Dispersion relations and radiative corrections for the two-pion channel

Peter Stoffer

Physik-Institut, University of Zurich, and Paul Scherrer Institut

in collaboration with G. Colangelo and M. Hoferichter,

JHEP **02** (2019) 006; arXiv:2308.04217 [hep-ph]

with G. Colangelo, M. Hoferichter, and B. Kubis,

JHEP **10** (2022) 032

with **J. Lüdtke** and M. Procura,

JHEP **04** (2023) 125

and with **N. Geralis**, **E. Kaziukėnas**, **J.-N. Toelstede**, and **T. Leplumey**

work in progress

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University of Zurich^{UZH}

PAUL SCHERRER INSTITUT PSI

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Outline

Overview

1. Introduction
2. Dispersive analysis of pion VFF
3. Zeros in the form factor
4. Structure-dependent radiative corrections
5. Summary

Two-pion contribution to HVP

- $\pi\pi$ contribution amounts to more than 70% of HVP contribution
- dominant source of HVP uncertainty
- can be expressed in terms of **pion vector form factor** $\Rightarrow$ constraints from **analyticity and unitarity**

A multitude of puzzles in HVP

- tension between BMWc **lattice-QCD** and dispersive evaluations based on older e^+e^- **cross sections**
- discrepancy between **CMD-3** and all previous e^+e^- experiments
- ongoing scrutiny of both lattice and dispersive evaluations
- role of **radiative corrections**?

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2. **Dispersive analysis of pion VFF**
3. Zeros in the form factor
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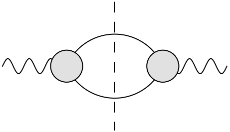
Unitarity and analyticity

implications of unitarity (two-pion intermediate states):

1. $\pi\pi$ contribution to HVP—pion vector form factor (VFF)
2. pion VFF—$\pi\pi$ scattering
3. $\pi\pi$ scattering—$\pi\pi$ scattering



: $\sigma(e^+e^- \to \pi^+\pi^-) \propto |F_\pi^V(s)|^2$

analyticity ⇒ dispersion relation for HVP contribution

Unitarity and analyticity

implications of unitarity (two-pion intermediate states):

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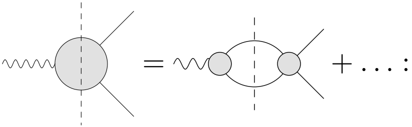

$$F_\pi^V(s) = |F_\pi^V(s)|e^{i\delta_1^1(s)+\ldots}$$

analyticity ⇒ dispersion relation for pion VFF

Unitarity and analyticity

implications of unitarity (two-pion intermediate states):

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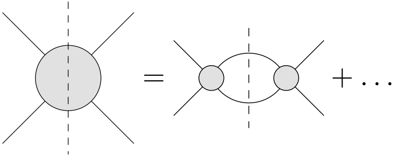
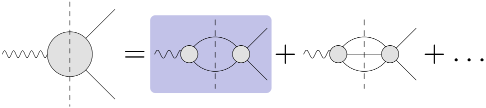


analyticity, crossing, PW expansion ⇒ Roy equations

Dispersive representation of pion VFF

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006



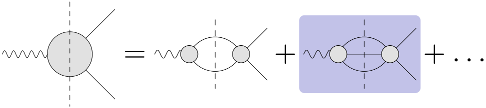
$$F_\pi^V(s) = \boxed{\Omega_1^1(s)} \times G_\omega(s) \times G_{\text{in}}^N(s)$$

- Omnès function with elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ as input:

$$\Omega_1^1(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$$

Dispersive representation of pion VFF

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006



$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\text{in}}^N(s)$$

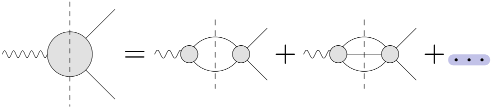
- isospin-breaking 3π intermediate state: negligible apart from ω resonance (ρ–ω interference effect)

