Model-independent extrapolation of MUonE data with D-Log approximants

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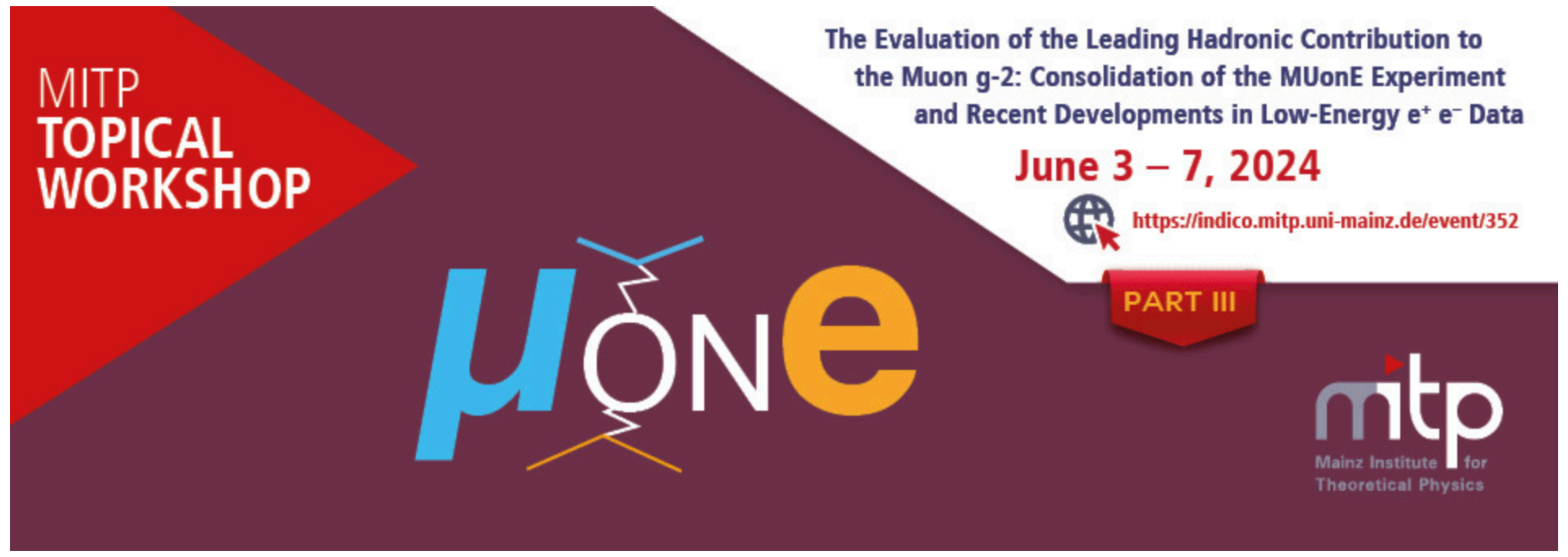
Based on arix 2405.13638

Work in collaboration with Diogo Boito, Cristiane Y. London, Camilo Rojas



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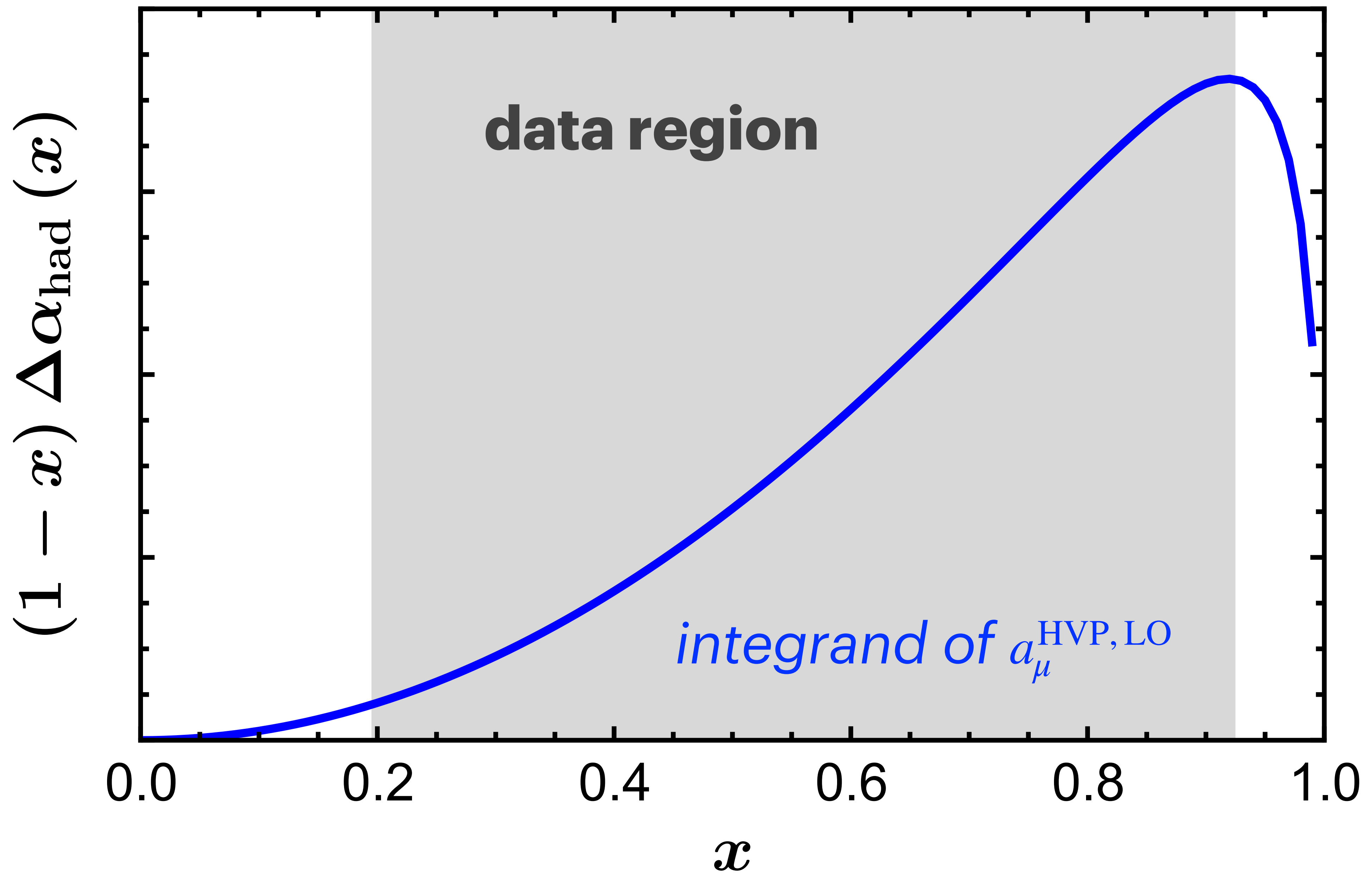
Motivation



Problem

Finding a reliable method to fit the data + good extrapolation outside data region without using external information

