NLO and NNLO HVP contributions to the muon g-2

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The Evaluation of the Leading Hadronic Contribution to the Muon g-2:

Consolidation of the MUonE Experiment and Recent Developments in Low Energy e^+e^- Data

MITP, Mainz

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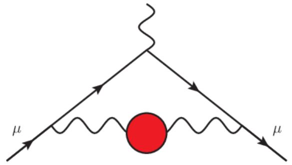
- exact NLO spacelike kernels
- alternative NLO calculation (see also Riccardo Pilato's talk for alternative LO calculation)
- approximate NNLO spacelike kernels
- NLO time-kernel: series expansions



Leading order (LO) hadronic vacuum polarization contribution to muon g-2.

timelike dispersive integral — spacelike dispersive integral

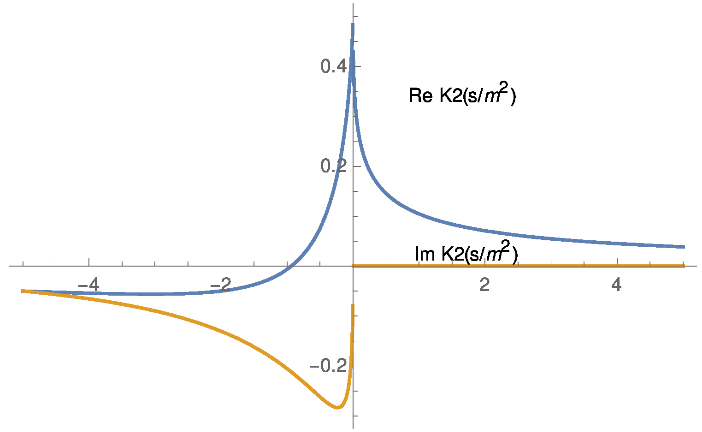
$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi^2}\int_{s_0=m^2_{\pi^0}}^{\infty}\frac{ds}{s}\,K^{(2)}(s/m_\mu^2)\,{\rm Im}\Pi(s) = -\frac{\alpha}{\pi^2}\int_{-\infty}^{0}\frac{dt}{t}\,\Pi(t)\,{\rm Im}K^{(2)}(t/m_\mu^2) = 6931(40)\times 10^{-11}\ ({\rm WP20})$$

$K^{(2)}(s/m_\mu^2)$: 1-loop QED g-2 contribution with a massive photon of mass $\sqrt{s}$

$$K^{(2)}(z) = \frac{1}{2} - z + \left(\frac{z^2}{2} - z\right)\ln z + \frac{\ln y(z)}{\sqrt{z(z-4)}}\left(z - 2z^2 + \frac{z^3}{2}\right)$$

$${\rm Im}K^{(2)}(z+i\epsilon) = \pi\theta(-z)\left[\frac{z^2}{2} - z + \frac{z - 2z^2 + \frac{z^3}{2}}{\sqrt{z(z-4)}}\right]$$

