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Uncertainty Quantification for Nucleon Form Factors from Lattice QCD

Dalibor Djukanovic

Actually, I never made it beyond ONE nucleon …

UQ - in FF from LQCD

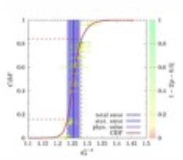
Editors' Suggestion

PDF HTML

Improved analysis of isovector nucleon matrix elements with $N_f = 2+1$ flavors of $\mathcal{O}(a)$ improved Wilson fermions

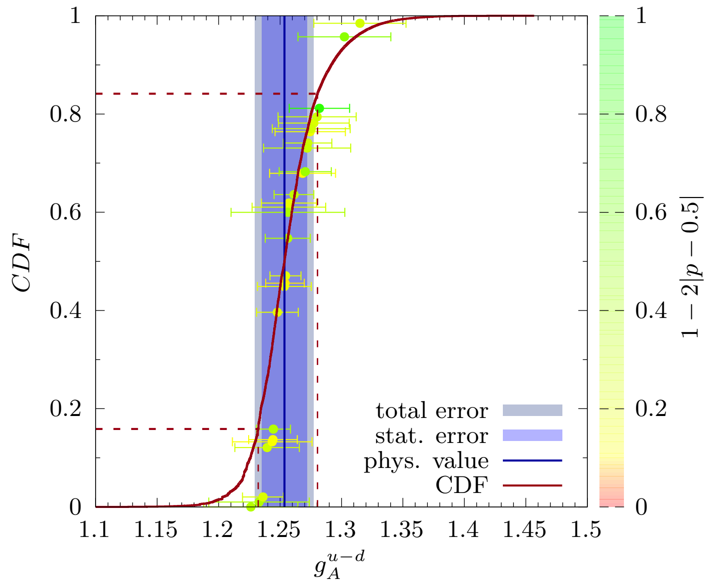
Dalibor Djukanovic, Georg von Hippel, Harvey B. Meyer, Konstantin Ottnad, and Hartmut Wittig

Phys. Rev. D **109**, 074507 (2024) – Published 16 April 2024



The authors present an improved analysis of a lattice determination of isovector nucleon matrix elements. With additional ensembles they achieve excellent control of all systematic errors and obtain the most precise results to date.

Show Abstract +



- Topic of this talk is plots like on the left
- Recent results from Mainz (for more details)

DD, von Hippel, Meyer, Ottnad, Wittig, „Improved analysis of isovector nucleon matrix elements with Nf=2+1 flavors of O(a)improved Wilson fermions“, *Phys.Rev.D* 109 (2024) 7, 074507

DD, von Hippel, Meyer, Ottnad, Salg, Wittig, „Zemach and Friar radii of the proton and neutron from lattice QCD,“ to appear in PRD

DD, von Hippel, Meyer, Ottnad, Salg, Wittig, „Precision Calculation of the Electromagnetic Radii of the Proton and Neutron from Lattice QCD“, Phys.Rev.Lett. 132 (2024) 21, 21

DD, von Hippel, Meyer, Ottnad ,Salg, Wittig, „Electromagnetic form factors of the nucleon from Nf=2+1 lattice QCD“, *Phys.Rev.D* 109 (2024) 9, 9

Agadjanov, DD, von Hippel, Meyer, Ottnad, Wittig, „Nucleon Sigma Terms with Nf=2+1 Flavors of O(a)-Improved Wilson Fermions“, Phys.Rev.Lett. 131 (2023) 26, 26

DD, von Hippel, Kopoonen, Meyer, Ottnad Wittig, „Isovector axial form factor of the nucleon from lattice QCD“, Phys.Rev.D 106 (2022) 7, 074503

UQ - Sources of Errors

„There are known knowns ... But there are also unknown unknowns“

Rumsfeld during a Pentagon news briefing in February 2002

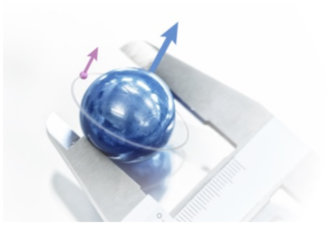
Rumsfeld-Classification	Knowns	Unknowns
Known	ME can be calculated using Lattice - Pathintegral	Statistical Errors, Dealing with Correlations
Unkknown	Systematics – excited states, finite lattice spacing, finite volume	?

UQ may mean one of two things in this talk (keep count tell me later):

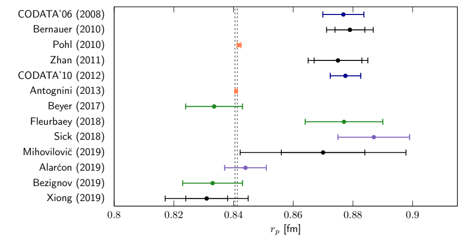
- Uncertainty Quantification
- Un Qualified comment

Impact of Form Factors

- Proton Radius Puzzle

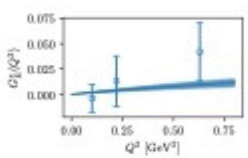


» Provide ab-initio calculation



Taken from Bernauer,
EPJ Web Conf. 234 (2020)

- Precision Tests of SM



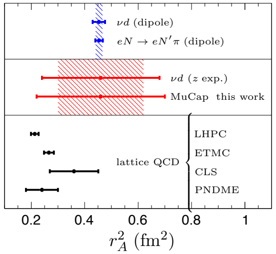
Taken from D.D. et al.
Phys.Rev.Lett. 123 (2019) 21, 212001

» Via strangeness FF → Parity Violation Experiments

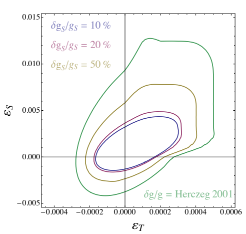
- Lattice determinations of strange FF very precise

» Via axial FF → Vital input to neutrino-nucleus scattering

- Lattice competitive to z-exp extractions of experiments



Taken from A. Kronfeld, et al.
Eur. Phys. J. A **55,** 196 (2019)



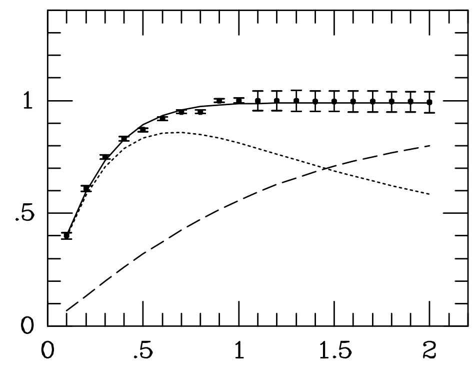
Taken from T. Bhattacharya et al.,
Phys. Rev. D85, 054512 (2012)

» Via Charges → Constraining BSM EFT couplings

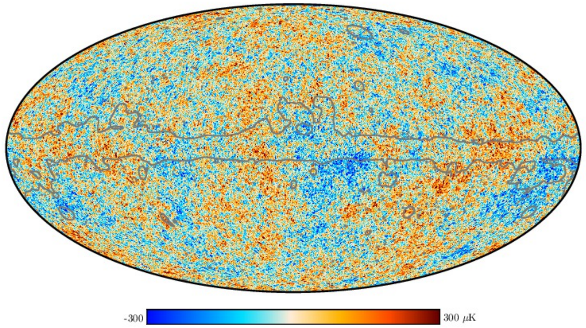
UQ - Motivation Dark Matter

Rotation Curves of Galaxies

Plot from *arXiv:2104.11488*



Evidence for Dark Matter

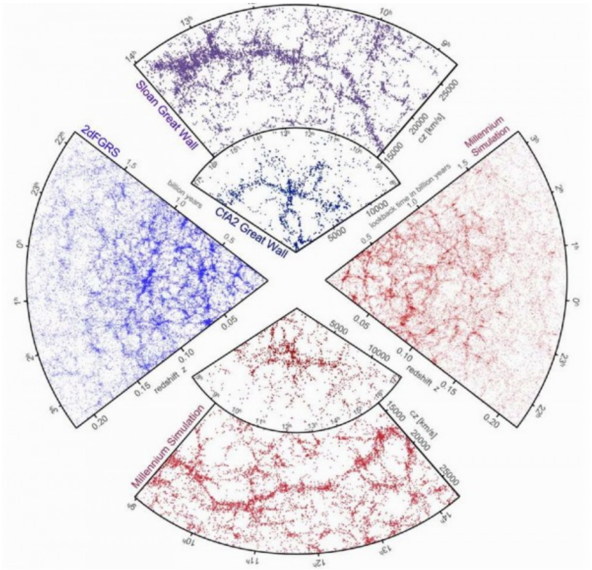


Cosmic Microwave Background

Plot from Y. Akrami, et al., Planck 2018 results. I. Overview and the cosmological legacy of Planck, arXiv:1807.06205.

Gravitational Lensing

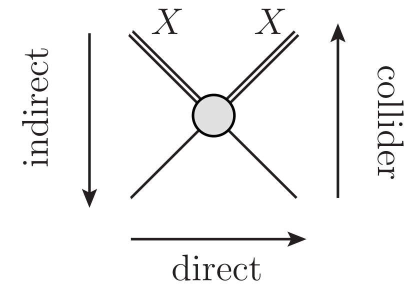
Credit: NASA/CXC/CfA/M. Markevitch et al.; NASA/STScI; Magel- lan/U.Arizona/D. Clowe et al.; NASA/STScI; ESO WFI



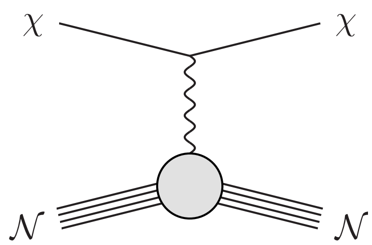
Simulation of Galaxy Structures

Plot from V. Springel, C. S. Frenk, S. D. White, The large-scale structure of the Universe, Nature, 440 (2006) 1137. arXiv:astro-ph/0604561

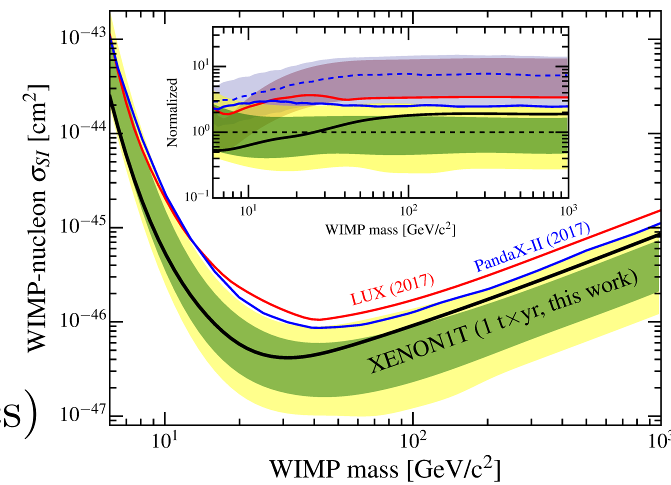
UQ - DM Direct Searches



- DM Candidate: WIMP
- WIMP-Nucleus-Scattering



Plot from E. Aprile et al,
Dark Matter Search Results from a One Tonne×Year Exposure of XENON1T
arXiV:1805.12562

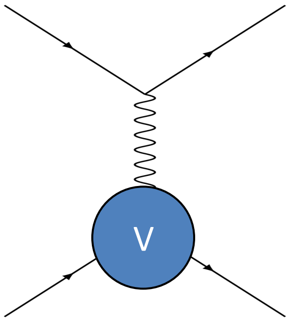


- Rate WIMP-Nucleus Scattering

$$R \sim \frac{\sigma^{\mathrm{SI}}}{M_\chi \mu_N^2} \times (\text{Nuclear Physics}) \times (\text{Astrophysics})$$

- σ^{SI} depends on Sigma-Term
- Crucial input for interpretation of experiments

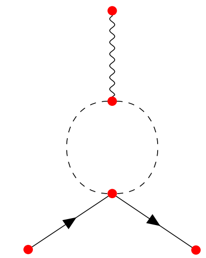
Nucleon Form Factors



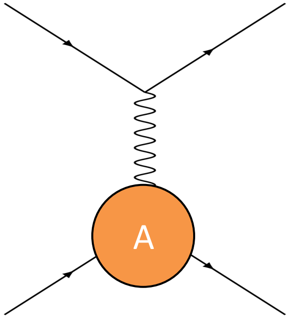
$$\langle N(p')|\bar{q}\gamma_\mu q|N(p)\rangle$$
$$= \bar{u}(p')\left[\gamma_\mu F_1^q(Q^2=-q^2) + i\,\frac{\sigma_{\mu\nu}}{2m_N}\,q^\nu F_2^q(Q^2)\right]u(p)$$
$$G_E(Q^2) = F_1^q(Q^2) - \frac{Q^2}{2m_N}F_2^q(Q^2) = \left(1 - \frac{1}{6}\langle r_E^2\rangle Q^2 + \cdots\right)$$
$$G_M(Q^2) = F_1^q(Q^2) + F_2^q(Q^2) = \mu\left(1 - \frac{1}{6}\langle r_M^2\rangle Q^2 + \cdots\right)$$