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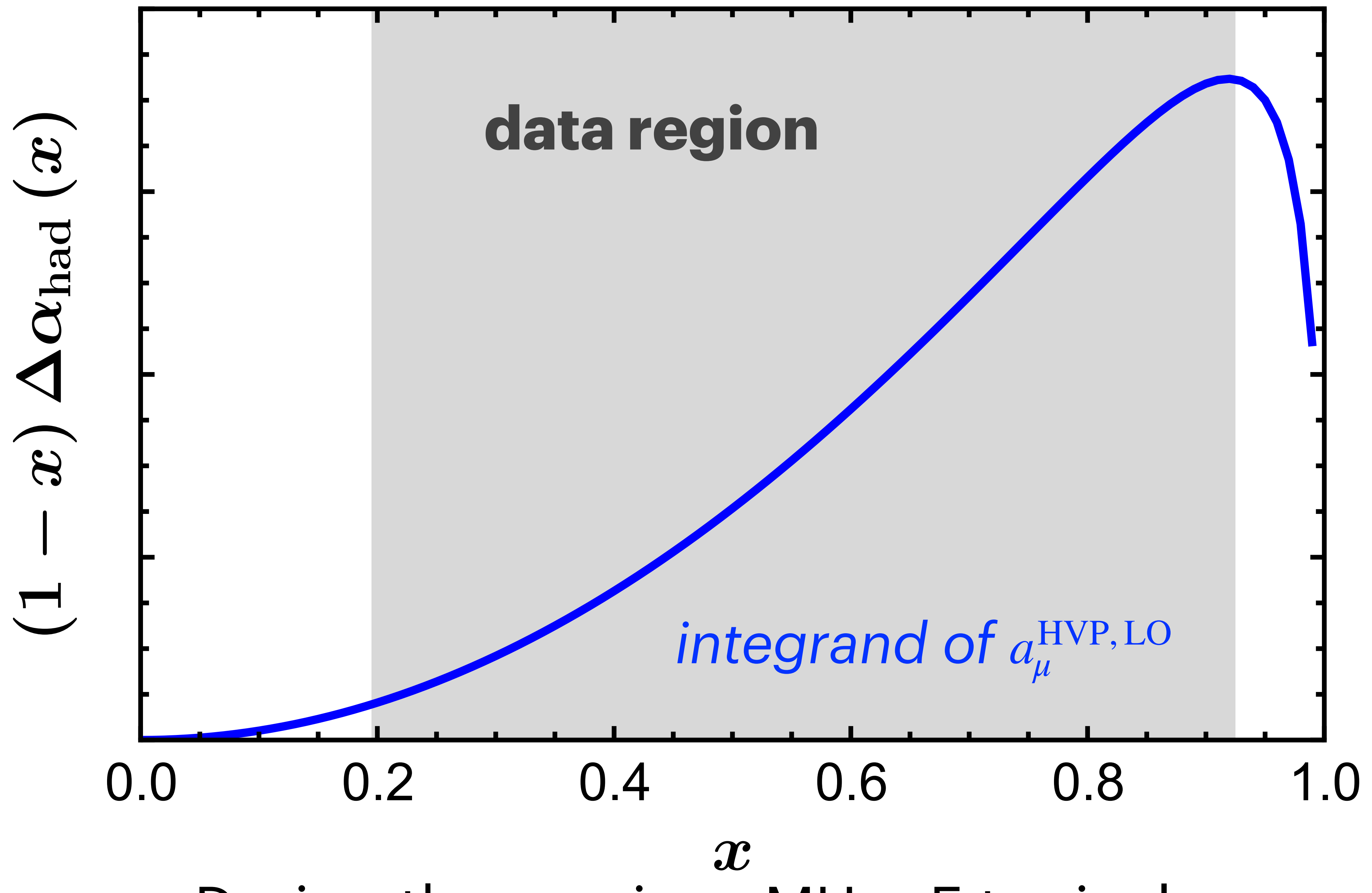


During the previous MUonE topical workshop at Mainz, we presented the Padé approximants' method:

Padé Approximants

$$P_N^M(z) = \frac{Q_M(z)}{R_N(z)} = \frac{q_0 + q_1\, z + \cdots + q_M\, z^M}{1 + r_1\, z + \cdots + r_N\, z^N}$$

Problem

Finding a reliable method to fit the data + good extrapolation outside data region without using external information

Advantages of PAs

- Systematic and model-independent method
- Partial reconstruction of analytic (physical) properties
- Efficient approximation
- It is possible to estimate the **convergence** (systematic) error

$$P_1^1(t) \leq P_2^2(t) \leq \ldots \leq \Delta\alpha_{\mathrm{had}} \leq \ldots \leq P_2^3(t) \leq P_1^2(t)$$

Motivation

Further motivation

Beyond fitting and extrapolation method, can we answer
"Is MUonE sensitive to characteristics of the $\Delta\alpha(t)$ function?"

($\pi^+\pi^-$ or $\pi^0\gamma$ production thresholds, ρ-meson?)

(Long-distance window or beyond?)

D-Log Padé approximants

- Again: Systematic and model-independent method
- Again: Partial reconstruction of analytic (physical) properties
- Again: Efficient approximation
- It is possible to estimate the **convergence** (systematic) error

Outline

- D-Log Padé approximants for MUonE
- The role of x_{max}
- Conclusions

Padé Theory

Stieltjes function

$$f(z) = \int_0^\infty \frac{\mathrm{d}\phi(u)}{1 + zu} \qquad \phi(u) \text{ is a measure in } u \in [0, \infty)$$

- $\Delta\alpha_{\text{had}}(t)$ is a Stieltjes function! [Masjuan, Peris '09], [Aubin, Blum, Golterman, Peris '12]
- There are convergence theorems for PAs to Stieltjes functions [Baker '96]
- Some convergence properties (thanks to analyticity and unitarity):
  - ✱ poles of P_N^{N+k}, $k \geq -1$, are located in the positive real axis;
  - ✱ PA sequences uniformly converge to the original function;
  - ✱ PA sequences act as bounds to the function

$$P_1^1(t) \leq P_2^2(t) \leq \cdots \leq \Delta\alpha_{\text{had}}(t) \leq \cdots \leq P_1^2(t) \leq P_0^1(t)$$

D-Log Padé Approximants

1) Imagine a function $f(z)$:

$$f(z) = A(z)\,\frac{1}{(\mu - z)^{\gamma}} + B(z),$$

2) Perform its logarithmic derivative:

$$F(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln f(z) \approx \frac{\gamma}{(\mu - z)}$$

3) Approach $F(z)$ with Padé sequences and then unfold:

- Integrate, exponentiate, and normalize:

$$D^N_M(z) = f(0) \exp\left[\int \mathrm{d}z\, \bar{P}^N_M(z)\right]$$

$D^N_M(t)$	
D^1_2	$\frac{-f_0\, t}{(r_1 - t)^{\gamma_1}}$
D^2_2	$\frac{-f_0\, t\, e^{\beta t}}{(r_1 - t)^{\gamma_1}}$
D^2_3	$\frac{-f_0\, t}{(r_1 - t)^{\gamma_1} (r_2 - t)^{\gamma_2}}$
D^3_3	$\frac{-f_0\, t\, e^{\beta t}}{(r_1 - t)^{\gamma_1} (r_2 - t)^{\gamma_2}}$

✦ Little is know about D-Logs, they work and we have a convergence conjecture. We find:

$$D^1_1(t) \leq D^2_2(t) \leq \ldots \leq \Delta\alpha_{\text{had}} \leq \ldots \leq D^2_3(t) \leq D^1_2(t)$$

Examples of PAs and DLogs

- After imposing some knowledge about $f(q^2)$ you want to approach:

Padé approximants

- VMD and simplest BW are a sort of PA:

$$F(q^2) = \sum_{v=0}^{N} c_v \frac{m_v^2}{m_v^2 - q^2} \quad \text{with} \quad \sum_{v=0}^{N} c_v = 1$$

$$\mathrm{BW_V}(q^2) = \frac{m_v^2}{m_v^2 - q^2 - i m_v \Gamma_v} \qquad v = \omega, \phi$$

DLog Padé approximants

- GS and Omnès are a sort of DLog:

$$\mathrm{BW}_v^{\mathrm{GS}}(q^2) = \frac{m_v^2}{m_v^2 - q^2 + f(q^2, m, \Gamma) - i m_v \Gamma_v(q^2, m, \Gamma)} \qquad v = \rho, \rho' \ldots$$

$$F(s) = P(s)\Omega(s)$$

$$\Omega(s) = \exp\left\{ \frac{s}{\pi} \int_{m_\pi^2}^{\infty} \mathrm{d}s' \frac{\delta(s')}{s'(s'-s)} \right\}$$

