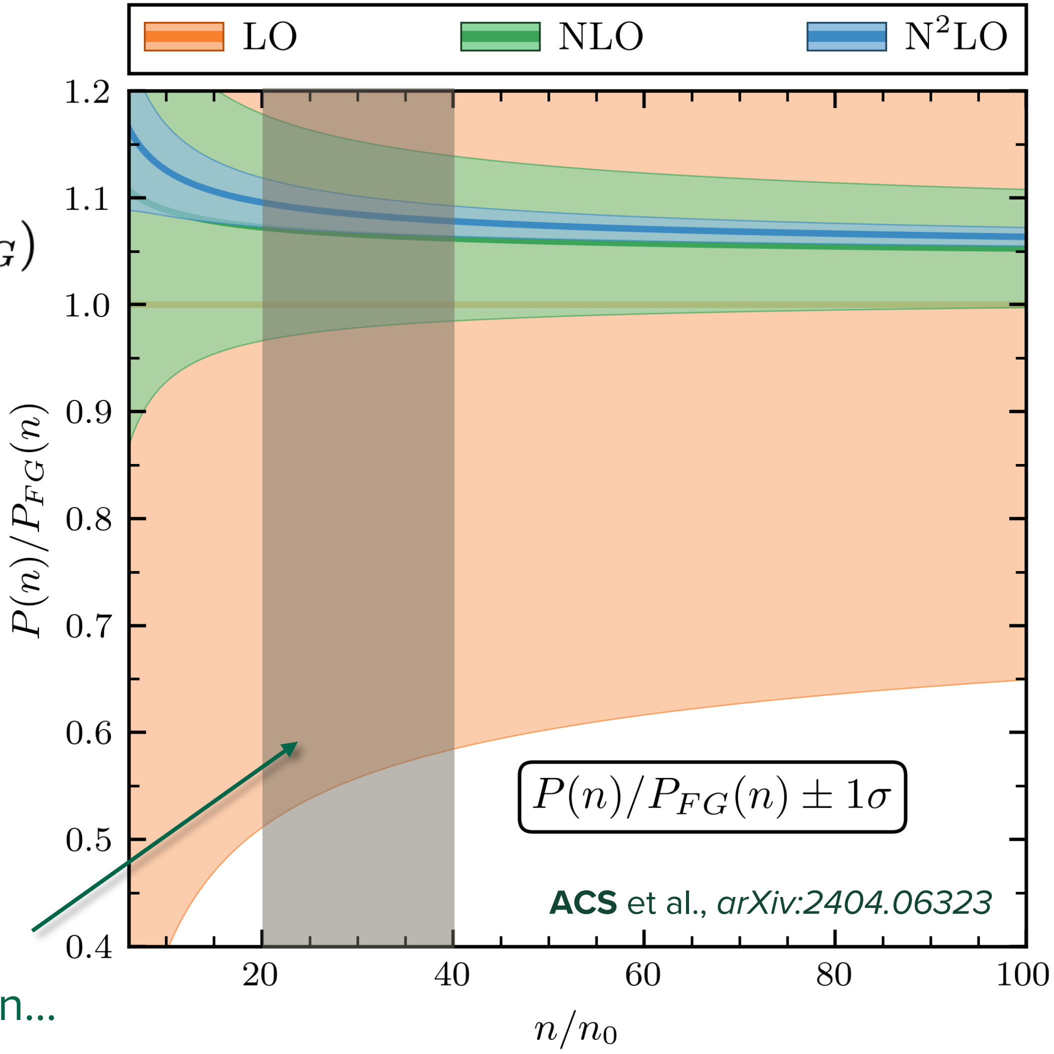
$$P(n) = P_{FG}(n)\Big[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n) Q^2(\bar{\Lambda}_{FG}) + \ldots\Big]$$

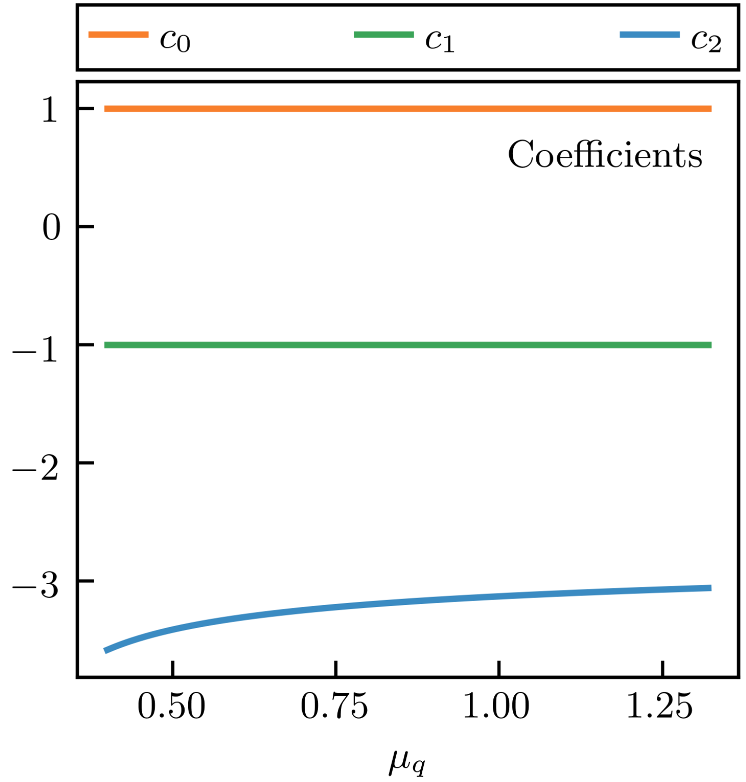
Apply **gsum** to **P(n)**

Q = $\frac{N_f}{\pi}\alpha_s(\bar{\Lambda})$

$y_{ref}$ = $P_{FG}(n)$

Nonperturbative effects under (20-40) $n_0$: pairing, hadronization…




**Truncation error** often assessed by varying renormalization scale to obtain band---not statistically rigorous

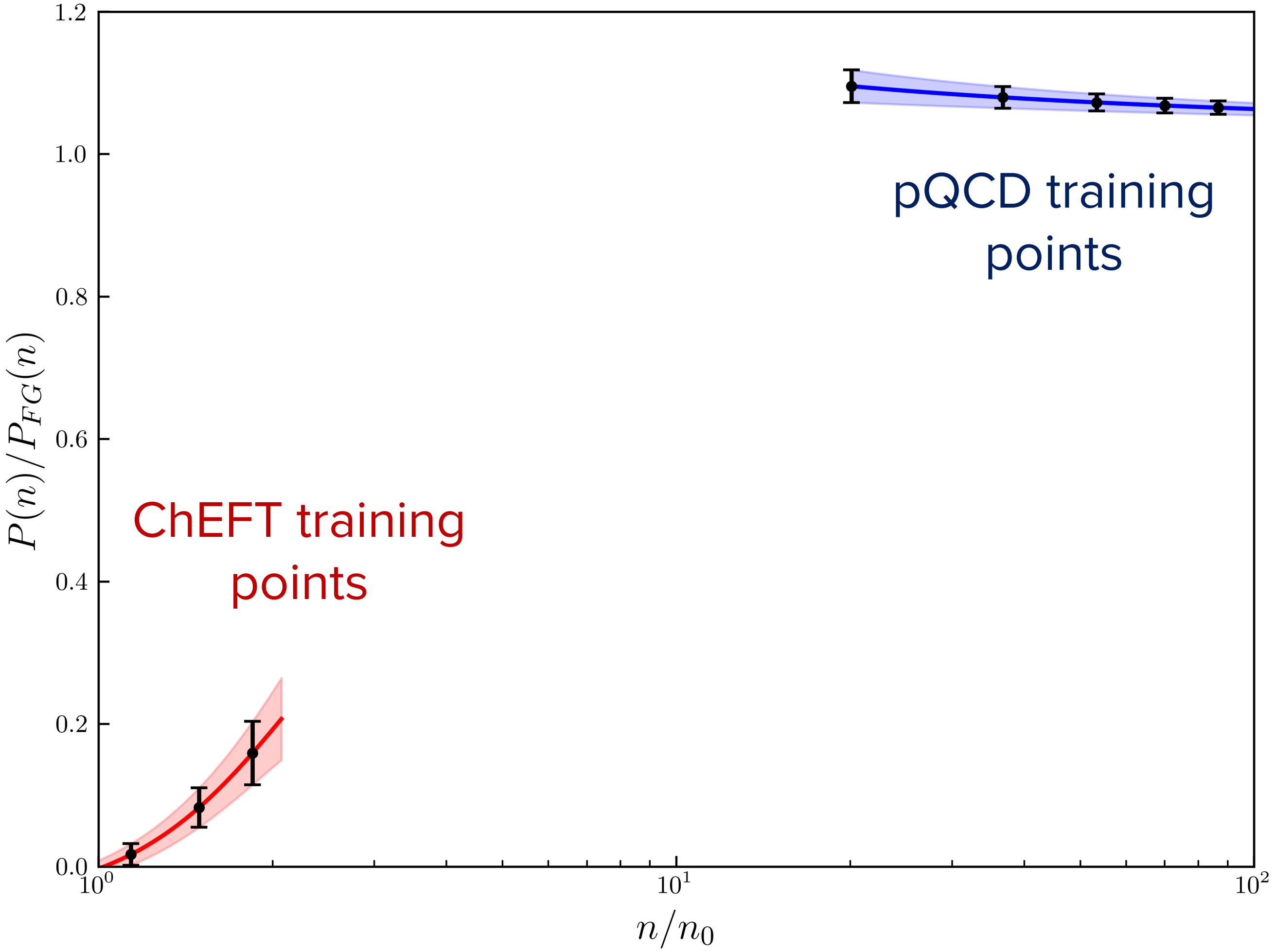
* See newer works by Gorda et al. (2022, 2023), MiHO model for higher orders



Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# GPs: a correlated approach to BMM

*Mixing random variables "curvewise"*



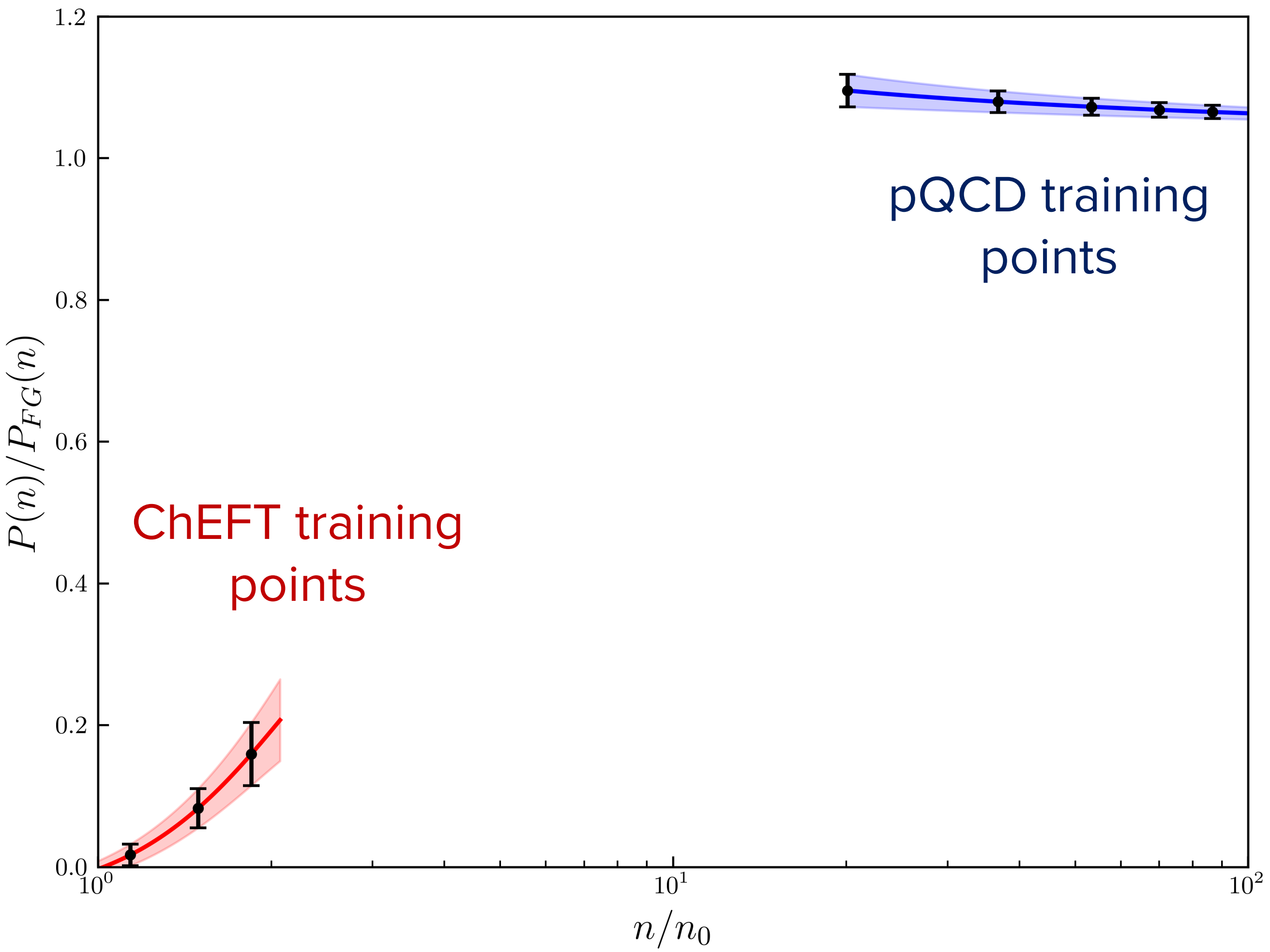

**Underlying theory (QCD)**

$$Y_i = F + \delta Y_i, \qquad i \in [0, M]$$

OHIO UNIVERSITY

# GPs: a correlated approach to BMM

*Mixing random variables "curvewise"*




**Underlying theory (QCD)**

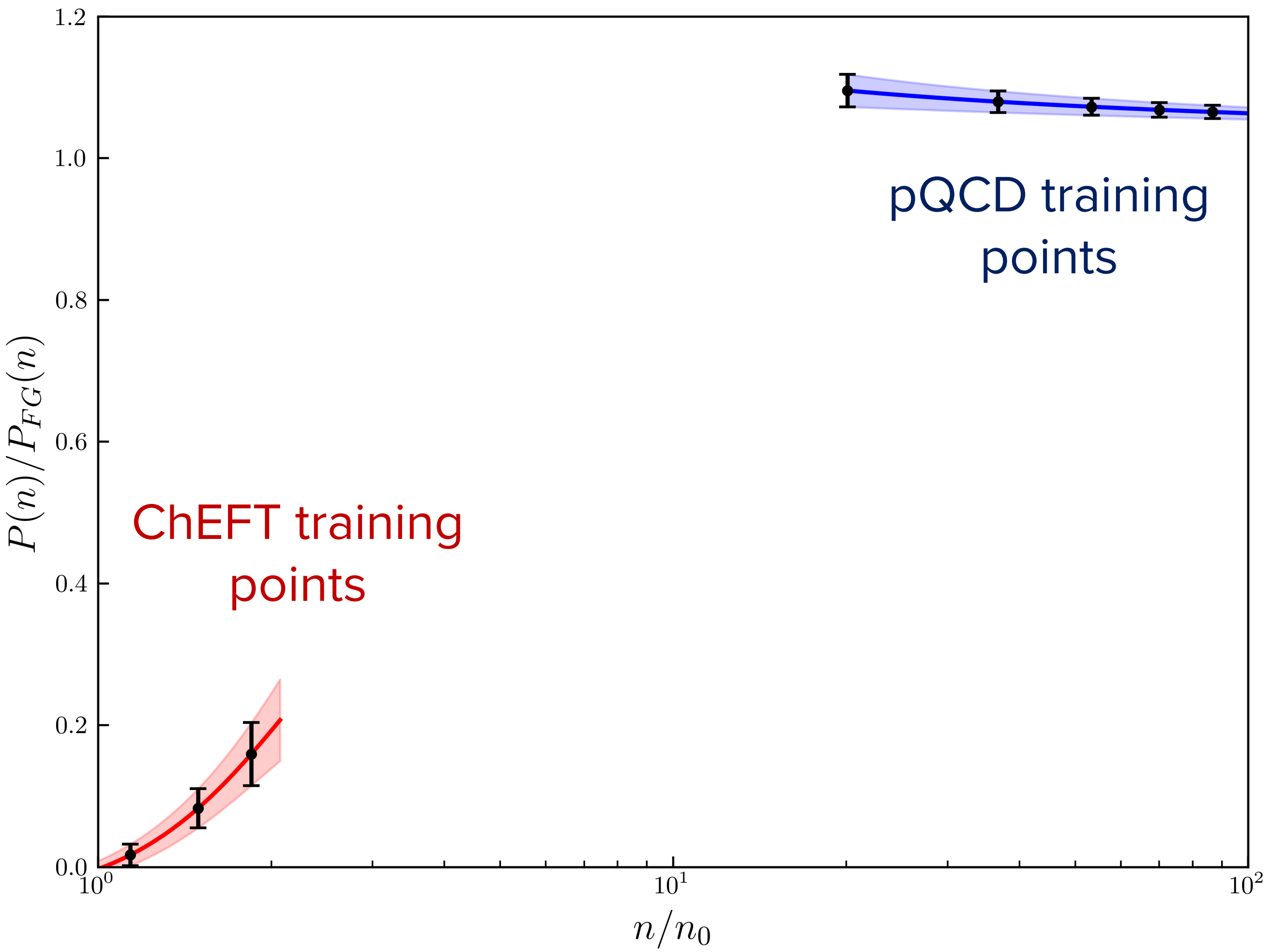
$$Y_i = F + \delta Y_i, \qquad i \in [0, M]$$

$$\sim \mathrm{GP}[0, \kappa_y^{(i)}(x, x')]$$

[Full covariance matrix]