$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^\infty ds' \frac{\text{Im} g_\omega(s')}{s'(s'-s)} \left(\frac{1 - \frac{9M_\pi^2}{s'}}{1 - \frac{9M_\pi^2}{M_\omega^2}} \right)^4 ,$$

$$g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}$$

Dispersive representation of pion VFF

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006



$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times \boxed{G_{\text{in}}^N(s)}$$

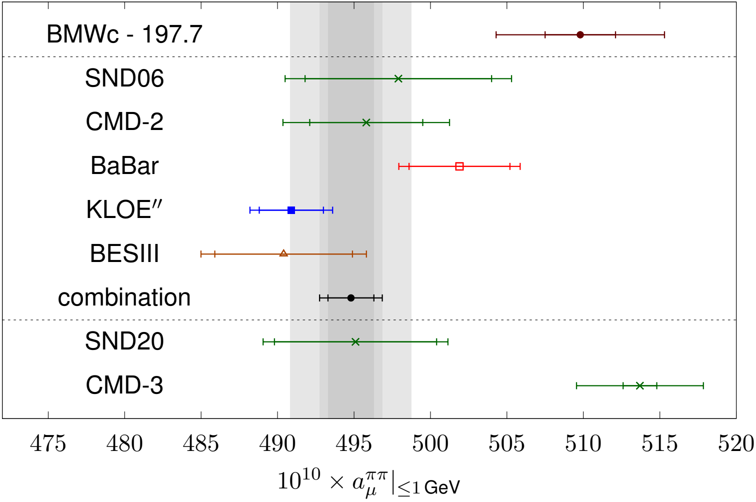
- heavier intermediate states: 4π (mainly $\pi^0\omega$), $\bar{K}K$, $\ldots$
- described in terms of a conformal polynomial with cut starting at $\pi^0\omega$ threshold

$$G_{\text{in}}^N(s) = 1 + \sum_{k=1}^{N} c_k (z^k(s) - z^k(0))$$

- correct P-wave threshold behavior imposed
- potentially leads to **zeros of the form factor**

Result for $a_\mu^{\mathrm{HVP},\pi\pi}$ below 1 GeV

→ Colangelo, Hoferichter, Stoffer, JHEP **02** (2019) 006 and 2308.04217 [hep-ph]
Colangelo, Hoferichter, Kubis, Stoffer, JHEP **10** (2022) 032



CMD-3 vs. all the rest

→ Colangelo, Hoferichter, Stoffer, 2308.04217 [hep-ph]

discrepancy	$a_\mu^{\pi\pi}\big\|_{[0.60,0.88]\text{ GeV}}$	$a_\mu^{\pi\pi}\big\|_{\leq 1\text{ GeV}}$	int window
SND06	1.8σ	1.7σ	1.7σ
CMD-2	2.3σ	2.0σ	2.1σ
BaBar	3.3σ	2.9σ	3.1σ
KLOE$''$	5.6σ	4.8σ	5.4σ
BESIII	3.0σ	2.8σ	3.1σ
SND20	2.2σ	2.1σ	2.2σ
Combination	4.2σ (6.1σ)	3.7σ (5.0σ)	3.8σ (5.7σ)

(discrepancies in brackets exclude systematic effect due to BaBar–KLOE tension)

- p-value of fit to CMD-3: 20%
- $\pi\pi$ phase shifts reasonable, main effect in conformal polynomial
- effect on charge radius as expected for rather uniform cross-section shift

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Zeros in the pion VFF?

- presence of zeros can in principle be tested with modulus sum rule → H. Leutwyler, arXiv:hep-ph/0212324

$$\psi(s) := \frac{1}{(s_0 - s)^{3/2}} \log \frac{F_\pi^V(s)}{F_\pi^V(s_0)}\,, \qquad s_0 = 4M_\pi^2$$

⇒ check if $\psi(s)$ fulfills unsubtracted dispersion relation

$$\psi(s) \overset{?}{=} \frac{1}{\pi}\int_{s_0}^{\infty} ds' \frac{\mathrm{Im}\psi(s')}{s'-s}\,, \qquad \mathrm{Im}\psi(s) = -\frac{1}{(s-s_0)^{3/2}} \log \left| \frac{F_\pi^V(s)}{F_\pi^V(s_0)} \right|$$

- only need **modulus of form factor** ⇒ experiment

Zeros in the pion VFF?