Dirac Pauli

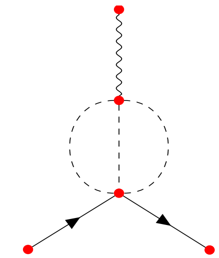
$$\langle r_{E/M}^2\rangle \xrightarrow{M_\pi\to 0} \{\ln M_\pi, \frac{1}{M_\pi}\}$$



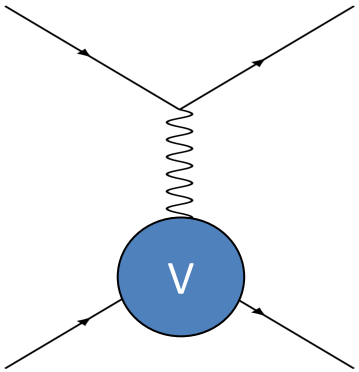
$$\langle r_A^2\rangle \xrightarrow{M_\pi\to 0} const.$$

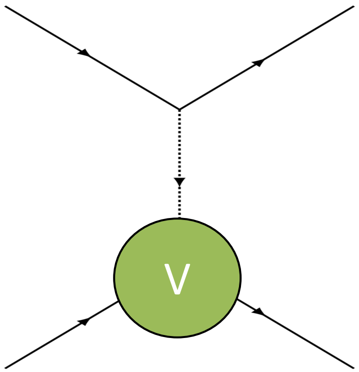


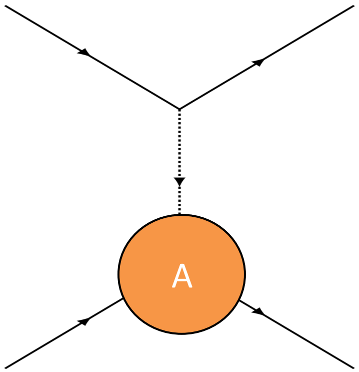
$$\langle N(p')|\bar{q}\gamma_\mu\gamma_5 q|N(p)\rangle$$
$$= \bar{u}(p')\left[\gamma_\mu G_A(Q^2) + i\,\frac{q^\mu}{2m_N}\,G_P(Q^2)\right]\gamma_5\frac{\tau^3}{2}u(p)$$
$$G_A(Q^2) = g_A\left(1 - \tfrac{1}{6}\langle r_A^2\rangle Q^2 + \cdots\right)$$
$$2m_N G_A(Q^2) - \frac{Q^2}{2\,m_N}G_P(Q^2) = \frac{2\,M_\pi^2 F_\pi}{M_\pi^2+Q^2}G_{\pi N}(Q^2)$$ (PCAC)



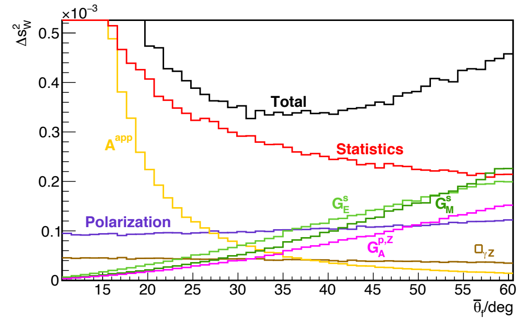
Parity Violation







Projected Error Budget for P2:



Taken from arxiv:1802.04759

$$A = \frac{d\sigma_R - d\sigma_L}{d\sigma_R + d\sigma_L} = -\frac{G_F Q^2}{4\sqrt{2}\pi\alpha}\,\frac{\epsilon\, G_E^{\gamma} G_E^{Z} + \tau G_M^{\gamma} G_M^{Z} - (1 - 4\sin^2\theta_W)\,\epsilon' G_M^{\gamma}\, G_A}{\epsilon\left(G_E^{\gamma}\right)^2 + \tau\left(G_M^{\gamma}\right)^2}$$

$$\tau = \frac{Q^2}{4\,m_N^2}, \epsilon = \left(1 + 2(1+\tau)\tan^2\frac{\theta}{2}\right)^{-1}, \epsilon' = \sqrt{\tau(1+\tau)(1-\epsilon^2)}$$

$$G_{E/M}^{Z,p} = (1 - 4\sin^2\theta_W)\, G_{E/M}^{\gamma,p} - G_{E/M}^{\gamma,n} - G_{E/M}^{s}$$

- Sensitive to the Weak Charge
  Test of SM at low energies
- Need e/m FF (strange)
- Need axial FF (strange)

(Decomposition assuming Isospin Symmetry)

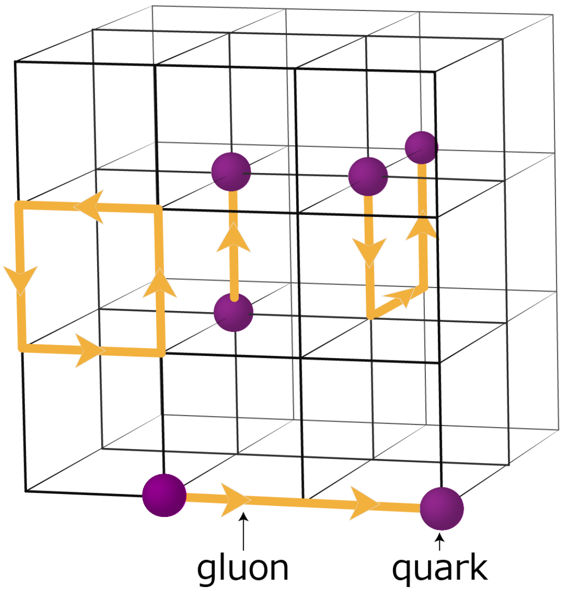
Lattice – Oneslide Intro

- Discretize Space Time
- Lattice action

$$S^{Lat}[U,\Psi,\overline{\Psi}] = S_G^{Lat}[U] + S_F^{Lat}[U,\Psi,\overline{\Psi}]$$

$$\langle\Omega\rangle = \frac{1}{Z}\int \prod_{x,\mu} dU_\mu(x)\,\Omega \prod_{f=u,d,s} \det\left[D + m_f\right] e^{-S_G}$$

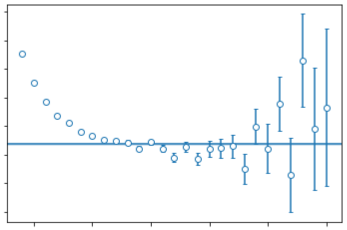
- $\langle\Omega\rangle$ evaluated stochastically (MC-HMC)
- Challenges
  - Need to extrapolate to continuum
    - In lattice spacing
    - In lattice Volume
  - Need to extrapolate to physical quark masses (Chiral EFT)
  - Need to control excited states

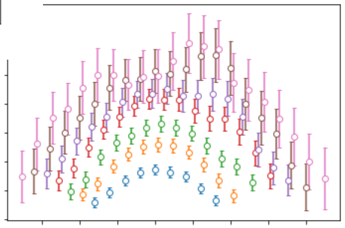


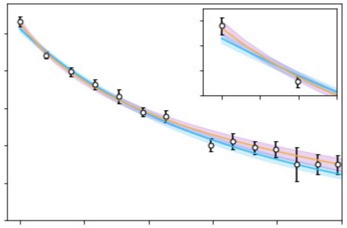
Source: JICFuS, Tsukuba

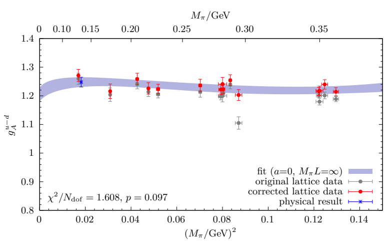
Sources of Uncertainty

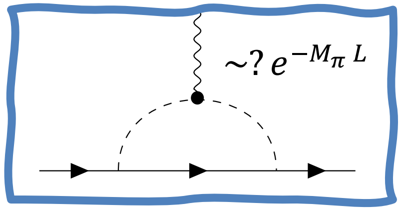
- Statistical Accuracy
- Excited State Contamination
- Model Dependence
- Extrapolations
  - Chiral
  - Continuum
  - Finite Size



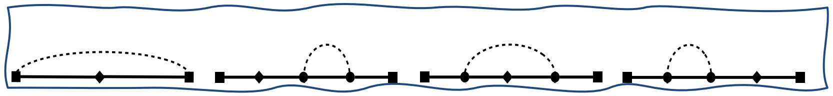








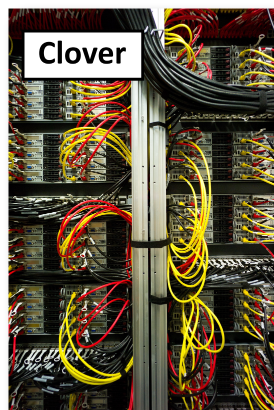
Aside: Can calculate Excited States directly in ChPT even for FF (reliable for large t_{sep})



Taken from O. Bär, H.Colic, *Phys.Rev.D* 103 (2021) 11, 114514

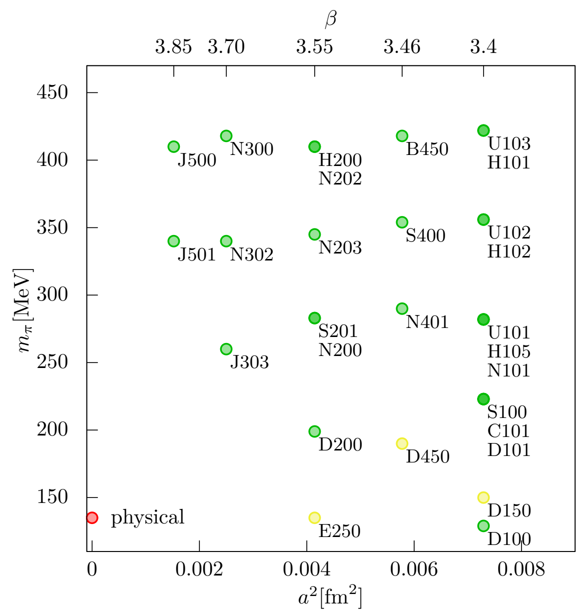
Lattice

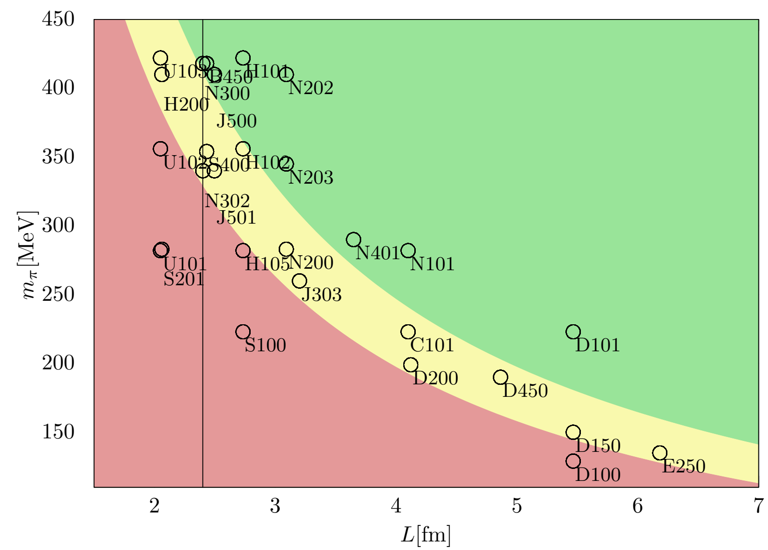
- Discretization not unique:
  Wilson, DWF, HISQ …
- $N_f = 2 + 1$ (2 degenerate u/d + s)
- Gauge ensembles produced within **C**oordinated **L**attice **S**imulations



Luscher and Schaefer, "Lattice QCD without topology barriers", JHEP 07 (2011) 036
Bruno, DD, Engel et al, „Simulation of QCD with N_f=f= 2 + 1 flavors of non-perturbatively improved Wilson fermions", JHEP 02 (2015) 043