# GPs: a correlated approach to BMM

*Mixing random variables "curvewise"*




**Underlying theory (QCD)**

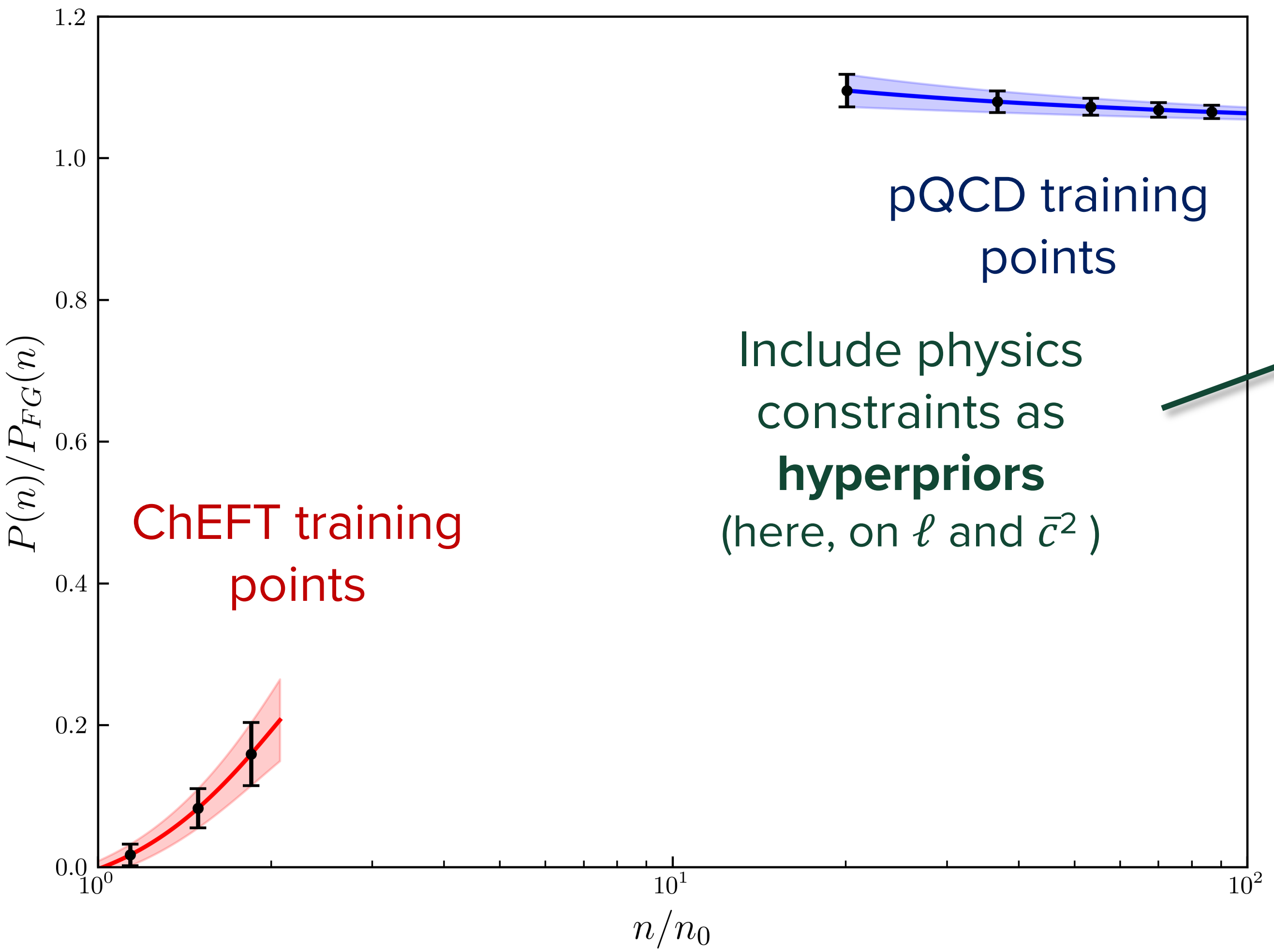
$$Y_i = F + \delta Y_i, \qquad i \in [0, M]$$

$\sim \mathrm{GP}[0, \kappa_f(x, x')]$ [Kernel choice]

$\sim \mathrm{GP}[0, \kappa_y^{(i)}(x, x')]$ [Full covariance matrix]

# GPs: a correlated approach to BMM

*Mixing random variables "curvewise"*




Include physics constraints as **hyperpriors** (here, on $\ell$ and $\bar{c}^2$ )

**Underlying theory (QCD)**

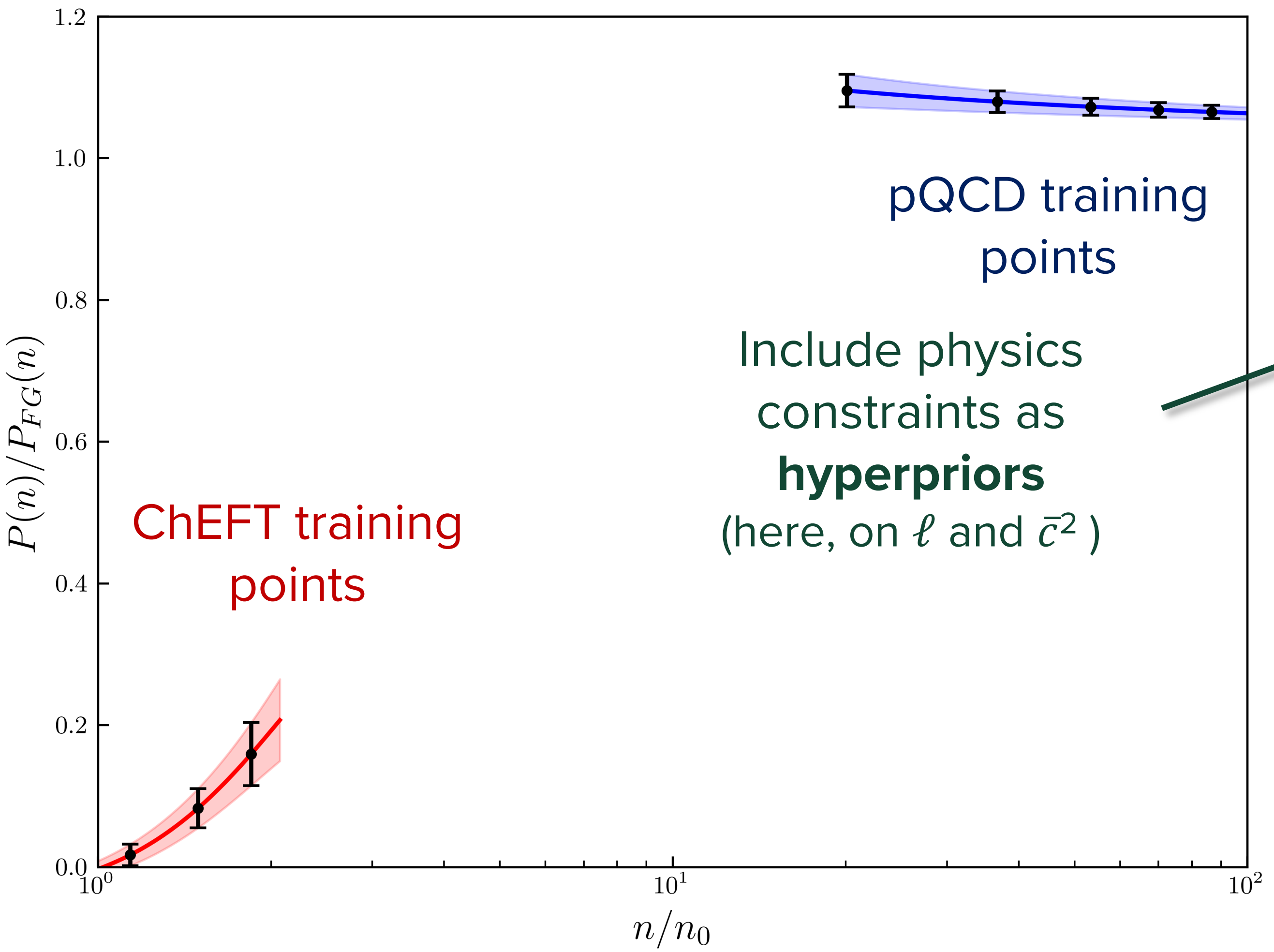
$$Y_i = F + \delta Y_i, \qquad i \in [0, M]$$

$\sim \mathrm{GP}[0, \kappa_f(x, x')]$ [Kernel choice]

$\sim \mathrm{GP}[0, \kappa_y^{(i)}(x, x')]$ [Full covariance matrix]

# GPs: a correlated approach to BMM

*Mixing random variables "curvewise"*




Include physics constraints as **hyperpriors** (here, on $\ell$ and $\bar{c}^2$ )

**Underlying theory (QCD)**

$$Y_i = F + \delta Y_i, \qquad i \in [0, M]$$

$\sim \mathrm{GP}[0, \kappa_f(x, x')]$ [Kernel choice]

$\sim \mathrm{GP}[0, \kappa_y^{(i)}(x, x')]$ [Full covariance matrix]

Therefore, assuming a **Gaussian form** and using Bayes' theorem, again determine a **common mean**

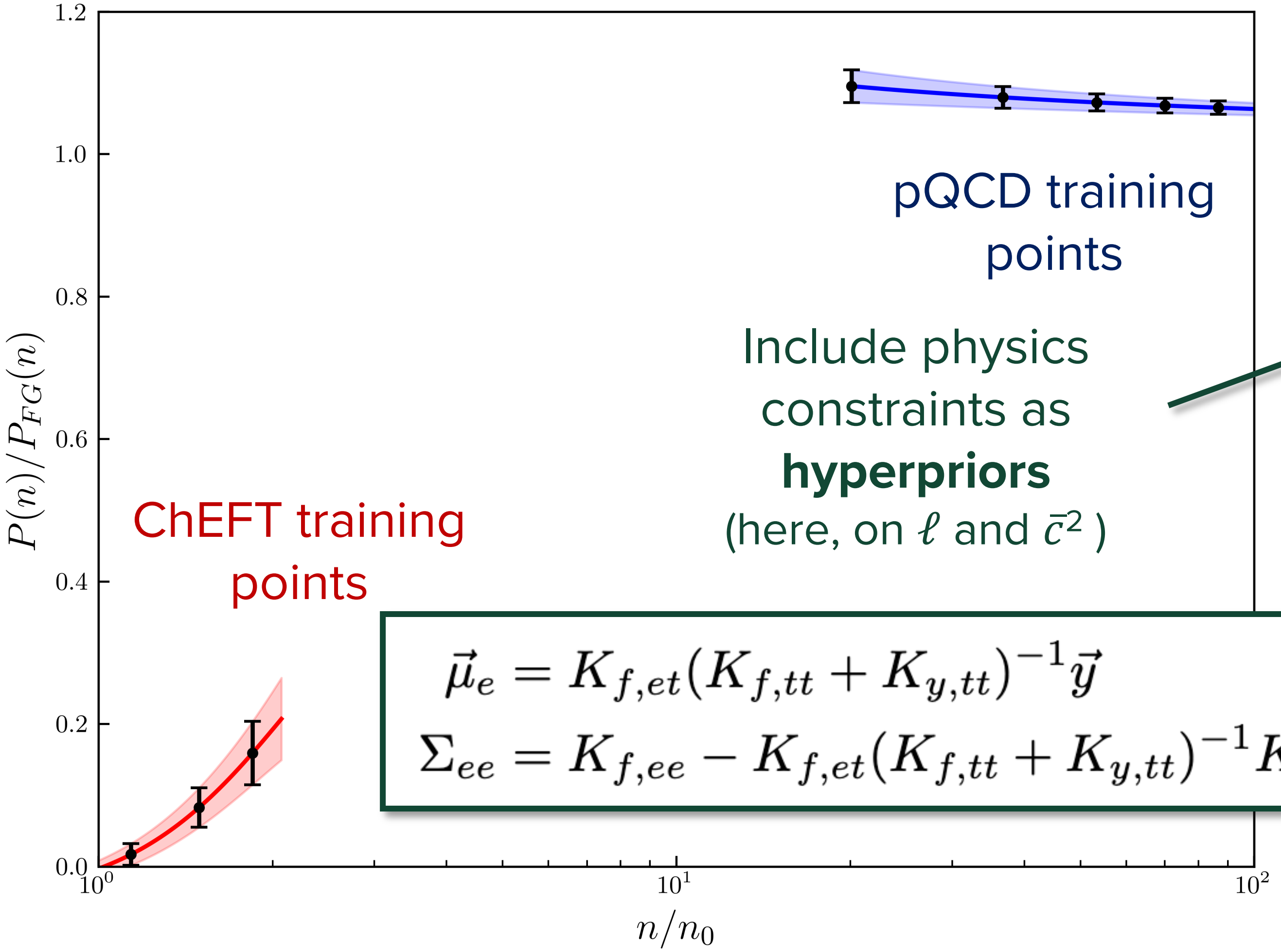
$$F \,|\, \vec{y}, K_y, K_f \sim \mathcal{N}[\mu, \Sigma]$$

$$\vec{\mu} \equiv \Sigma B_t^T K_y^{-1} \vec{y}$$

$$\Sigma \equiv (K_f^{-1} + B_t^T K_y^{-1} B_t)^{-1}$$

# GPs: a correlated approach to BMM

*Mixing random variables "curvewise"*




Include physics constraints as **hyperpriors** (here, on $\ell$ and $\bar{c}^2$ )

$$\vec{\mu}_e = K_{f,et}(K_{f,tt} + K_{y,tt})^{-1}\vec{y}$$
$$\Sigma_{ee} = K_{f,ee} - K_{f,et}(K_{f,tt} + K_{y,tt})^{-1}K_{f,te}$$