$z(s)$ conformal mappings

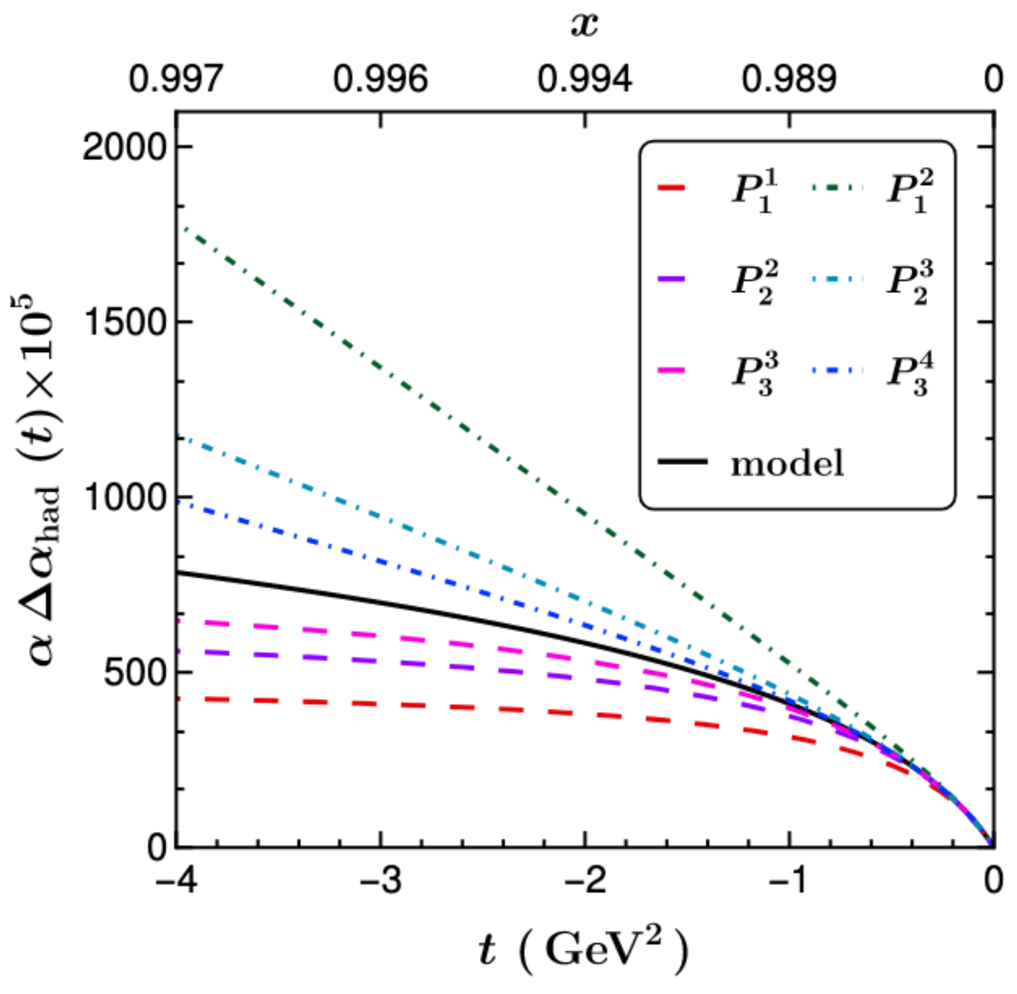
✦ However, the method requires sequences of approximants:

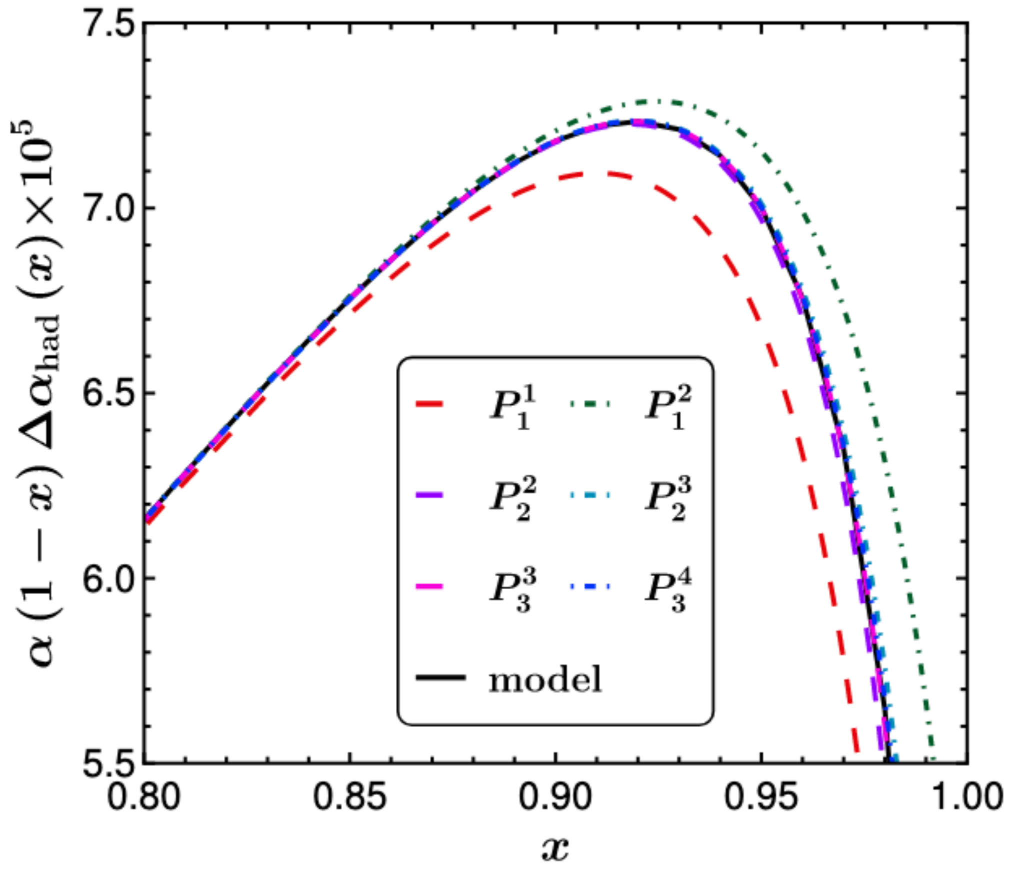
$$P_1^1(t) \leq P_2^2(t) \leq \ldots \leq \Delta\alpha_{\text{had}} \leq \ldots \leq P_2^3(t) \leq P_1^2(t)$$

$$D_1^1(t) \leq D_2^2(t) \leq \ldots \leq \Delta\alpha_{\text{had}} \leq \ldots \leq D_3^2(t) \leq D_2^1(t)$$