Landscape of CLS ensembles





Red = $m_\pi L < 4$

Yellow = $4 < m_\pi L < 5$

Taken from:
D.Mohler et al, EPJ Web of Conferences **175**, 02010 (2018)

Lattice Setup

ID	a [fm]	T/a	L/a	M_π [MeV]	$M_\pi L$	t_{sep} [fm]	N_{cfg}
H102	0.086	96	32	354	4.96	0.35, 0.43, 0.52, 0.6, 0.69, 0.78, 0.86, 0.95, 1.04, 1.12, 1.21, 1.3, 1.38, 1.47	2005
H105		96	32	280	3.93		1027
C101		96	48	225	4.73		2000
N101		128	48	281	5.91		1596
S400	0.076	128	32	350	4.33	0.31, 0.46, 0.61, 0.76, 0.92, 1.07, 1.22, 1.37, 1.53	2873
N451		128	48	286	5.31		1011
D450		128	64	216	5.35		500
D452		128	64	153	3.79		1000
N203	0.064	128	48	346	5.41	0.26, 0.39, 0.51, 0.64, 0.77, 0.9, 1.03, 1.16, 1.29, 1.41	1543
N200		128	48	281	4.39		1712
D200		128	64	203	4.22		2000
E250		192	96	129	4.04		400
S201		128	32	293	3.05		2093
N302	0.050	128	48	348	4.22	0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1., 1.1, 1.2, 1.3, 1.39	2201
J303		192	64	260	4.19		1073
E300		192	64	174	4.21		570

- Enlarged range in t_{sep}
  → Monitor excited state contribution
- Roughly same statistics at every t_{sep}
  → Number of sources adapted to t_{sep}
- Chiral/Continuum/Finite-Size extrapolation possible

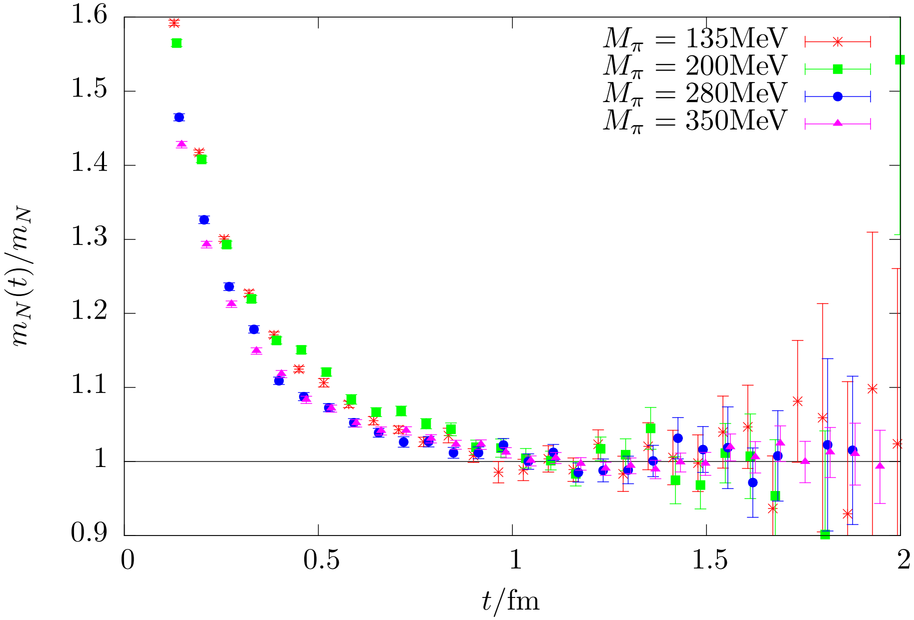
Physics from the Lattice

- Physics contained in correlation functions

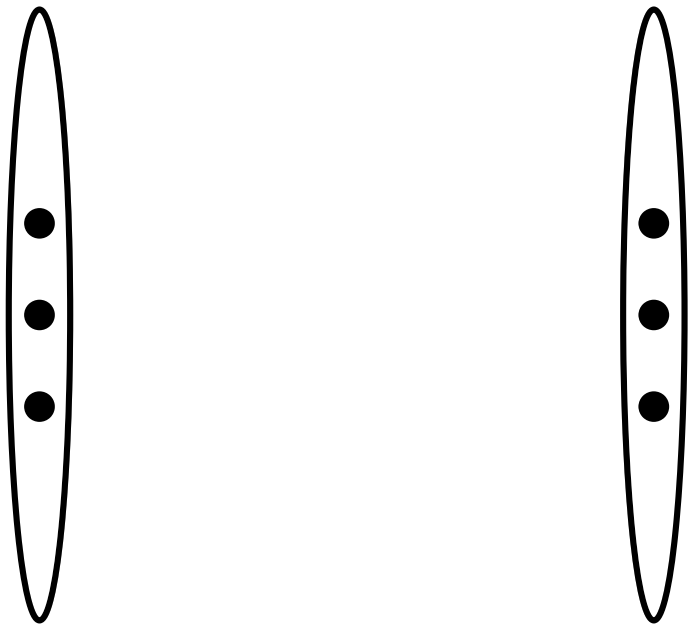
$$\sum e^{i\boldsymbol{p}(\boldsymbol{y}-\boldsymbol{x})} \langle \mathcal{O}_N(x)\mathcal{O}_N(y)^\dagger \rangle = \sum a_n(\boldsymbol{p}) e^{-E_n(p)(y_0-x_0)}$$

$$\xrightarrow{(y_0-x_0)\to\infty} a_0(\boldsymbol{p}) e^{-E_0(y_0-x_0)}$$

- $\mathcal{O}_N$: Nucleon interpolating o
- Ground state dominates for
- Challenges:
  - – Signal to noise det
  - – Need to control ex



UQ - Infamous S/N Nucleons

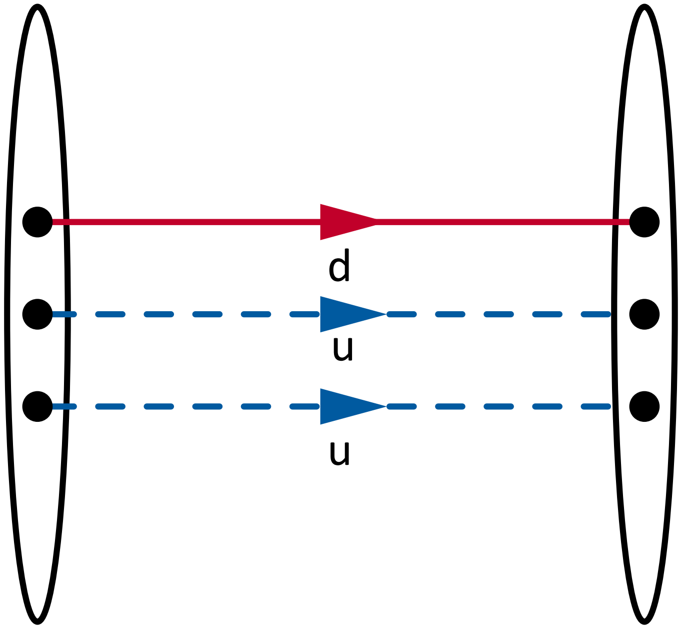


Nucleon interpolating operators

$$\Psi(x) = \epsilon_{abc} u_a^T(x) C\gamma_5 d_b(x) u_c(x)$$

UQ - Infamous S/N Nucleons

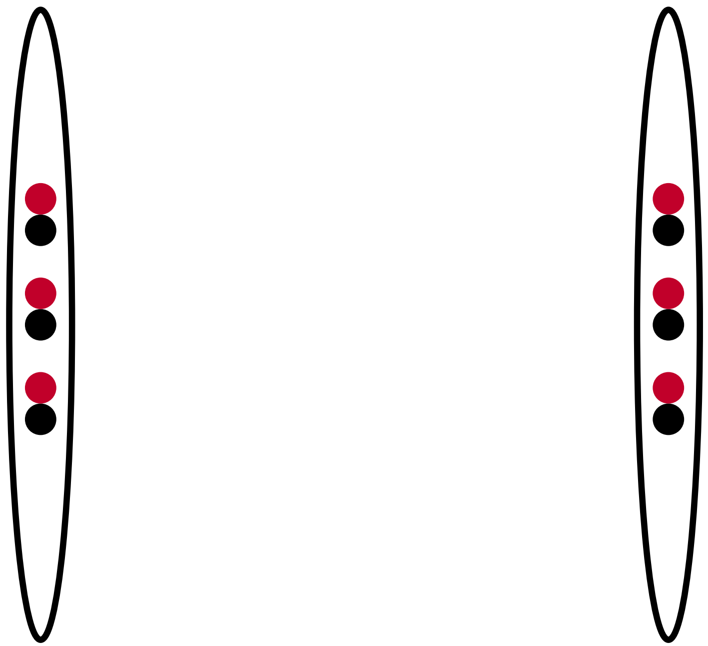
$$\Psi(x) = \epsilon_{abc} u_a^T(x) C\gamma_5 d_b(x) u_c(x)$$



Wick Contraction

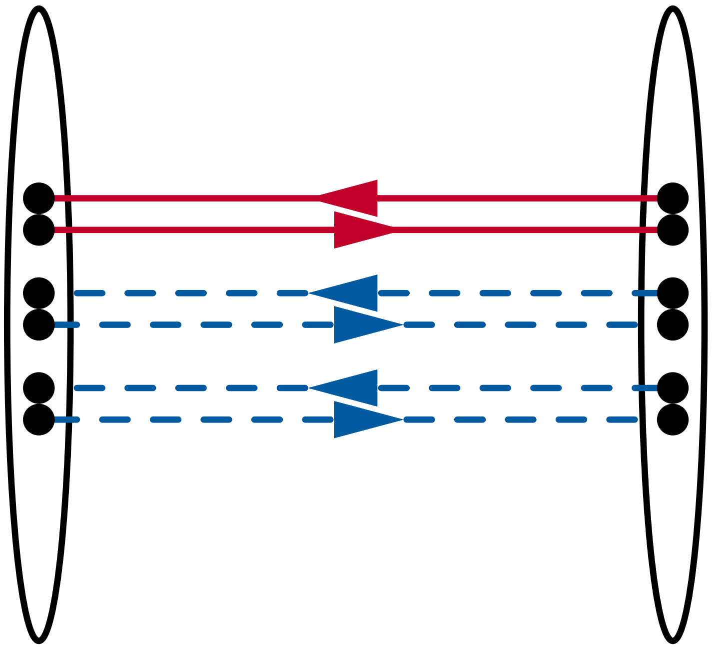
$$\langle \Psi(x)\bar{\Psi}(y)\rangle \sim e^{-m_N t}$$