**Underlying theory (QCD)**

$$Y_i = F + \delta Y_i, \qquad i \in [0, M]$$

$\sim \mathrm{GP}[0, \kappa_f(x, x')]$ [Kernel choice]

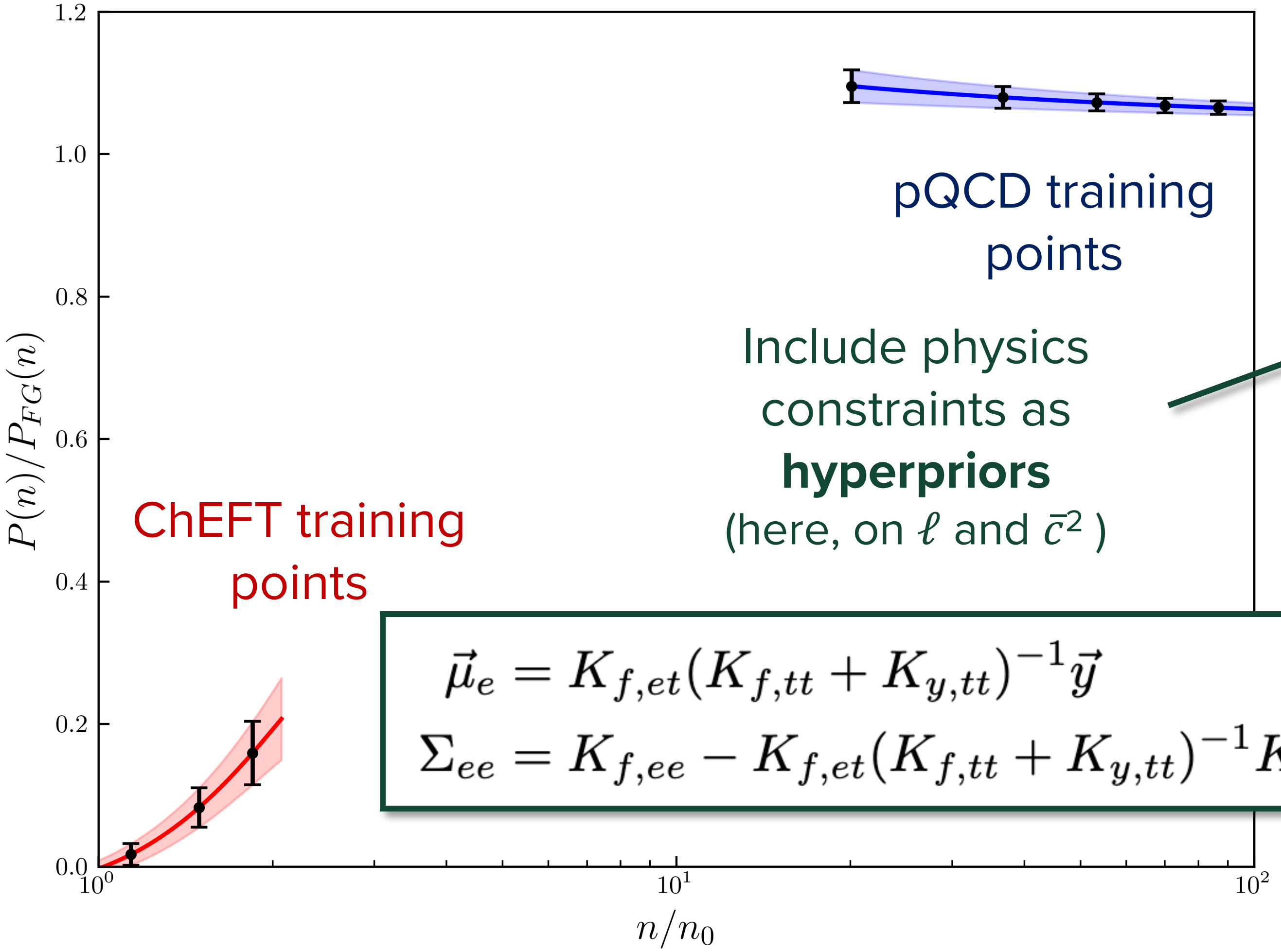
$\sim \mathrm{GP}[0, \kappa_y^{(i)}(x, x')]$ [Full covariance matrix]

Therefore, assuming a **Gaussian form** and using Bayes' theorem, again determine a **common mean**

$$F \,|\, \vec{y}, K_y, K_f \sim \mathcal{N}[\mu, \Sigma]$$
$$\vec{\mu} \equiv \Sigma B_t^T K_y^{-1}\vec{y}$$
$$\Sigma \equiv (K_f^{-1} + B_t^T K_y^{-1} B_t)^{-1}$$

# GPs: a correlated approach to BMM

*Mixing random variables "curvewise"*




pQCD training points

ChEFT training points

Include physics constraints as **hyperpriors** (here, on $\ell$ and $\bar{c}^2$ )

$$\vec{\mu}_e = K_{f,et}(K_{f,tt} + K_{y,tt})^{-1}\vec{y}$$

$$\Sigma_{ee} = K_{f,ee} - K_{f,et}(K_{f,tt} + K_{y,tt})^{-1}K_{f,te}$$

**Underlying theory (QCD)**

$$Y_i = F + \delta Y_i, \qquad i \in [0, M]$$

$\sim \mathrm{GP}[0, \kappa_f(x, x')]$ [Kernel choice]

$\sim \mathrm{GP}[0, \kappa_y^{(i)}(x, x')]$ [Full covariance matrix]

Therefore, assuming a **Gaussian form** and using Bayes' theorem, again determine a **common mean**

$$F \,|\, \vec{y}, K_y, K_f \sim \mathcal{N}[\mu, \Sigma]$$

$$\vec{\mu} \equiv \Sigma B_t^T K_y^{-1}\vec{y}$$

$$\Sigma \equiv (K_f^{-1} + B_t^T K_y^{-1} B_t)^{-1}$$

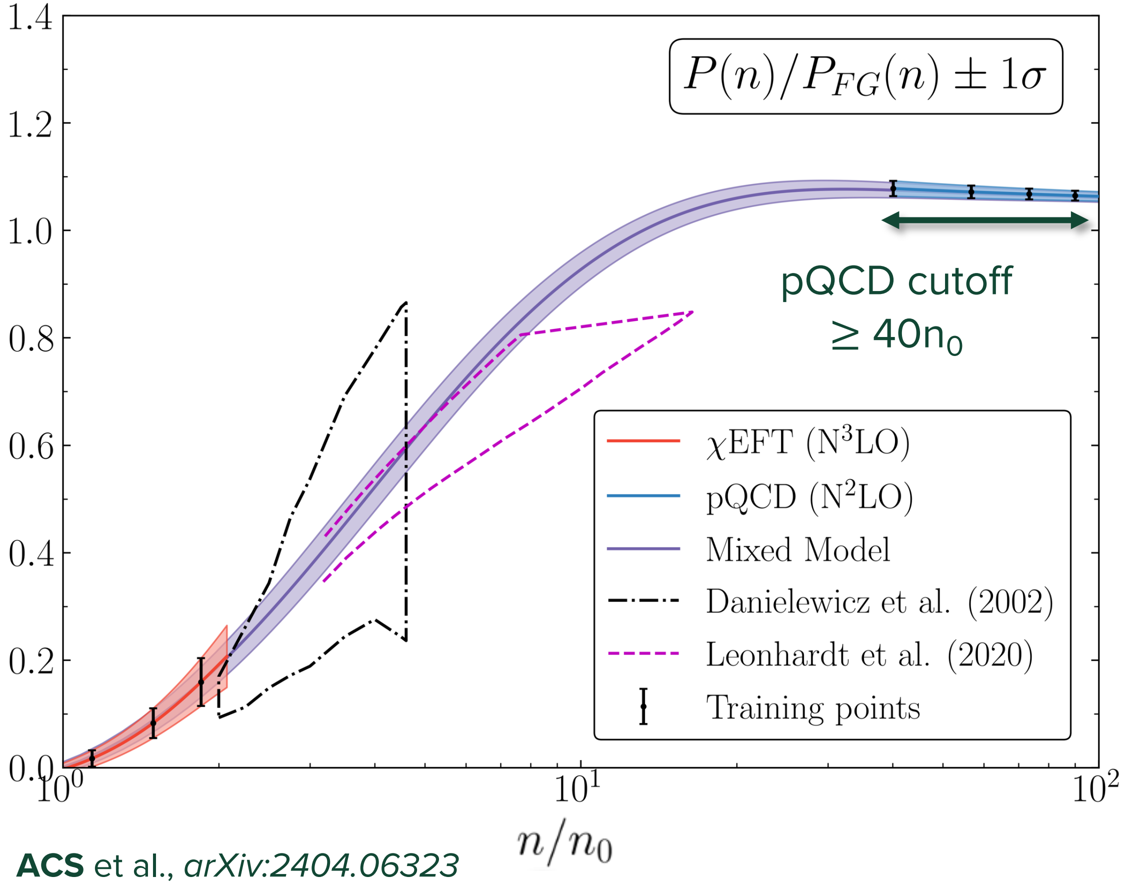
**GP equations are the mixed model result!**

# Results: pressure

# Results: pressure

Unconstrained prior

GP learns very long lengthscale (~1.9 in *ln(n)* space) => induces strong pQCD influence in chiral EFT region => **small bands**

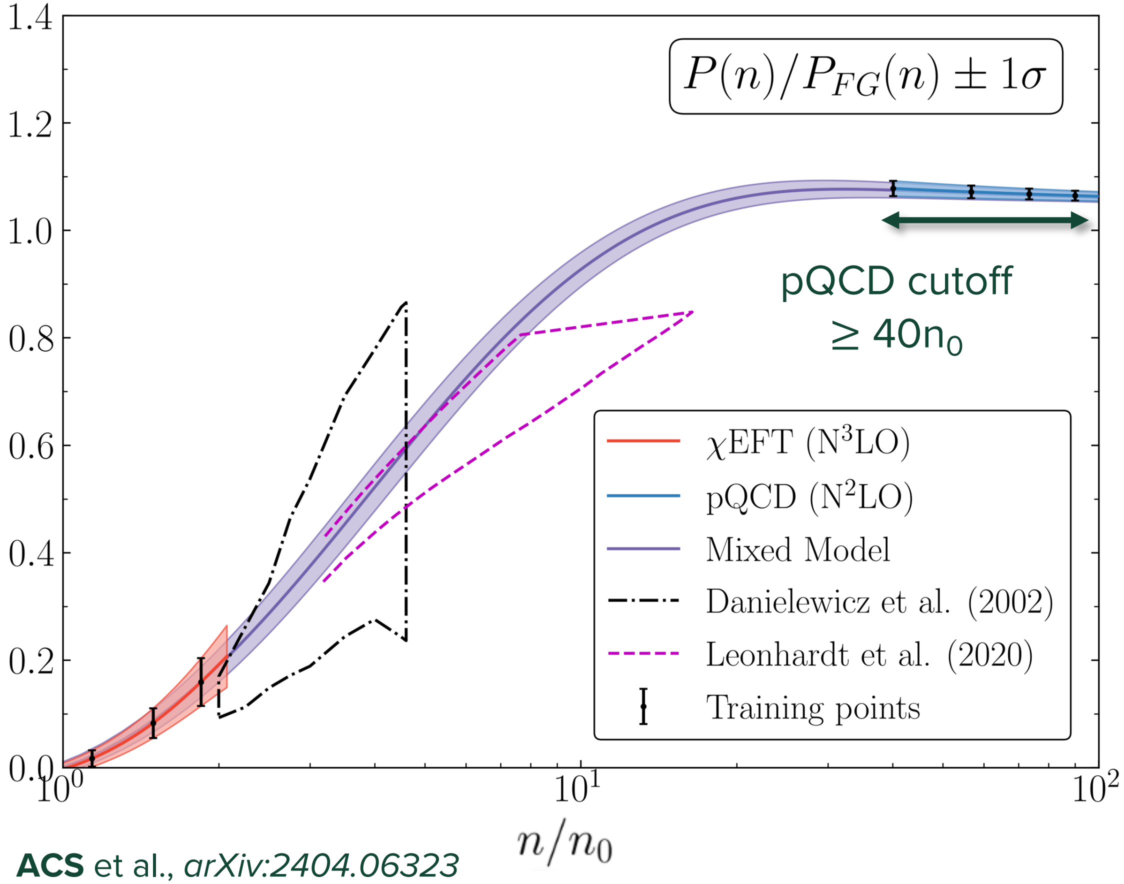



**ACS** et al., *arXiv:2404.06323*

# Results: pressure

## Unconstrained prior

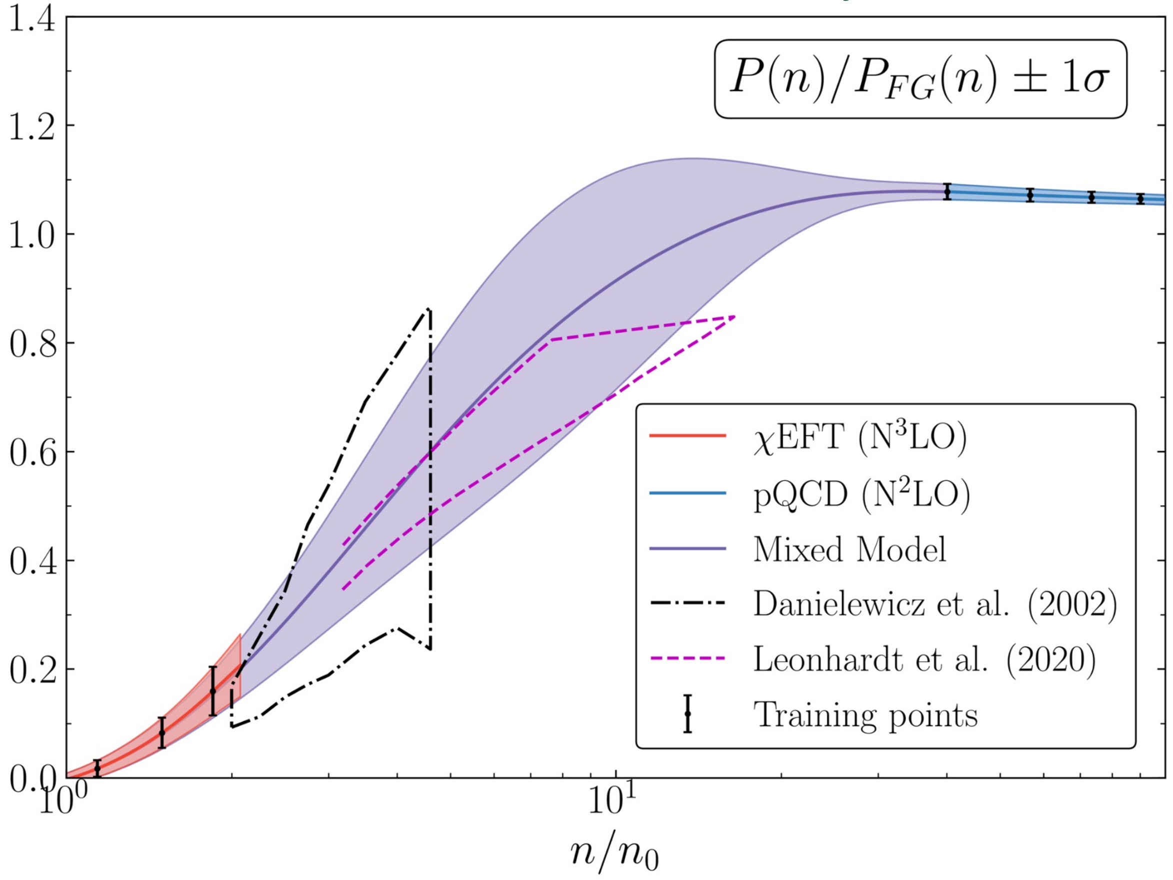
GP learns very long lengthscale (~1.9 in *ln(n)* space) => induces strong pQCD influence in chiral EFT region => **small bands**