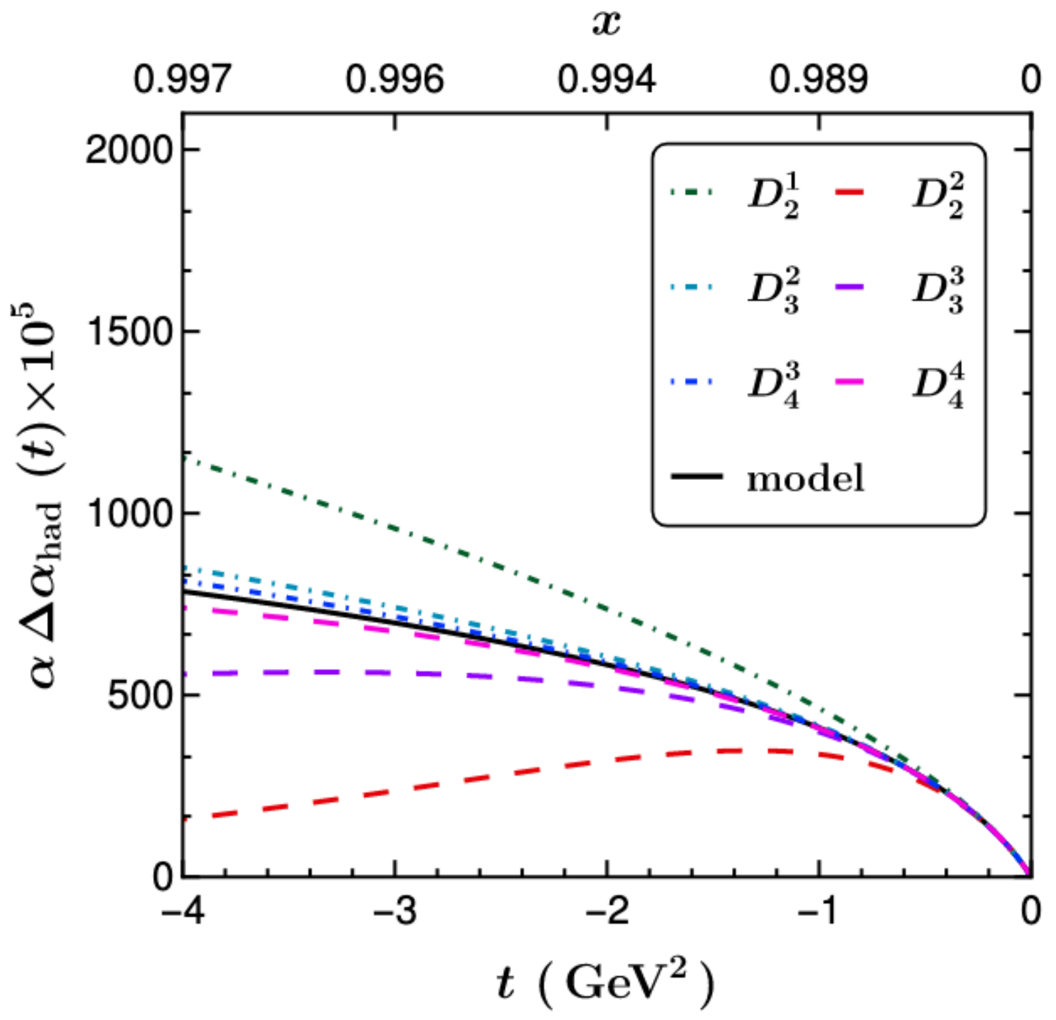
From Taylor expansion

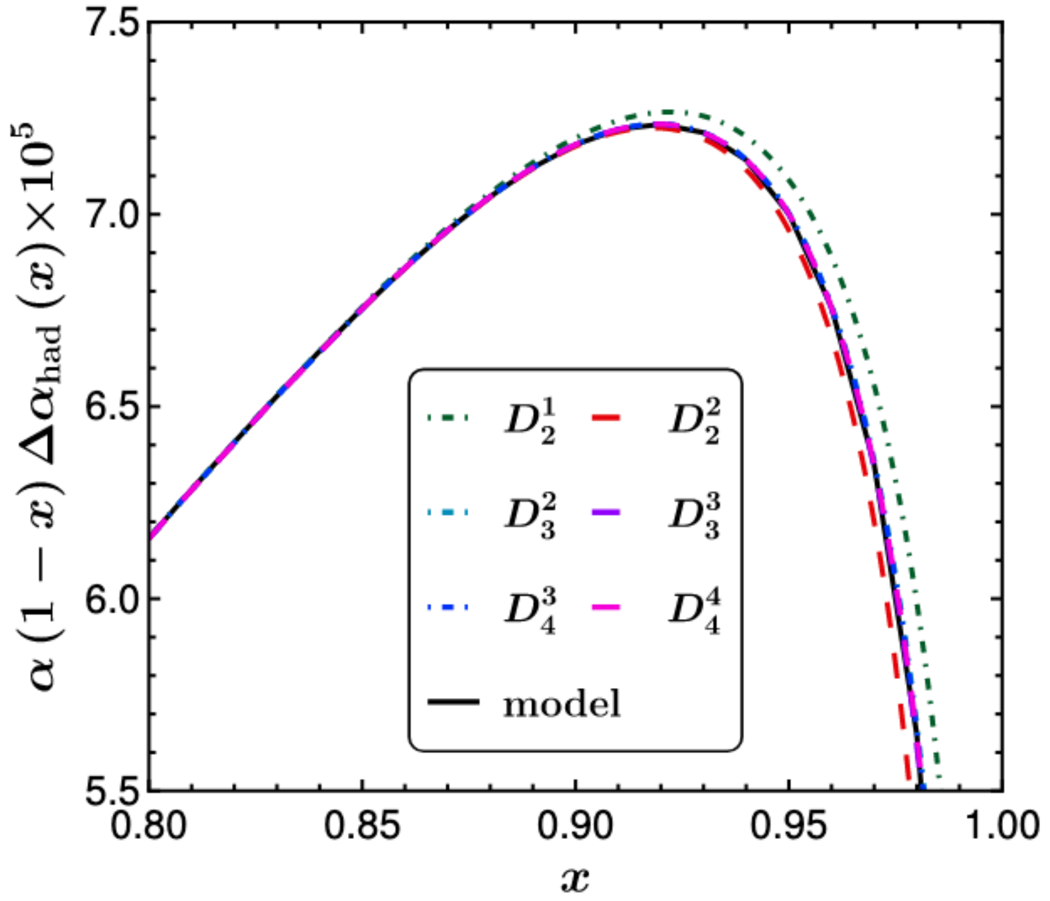
Padé Approximants

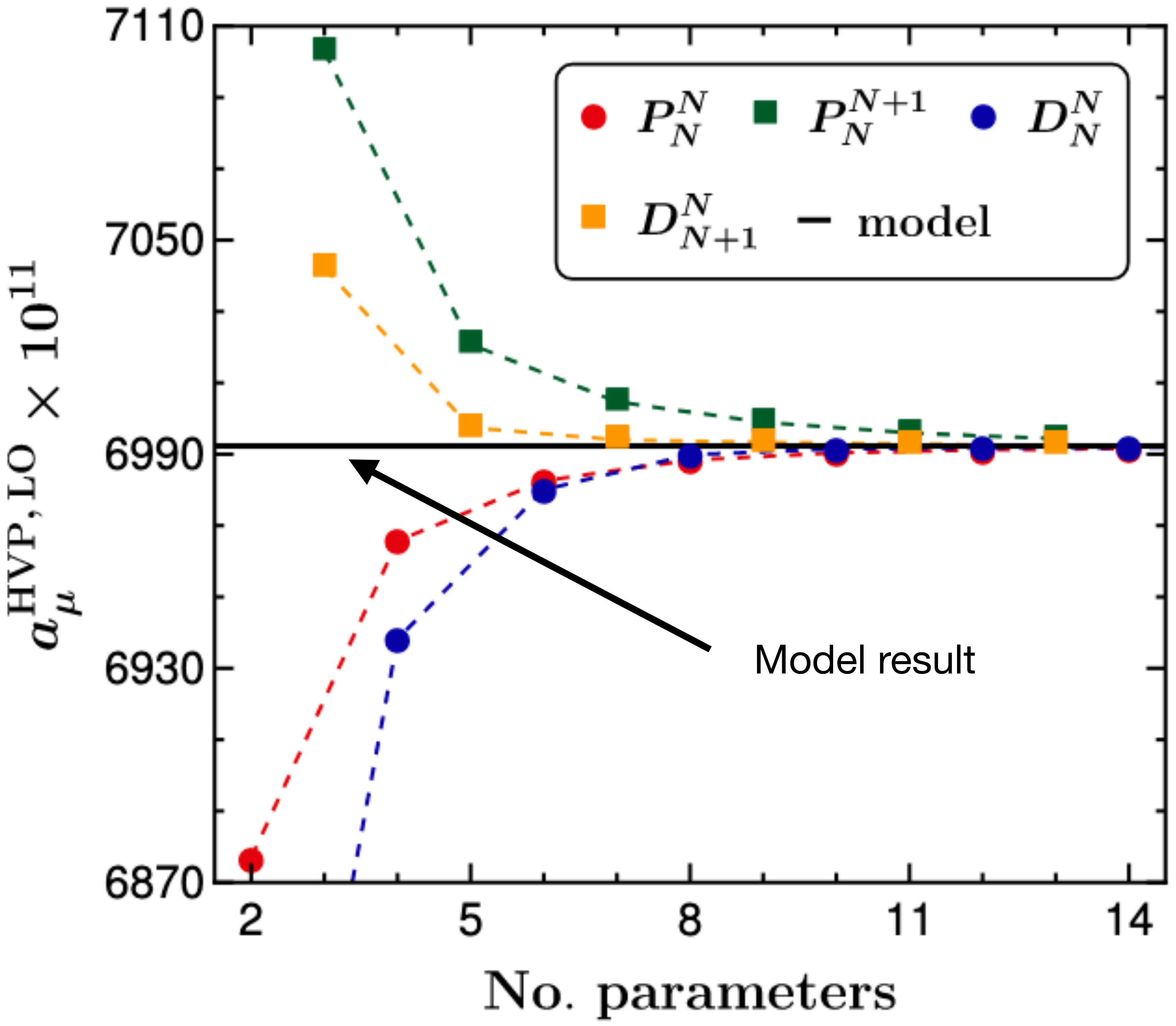




D-Log Padé Approximants







Taylor expansion from the model @ Greynat and de Rafael '22

Fitting Method

- Data generation — simple model for $\Delta\alpha_{\rm had}(t)$ as a Stieltjes function

Greynat, de Rafael (2022)

$$(x, \alpha\,\Delta\alpha_{\rm had}(x) \times 10^5)$$
$$0.2 \leq x \leq 0.93$$

30 equally spaced bins — mean value of each bin

- Fitting parameters: unknown Taylor series coefficients of $\Delta\alpha_{\rm had}(t)$

$$\Delta\alpha_{\rm had}(t) = a_1\, t + a_2\, t^2 + a_3\, t^3 + \cdots$$

Example PA:

$$P_1^1(x) = -\frac{b_1\, m_\mu^2\, x^2}{1 - x + b_2\, m_\mu^2\, x^2} \qquad t = -\frac{x^2\, m_\mu^2}{1-x}$$

$$b_1 = a_1 < 0 \qquad b_2 = \frac{a_2}{a_1} > 1$$

model-independent constraints

- PA parameters: χ^2 minimization

Fitting Method

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$$\Delta\alpha_{\rm had}(t) = a_1\, t + a_2\, t^2 + a_3\, t^3 + \cdots$$

Example DLog:

$$D_2^1(t) = \frac{-f_0\, t}{(r_1 - t)^{\gamma_1}} \to D_2^1(x) = \frac{f_0\, m_\mu^2\, x^2 (1-x)^{-1+\gamma_1}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1}}$$