- no zeros possible in the region of validity of χPT

  → H. Leutwyler, arXiv:hep-ph/0212324

- zeros in low-energy region excluded via unitarity/analyticity

  → B. Ananthanarayan, I. Caprini, I. Sentitemsu Imsong, PRD **83** (2011) 096002

- zeros excluded at large values of $|s|$ from asymptotic behavior → G. P. Lepage, S. J. Brodsky, PLB **87** (1979) 359
- use VFF parametrization to **test presence of zeros**:
  - fits lead to $G_{\text{in}}^N(s)$ free of zeros for $N \leq 4$
  - for $N > 4$, zeros show up, accompanied by **fit instabilities**
  - zeros for $N > 4$ source of **main systematic uncertainty** in our representation

Constrained fits without zeros

→ work in progress with **Thomas Leplumey** (ETH master student)

- impose absence of zeros, either via explicit parametrization, or **sum-rule constraint**

$$\log G_{\text{in}}^N(s_{\text{in}}) = \frac{1}{\pi}\int_{s_{\text{in}}}^{\infty}\frac{ds'}{s'}\frac{s_{\text{in}}^{3/2}}{(s'-s_{\text{in}})^{3/2}}\log\left|\frac{G_{\text{in}}^N(s')}{G_{\text{in}}^N(s_{\text{in}})}\right|$$

- observe stabilization of fits for larger $N \Rightarrow$ **main source of uncertainty eliminated**

Constrained fits without zeros

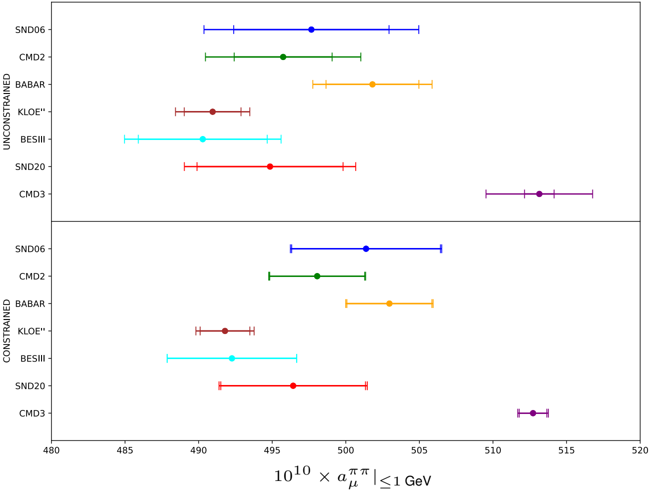
→ work in progress with **Thomas Leplumey** (ETH master student)

- **marginal impact** on χ^2/dof of fit, ω mass and mixing parameter, **central values** of $\pi\pi$ phase
- systematic **uncertainties much reduced** for $\pi\pi$-phase δ_1^1, $a_\mu^{\pi\pi}$, and pion charge radius $\langle r_\pi^2\rangle$
- fits now lead to results for $\langle r_\pi^2\rangle$ that could be used to discriminate between experiments ⇒ opportunity for **independent lattice-QCD checks**

  → Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

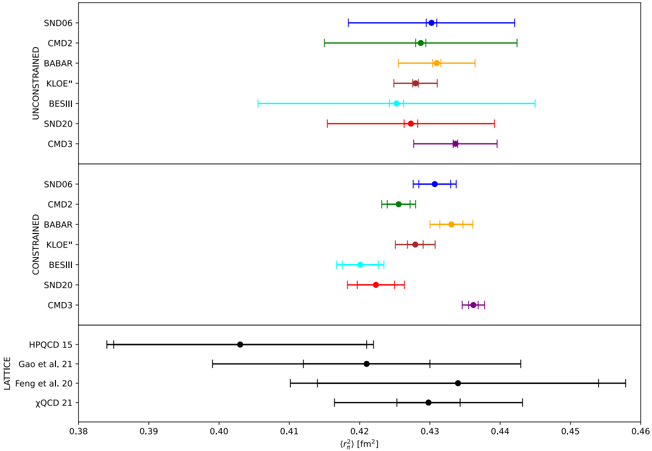
Constrained fits without zeros: $a_\mu^{\pi\pi}$

→ work in progress with **Thomas Leplumey** (ETH master student)



Constrained fits without zeros: pion charge radius $\langle r_\pi^2 \rangle$

→ work in progress with **Thomas Leplumey** (ETH master student)



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How can we resolve the discrepancies?

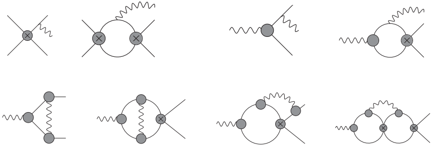
- CMD-3 vs. CMD-2 (and SND): experimental issue?
- SND20: **incompatible** with unitarity/analyticity constraints (p-value: 3.8×10^{-3})
- CMD-3 vs. radiative-return experiments: **model-dependent theory input**—how reliable are uncertainties?

How can we resolve the discrepancies?

- certainly **no conceptual problem** with dispersive approach per se
- dispersive approach relies on **data input**
- but experiments require **theory input**
  $\Rightarrow$ try to **reduce model dependence** in that theory input
  $\Rightarrow$ need **more dispersion theory**, not less!

Dispersive approach to isospin corrections in $\pi\pi$ scattering and F_π^V → talk by G. Colangelo at Zurich WorkStop 2023

→ G. Colangelo, M. Cottini, J. Monnard, J. Ruiz de Elvira, work in progress



- pion-mass difference in Roy equations
- photonic corrections (real + virtual) to $\pi\pi$ scattering and pion vector form factor

Dispersive approach to isospin corrections in $\pi\pi$ scattering and F_π^V → talk by G. Colangelo at Zurich WorkStop 2023

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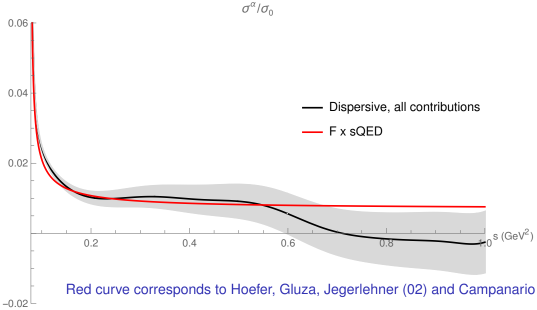


Red curve corresponds to Hoefer, Gluza, Jegerlehner (02) and Campanario et al. (19)

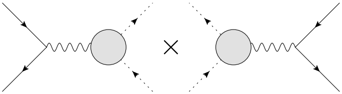
→ J. Monnard, PhD thesis (2021)

⇒ no dramatic effects found

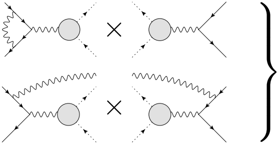
Comparison of MCs → talks by A. Signer, Y. Ulrich

- radiative-return experiments: PHOKHARA
  - FSR from pointlike pions
  - boxes, pentagons with vector form factor *outside* loop integral
- direct scan experiments: MCGPJ
  - FSR from pointlike pions
  - box diagrams for asymmetry with vector form factor inside loop

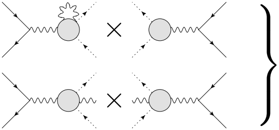
Direct scan experiments: LO



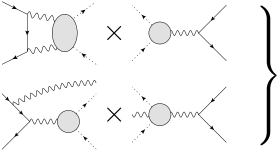
Direct scan experiments: NLO



pure QED



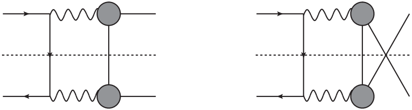
included in generators in terms of sQED
dispersive approach by Colangelo et al.