**ACS** et al., *arXiv:2404.06323*

## Constrained prior on lengthscale

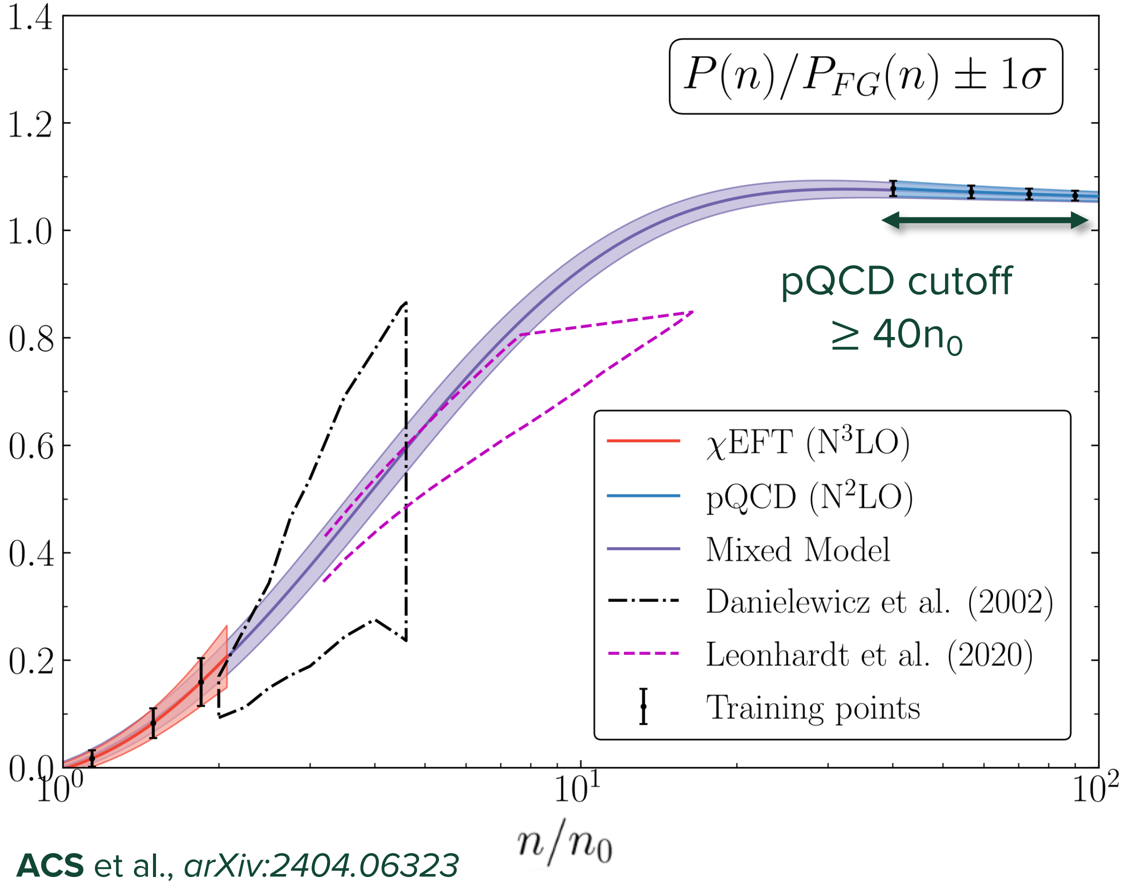
Reduction of the correlation length between the two EOSs => wider uncertainty bands



# Results: pressure

## Unconstrained prior

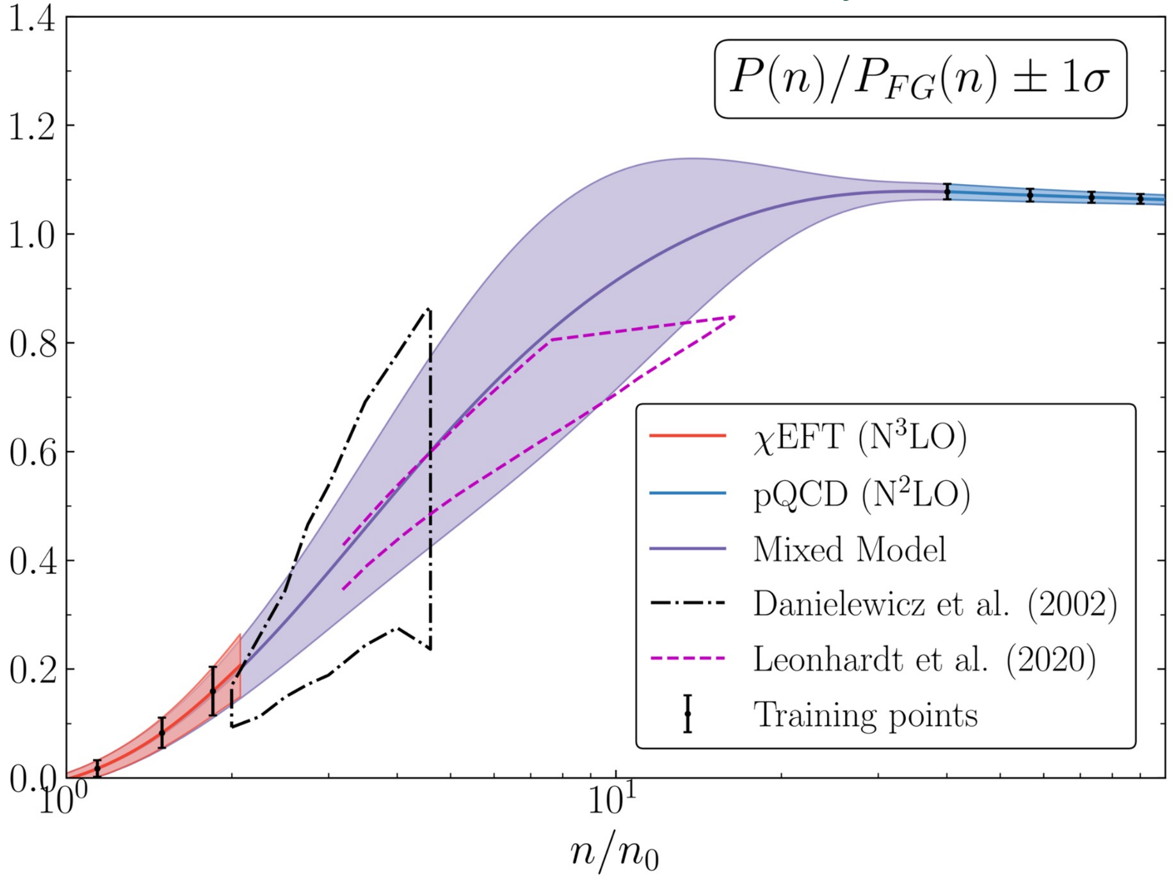
GP learns very long lengthscale (~1.9 in *ln(n)* space) => induces strong pQCD influence in chiral EFT region => **small bands**




**ACS** et al., *arXiv:2404.06323*

## Constrained prior on lengthscale

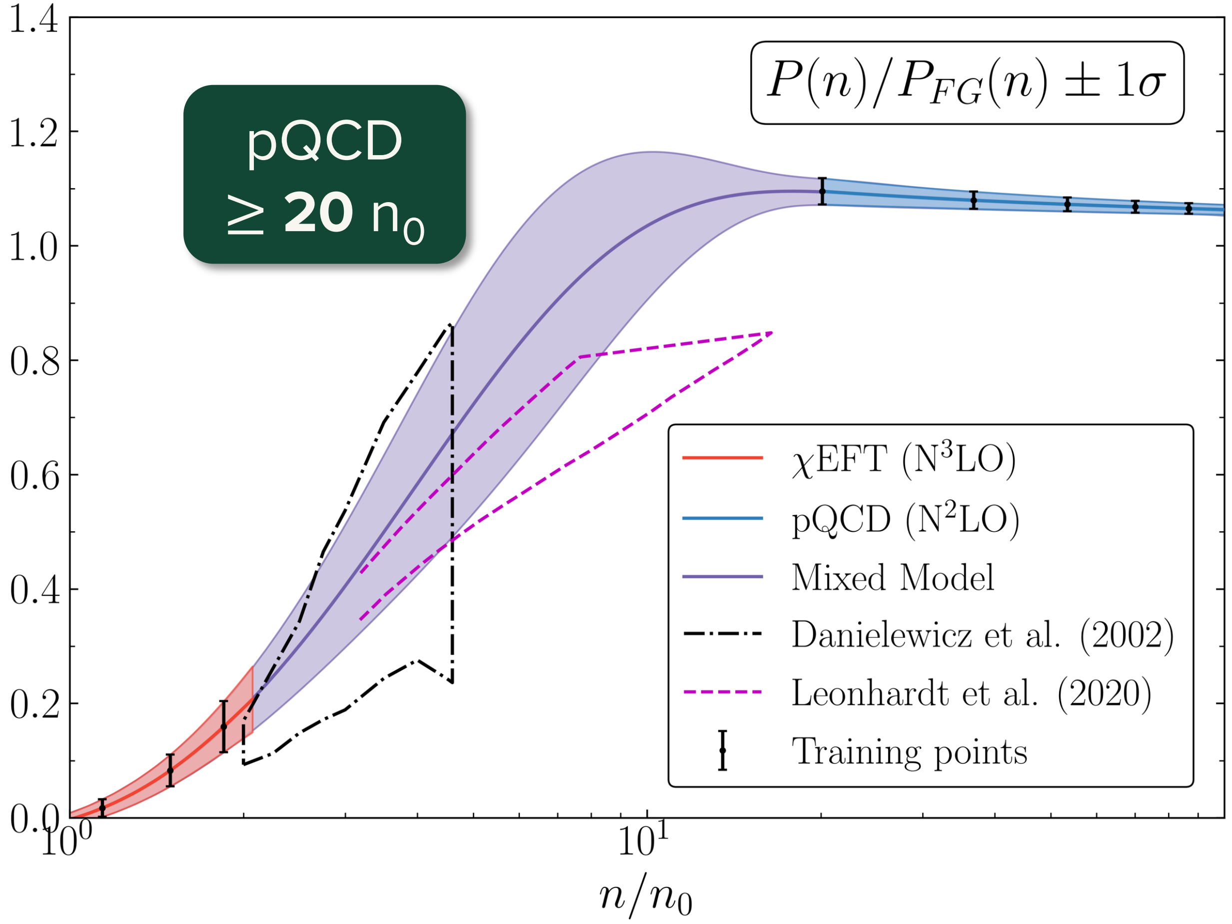
Reduction of the correlation length between the two EOSs => wider uncertainty bands

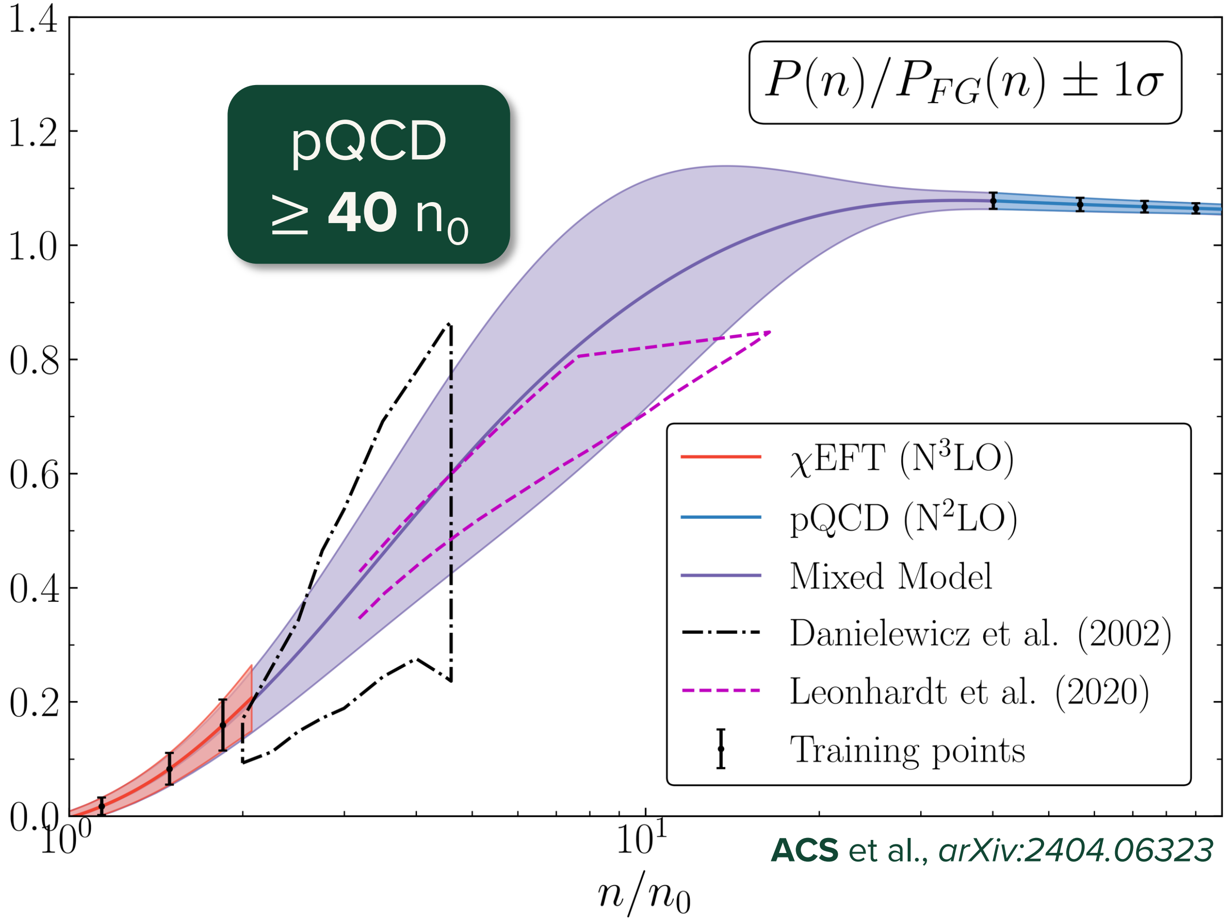



**Physics-informed priors on the mixed model GP have a very large influence on the uncertainties obtained**

# Results: pressure

Unknown cutoff of pQCD validity = testing with **two cases**:
pQCD cutoffs for ≥ **20 $n_0$** and ≥ **40 $n_0$**