$$D_2^2(t) = \frac{-f_0\, t\, e^{\beta t}}{(r_1 - t)^{\gamma_1}} \to D_2^2(x) = \frac{f_0\, m_\mu^2\, x^2 (1-x)^{-1+\gamma_1}}{(r_1 - r_1 x + m_\mu^2 x^2)^{\gamma_1}} e^{\beta \frac{m_\mu^2 x^2}{(x-1)}}$$

- DLog parameters: χ^2 minimization

model-independent constraints for $\beta, r_1, \gamma_1, \ldots$

Realistic Data

- 1000 toy data sets
- $(x, \alpha\, \Delta\alpha_{\rm had}(x) \times 10^5)$ – 30 data points equally spaced in $0.2 \leq x \leq 0.93$
- Central value randomly chosen from a gaussian distribution with expected error of MUonE experiment

  private communication with Abbiendi, Carloni Calame, Venanzoni

- Analysis of the fits for each Padé and DLog

$$\Delta\alpha_{\rm had}(t) = \sum_{i=1}^{\infty} a_i\, t^i$$

$$0 < a_i < a_{i+1}$$

- χ^2 penalties (θ functions) if coefficients do not follow the expected hierarchy
- Value of $a_\mu^{\rm HVP,LO}$ calculated for each data set (extrapolation with exact model)

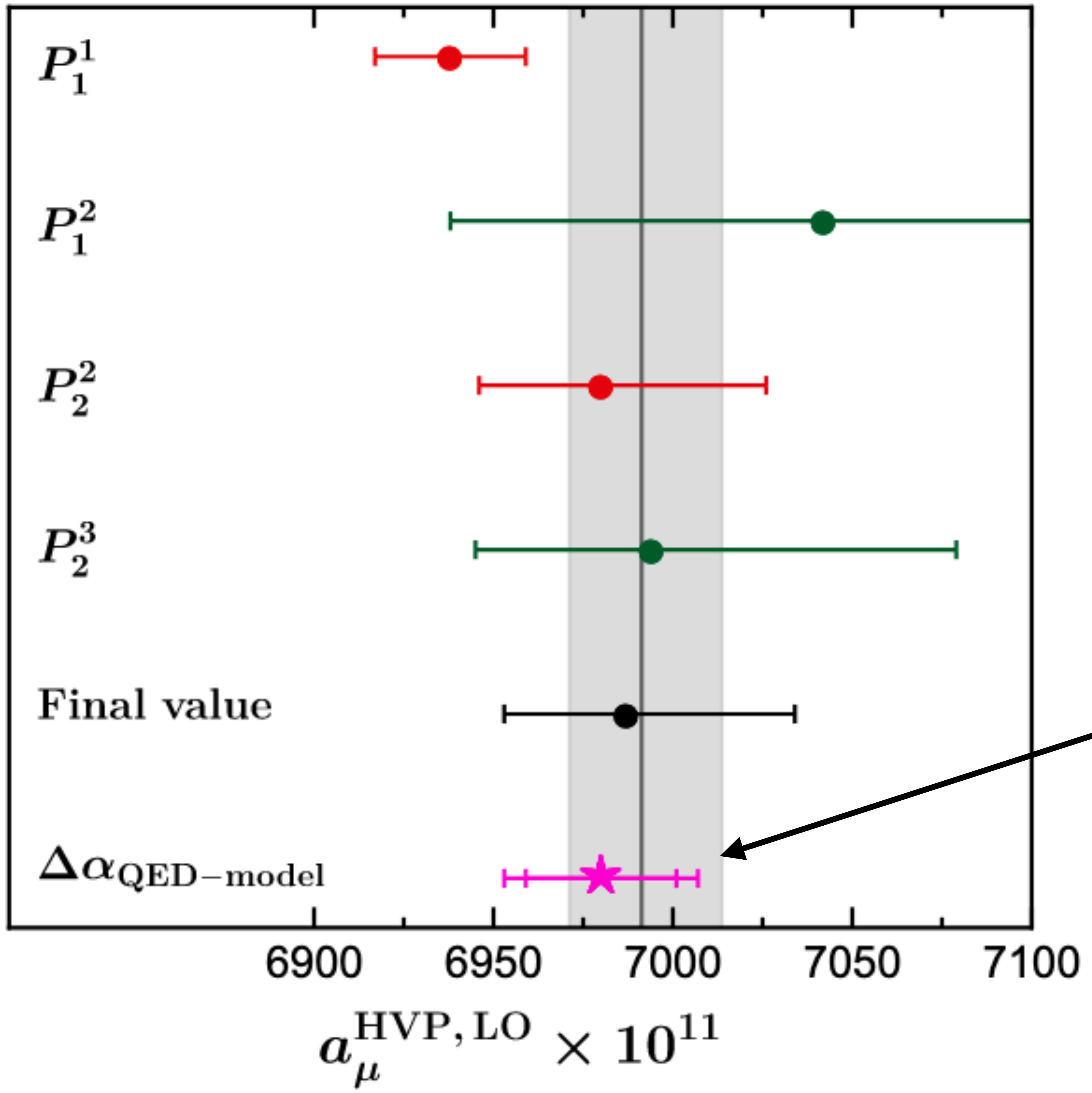
Toy model result:

$$a_\mu^{\rm HVP,LO} = (6991^{+22}_{-20}) \times 10^{-11}$$

best result we can expect from PAs and DLogs predictions

Realistic Data

$$P_1^1(t) \leq P_2^2(t) \leq \ldots \leq \Delta\alpha_{\rm had} \leq \ldots \leq P_2^3(t) \leq P_1^2(t)$$