$$y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}}$$

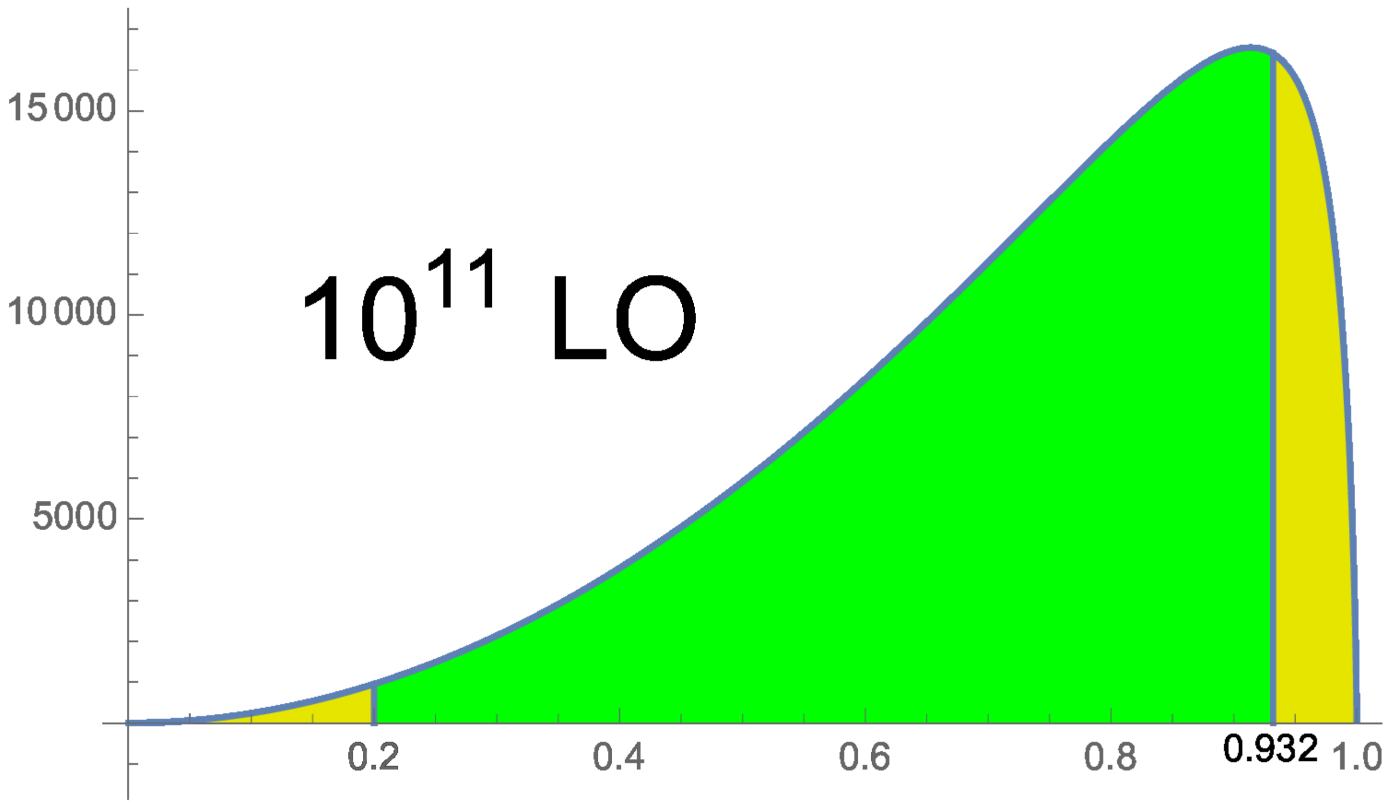


changing variable in the dispersive integral $t \to x(y(t/m_\mu^2)) = 1 + 1/y(t/m_\mu^2)$

$$a_\mu^{\rm HVP}({\rm LO}) = \frac{\alpha}{\pi}\int_0^1 dx\,\kappa^{(2)}(x)\,\Delta\alpha_{\rm had}(t(x))$$

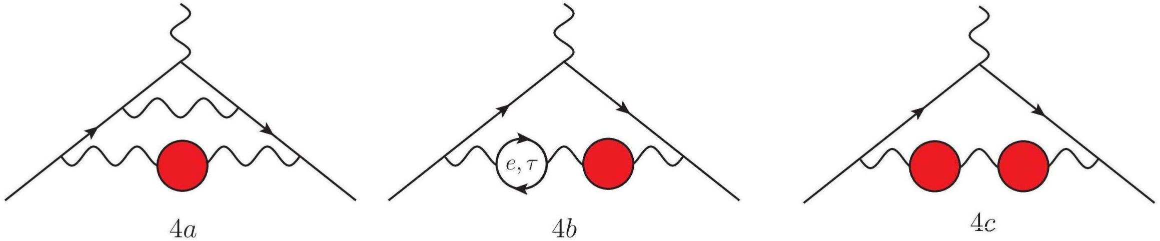
Lautrup, Peterman, deRafael 1972, Carloni Passera Trentadue Venanzoni 2015

$$\boxed{\kappa^{(2)}(x) = 1 - x} \qquad \Delta\alpha_{\rm had}(t) = -\Pi(t) \qquad t(x) = m_\mu^2\frac{x^2}{x-1}$$