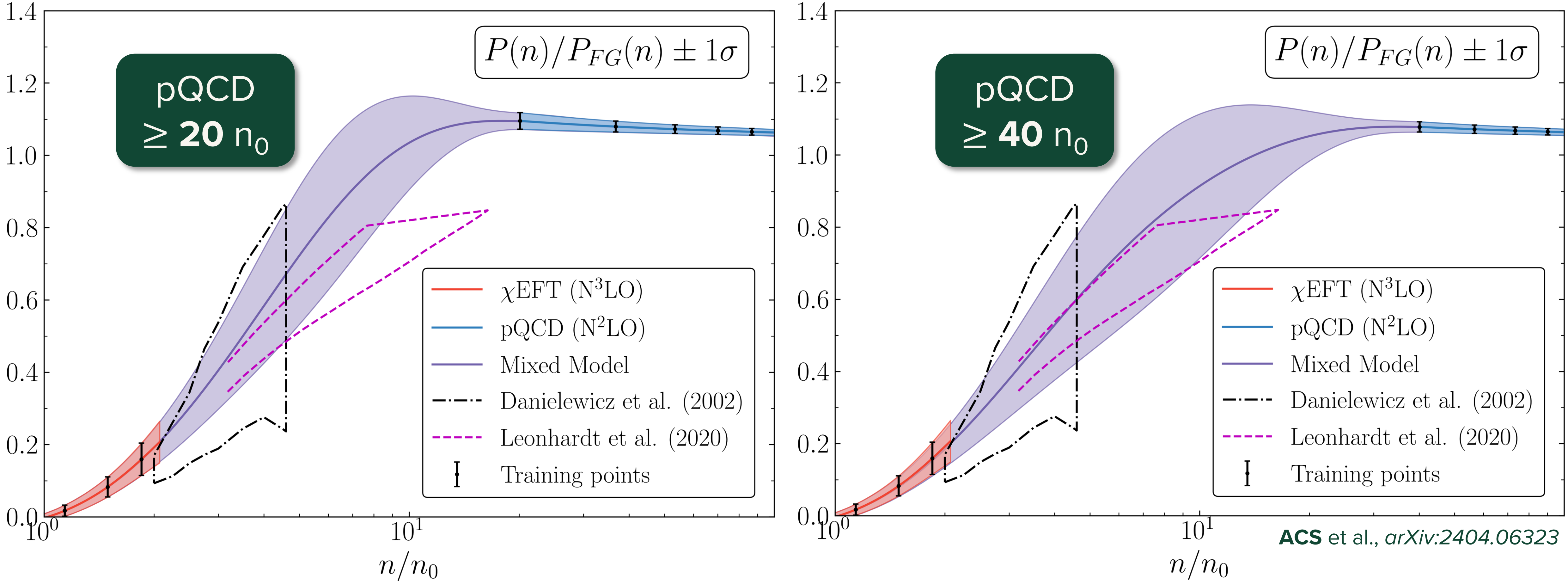



**ACS** et al., *arXiv:2404.06323*

# Results: pressure

Unknown cutoff of pQCD validity = testing with **two cases**:
pQCD cutoffs for ≥ **20 $n_0$** and ≥ **40 $n_0$**

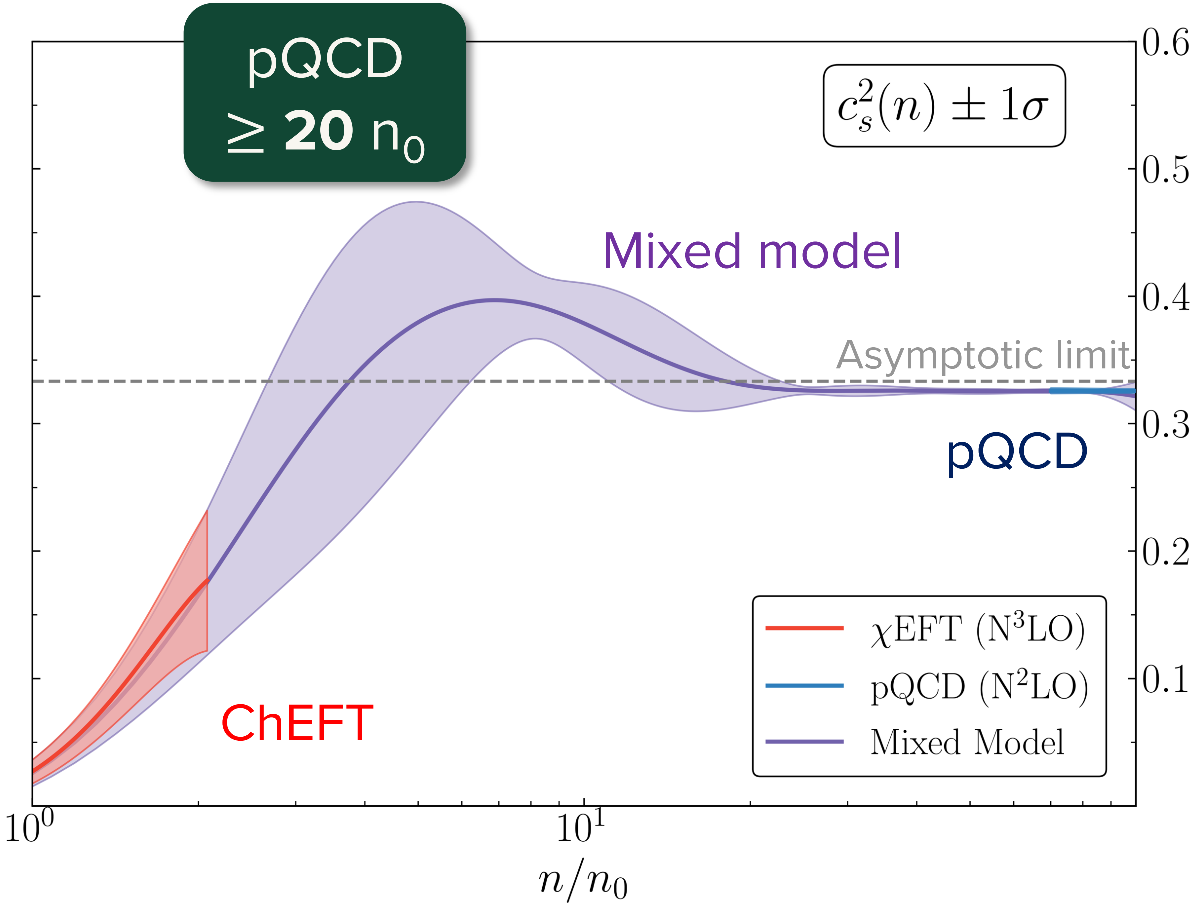


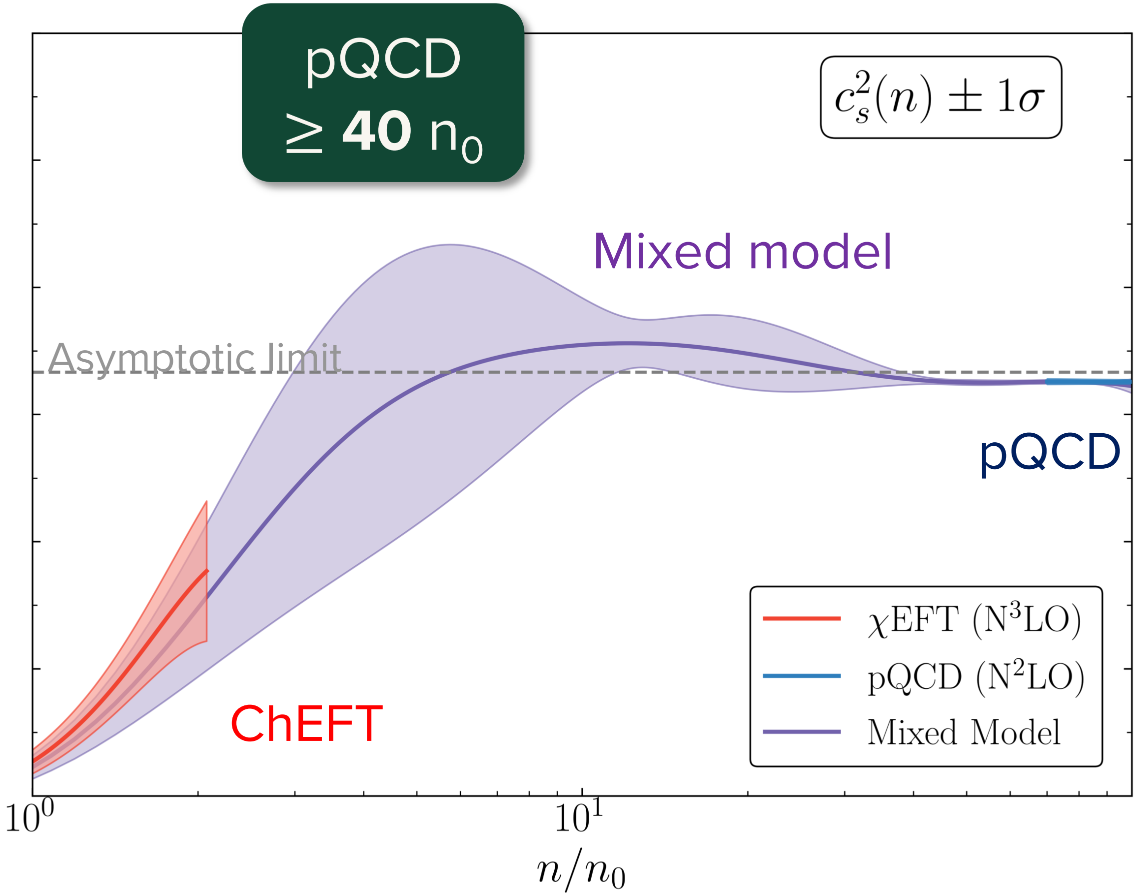

**ACS** et al., *arXiv:2404.06323*

**Constraining the correlation length between chiral EFT & pQCD is crucial to avoid unphysical model correlations at low densities**

# Results: speed of sound

Mixed models both approaching 0.33 asymptotically (from below)



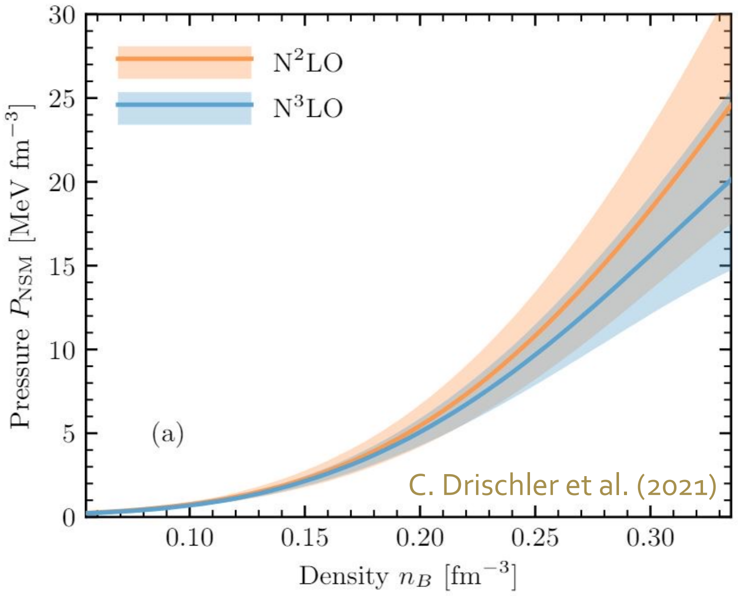




**ACS** et al., *arXiv:2404.06323*

**Causal, stable EOS**

# Discussion: BMM for nuclear physics

## Next steps: neutron-rich matter




Extending **BMM** calculations to neutron-rich matter

Goal: global, microscopic, QCD-based EOSs for merger simulations

Inclusion of important **astrophysical observations** (NICER, …) and **experimental constraints** (FRIB400, …)

## Opportunities for discussion…

**UQ of the pQCD expansion**
Scale variation techniques vs. missing higher orders UQ => possibly double-counting information if using both

**Exploring kernel design**
EOS as a case study: constraints from astro./exp. could inform choices via type of phase transition allowed

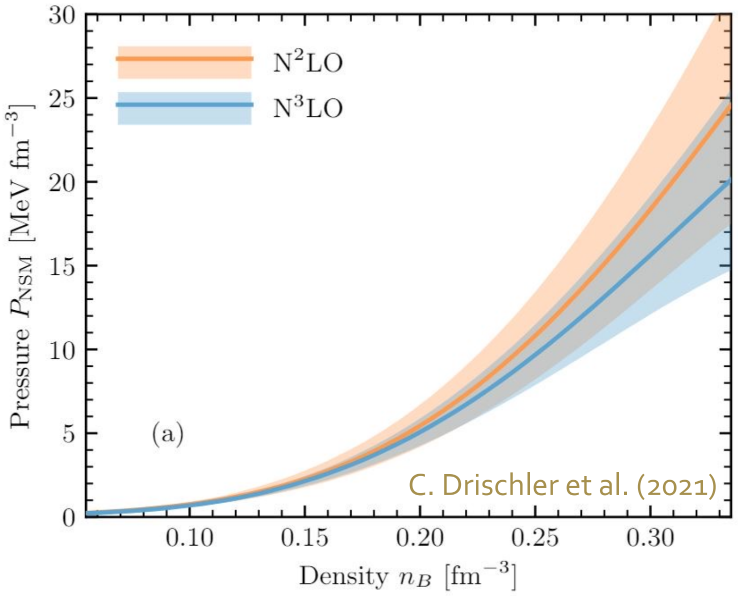
## … and some challenges

**Extending this version of BMM to multiple dimensions**
Crucial for performing UQ in any multi-dimensional nuclear physics problem

**Prior choices**
How to best incorporate known physics => priors other than GPs

# Discussion: BMM for nuclear physics

## Next steps: neutron-rich matter




Extending **BMM** calculations to neutron-rich matter

Goal: global, microscopic, QCD-based EOSs for merger simulations

Inclusion of important **astrophysical observations** (NICER, …) and **experimental constraints** (FRIB400, …)

## Opportunities for discussion…

**UQ of the pQCD expansion**
Scale variation techniques vs. missing higher orders UQ => possibly double-counting information if using both

**Exploring kernel design**
EOS as a case study: constraints from astro./exp. could inform choices via type of phase transition allowed

**This BMM framework is *not* restricted to the EOS!**
**So, you might ask…**

## … and some challenges

**Extending this version of BMM to multiple dimensions**
Crucial for performing UQ in any multi-dimensional nuclear physics problem

**Prior choices**
How to best incorporate known physics => priors other than GPs

?

**How do I use this BMM framework for *my* particular problem?**

# Taweret: BMM software

**DOI:** 10.21105/joss.06175

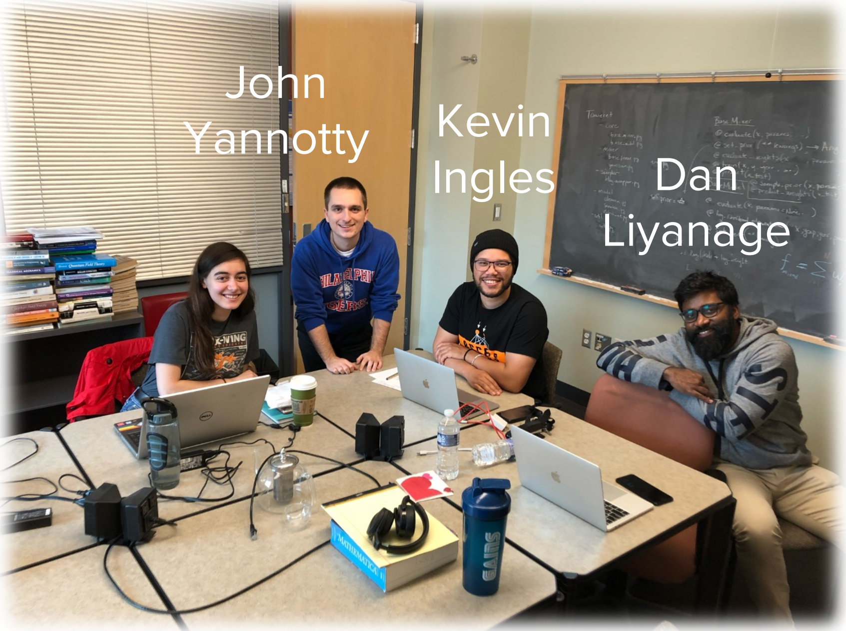
**Open-source** repository:
https://github.com/bandframework/Taweret