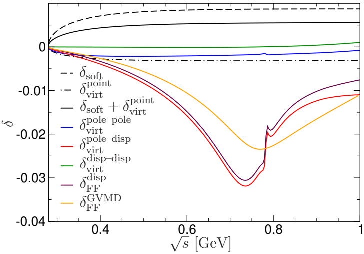
contributes only to asymmetry;

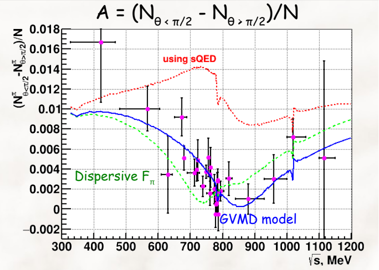
only pole terms:
→ Ignatov, Lee (2022)
→ Colangelo, Hoferichter, Monnard, Ruiz de Elvira (2022)

Forward-backward asymmetry



→ talks by G. Colangelo and F. Ignatov at Zurich WorkStop 2023



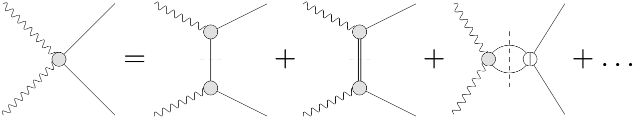


→ Colangelo, Hoferichter, Monnard, Ruiz de Elvira (2022)

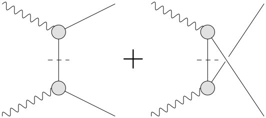
→ Ignatov, Lee (2022)

Dispersion relations for $\gamma^*\gamma^* \to \pi^+\pi^-$

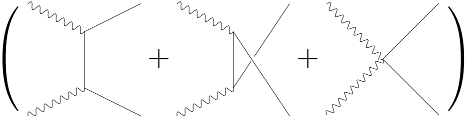
→ Colangelo, Hoferichter, Procura, Stoffer (2015), Hoferichter, Stoffer (2019)



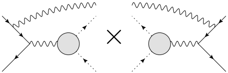
pole term = FsQED



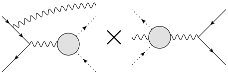
$$= \quad F_\pi^V(q_1^2)F_\pi^V(q_2^2) \times$$



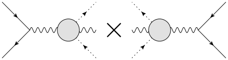
Radiative-return experiments: LO



pure QED

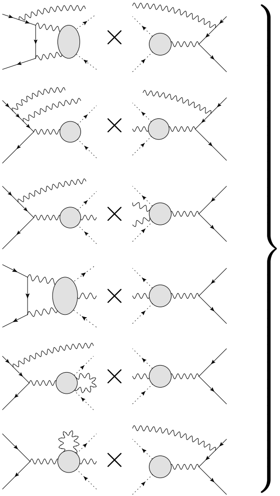


contributing only to charge asymmetry



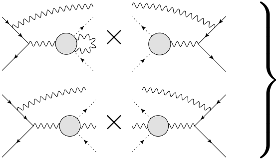
suppressed by cuts?
PHOKHARA: sQED + resonance approximations
dispersive approach by Colangelo et al.

Radiative-return experiments: NLO (omitting pure QED corrections to LO)



contributes only to asymmetry

Radiative-return experiments: NLO (omitting pure QED corrections to LO)



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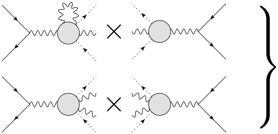
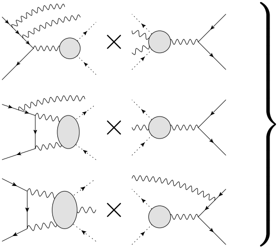
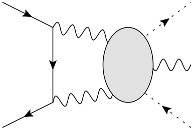


contained in PHOKHARA
pure FSR: sufficiently suppressed by experimental cuts?