UQ - Infamous S/N Nucleons



Part of Variance is $\langle \Psi(x)\Psi(x)\bar{\Psi}(y)\bar{\Psi}(y) \rangle$

UQ - Infamous S/N Nucleons

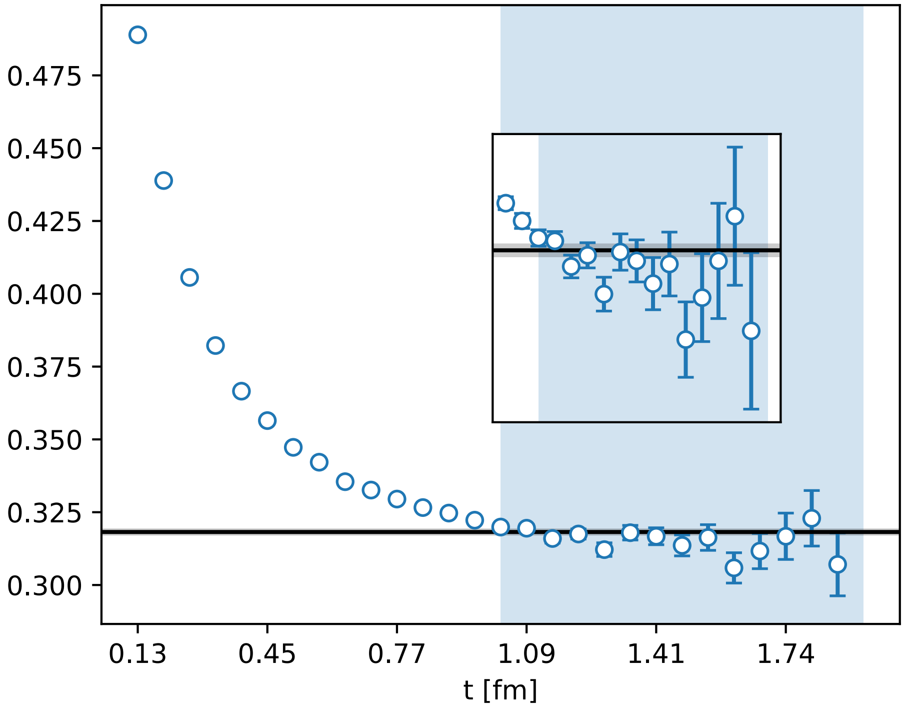


Part of Variance is $\langle \Psi(x)\Psi(x)\bar{\Psi}(y)\bar{\Psi}(y) \rangle$

Noise $\sim e^{-3m_\pi t}$ vs Signal $\sim e^{-m_N t}$

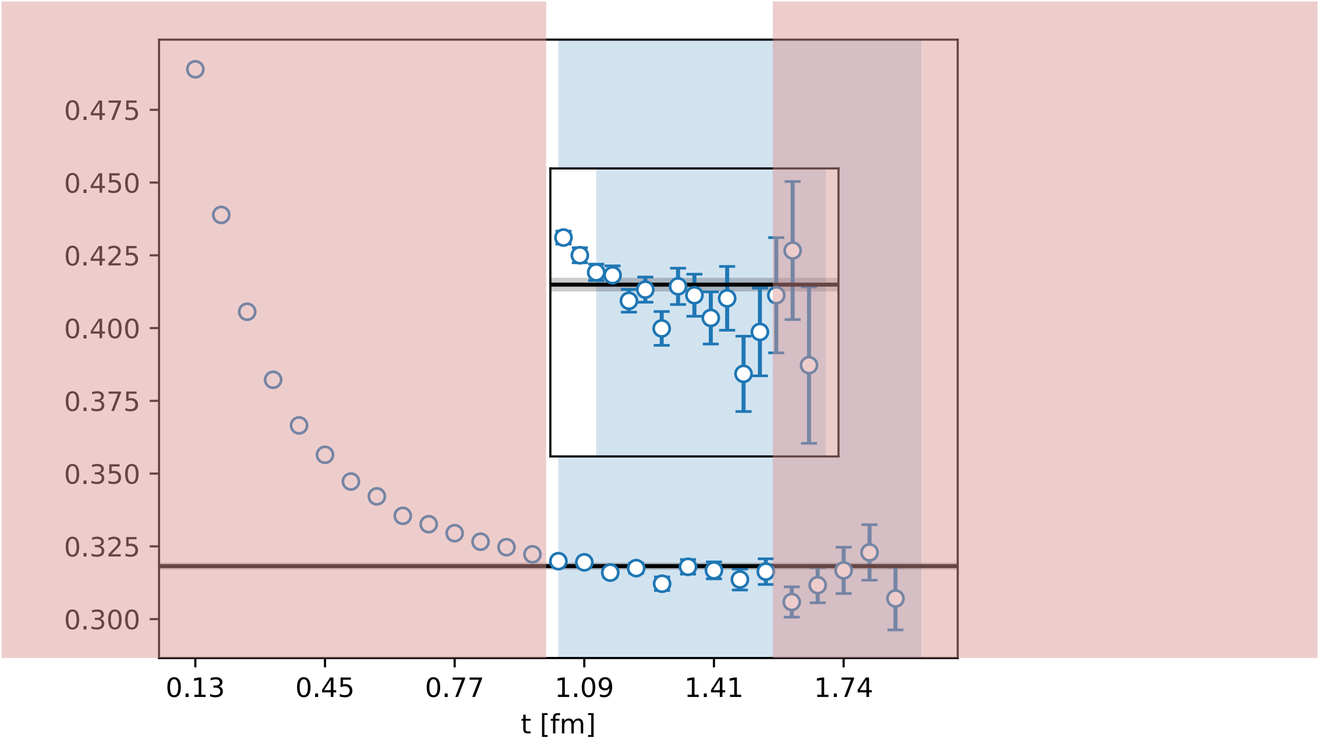
Noise wins at large times

UQ - Infamous S/N Nucleons



Effective mass

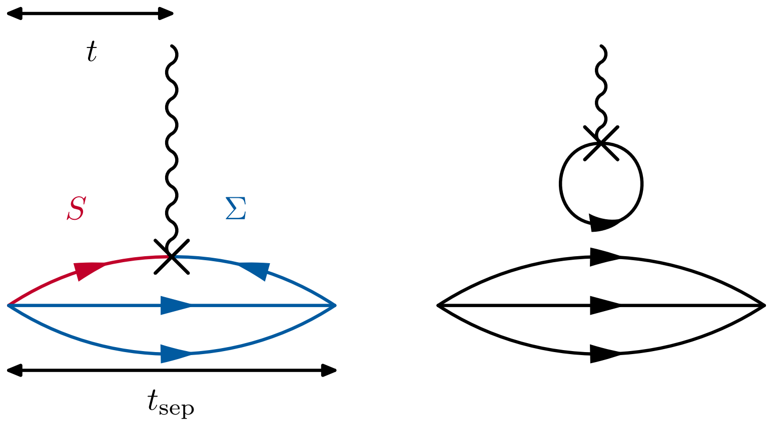
UQ - Infamous S/N Nucleons



At early times excited states contribute significantly

Noise wins at large times

3pt Functions Distances



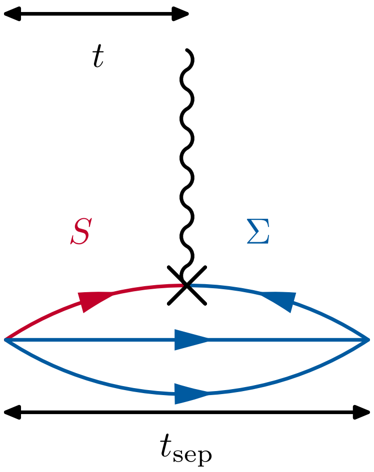
- In 3pt-functions we have two distances between
  - Source and current insertion time
  - Source and current insertion time (or source-sink separation)
- Very hard to make both large at the same time
- Excited-state problem is exacerbated
- Additional problem from Quark Disconnected Diagrams (notoriously hard to evaluate)

Direct Determination

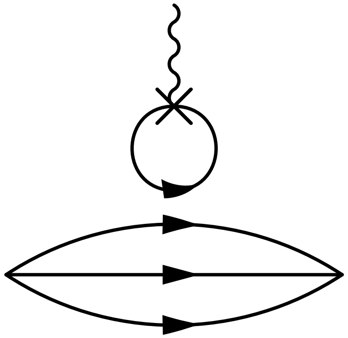
- Connected part
  - Sequential Source
  - Zero Momentum at sink

$$C_2(t;\mathbf{p}) = \Gamma_{\alpha\beta} \sum_{\mathbf{x}} e^{-i\mathbf{px}} \left\langle \Psi_\beta(\mathbf{x},t)\overline{\Psi}_\alpha(0)\right\rangle,$$

$$C_3(t,t_s;\mathbf{q}) = \Gamma'_{\alpha\beta} \sum_{\mathbf{x},\mathbf{y}} e^{i\mathbf{qy}} \left\langle \Psi_\beta(\mathbf{x},t_s)\mathcal{O}_{\mathrm{S}}(\mathbf{y},t)\overline{\Psi}_\alpha(0)\right\rangle,$$



- Disconnected part
  - Loops All-to-All: OET+HPE+HP
  - Still Noisy:
    → Additional two-point functions

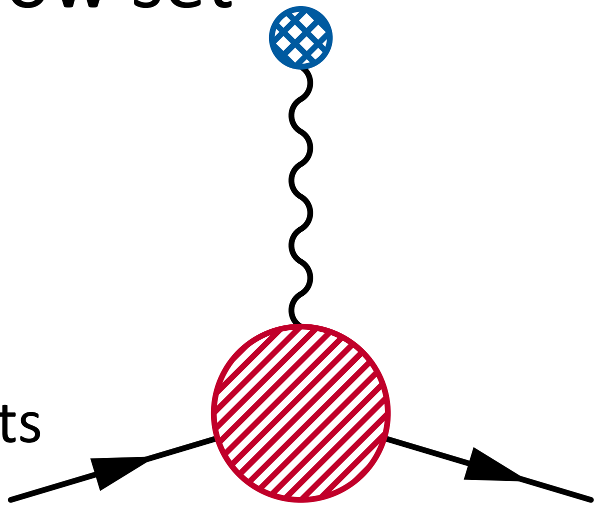


$$C_3^{\mathrm{disc}}(t,t_s;\mathbf{0}) = \left\langle L_S(\mathbf{0},z_0)\cdot C_2(\mathbf{p}',y_0,x;\Gamma')\right\rangle - \left\langle L_S(\mathbf{0},z_0)\right\rangle \cdot \left\langle C_2(\mathbf{p}',y_0,x;\Gamma')\right\rangle$$

Form Factors on the Lattice

- Computational Frame Work is now set
- One needs to
  - Plugin the desired current
  - Deal with the result
    - Extract ground-state matrix elements

$$R^s_{V_\mu}(z_0, \boldsymbol{q}; y_0, \boldsymbol{p}'; \Gamma_\nu) = \frac{C^s_{3,V_\mu}(\boldsymbol{q}, z_0; \boldsymbol{p}', y_0; \Gamma_\nu)}{C_2(\boldsymbol{p}', y_0)} \times \sqrt{\frac{C_2(\boldsymbol{p}', y_0) C_2(\boldsymbol{p}', z_0) C_2(\boldsymbol{p}'\text{-}\boldsymbol{q}, y_0\text{-}z_0)}{C_2(\boldsymbol{p}'\text{-}\boldsymbol{q}, y_0) C_2(\boldsymbol{p}'\text{-}\boldsymbol{q}, z_0) C_2(\boldsymbol{p}', y_0\text{-}z_0)}}.$$



    - Perform CCF extrapolation
    - Give best estimate of the error from the above
- Lets start with scalar current

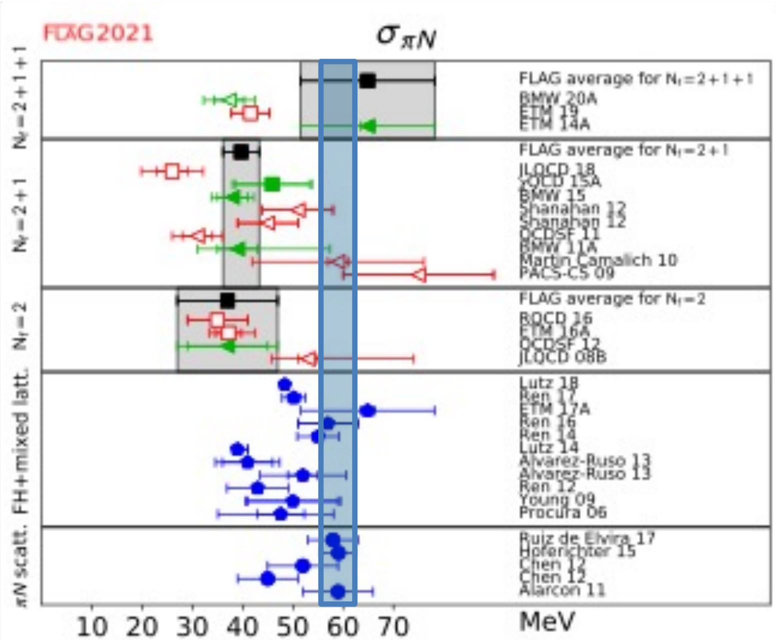
Sigma-Term

- Phenomenologically via Pion-Nucleon-Scattering (Chang-Dashen-Theorem + extrap.)
- Lattice calculation

$$\sigma_{\pi N} = m_l \langle N | \bar{u}u + \bar{d}d | N \rangle = m_l \frac{\partial m_N}{\partial m_l}$$