*Published in* JOSS — The Journal of Open Source Software

Python interface for user to use BMM methods on **their own models**

Present: our 3 methods + toy models

**Near future**: *add your own BMM method!*




Picture credit: Dick Furnstahl

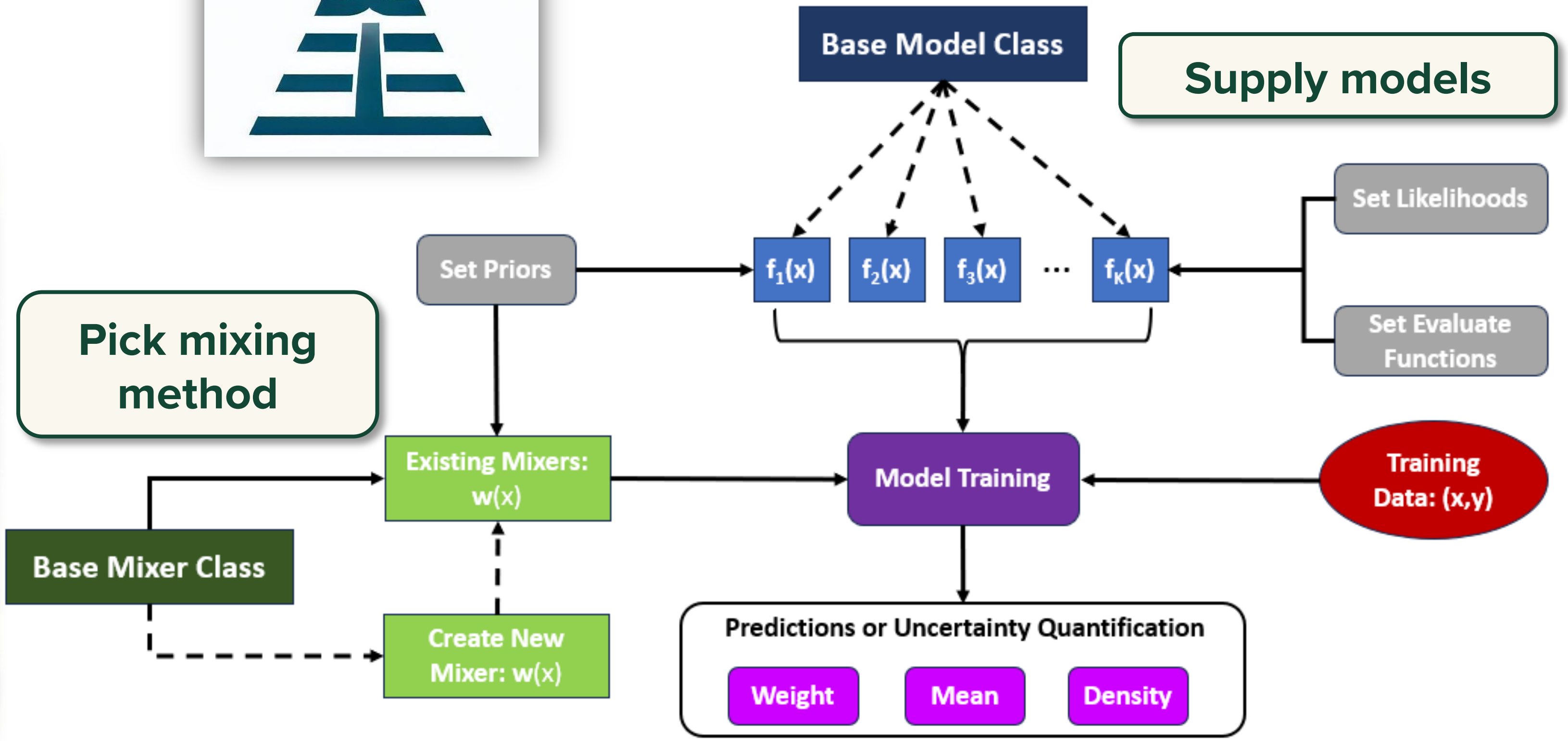



Figure credit: John Yannotty



**Open-source repository for the EOS work coming soon!**

OHIO UNIVERSITY

# Thank you!

Daniel Phillips (OU)

Dick Furnstahl (OSU)

Christian Drischler (OU)

Jordan Melendez

BAND Bayesian Analysis of Nuclear Dynamics

U.S. DEPARTMENT OF ENERGY

NSF

ASCSN

BUQEYE Collaboration

INSTITUTE OF NUCLEAR & PARTICLE PHYSICS INPP @OHIO UNIVERSITY

ACS supported by: US DOE, contract DE-FG02-93ER-40756, NSF CSSI program, award number OAC-2004601

# Backup slides

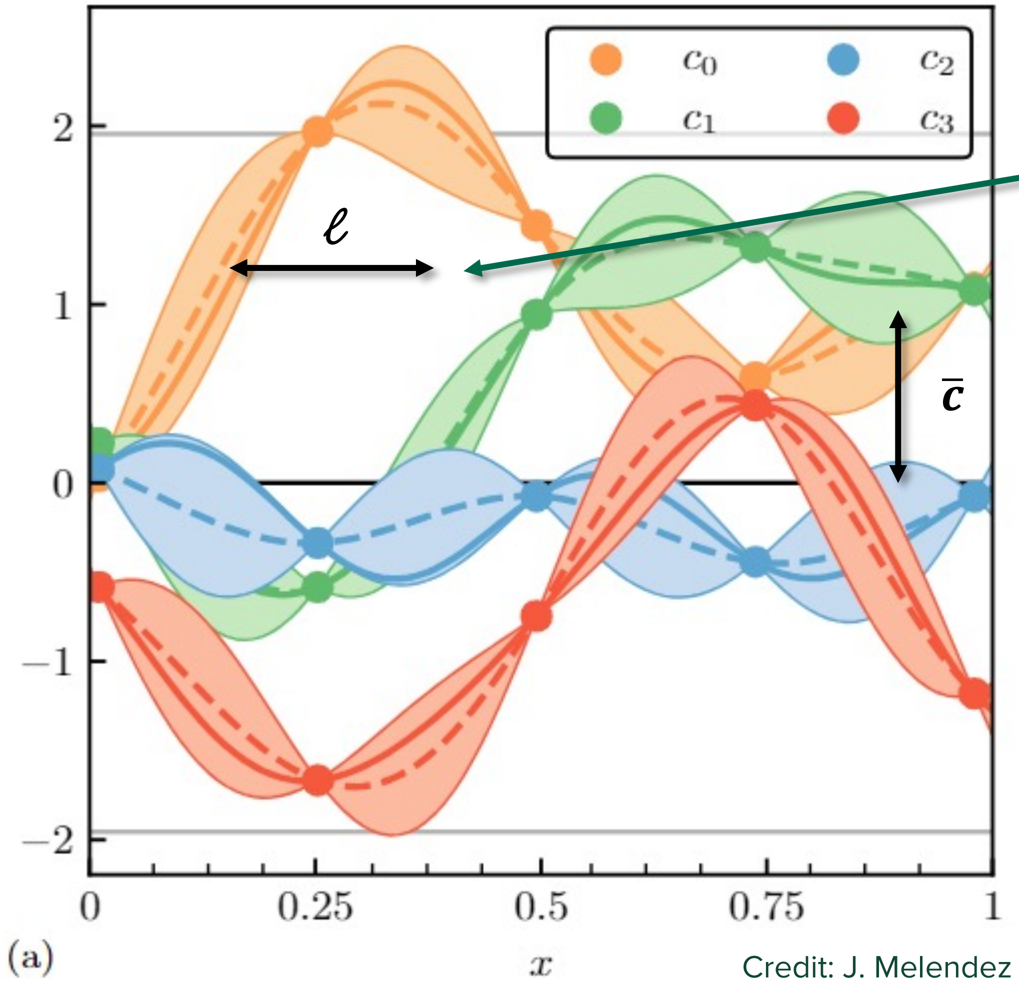
# Crash course: Gaussian Processes (GPs)

"Set of random variables, any subset of which possesses a Gaussian distribution"

Less abstract:
Defined by mean function and covariance function (*kernel*)

$$f(x) \sim \mathcal{GP}[m(x), \kappa(x, x')]$$

Contains dependence on variance and lengthscale (RBF, Matérn, etc.)




Credit: J. Melendez et al. (2019)

Correlation structure parameterized by a lengthscale and variance

Hyperparameters that *can* be determined by Bayesian parameter estimation

Can be fitted to **data + uncertainties** and used to predict at new points

# High densities: pQCD EOS

Invoking the Kohn-Luttinger-Ward inversion theorem

$$P(\mu) = P_{FG}(\mu)\left[c_0 + c_1 Q(\bar{\Lambda}) + c_2(\mu) Q^2(\bar{\Lambda})\right]$$

**Goal**: $P(\mu) \to P(n)$

1 $\mu = \mu_{FG} + \mu_1 + \mu_2$ ← Perturbative expansion

2 Taylor expand => $n(\mu) = \frac{\partial P(\mu)}{\partial \mu} \longrightarrow n(\mu_{FG}) \equiv n(\mu_{FG} + \mu_1 + \mu_2) \equiv \bar{n}$ Input number density

3 Equate terms by counting powers of $\alpha_s$

$$\bar{n}(\mu) = c_0(\mu) \frac{\partial P_{FG}(\mu)}{\partial \mu}\bigg|_{\mu=\mu_{FG}}, \quad \mu_1 = -\frac{c_1 Q(\bar{\Lambda}) \frac{\partial P_{FG}(\mu)}{\partial \mu}}{c_0 \frac{\partial^2 P_{FG}(\mu)}{\partial \mu^2}}\Bigg|_{\mu=\mu_{FG}}$$

+ $\mu_2$ expression

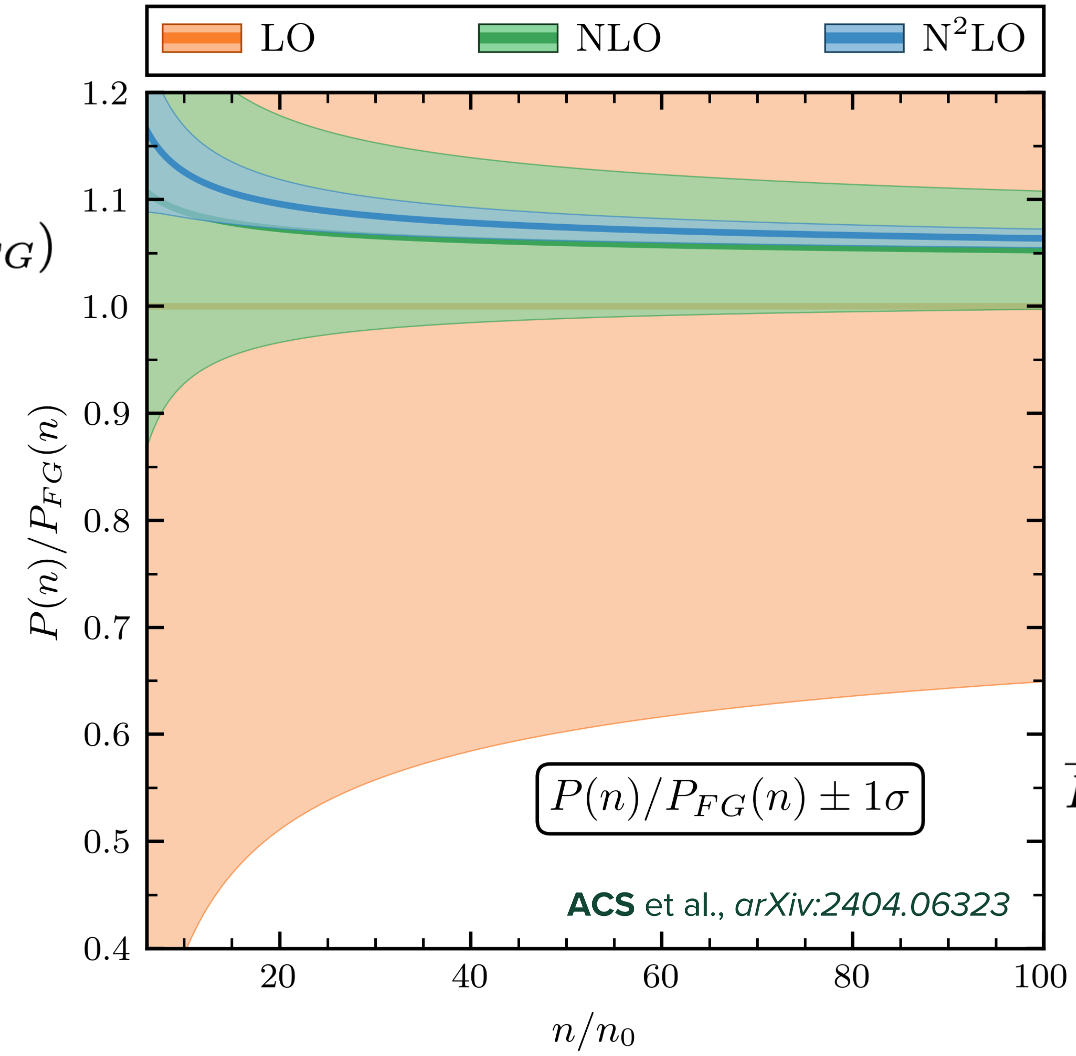
4 Expand P(mu), insert terms, keep up to second order in $\alpha_s$

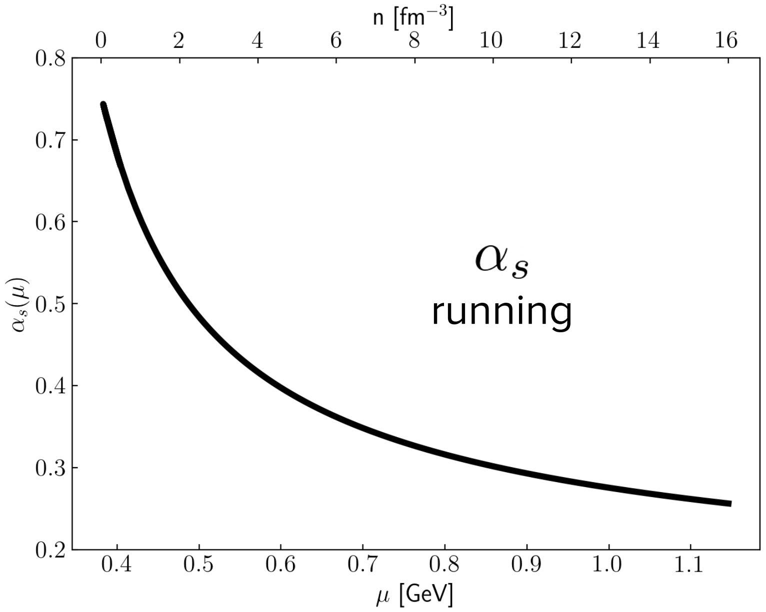
$$\frac{P(n)}{P_{FG}(n)} = 1 + \frac{2}{3\pi}\alpha_s(\bar{\Lambda}_{FG}) + \frac{8}{9\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{N_f^2}{3\pi^2} c_2(\mu_{FG}) \alpha_s^2(\bar{\Lambda}_{FG}) - \frac{\beta_0}{3\pi^2}\alpha_s^2(\bar{\Lambda}_{FG})$$

# High densities: pQCD EOS

Invoke the Kohn-Luttinger-Ward inversion theorem

$$P(n) = P_{FG}(n)\Big[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n) Q^2(\bar{\Lambda}_{FG}) + \dots\Big]$$




P(n) scaled by FG pressure:

$$\frac{P(n)}{P_{FG}(n)} = 1 + \frac{2}{3\pi}\alpha_s(\bar{\Lambda}_{FG}) + \frac{8}{9\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{\beta_0}{3\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{N_f^2}{3\pi^2} c_2(\mu_{FG})\alpha_s^2(\bar{\Lambda}_{FG})$$



Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# High densities: pQCD EOS

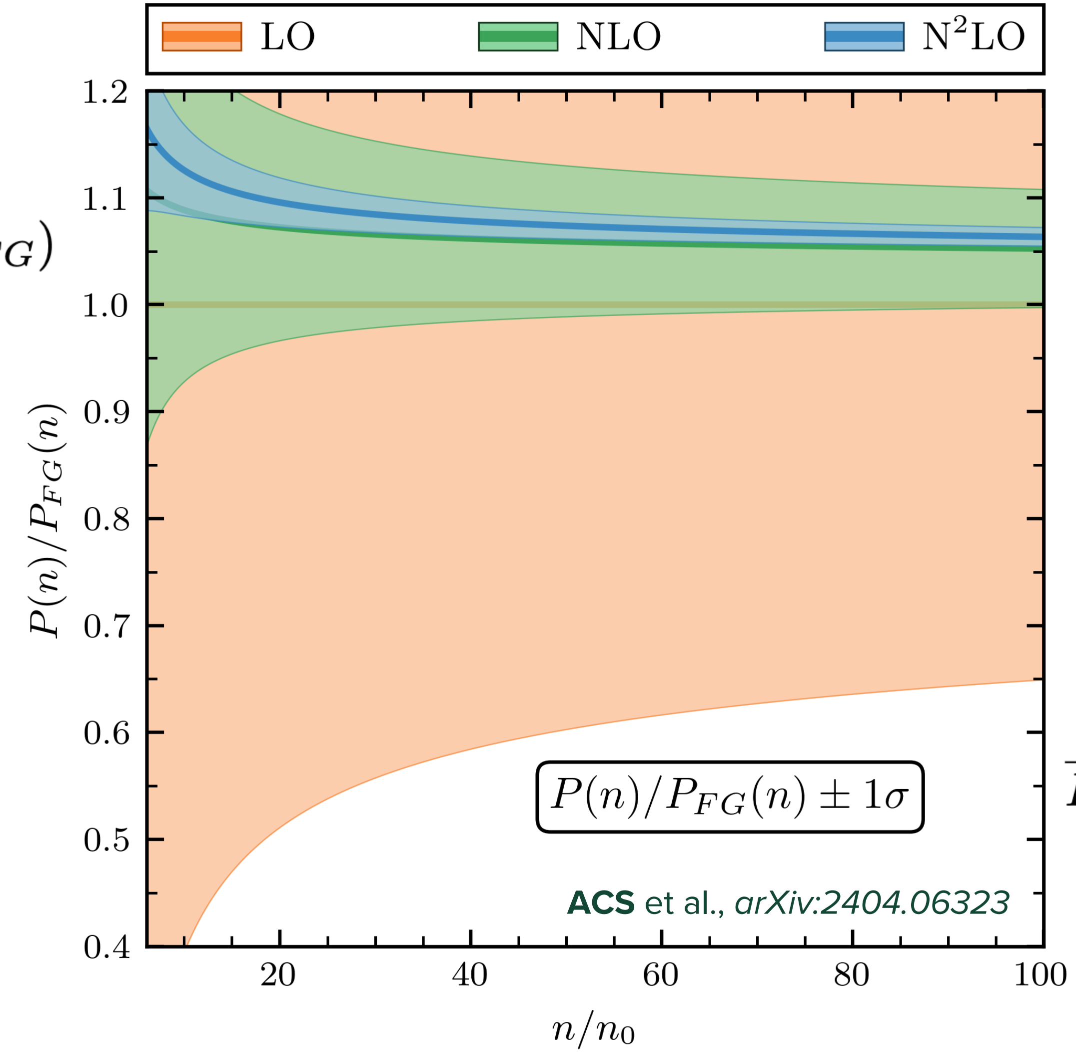
Invoke the Kohn-Luttinger-Ward inversion theorem

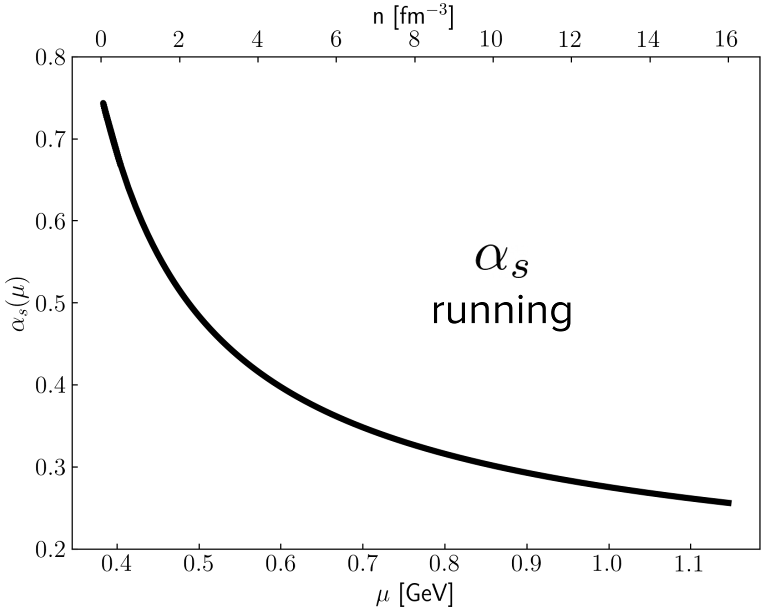
$$P(n) = P_{FG}(n)\left[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n) Q^2(\bar{\Lambda}_{FG}) + \ldots\right]$$

Apply **gsum** to **P(n)**

Q = $\frac{N_f}{\pi}\alpha_s(\bar{\Lambda})$

$y_{ref}$ = $P_{FG}(n)$




P(n) scaled by FG pressure:

$$\frac{P(n)}{P_{FG}(n)} = 1 + \frac{2}{3\pi}\alpha_s(\bar{\Lambda}_{FG}) + \frac{8}{9\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{\beta_0}{3\pi^2}\alpha_s^2(\bar{\Lambda}_{FG}) - \frac{N_f^2}{3\pi^2}c_2(\mu_{FG})\alpha_s^2(\bar{\Lambda}_{FG})$$

Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# High densities: pQCD EOS

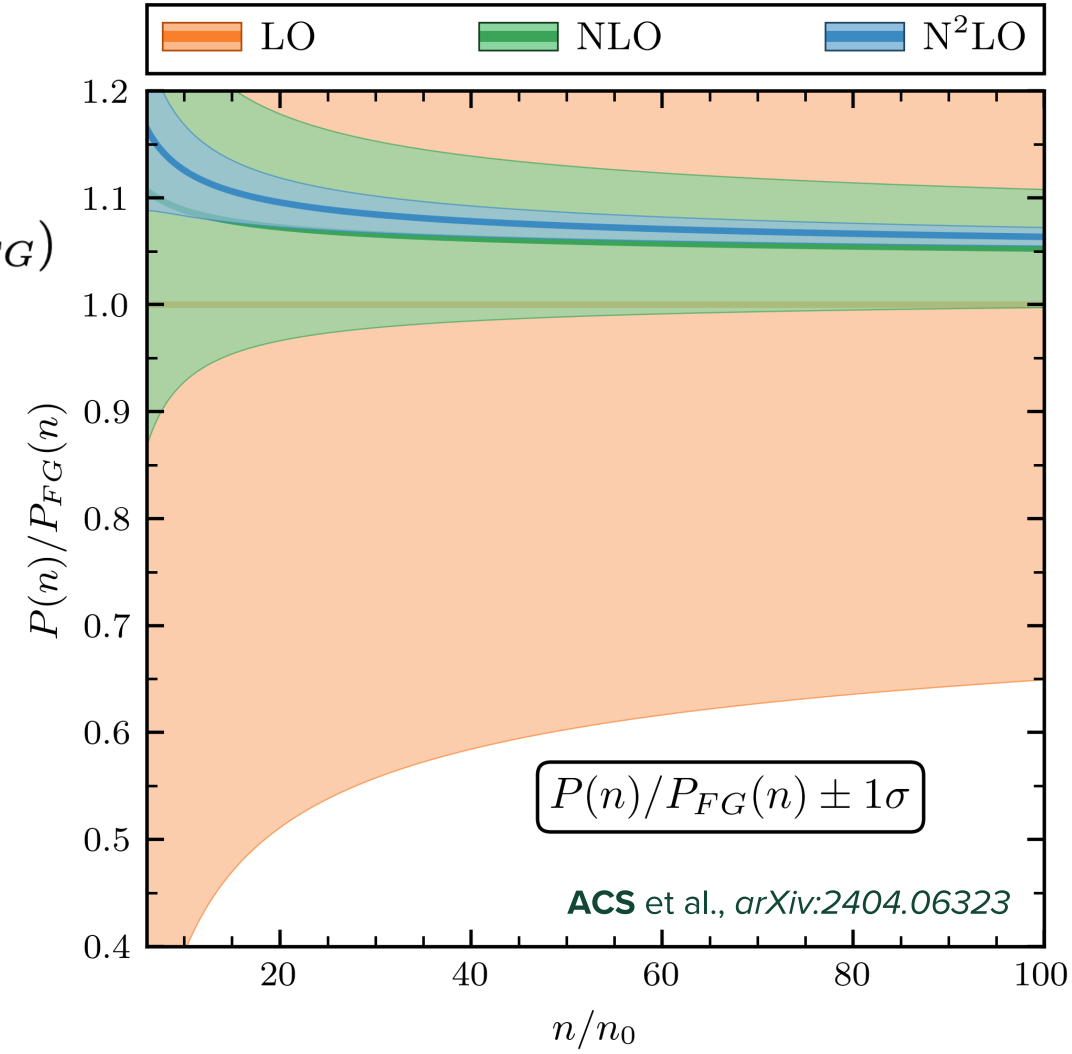
Invoke the Kohn-Luttinger-Ward inversion theorem

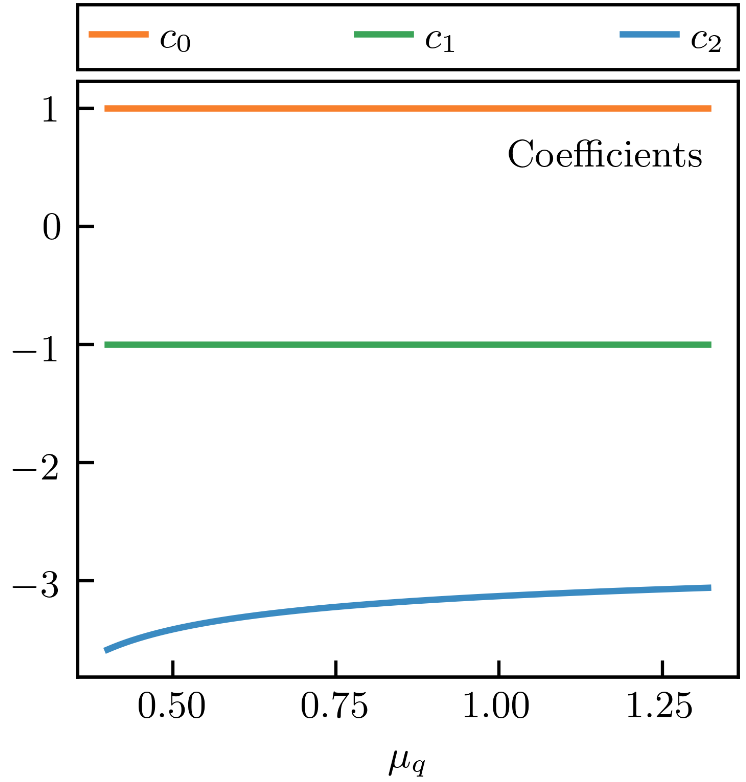
$$P(n) = P_{FG}(n)\Big[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n) Q^2(\bar{\Lambda}_{FG}) + \ldots\Big]$$

Apply **gsum** to **P(n)**

Q = $\frac{N_f}{\pi}\alpha_s(\bar{\Lambda})$

$y_{ref}$ = $P_{FG}(n)$




**Truncation error** *historically* not assessed in a Bayesian way* -> vary renormalization scale to obtain band

* See newer works by Gorda et al. (2022, 2023), MiHO model for higher orders



Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# High densities: pQCD EOS

Invoke the Kohn-Luttinger-Ward inversion theorem

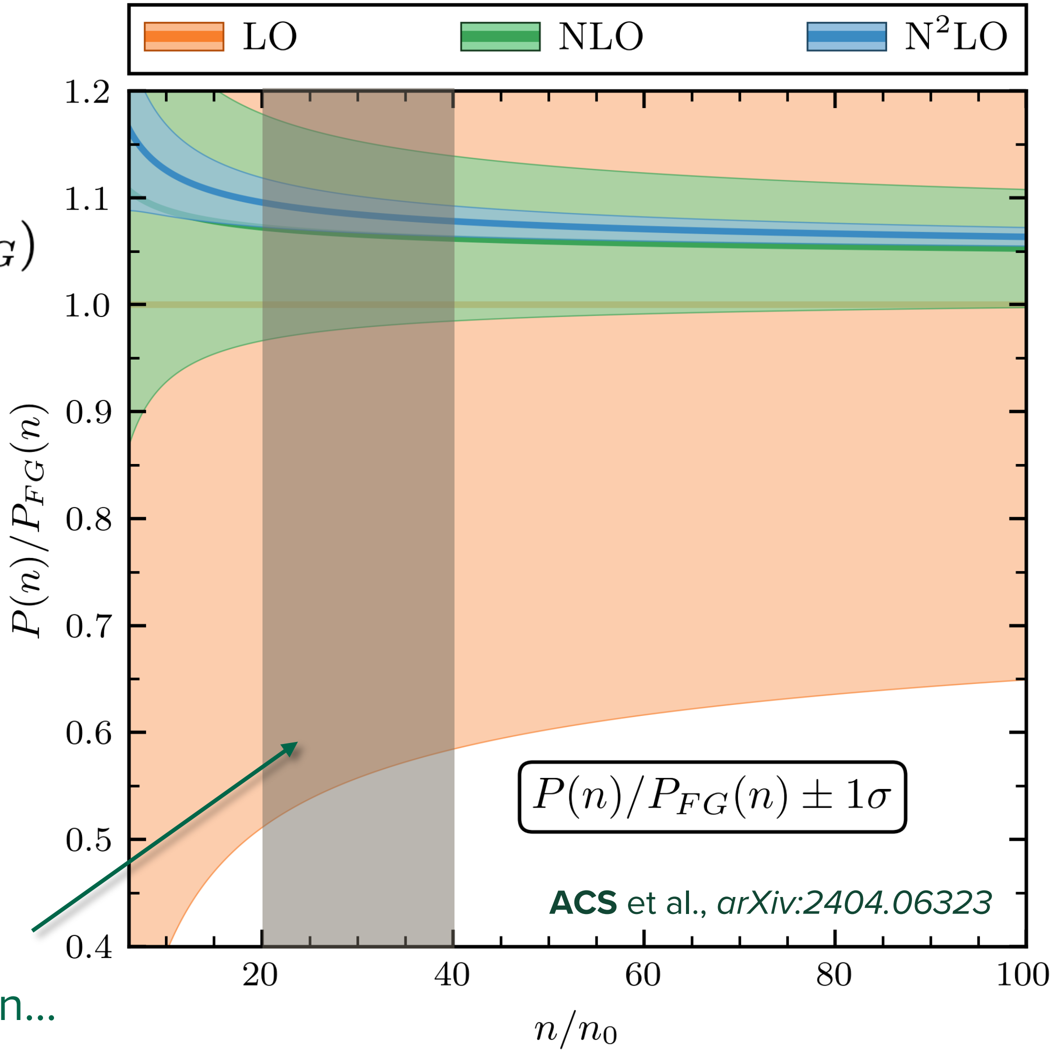
$$P(n) = P_{FG}(n)\Big[c_0 + c_1 Q(\bar{\Lambda}_{FG}) + c_2(n) Q^2(\bar{\Lambda}_{FG}) + \ldots\Big]$$

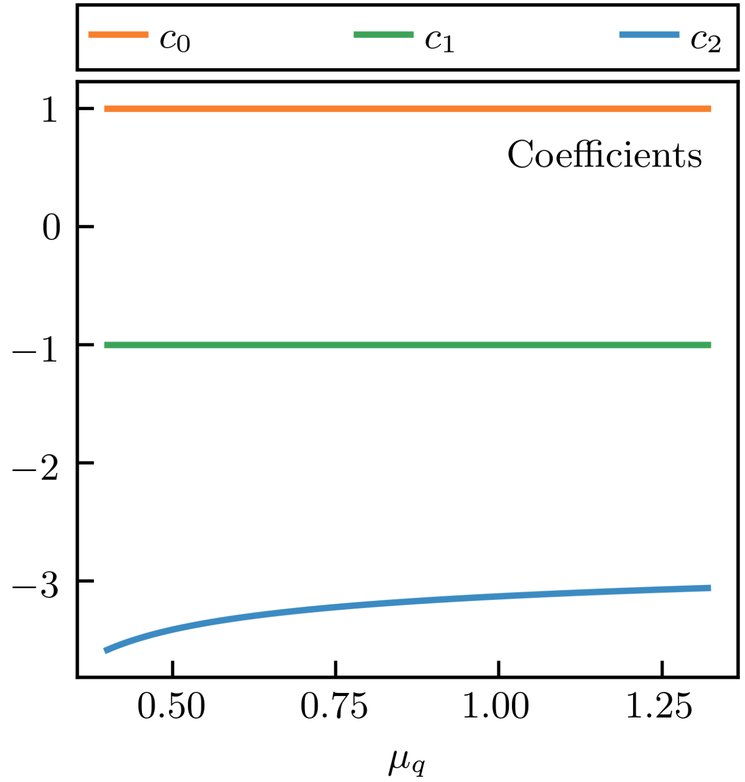
Apply **gsum** to **P(n)**

Q = $\frac{N_f}{\pi}\alpha_s(\bar{\Lambda})$

$y_{ref}$ = $P_{FG}(n)$

Nonperturbative effects under (20-40) $n_0$: pairing, hadronization...