Plot of spacelike LO integrand $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\mathrm{had}}(t(x))$ peak at $x = 0.914$

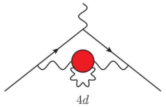
- with $E_\mu = 150 GeV$ MUonE will directly scan the region $0.2 < x < 0.932$
- Green=LO directly scanned by MUonE= 84% of $a_\mu^{\mathrm{HVP}}(\mathrm{LO})$ *84 → 99% alternative LO approach*



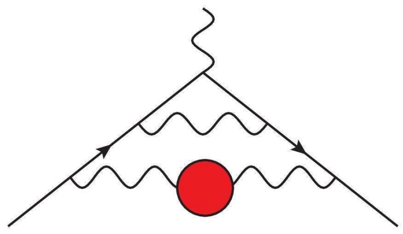
- Class a: 1 HVP insertion in one photon line of 2-loop QED vertex diagrams
- Class b: 1 HVP insertion in the photon line of 2-loop QED vertex with one electron vacuum polarization
- Class c: 2 HVP insertion in the 1-loop QED vertex diagram

$$a_\mu^{\rm HVP}({\rm NLO}; 4a) = -209.0 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; 4b) = +106.8 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; 4c) = +3.5 \times 10^{-11}$$
$$a_\mu^{\rm HVP}({\rm NLO}; total) = -98.7(9) \times 10^{-11}$$

(Krause 1996, Hagiwara Liao Martin Nomura Toebner 2011, Kurz Liu Marquard Steinhauser 2014)



HVP insertion with internal corrections already incorporated in LO



timelike and spacelike integral:

$$a_\mu^{\rm HVP}({\rm NLO};4a)=\frac{\alpha^2}{\pi^3}\int_{s_0}^{\infty}\frac{ds}{s}\,2K^{(4)}(s/m_\mu^2)\,{\rm Im}\Pi(s)=-\frac{\alpha^2}{\pi^3}\int_{-\infty}^{0}\frac{dt}{t}\Pi(t)\,{\rm Im}2K^{(4)}(t/m_\mu^2)$$

$2K^{(4)}(s/m_\mu^2)$: 2-loop QED g-2 contribution from diagrams with one massive photon of mass $\sqrt{s}$ and one massless photon (factor 2 due to normalization chosen)

Dispersion relations: $K^{(4)}(z)=\frac{1}{\pi}\int_{-\infty}^{0}dz'\frac{{\rm Im}K^{(4)}(z')}{z'-z},\ z>0 \qquad \frac{1}{\pi}\int_{s_0}^{\infty}\frac{ds}{s}\frac{{\rm Im}\Pi(s)}{s-q^2}=\frac{\Pi(q^2)}{q^2}\ , \quad q^2<0$

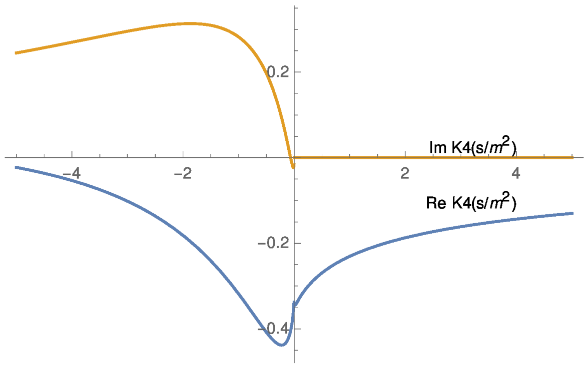
$$K^{(4)}(z) = \left(\frac{z^2}{2} - \frac{7z}{6} + \frac{1}{2}\right)\left[-3\mathrm{Li}_3(-y) - 6\mathrm{Li}_3(y) + 2\left(\mathrm{Li}_2(-y) + 2\mathrm{Li}_2(y)\right)\ln y + \frac{1}{2}\left(\ln^2 y + \pi^2\right)\ln(y+1) + \ln(1-y)\ln^2 y\right]$$
$$+\frac{\left(-\frac{z^3}{6}+\frac{z^2}{4}-\frac{7z}{6}-\frac{4}{z-4}+\frac{13}{3}\right)\left(\mathrm{Li}_2(-y)+\frac{\ln^2 y}{4}+\frac{\pi^2}{12}\right)}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{12}+\frac{17z^2}{6}-2z\right)\left(\mathrm{Li}_2(y)-\frac{1}{4}\ln^2 y+\ln(1-y)\ln y-\frac{\pi^2}{6}\right)}{\sqrt{(z-4)z}}$$
$$+\left(-\frac{29z^2}{96}+\frac{53z}{48}+\frac{2}{z-4}-\frac{1}{3z}+\frac{19}{24}\right)\ln^2 y + \frac{\left(\frac{23z^3}{144}-\frac{115z^2}{72}+\frac{127z}{36}-\frac{4}{3}\right)\ln y}{\sqrt{(z-4)z}} + \frac{\left(-\frac{7z^3}{48}+\frac{17z^2}{24}-\frac{z}{2}\right)\ln y\ln z}{\sqrt{(z-4)z}}$$
$$+\frac{1}{6}\pi^2\left(-\frac{z^2}{2}+\frac{5z}{24}-\frac{2}{z}+\frac{9}{4}\right)+\frac{5}{96}z^2\ln^2 z+\left(\frac{23z^2}{144}-\frac{7z}{36}+\frac{1}{z-4}+\frac{19}{12}\right)\ln z+\frac{115z}{72}-\frac{139}{144}$$