Directly or via Mass

- Some tension between Roy-Steiner based estimate and Lattice

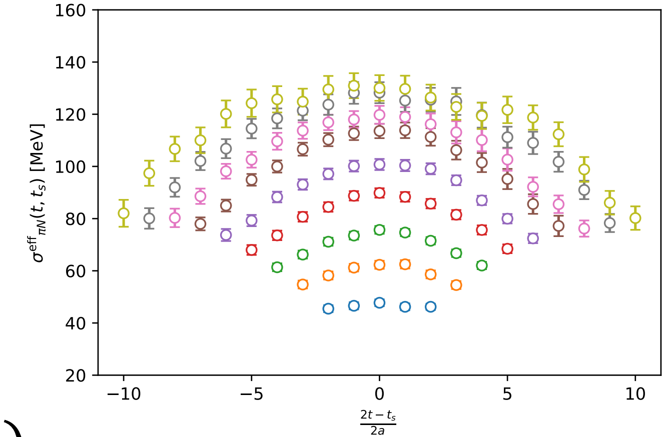


Excited States – Summation

- Usual Ratio (forward limit):

$$R(t,t_s) = \frac{C_3(t,t_s)}{C_2(t_s)} \qquad \mathrm{Re}\,R(t,t_s) \xrightarrow{t,(t_s-t)\gg 0} G_\mathrm{S}$$

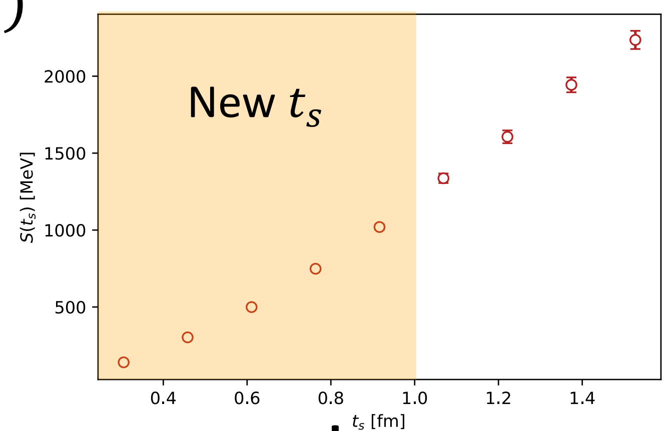
$$G_\mathrm{S}^\mathrm{eff}(t,t_s) = \mathrm{Re}\,R(t,t_s)$$



Excited states $\sim e^{-\Delta t}, e^{-\Delta(t_s-t)}$

- Summed correlator:

$$S(t_s) = \sum_{t=t_c}^{t_s-t_c} \sigma_{\pi N}^\mathrm{eff}(t,t_s)$$



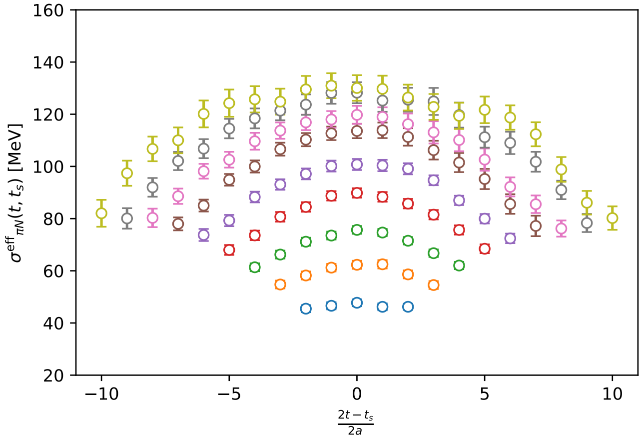
Excited states parametrically suppressed

Excited States – Summation

- Usual Ratio (forward limit):

$$R(t,t_s) = \frac{C_3(t,t_s)}{C_2(t_s)} \qquad \mathrm{Re}\,R(t,t_s) \xrightarrow{t,(t_s-t)\gg 0} G_\mathrm{S}$$

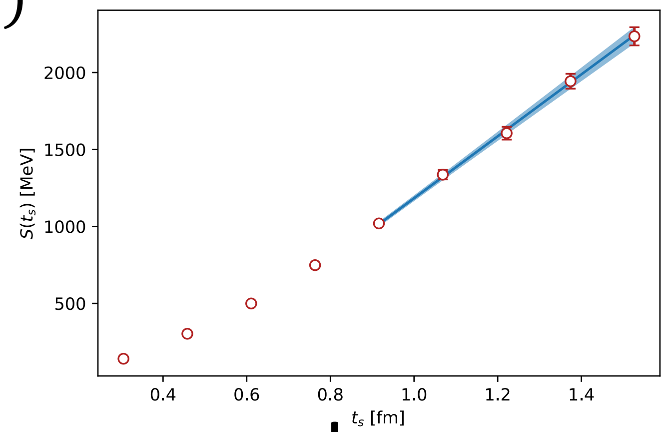
$$G_\mathrm{S}^\mathrm{eff}(t,t_s) = \mathrm{Re}\,R(t,t_s)$$



Excited states $\sim e^{-\Delta t}, e^{-\Delta(t_s - t)}$

- Summed correlator:

$$S(t_s) = \sum_{t=t_c}^{t_s-t_c} \sigma^\mathrm{eff}_{\pi N}(t,t_s)$$



Excited states parametrically suppressed

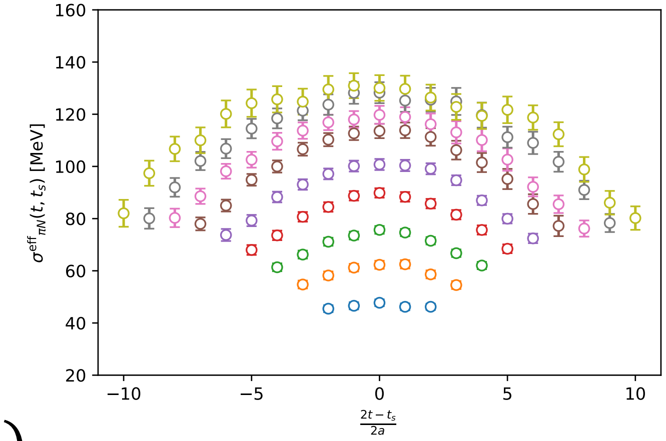
$$S(t_s) = (\sigma_{\pi N} \qquad\qquad) (1 + t_s - 2t_c) \qquad\qquad \cdot \ldots$$

Excited States – Summation

- Usual Ratio (forward limit):

$$R(t,t_s) = \frac{C_3(t,t_s)}{C_2(t_s)} \qquad \mathrm{Re}\,R(t,t_s) \xrightarrow{t,(t_s-t)\gg 0} G_\mathrm{S}$$

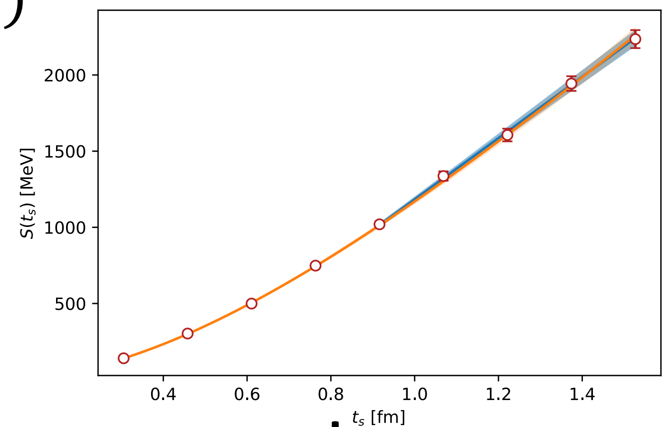
$$G_\mathrm{S}^\mathrm{eff}(t,t_s) = \mathrm{Re}\,R(t,t_s)$$



Excited states $\sim e^{-\Delta t}, e^{-\Delta(t_s-t)}$

- Summed correlator:

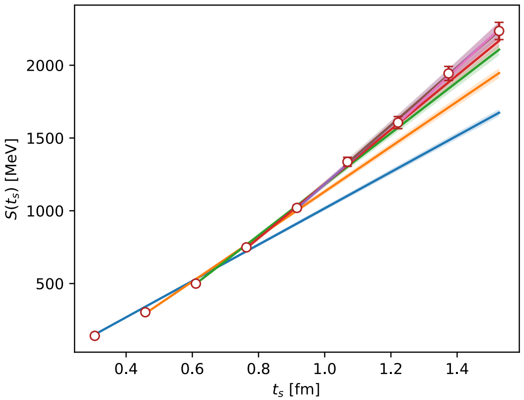
$$S(t_s) = \sum_{t=t_c}^{t_s-t_c} \sigma_{\pi N}^\mathrm{eff}(t,t_s)$$

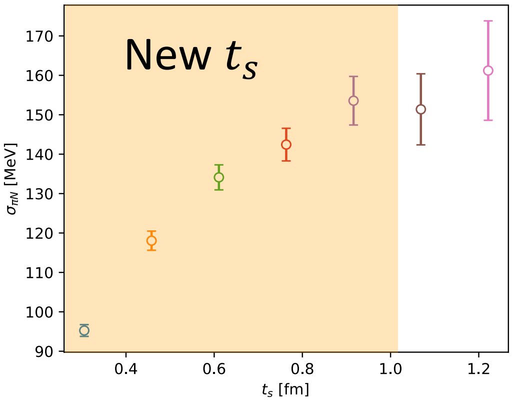


Excited states parametrically suppressed

$$S(t_s) = \left(\sigma_{\pi N} \boxed{+ m_{11} e^{-\Delta t_s}}\right)(1 + t_s - 2t_c) + e^{-\Delta t_s}\frac{2m_{10}\left(e^{\Delta(1-t_c+t_s)} - e^{\Delta t_c}\right)}{e^{\Delta}-1} + \ldots$$

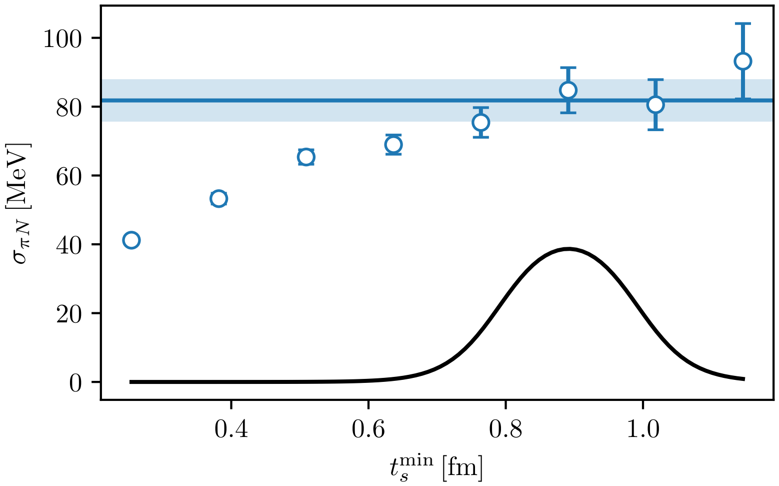
Excited States – Summation

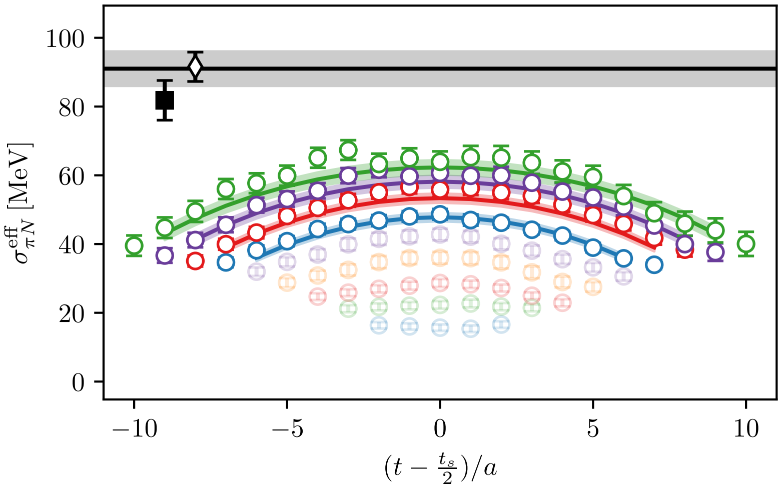




- Excited State Fits need priors for gap Δ (like explicit 2-state-Fit)
- Linear Fits:
  - Not trustworthy for small t_s
  - Error increases with larger starting t_s
  - Several possibilities
    - Choose one, use weights e.g. AIC, p-values, …
    - Define a window in physical units and average