???
PHOKHARA: sQED, multiplied by form factors *outside* loop
ISR–FSR interference
potential red flag identified during 2023 WorkStop

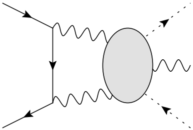
Most difficult sub-process: $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$



- PHOKHARA: sQED × $F_\pi^V(s)$ (s: e^+e^- invariant squared energy)—model prescription, which achieves cancellation of IR singularities
- **not FsQED** (= dispersive pole terms): lesson learnt from asymmetry might raise concerns
- here: **dispersive pole terms expected to be bad approximation**: $\pi\pi$ system in p-wave, ρ resonance in rescattering

Most difficult sub-process: $\gamma^*\gamma^*\gamma \to \pi^+\pi^-$

→ work in progress with **Emilis Kaziukėnas**, **Nikolas Geralis** (ETH master students), **J.-N. Toelstede**



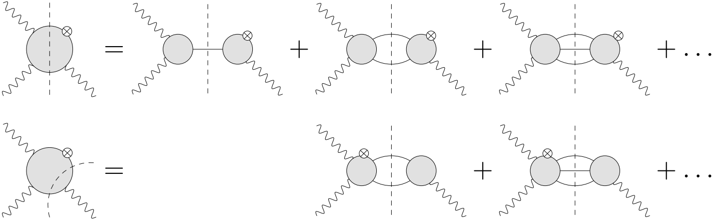
- goal: dispersive treatment of $\gamma^*\gamma^*\gamma \to \pi\pi$
- synergies with **dispersive approach to HLbL** in triangle kinematics → talk by M. Hoferichter
- warm-up: $\gamma^*\gamma^*\gamma \to \pi\pi$ at NLO in $SU(2)$ χPT computed

  → work in progress with **Emilis Kaziukėnas**

  ⇒ useful to understand **analytic structure** and for fixing **subtraction constants**

HLbL in triangle kinematics

→ Lüdtke, Procura, Stoffer, JHEP **04** (2023) 125



- same sub-process
- for HLbL: only soft-photon limit required
- beyond soft limit: **ambiguities in tensor decomposition** need to be addressed for $e^+e^- \to \pi^+\pi^-\gamma$
- dispersive definition of **pole terms** non-trivial

  → work in progress with **Emilis Kaziukėnas**

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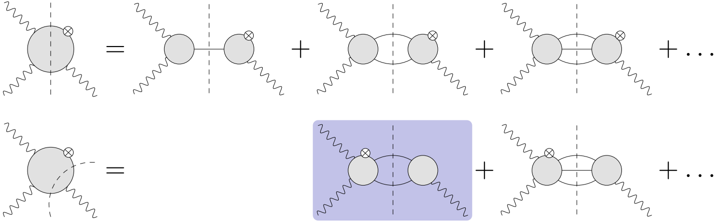

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Overview

Summary

- systematic uncertainties in pion VFF drastically reduced if **zeros are excluded**
- **data do not prefer zeros**: presence of zeros in fit connected with instabilities
- reduce model dependence in radiative corrections: **rely on dispersion theory**
- $\gamma^*\gamma^* \to \pi\pi$ sub-process: well understood **pion-pole terms**, rescattering could be included
- $\gamma^*\gamma^*\gamma \to \pi\pi$ sub-process: **very difficult**, but synergies with new dispersive approach to HLbL; pole terms not enough

Backup

Tension with lattice QCD

→ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

- force a different HVP contribution in VFF fits by including "lattice" datum with tiny uncertainty
- three different scenarios:
    - "low-energy" physics: $\pi\pi$ phase shifts
    - "high-energy" physics: inelastic effects, c_k
    - all parameters free
- study effects on pion charge radius, hadronic running of $\alpha_{\mathrm{QED}}^{\mathrm{eff}}$, phase shifts, cross sections