Barbieri Remiddi 1975

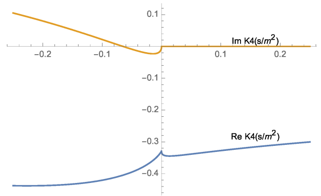
$$K^{(4)}(0) = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta(3) = -0.328479$$ 2-loop g-2 $$K^{(4)}(z \gg 1) \to \frac{1}{z}\left(-\frac{23\ln(z)}{36} - \frac{\pi^2}{3} + \frac{223}{54}\right)$$

$$\mathrm{Im}K^{(4)}(z+i\epsilon) = \pi\theta(-z)F^{(4)}(1/y(z)) \qquad y(z) = \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}} < -1$$

$$F^{(4)}(u) = \frac{-3u^4-5u^3-7u^2-5u-3}{6u^2}\left(2\mathrm{Li}_2(-u)+4\mathrm{Li}_2(u)+\ln(-u)\ln\left((1-u)^2(u+1)\right)\right)$$
$$+\frac{(u+1)\left(-u^3+7u^2+8u+6\right)}{12u^2}\ln(u+1)+\frac{\left(-7u^4-8u^3+8u+7\right)}{12u^2}\ln(1-u)$$
$$+\frac{23u^6-37u^5+124u^4-86u^3-57u^2+99u+78}{72(u-1)^2u(u+1)}$$
$$+\frac{12u^8-11u^7-78u^6+21u^5+4u^4-15u^3+13u+6}{12(u-1)^3u(u+1)^2}\ln(-u)$$