Convergence pattern preserved for the central values

- Good fit qualities
- Statistical and theoretical error of the same order but statistical is higher

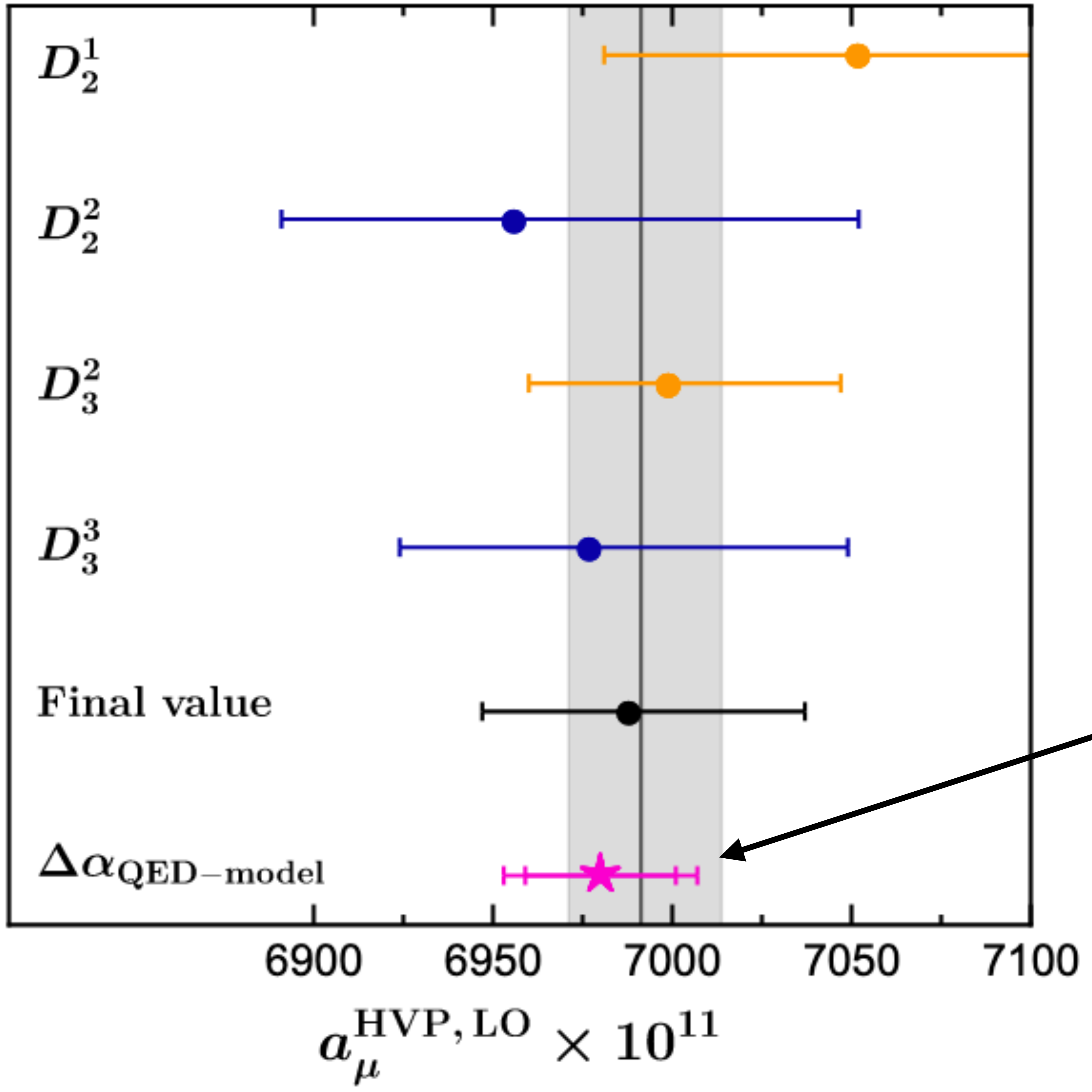
$$\Delta\alpha_{\rm QED-model}(t) = KM\left[-\frac{5}{9} - \frac{4M}{3t} + \frac{2\left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right]$$

Inner error bar — statistical error

Exterior error bar — statistical and systematic errors added in quadrature

Realistic Data

$$D_1^1(t) \leq D_2^2(t) \leq \ldots \leq \Delta\alpha_{\rm had} \leq \ldots \leq D_3^2(t) \leq D_2^1(t)$$



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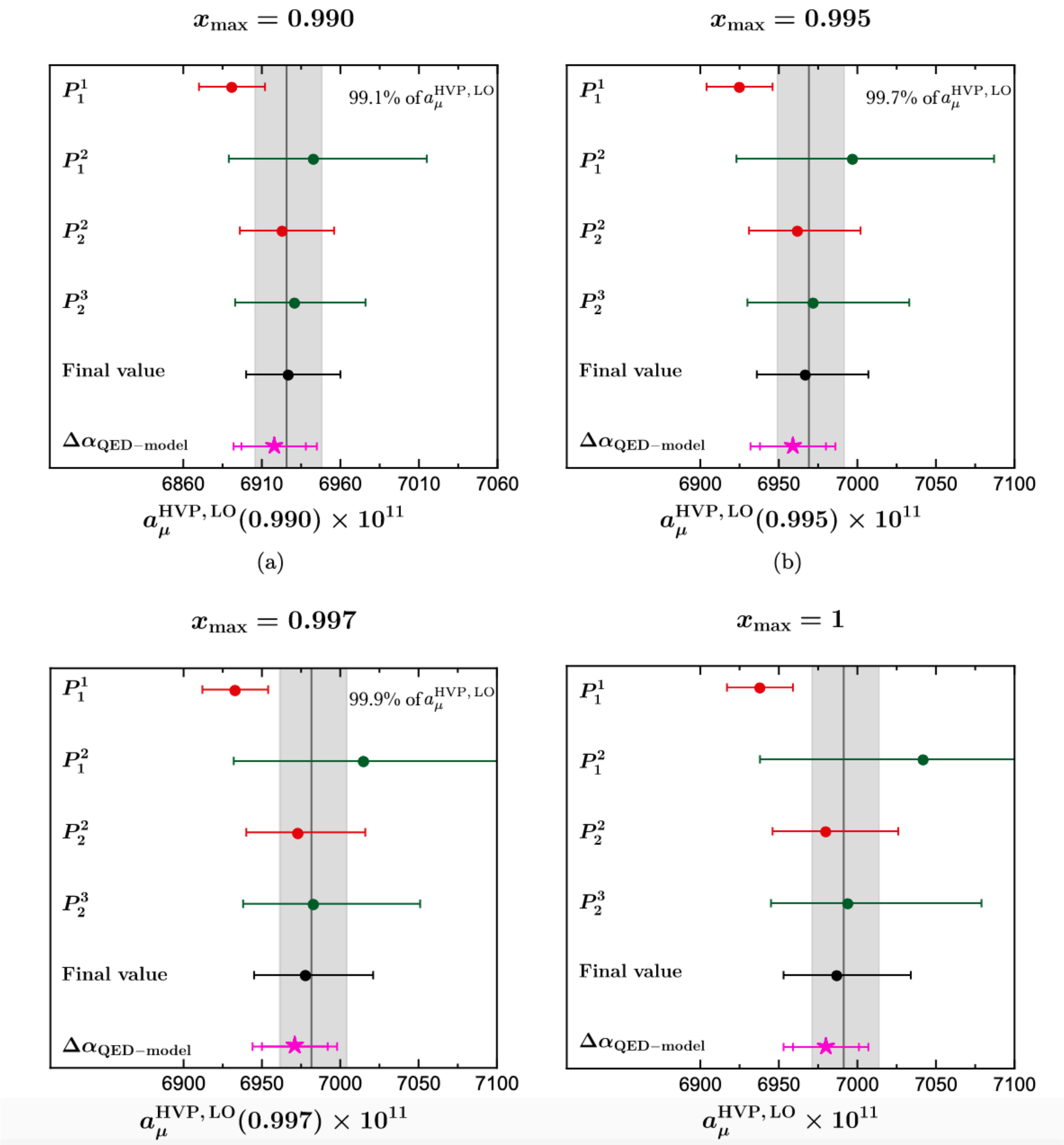
Realistic Data: extrapolation up to x_{max}