Excited-State Contamination



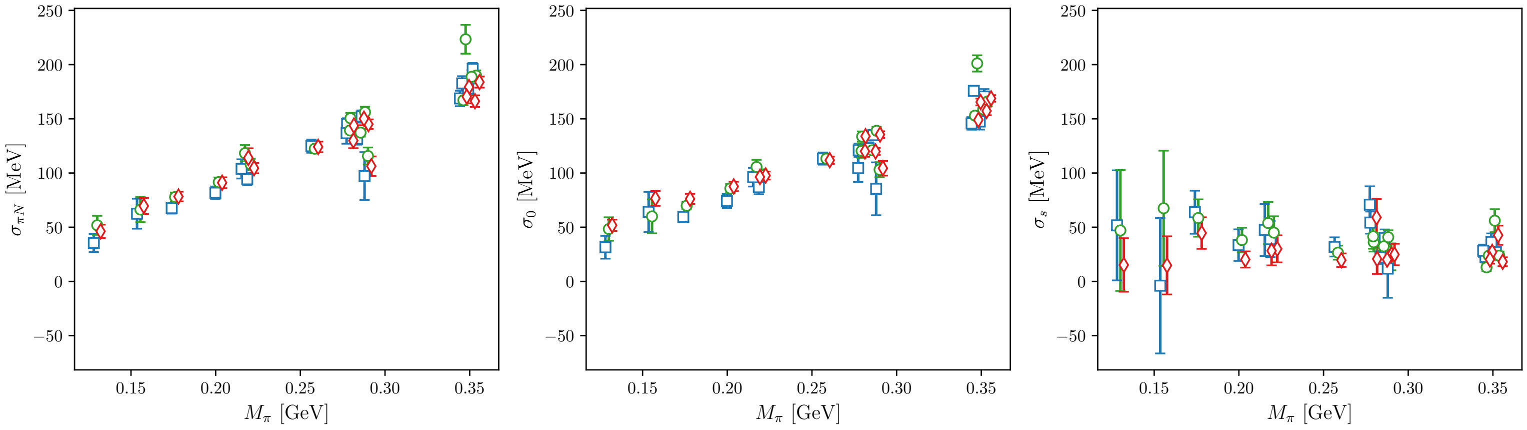


- Left: Blue data is linear fits to summation data at starting at t indicated on x-axis
  Black profile is window function

$$w_i = \frac{1}{2}\tanh\frac{t_s - t_{\text{lo}}}{\Delta t} - \frac{1}{2}\tanh\frac{t_s - t_{\text{up}}}{\Delta t}$$

  Blue band is weighted average of profile and data point
- Right: Effective FF data for different source-sink separations
  Black band is explicit two-state fit
- Have two separate ways for extracting the matrix element

Sigma Term



- Chiral expansion based on SU(3)-BChPT

$$m_N = m_0 - \underbrace{(2b_0 + 4b_f)}_{\hat{b}_0} M_\pi^2 - \underbrace{(4b_0 + 4b_d - 4b_f)}_{\hat{b}_1} M_K^2$$
$$+ \mathcal{F}_\pi I_{MB}(M_\pi) + \mathcal{F}_K I_{MB}(M_K) + \mathcal{F}_\eta I_{MB}(M_\eta),$$

$$\sigma_{\pi N} = \frac{M_\pi}{2}\frac{\partial m_N}{\partial M_\pi} + \frac{M_\pi^2}{4M_K}\frac{\partial m_N}{\partial M_K} + \frac{M_\pi^2}{6M_\eta}\frac{\partial m_N}{\partial M_\eta}$$
$$\sigma_s = \frac{2M_K^2 - M_\pi^2}{4M_K}\frac{\partial m_N}{\partial M_K} + \frac{2M_K^2 - M_\pi^2}{3M_\eta}\frac{\partial m_N}{\partial M_\eta},$$
$$\sigma_0 = \sigma_{\pi N} - \frac{2M_\pi^2}{2M_K^2 - M_\pi^2}\sigma_s.$$

$$\mathcal{F}_\pi = -\frac{3}{4}\left(D^2 + 2DF + F^2\right),$$
$$\mathcal{F}_K = -\left(\frac{5}{6}D^2 - DF + \frac{3}{2}F^2\right),$$
$$\mathcal{F}_\eta = -\frac{1}{2}\left(\frac{1}{6}D^2 - DF + \frac{3}{2}F^2\right),$$
$$I_{MB}(M) = \frac{M^3}{8F_\phi^2 m_0 \pi^2}\left(M \log\frac{M}{m_0} + \sqrt{4 - \frac{M^2}{m_0^2}}\, m_0 \arccos\left(\frac{M}{2m_0}\right)\right).$$

B. C. Lehnhart, J. Gegelia, and S. Scherer, J. Phys. G 31, 89 (2005)

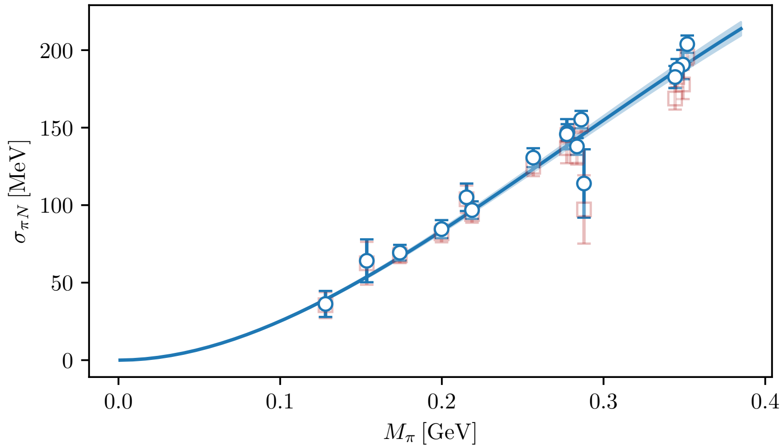
CCF

- Add terms for continuum and finite volume

$$\sigma_{\pi N/s} \to \sigma_{\pi N/s} + b_i \, \frac{a}{\sqrt{t_0}} \, M^2_{\pi/K}.$$

$$\sigma_{\pi N} \to \sigma_{\pi N} + b_L \Big(\frac{M_\pi^3}{M_\pi L} - \frac{M_\pi^3}{2} \Big) \exp\Big(-M_\pi L\Big).$$

- Example of a fit



Variation	$\sigma_{\pi N}$ [MeV]	σ_0 [MeV]	σ_s [MeV]	χ^2(dof)	weight in %
$M_\pi < 220$ MeV	42.04(1.27)	38.70(1.35)	43.18(9.20)	4.0(10)	1
$M_\pi < 285$ MeV	41.89(67)	38.98(69)	37.56(4.74)	20.5(18)	0
no cut in M_π	41.67(44)	38.91(41)	35.62(3.09)	42.9(30)	1
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$	41.58(6.58)	37.23(6.28)	56.36(24.19)	3.5(8)	0
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$	39.31(3.15)	37.05(3.06)	29.24(12.55)	19.6(16)	0
no cut in M_π+ $\mathcal{O}(a)$	37.55(1.82)	34.87(1.80)	34.68(6.69)	37.5(28)	2
$M_\pi < 220$ MeV+ $\mathcal{O}(e^{-mL})$	42.45(1.33)	39.10(1.40)	43.26(9.20)	3.8(9)	0
$M_\pi < 285$ MeV+ $\mathcal{O}(e^{-mL})$	42.43(79)	39.53(81)	37.52(4.74)	19.9(17)	0
no cut in M_π+ $\mathcal{O}(e^{-mL})$	42.87(59)	40.11(57)	35.78(3.09)	34.4(29)	26
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	42.69(6.68)	36.67(6.47)	77.88(45.65)	3.2(7)	0
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	39.38(3.35)	39.43(3.30)	-0.62(22.83)	16.7(15)	0
no cut in M_π+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	39.34(2.08)	37.61(2.04)	22.39(13.53)	31.1(27)	19
$M_\pi < 220$ MeV	46.81(1.14)	44.88(1.16)	24.92(5.61)	6.9(10)	27
$M_\pi < 285$ MeV	43.71(62)	42.02(63)	21.87(3.42)	27.8(18)	2
no cut in M_π	41.04(39)	39.32(39)	22.23(2.32)	92.3(30)	0
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$	51.38(5.87)	49.17(5.80)	28.65(16.12)	6.3(8)	5
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$	45.77(2.73)	44.14(2.73)	21.17(8.71)	27.2(16)	0
no cut in M_π+ $\mathcal{O}(a)$	40.38(1.65)	39.02(1.64)	17.62(4.73)	90.9(28)	0
$M_\pi < 220$ MeV+ $\mathcal{O}(e^{-mL})$	47.21(1.20)	45.28(1.22)	24.95(5.61)	6.8(9)	10
$M_\pi < 285$ MeV+ $\mathcal{O}(e^{-mL})$	44.44(76)	42.75(77)	21.79(3.42)	25.9(17)	2
no cut in M_π+ $\mathcal{O}(e^{-mL})$	42.79(56)	41.08(56)	22.15(2.32)	73.4(29)	0
$M_\pi < 220$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	52.26(5.93)	49.09(6.00)	41.03(32.57)	6.0(7)	2
$M_\pi < 285$ MeV+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	47.13(2.90)	46.07(2.99)	13.78(19.15)	24.6(15)	1
no cut in M_π+ $\mathcal{O}(a)$+ $\mathcal{O}(e^{-mL})$	43.83(1.87)	42.81(1.87)	13.24(10.25)	71.9(27)	0

UQ - Systematics and Errors

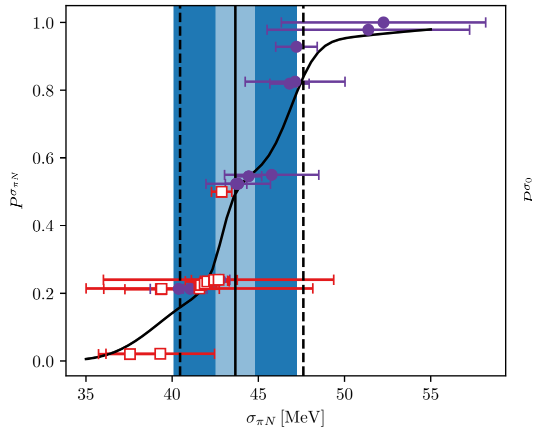
- With the two datasets perfrom all extrapolations
- Variations: Excited States, $M_\pi^2, \mathcal{O}(a^2), \mathcal{O}(e^{-M_\pi L})$
- No a priori best extrapolation
- Treat variations as Models
- Perform averages based on AIC weights

$$w_i^{\text{AIC}} = \frac{e^{-\frac{1}{2}\text{AIC}_i}}{\sum_j e^{-\frac{1}{2}\text{AIC}_j}},$$

- Treat model estimates as random variable with CDF

$$P^x(y) = \int_{-\infty}^{y} \sum_i^n w_i \mathcal{N}(y'; x_i, \sigma_i^2) dy'$$

Strategy from S. Borsanyi et al. (2020), *Nature* **593,** 51–55 (2021)



Akaike Information Criterion

- IC based on Kullback-Leibler Divergence

$$\mathrm{AIC} = -2\ln\hat{L} + 2k$$

- For least-square-fitting

$$\mathrm{AIC} = \chi^2(\hat{a}) + 2k$$

Fit parameters

- Including data selection

$$\mathrm{AIC} = \chi^2(\hat{a}) + 2k + 2d_c$$

Cut datapoints

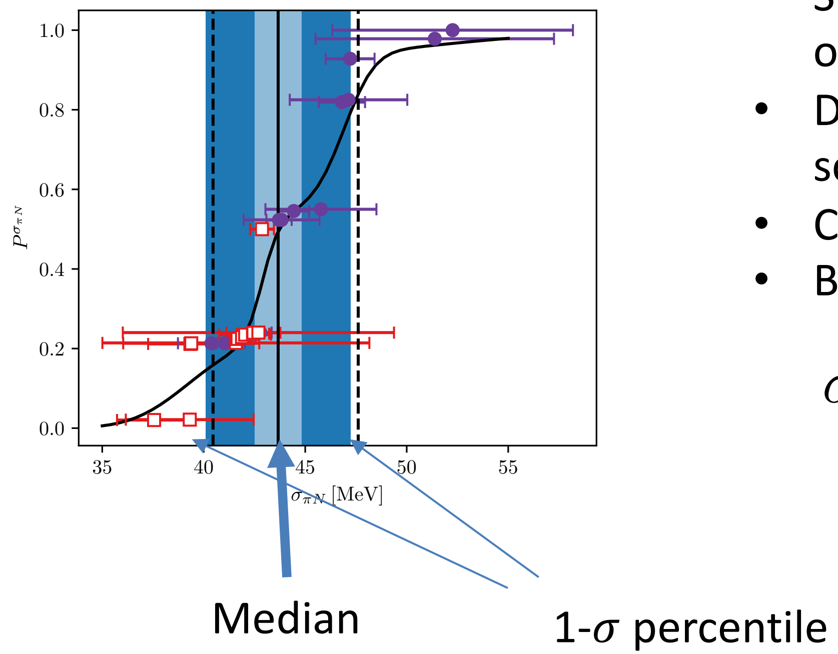
- Weights are higher for
  – Better fits
  – Less fit parameters
  – More Data used

$$w_i^{\mathrm{AIC}} = \frac{e^{-\frac{1}{2}\mathrm{AIC}_i}}{\sum_j e^{-\frac{1}{2}\mathrm{AIC}_j}},$$