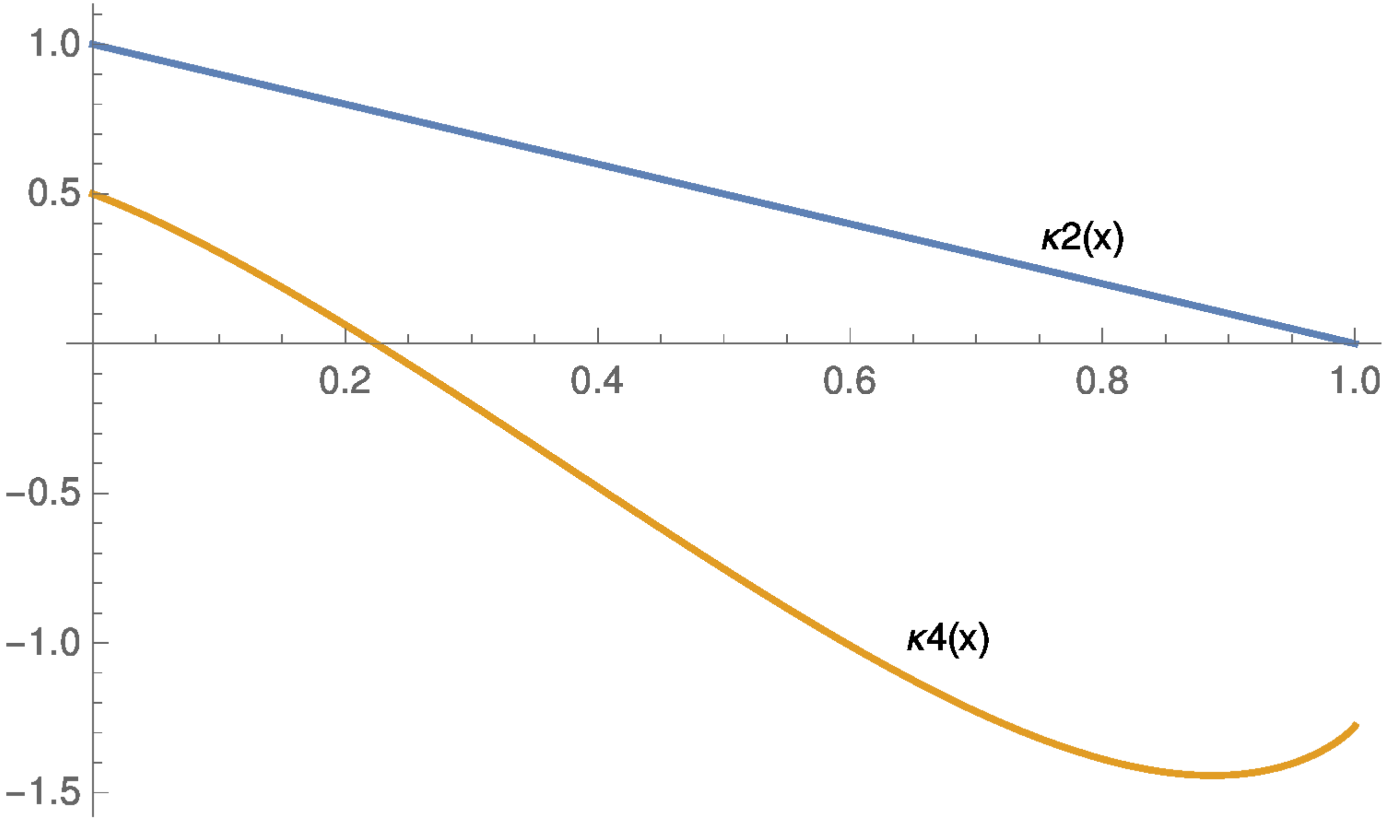
Balzani, S.L., Passera 2112.05704, Nesterenko 2112.05009.



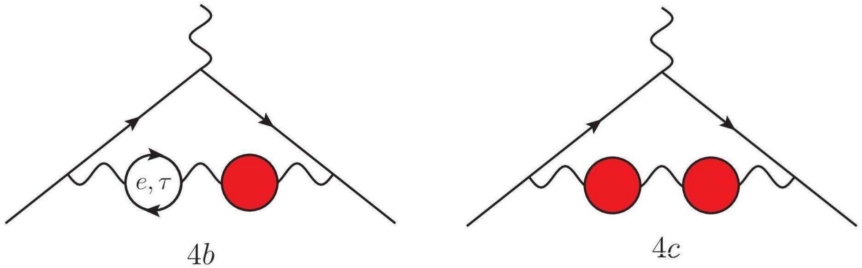
$$a_\mu^{\mathrm{HVP}}(\mathrm{NLO};4a) = \left(\frac{\alpha}{\pi}\right)^2\int_0^1 dx\,\kappa^{(4)}(x)\Delta\alpha_{\mathrm{had}}(t(x))$$

$z \to y \to x$ Space-like NLO kernel $\kappa^{(4)}(x)$

$$\kappa^{(4)}(x) = \frac{2(2-x)}{x(x-1)}F^{(4)}(x-1)$$



- $\kappa^{(4)}(x)$ changes sign at $x = 0.224212$ ($\to q^2 \approx -(23\text{MeV})^2$)
- $\kappa^{(4)}(1) = -\frac{23}{18}$, $\quad \kappa^{(4)}(0) = \frac{1}{2}$;
- $\kappa^{(4)}(x)$ provides stronger weight a large $q^2 < 0$ ($x \to 1$) than $\kappa^{(2)}(x)$