**Truncation error** *historically* not assessed in a Bayesian way* -> vary renormalization scale to obtain band

* See newer works by Gorda et al. (2022, 2023), MiHO model for higher orders

Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# High densities: pQCD EOS

Calculate speed of sound squared using KLW expansion:

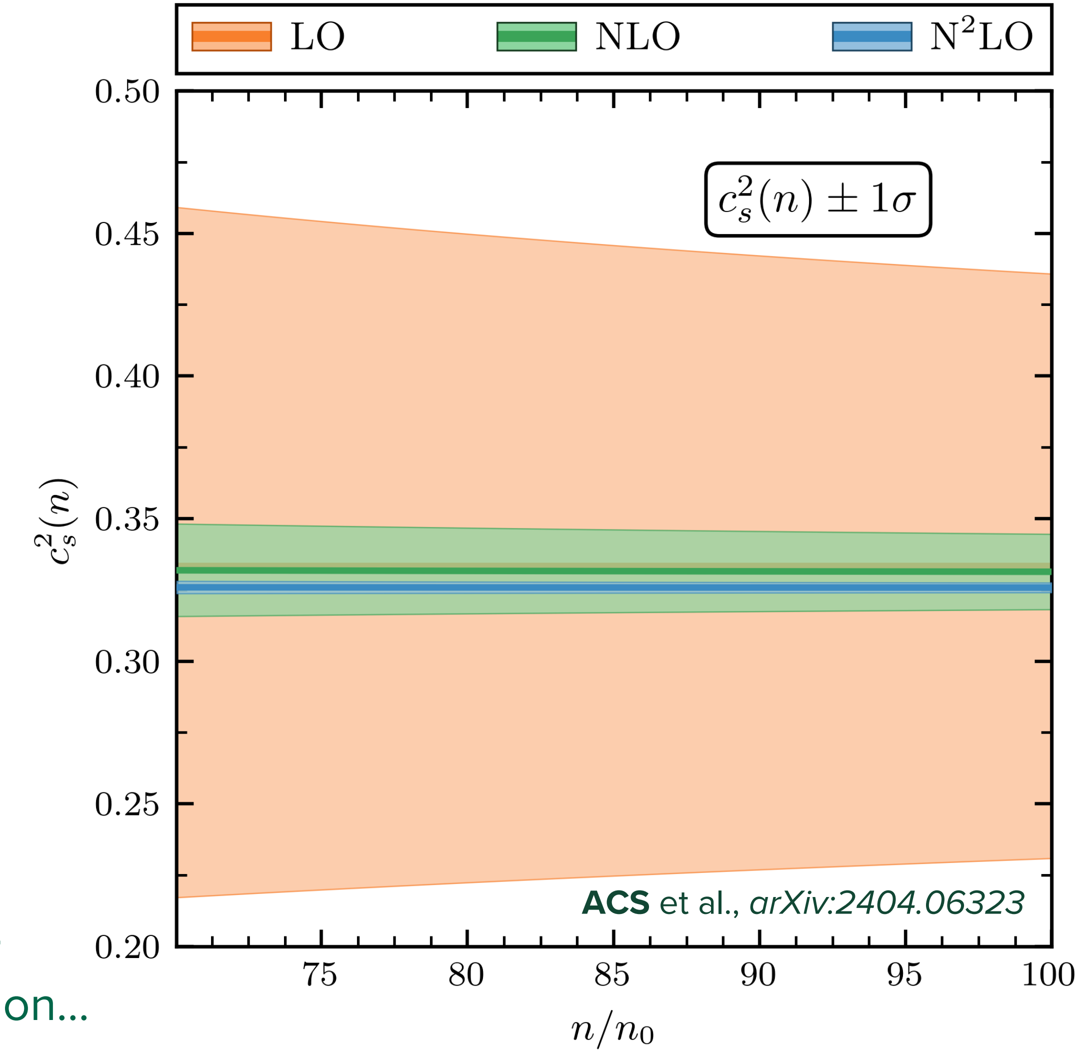
$$c_s^2(n) = n\frac{\partial \ln(\mu(n))}{\partial n}$$

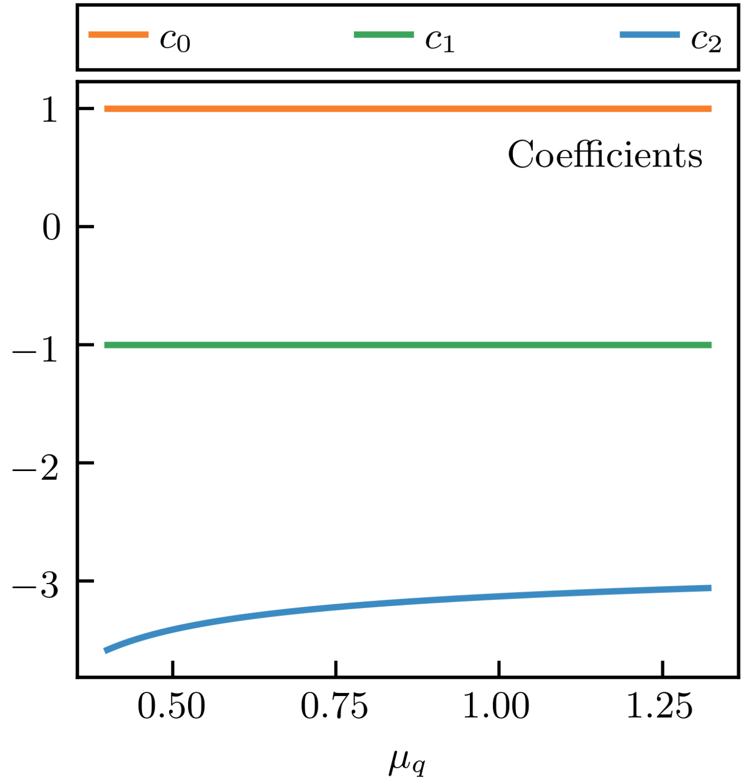
[**Equal** at N$^2$LO to the calculation of $c_s^2(\mu)$ using the pressure directly]

Apply **gsum** to **P(n)**

Q = $\frac{N_f}{\pi}\alpha_s(\bar{\Lambda})$

$y_{ref}$ = $P_{FG}(n)$




Nonperturbative effects under (20-40) $n_0$: pairing, hadronization…

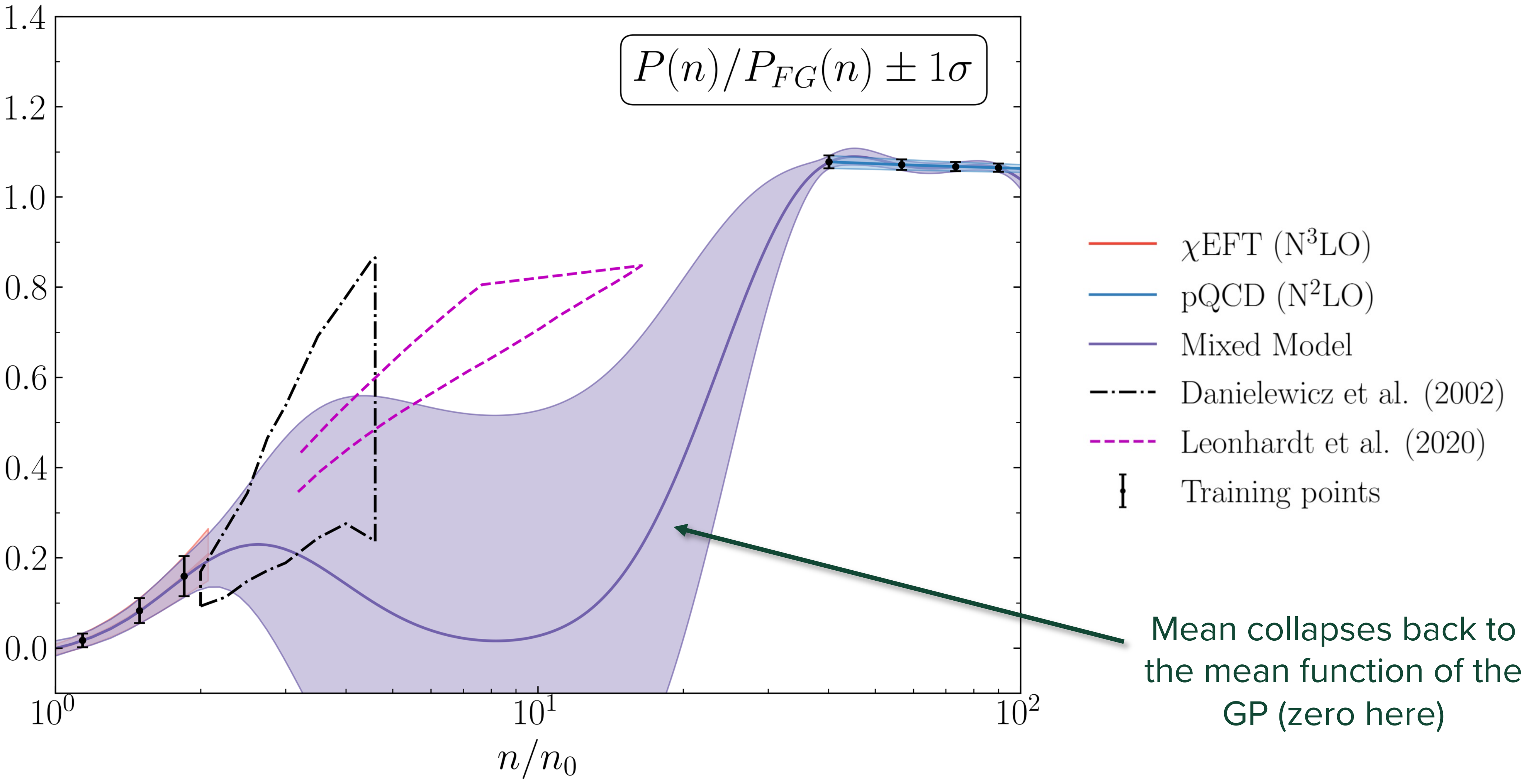
**Truncation error** *historically* not assessed in a Bayesian way* -> vary renormalization scale to obtain band

* See newer works by Gorda et al. (2022, 2023), MiHO model for higher orders



Original model: Tyler Gorda, Risto Paatelainen, Saga Säppi, and Kaapo Seppänen, Phys. Rev. Lett. **131**, 181902

# Extreme case: very short lengthscale




Mean collapses back to the mean function of the GP (zero here)

# BAND collaboration: efforts

Bayesian Analysis of Nuclear Dynamics

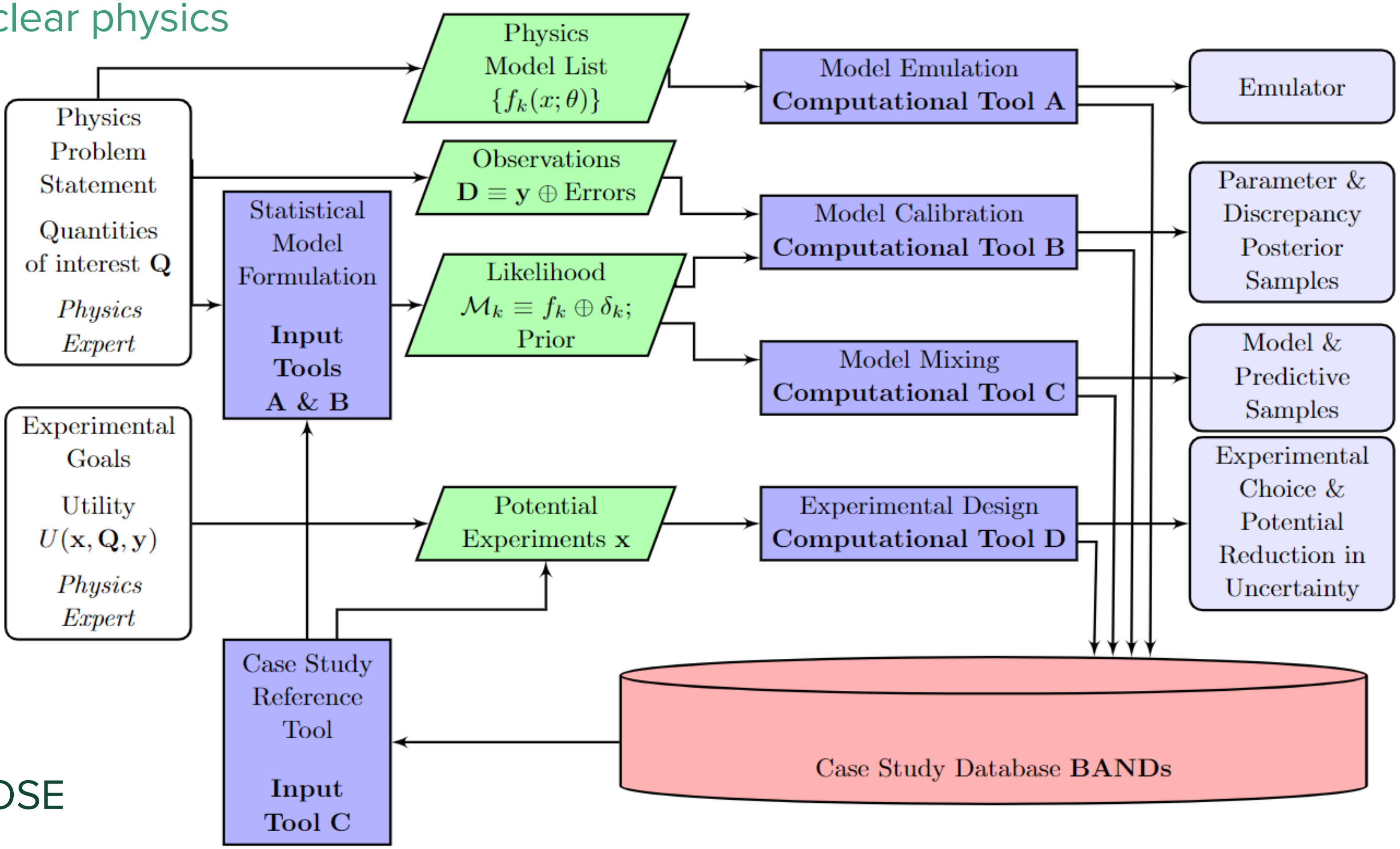
v0.3 of the BAND framework is out!

https://github.com/bandframework/bandframework

## Cyber-infrastructure

Develop tools for nuclear physics community

Test on toy models and real-world applications




ParMOO

BMEX

surmise

ROSE

Our model mixing software!

SAMBA

Taweret