Sigma Term Error Estimate

- Use the weights to define CDF as

$$P^x(y) = \int_{-\infty}^{y} \sum_{i}^{n} w_i \mathcal{N}(y'; x_i, \sigma_i^2) dy'$$

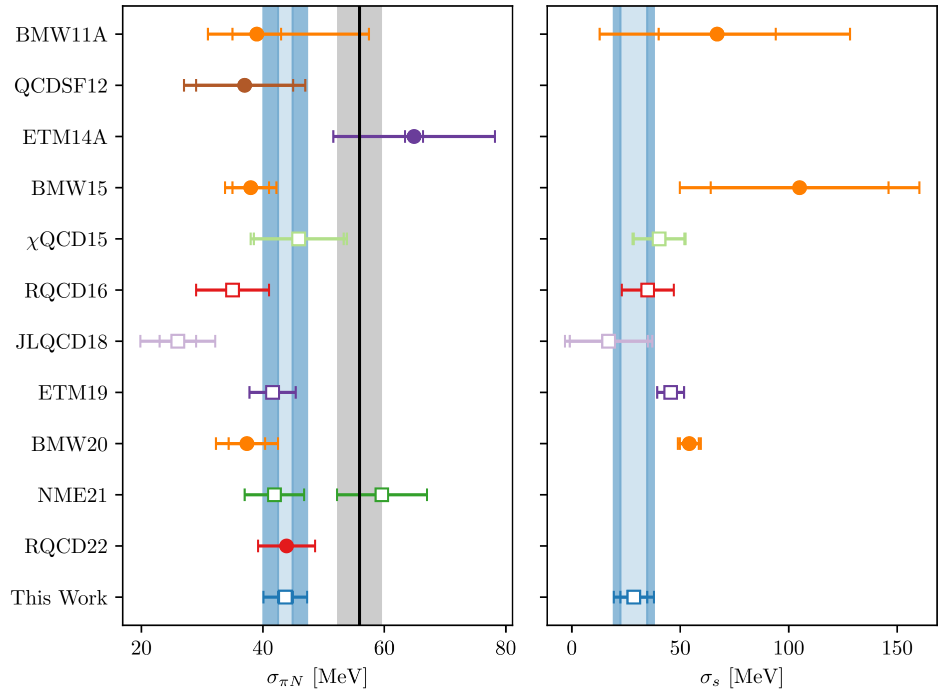


- Quoted value is the median of CDF
- Shaded area is symmetrized 1-σ interval of CDF
- Disentangle syst. from stat. error by scaling error in $\mathcal{N}(y'; x_i, \sigma_i^2) dy'$
- CDF smoothens rugged dsitribution
- Bulk of variations included in errorband

$$\sigma_{\pi N} = 43.7(1.2)(3.4)\,\mathrm{MeV}$$
$$\sigma_0 = 41.3(1.2)(3.4)\,\mathrm{MeV}$$
$$\sigma_s = 28.6(6.2)(7.0)\,\mathrm{MeV},$$

UQ - Comparison Sigma Term

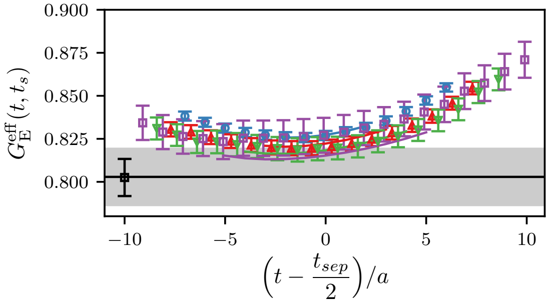
- Sigma Term is dominated by stystematic error
- Excited-state largest source of uncertainty

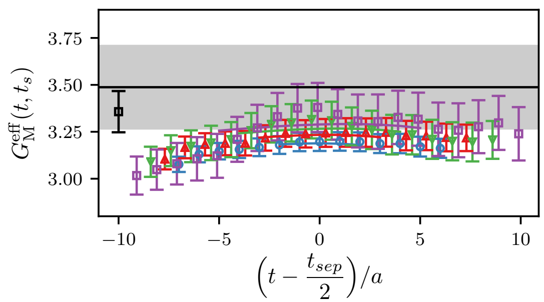


EM FF of the Proton

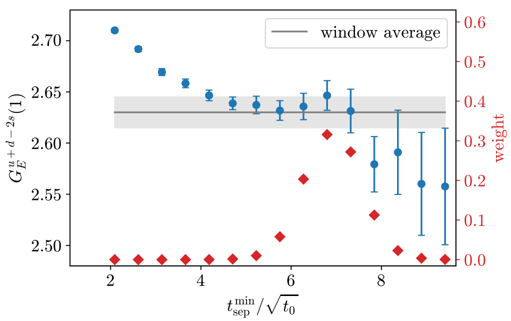
- Current is now γ_μ
- Effective FF

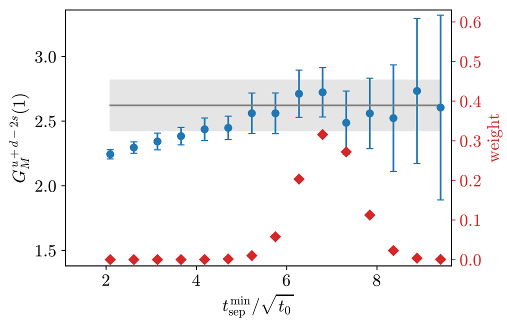
$$R^s_{V_\mu}(z_0,\boldsymbol{q};y_0,\boldsymbol{p}';\Gamma_\nu)=\frac{C^s_{3,V_\mu}(\boldsymbol{q},z_0;\boldsymbol{p}',y_0;\Gamma_\nu)}{C_2(\boldsymbol{p}',y_0)}\times\sqrt{\frac{C_2(\boldsymbol{p}',y_0)C_2(\boldsymbol{p}',z_0)C_2(\boldsymbol{p}'\text{-}\boldsymbol{q},y_0\text{-}z_0)}{C_2(\boldsymbol{p}'\text{-}\boldsymbol{q},y_0)C_2(\boldsymbol{p}'\text{-}\boldsymbol{q},z_0)C_2(\boldsymbol{p}',y_0\text{-}z_0)}}.$$





- Again use window-average of summed correlator

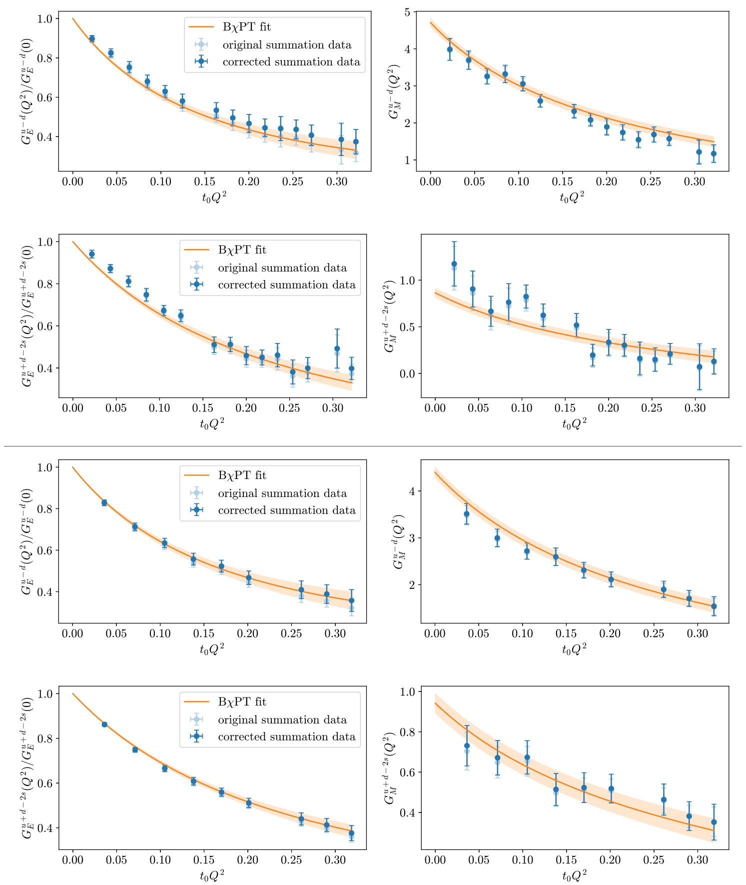




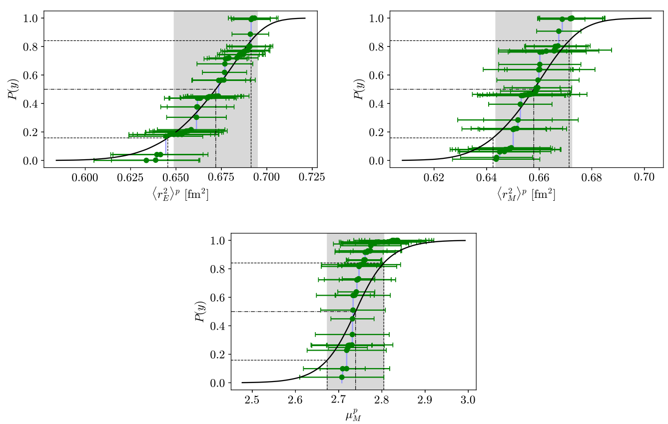
$$w_i=\tanh\frac{t_i-t_w^{\text{low}}}{\Delta t_w}-\tanh\frac{t_i-t_w^{\text{up}}}{\Delta t_w}.$$

FF of the Nucleon

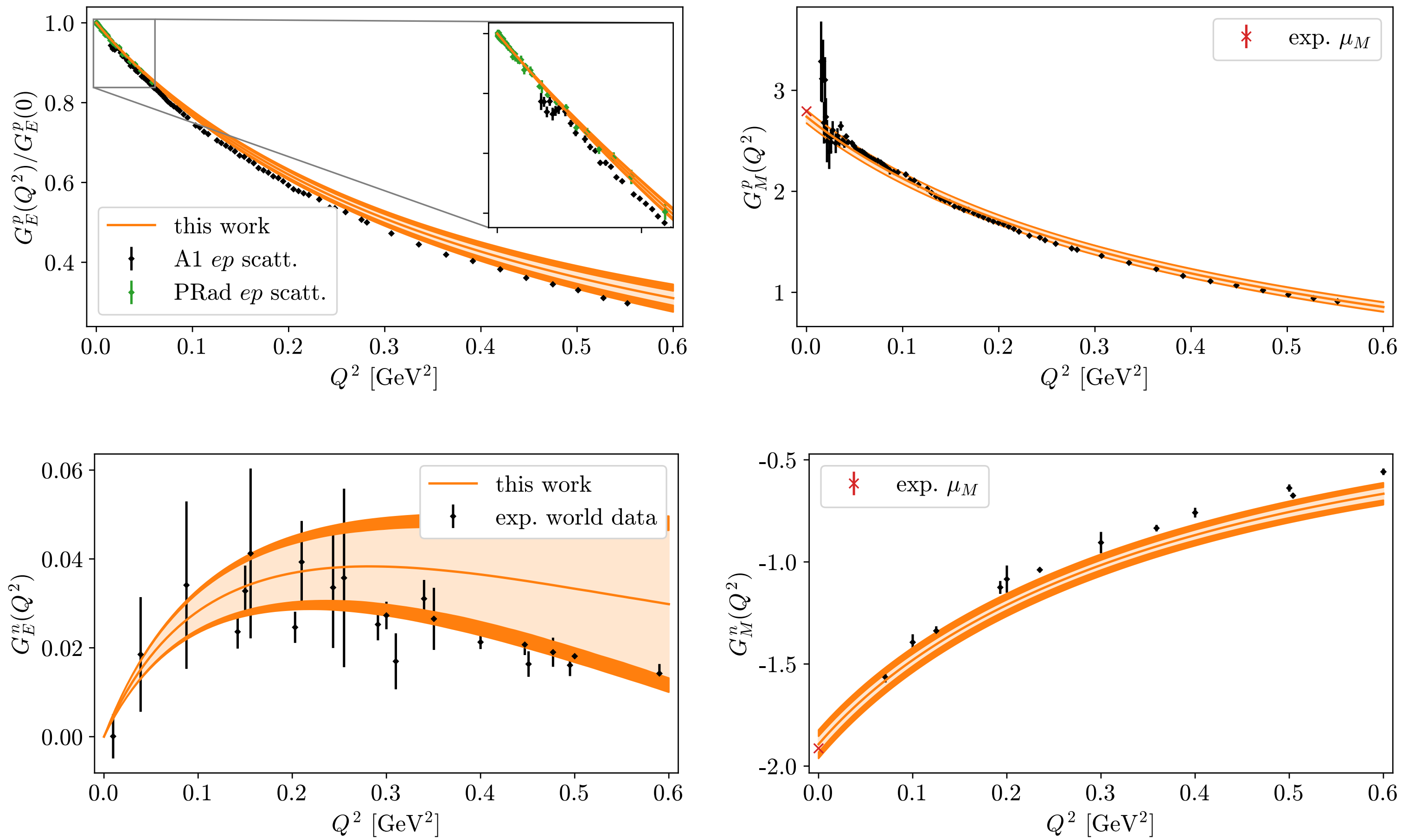
- Simultaneous fit to all ensembles using BChPT ammended with terms for continuum and finite volume T. Bauer, J. C. Bernauer, and S. Scherer, Phys. Rev. C86, 065206 (2012),
- Perform variations of the fits, e.g. cuts in momentum transfer, pion mass, etc.
- Calculate radii and magnetic moment