Both spacelike integrals contain the LO kernel $\kappa_2(x)$:

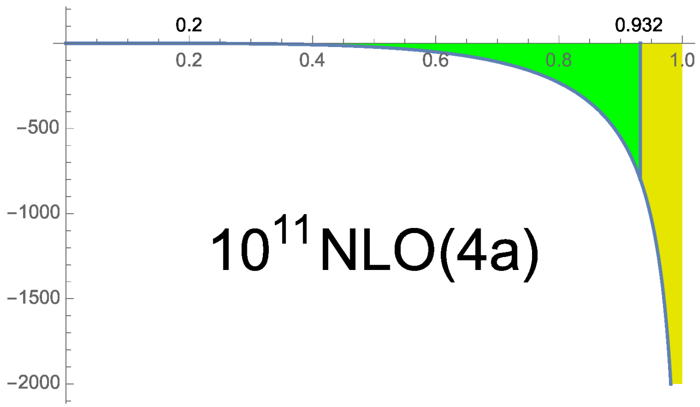
$$a_\mu^{\rm HVP}({\rm NLO};4b) = \frac{\alpha}{\pi}\int_0^1 dx\; \kappa^{(2)}(x)\Delta\alpha_{\rm had}(t(x))\; 2\left(\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x))\right)$$

$$a_\mu^{\rm HVP}({\rm NLO};4c) = \frac{\alpha}{\pi}\int_0^1 dx\; \kappa^{(2)}(x)\left(\Delta\alpha_{\rm had}(t(x))\right)^2$$

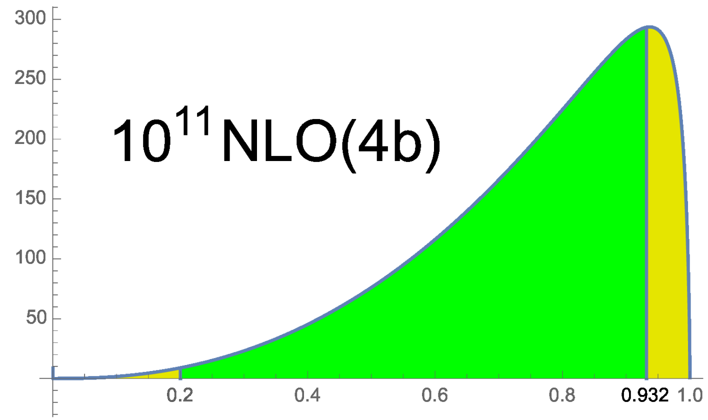
$\Delta\alpha_{\rm l}(t) = -\Pi_l^{(2)}(t)$ Π_l renormalized one-loop QED vacuum polarization function

$$\Pi_l^{(2)}(t) = \left(\frac{\alpha}{\pi}\right)\left[\frac{8}{9} - \frac{\beta_l^2}{3} + \beta_l\left(\frac{1}{2} - \frac{\beta_l^2}{6}\right)\ln\frac{\beta_l - 1}{\beta_l + 1}\right]\;,\quad \beta_l = \sqrt{1 - 4m_l^2/t}$$