Fits up to $x \sim 0.93$ and extrapolated up to x_{max}:



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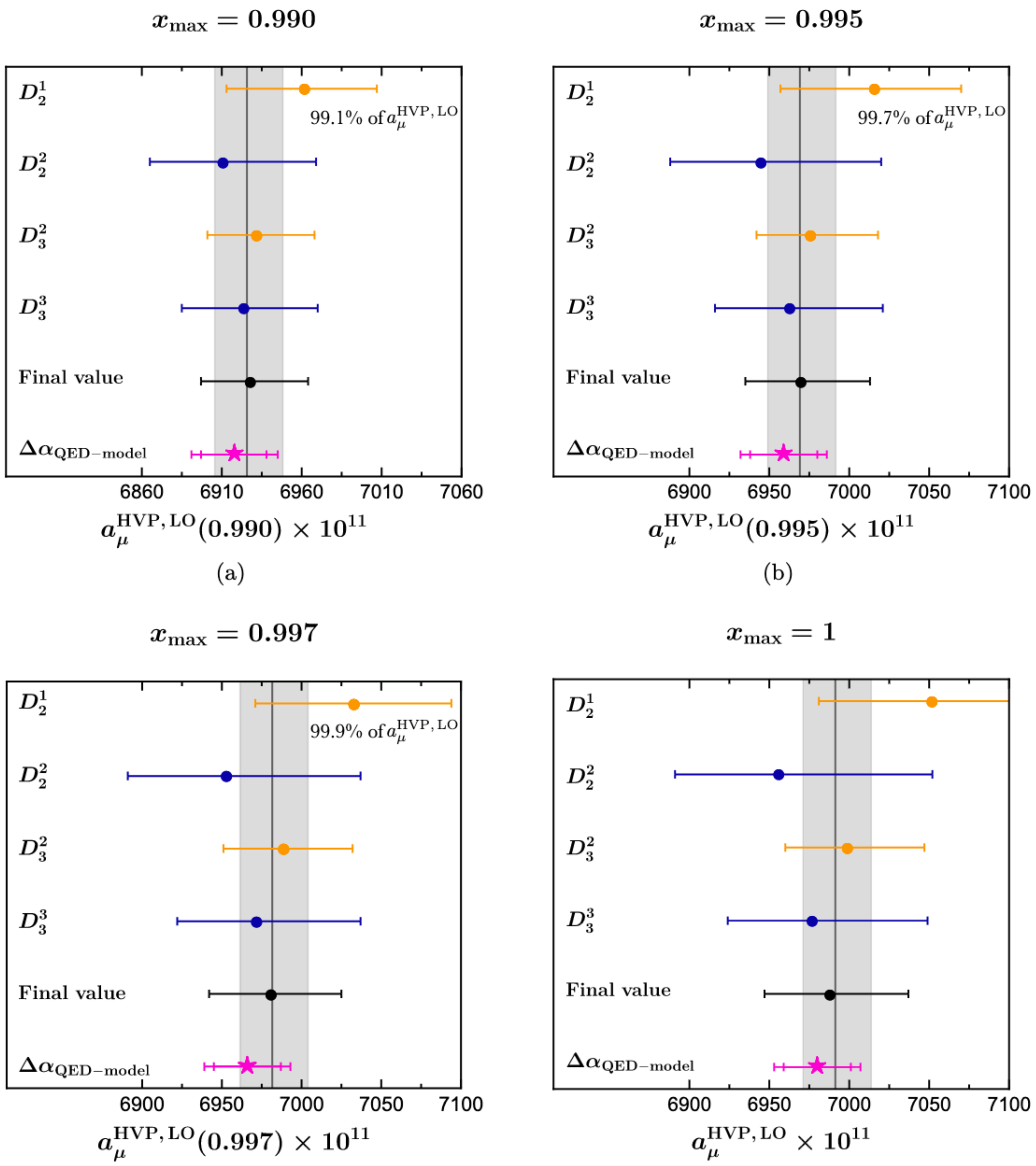

Realistic Data: extrapolation up to x_{max}

Fits up to $x \sim 0.93$ and extrapolated up to $x_{\max}$:

$x_{\max}$	$a_{\mu,\,\mathrm{PAs}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{Dlogs}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{QED-model}}^{\mathrm{HVP,LO}}$	$a_{\mu,\,\mathrm{data-sets}}^{\mathrm{HVP,LO}}$
0.990	$6927\,\left({}^{+33}_{-27}\right)(\pm 4)$	$6928\,\left({}^{+36}_{-31}\right)(\pm 4)$	$6918\,\left({}^{+21}_{-20}\right)(\pm 4)$	$6926\,\left({}^{+22}_{-20}\right)$
0.995	$6967\,\left({}^{+40}_{-31}\right)(\pm 5)$	$6970\,\left({}^{+42}_{-34}\right)(\pm 7)$	$6959(\pm 21)(\pm 17)$	$6969\,\left({}^{+22}_{-20}\right)$
0.997	$6978\,\left({}^{+43}_{-33}\right)(\pm 5)$	$6981\,\left({}^{+43}_{-38}\right)(\pm 9)$	$6971(\pm 21)(\pm 17)$	$6982\,\left({}^{+22}_{-20}\right)$
1.000	$6987\,\left({}^{+46}_{-34}\right)(\pm 7)$	$6988\,\left({}^{+48}_{-39}\right)(\pm 11)$	$6980(\pm 21)(\pm 17)$	$6991\,\left({}^{+22}_{-20}\right)$

First error - statistical error

Second error - systematic (extrapolation) error

$$\Delta\alpha_{\mathrm{QED-model}}(t) = KM\left[-\frac{5}{9} - \frac{4M}{3t} + \frac{2\left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right]$$

Realistic Data: extrapolation up to x_{max}

Fits up to $x \sim 0.93$ and extrapolated up to $x_{\max}$:

$x_{\max}$	$a_{\mu,\,\mathrm{PAs}}^{\mathrm{HVP,\,LO}}$	$a_{\mu,\,\mathrm{Dlogs}}^{\mathrm{HVP,\,LO}}$	$a_{\mu,\,\mathrm{QED-model}}^{\mathrm{HVP,\,LO}}$	$a_{\mu,\,\mathrm{data-sets}}^{\mathrm{HVP,\,LO}}$
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First error - statistical error

Second error - systematic (extrapolation) error

Extrapolation seems to be under control!

$$\Delta\alpha_{\mathrm{QED-model}}(t) = KM\left[-\frac{5}{9} - \frac{4M}{3t} + \frac{2\left(\frac{4M^2}{3t^2} + \frac{M}{3t} - \frac{1}{6}\right)}{\sqrt{1-\frac{4M}{t}}}\log\left|\frac{1-\sqrt{1-\frac{4M}{t}}}{1+\sqrt{1-\frac{4M}{t}}}\right|\right]$$

Conclusions

- D-Logs and Padé approximant sequences are a model-independent method to fit and extrapolate the data from the MUonE experiment. Approximants up to 6 coefficients seem ok!
- The method uses fundamental knowledge about the analytic structure of $\Delta\alpha_{\rm had}(t)$. It is a Stieltjes function (we use analyticity and unitarity, yet inclusive) ⇒ beyond the unitary cut!

⚠ PAs and DLogs can provide a lower and upper **bound** for the true value!

- Uncertainties may be reduced if:
  - Knowledge about the structure of $\Delta\alpha_{\rm had}(t)$ is included (or either extracted from fit)
  - Extrapolate to certain x_{max} (corresponding to "large enough energy") and then match to pQCD or e^+e^- (with DLogs we can access the time-like: $\pi^+\pi^-$ or $\pi^0\gamma$ production thresholds, ρ-meson!)