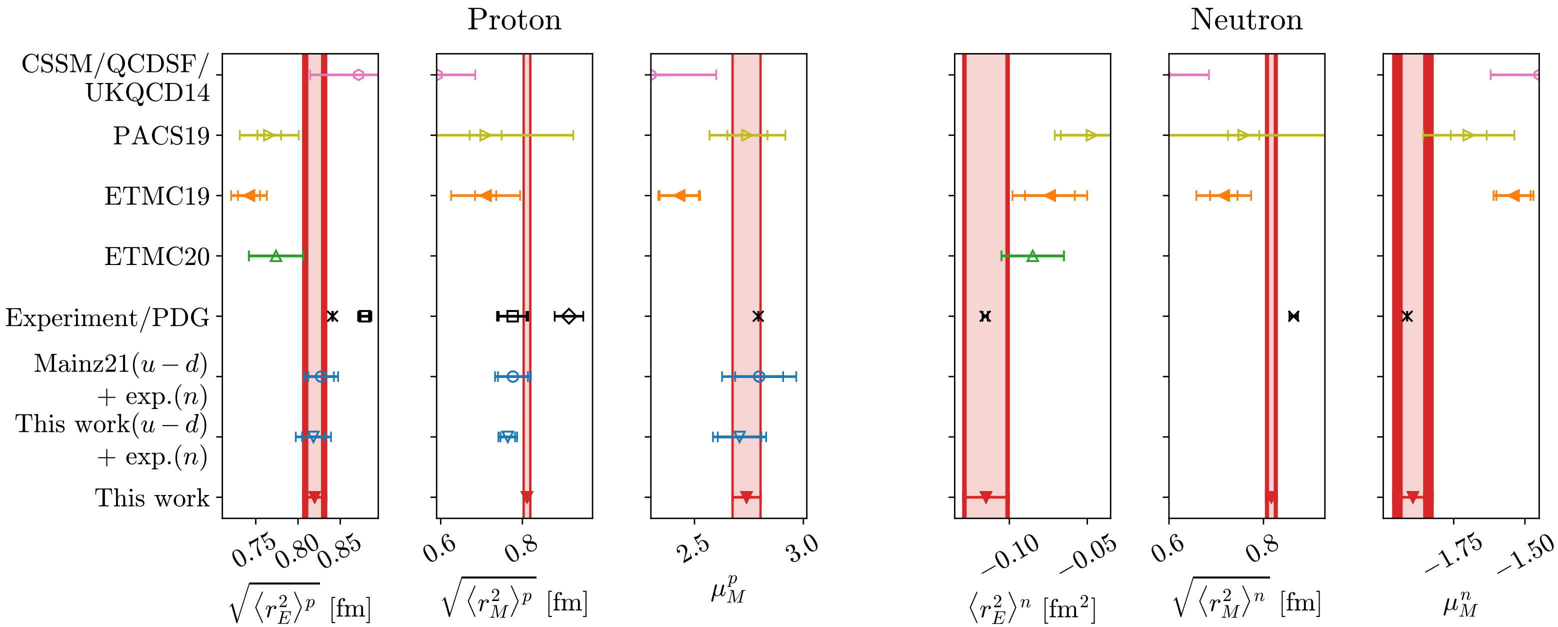
$$\langle r^2 \rangle = -\frac{6}{G(0)} \frac{\partial G(Q^2)}{\partial Q^2}\Big|_{Q^2=0} .$$



Model Average FF



UQ - Comparison to others FF



- Electric radius and magnetic moment consistent with experiment
- Some tension for magnetic radius
- Magnetic properties accessible in spectroscopy measurement of HFS

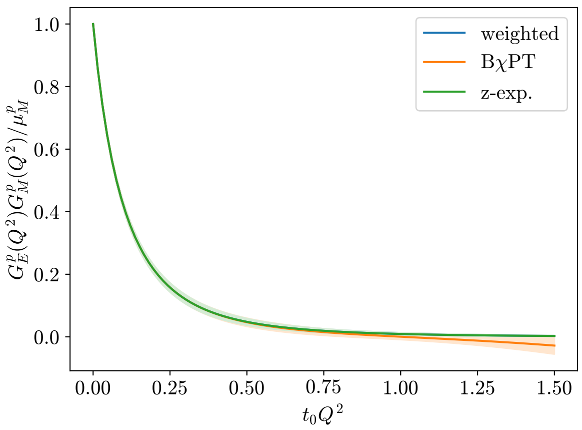
$$
\begin{aligned}
r_Z^p &= -\frac{4}{\pi}\int_0^\infty \frac{dQ}{Q^2}\left[\frac{G_E^p(Q^2)G_M^p(Q^2)}{\mu_M^p} - \frac{G_E^p(0)G_M^p(0)}{\mu_M^p}\right] \\
&= -\frac{2}{\pi}\int_0^\infty \frac{dQ^2}{(Q^2)^{3/2}}\left[\frac{G_E^p(Q^2)G_M^p(Q^2)}{\mu_M^p} - 1\right].
\end{aligned}
$$

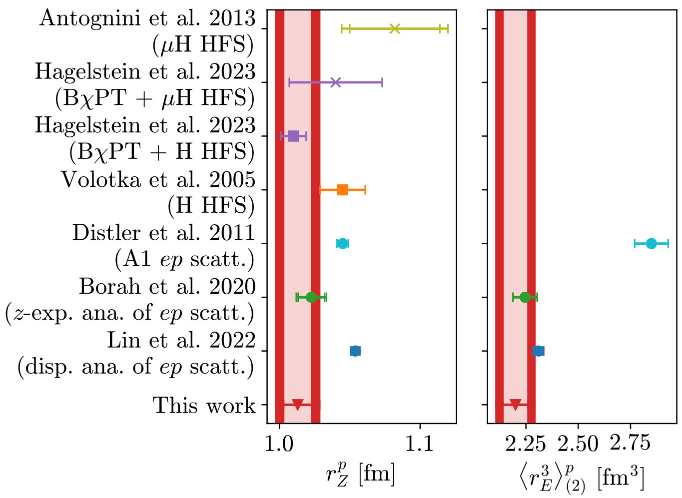
Zemachradius

Zemach

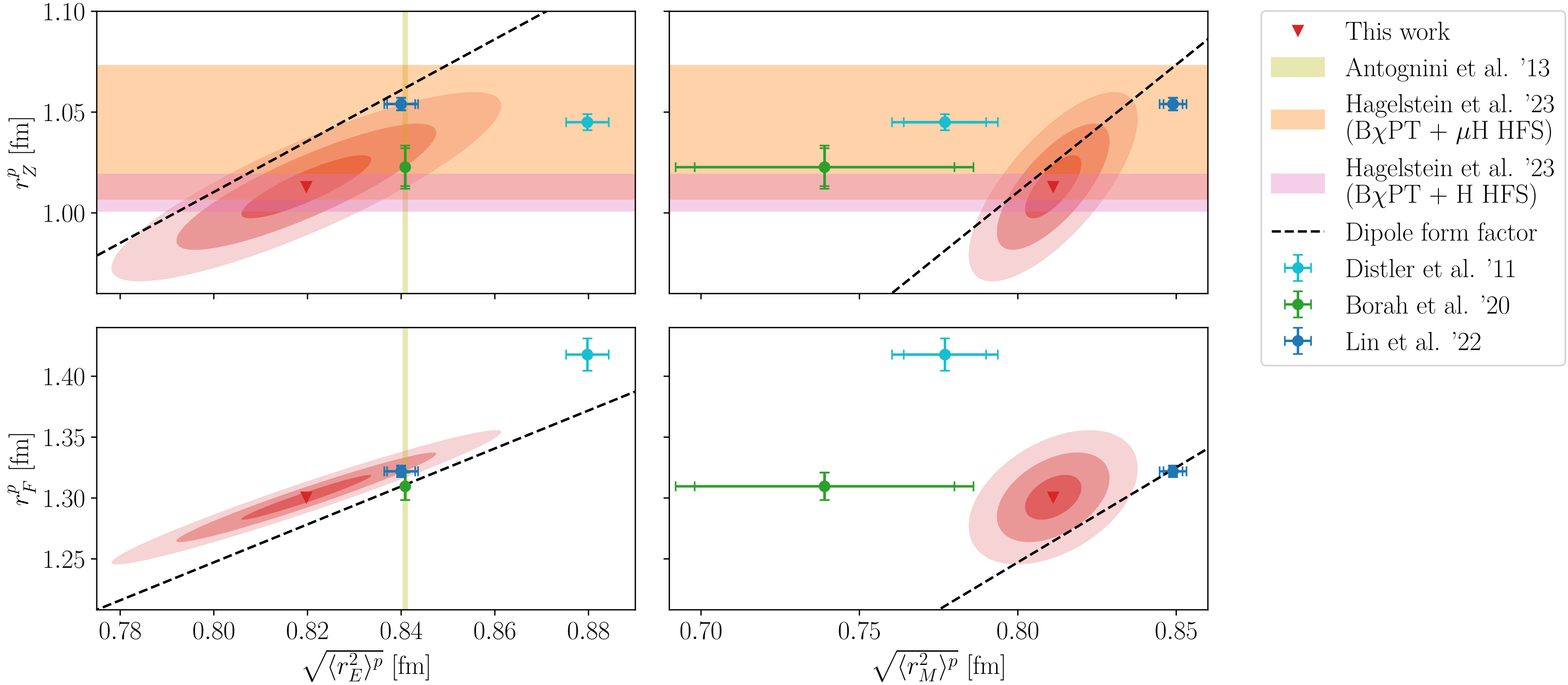
- Integral needs full range in momentum
- Smoothly interpolate between BChPT and z-expansio

$$F(Q^2) = \frac{1}{2}\left[1 - \tanh\left(\frac{Q^2 - Q^2_{\text{cut}}}{\Delta Q^2_w}\right)\right] F^\chi(Q^2) + \frac{1}{2}\left[1 + \tanh\left(\frac{Q^2 - Q^2_{\text{cut}}}{\Delta Q^2_w}\right)\right] F^z(Q^2),$$





Zemach and Friarradius



$$\langle r_E^3\rangle_{(2)}^p = \frac{48}{\pi}\int_0^\infty \frac{dQ}{Q^4}\left[(G_E^p(Q^2))^2 - (G_E^p(0))^2 - \frac{\partial (G_E^p(Q^2))^2}{\partial Q^2}\bigg|_{Q^2=0} Q^2\right]$$

$$= \frac{24}{\pi}\int_0^\infty \frac{dQ^2}{(Q^2)^{5/2}}\left[(G_E^p(Q^2))^2 - 1 + \frac{1}{3}\langle r_E^2\rangle^p Q^2\right].$$

$$r_F^p = \sqrt[3]{\langle r_E^3\rangle_{(2)}^p}.$$

Thank you!

My UQ count is 6