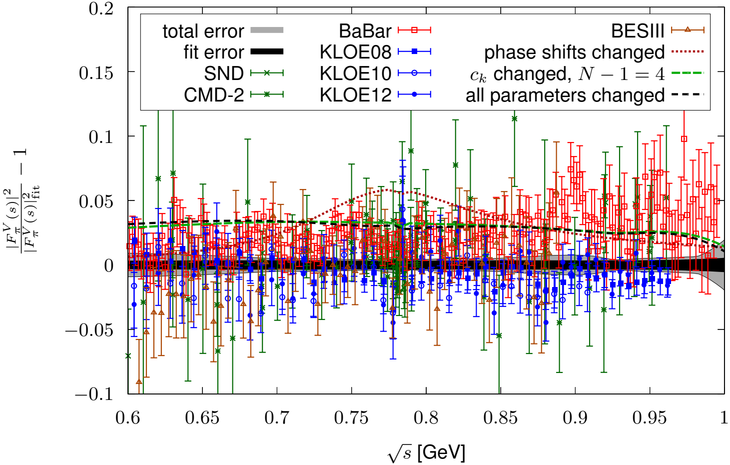
Modifying $a_\mu^{\pi\pi}\big|_{\leq 1\,\text{GeV}}$

→ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

- “low-energy” scenario requires large local changes in the cross section in the ρ region
- “high-energy” scenario has an impact on **pion charge radius** and the space-like VFF $\Rightarrow$ chance for independent lattice-QCD checks

Modifying $a_\mu^{\pi\pi}|_{\leq 1}$ GeV

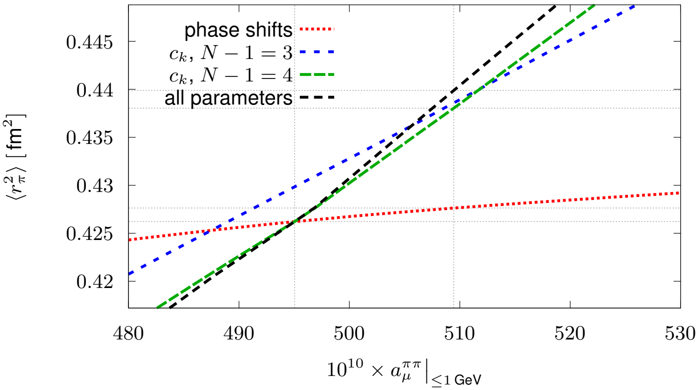
→ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073



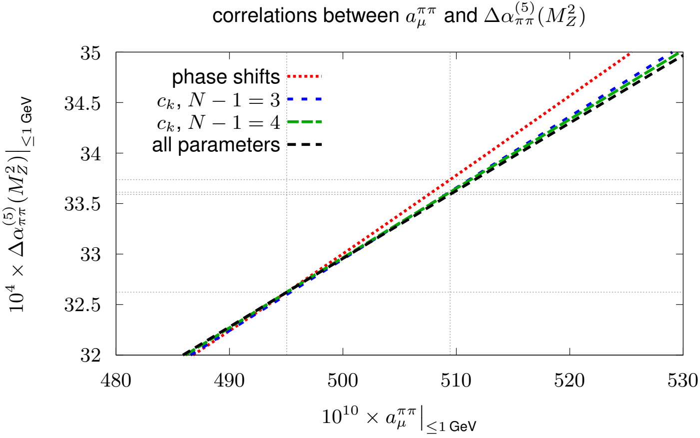
Modifying $a_\mu^{\pi\pi}\big|_{\leq 1\,\text{GeV}}$

→ Colangelo, Hoferichter, Stoffer, PLB **814** (2021) 136073

correlations between $a_\mu^{\pi\pi}$ and $\langle r_\pi^2\rangle$



Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$



Modifying $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$