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Thanks!

Stieltjes Functions

Stieltjes function

$$f(z) = \int_0^\infty \frac{\mathrm{d}\phi(u)}{1+zu} \qquad \phi(u) \text{ is a measure in } u \in [0,\infty)$$

$$f(z) = \sum_{i=0}^{\infty} f_i (-z)^i, \qquad f_i = \int_0^\infty u^i \, \mathrm{d}\phi(u)$$

Stieltjes series

$$\begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+n} \\ f_{m+1} & f_{m+2} & \cdots & f_{m+n+1} \\ \vdots & \vdots & & \vdots \\ f_{m+n} & f_{m+n+1} & \cdots & f_{m+2n} \end{vmatrix} > 0 \qquad m \geq 0 \qquad n \geq 0$$

determinant condition

Aubin, Blum, Golterman, Peris (2012) Masjuan, Peris (2009)

- $\Delta\alpha_{\mathrm{had}}(t)$ is a Stieltjes function in $t \in (-\infty, 0]$ since HVP correlator is a Stieltjes function

hierarchy

$$\Delta\alpha_{\mathrm{had}}(t) = \sum_{i=1}^{\infty} a_i \, t^i \longrightarrow 0 < a_i < a_{i+1}, \qquad i \in \mathbb{N}$$

Model of Greynat and de Rafael

$$\mathrm{Im}\,\Pi_{\mathrm{had}}(s) = \frac{1}{4\pi}\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\left(\frac{|F(s)|^2}{12} + \sum_{i=u,d,\dots} Q_i^2\,\Theta(s,s_c,\Delta)\right)\theta(s-4m_\pi^2)$$

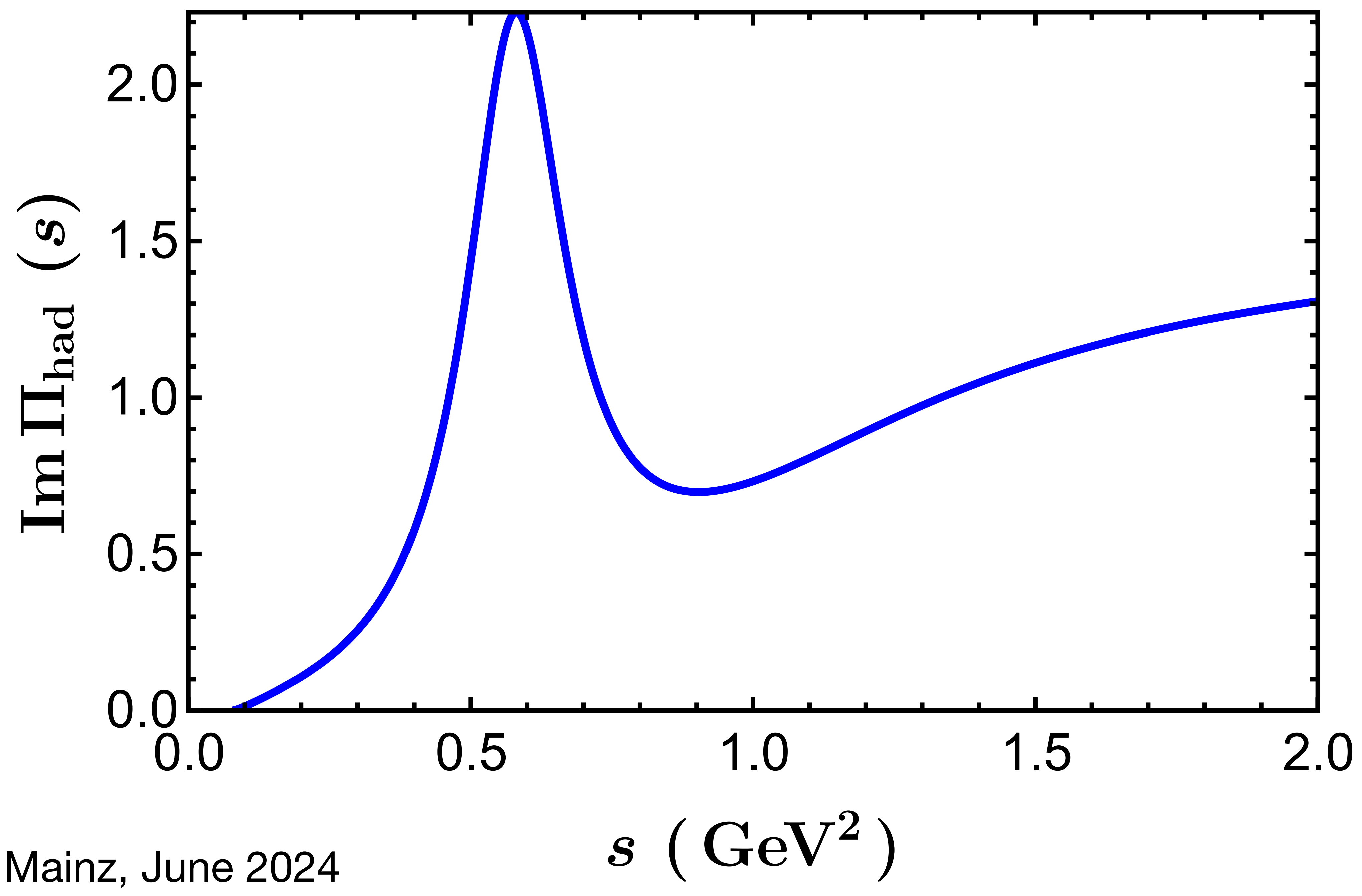
model used to generate toy data

Greynat, de Rafael (2022)

$$|F(s)|^2 = \frac{m_\rho^4}{(m_\rho^2-s)^2 + m_\rho^2\,\Gamma(s)^2}$$

$$\Gamma(s) = \frac{m_\rho\, s}{96\pi f_\pi^2}\left[\left(1-\frac{4m_\pi^2}{s}\right)^{3/2}\theta(s-4m_\pi^2) + \frac{1}{2}\left(1-\frac{4m_k^2}{s}\right)^{3/2}\theta(s-4m_k^2)\right]$$

$$\Theta(s) = \frac{2}{\pi}\left[\frac{\arctan\left(\frac{s-s_c}{\Delta}\right) - \arctan\left(\frac{4m_\pi^2-s_c}{\Delta}\right)}{\frac{\pi}{2} - \arctan\left(\frac{4m_\pi^2-s_c}{\Delta}\right)}\right]$$



Fitting Method

- Fitting function from PAs — example

matching coefficients

$$\Delta\alpha_{\rm had}(t) = a_1\, t + a_2\, t^2 + \dots$$ unknown Taylor series coefficients

$$P_1^1(t) = \frac{q_0 + q_1\, t}{1 + r_1\, t} \approx q_0 + (q_1 - q_0 r_1)t + (q_0 r_1^2 - q_1 r_1)t^2 + \dots$$

$$P_1^1(t) = \frac{a_1^2\, t}{1 - a_2\, t}$$

$$t = -\frac{x^2\, m_\mu^2}{1-x}$$

$$P_1^1(x) = -\frac{a_1^2\, m_\mu^2\, x^2}{a_1 - a_1\, x + a_2\, m_\mu^2\, x^2} = -\frac{b_1\, m_\mu^2\, x^2}{1 - x + b_2\, m_\mu^2\, x^2}$$

$$b_1 = a_1 < 0 \qquad b_2 = \frac{a_2}{a_1} > 1$$

model-independent constraints

- Modified χ^2 function with penalties

$$\chi^2 = \sum_{i,j=1}^{30} \left[\alpha\, \Delta\alpha_{\rm had}(x_i) \times 10^5 - P_N^M(x_i)\right] (C^{-1})_{ij} \left[\alpha\, \Delta\alpha_{\rm had}(x_j) \times 10^5 - P_N^M(x_j)\right] + n_{\rm dof} \sum_{i=2}^{N+M} \theta(a_i - a_{i-1})$$