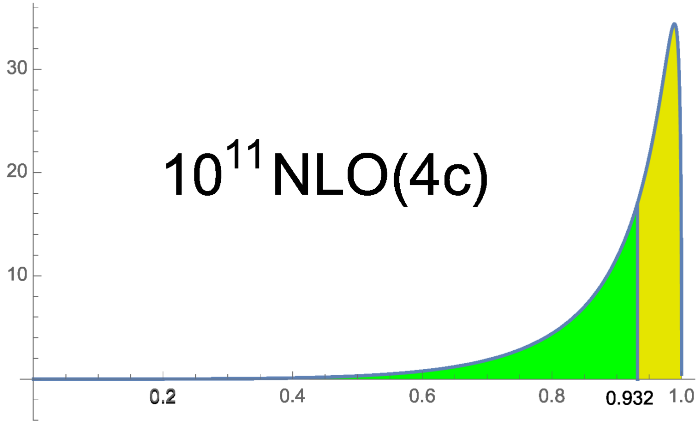
Plots of NLO integrands $4a$ $4b$ $4c$



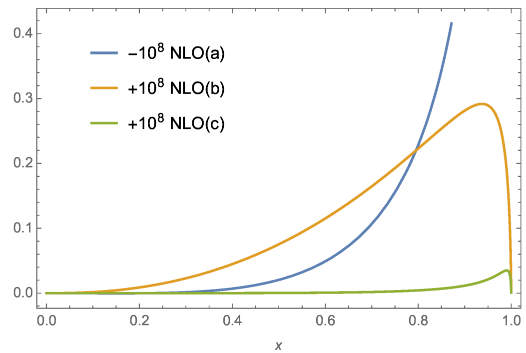
Plot of spacelike NLO($4a$) integrand $(\alpha/\pi)^2\kappa^{(4)}(x)\Delta\alpha_{\rm had}(t(x))$
logarithmic singularity for $x = 1$



Plot of spacelike NLO($4b$) integrand $(\alpha/\pi)\kappa^{(2)}(x)\Delta\alpha_{\rm had}(t(x))\ 2\left(\Delta\alpha_e^{(2)}(t(x)) + \Delta\alpha_\tau^{(2)}(t(x))\right)$
peak at $x = 0.936$



Plot of spacelike NLO($4c$) integrand $(\alpha/\pi)\kappa^{(2)}(x)\,(\Delta\alpha_{\rm had}(t(x)))^2$
peak at $x = 0.988$



the 3 NLO integrands

- with $E_\mu = 150 GeV$ MUonE will directly scan the region $0.2 < x < 0.932$
- Green=directly scanned by MUonE: 41% of $a_\mu^{\rm HVP}({\rm NLO}; 4a)$, 82% of $a_\mu^{\rm HVP}({\rm NLO}; 4b)$, 49% of $a_\mu^{\rm HVP}({\rm NLO}; 4c)$
- $a_\mu^{\rm HVP}({\rm NLO})$: can we apply the alternative approach?

Ignatov, Pilato, Teubner and Venanzoni, *Phys.Lett.***B** 848 (2024) 138344, arXiv:2309.14205 (see Riccardo's talk)

- splitting the *timelike* integral in low and high-energy regions
- fit approximations $K_1(s)$, $\tilde{K}_1(s)$ to the timelike kernel $K(s)$ in both regions
- split the integral and express integrals of fitting functions with derivatives of $\Delta\alpha_h(t)$ at $t=0$ (obtained from MUonE data) and contours integrals in the complex plane (obtained from pQCD).

$$a_\mu^{\mathrm{HVP;NLO}} = a_\mu^{\mathrm{HVP;NLO(I)}} + a_\mu^{\mathrm{HVP;NLO(II)}} + a_\mu^{\mathrm{HVP;NLO(III)}} + a_\mu^{\mathrm{HVP;NLO(IV)}}$$

$$a_\mu^{\mathrm{HVP;NLO(I)}} = -\left(\frac{\alpha}{\pi}\right)^{1+1} \sum \frac{c_n^{(\mathrm{NLO})}}{n!} \frac{d^n}{d\,t^n} \Delta\alpha_{had}(t)\Big|_{t=0}$$

$$a_\mu^{\mathrm{HVP;NLO(II)}} = -\left(\frac{\alpha}{\pi}\right)^{1+1} \frac{1}{2\pi i} \int\limits_{|s|=s_0} \frac{ds}{s} \left(K_1^{(\mathrm{NLO})}(s) - \tilde{K}_1^{(\mathrm{NLO})}(s)\right) \Pi_{had}(s)\Big|_{\mathrm{pQCD}}$$

$$a_\mu^{\mathrm{HVP;NLO(III)}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^{2+1} \int\limits_{s_{\mathrm{th}}}^{s_0} \frac{ds}{s} \left(K^{(\mathrm{NLO})}(s) - K_1^{(\mathrm{NLO})}(s)\right) R(s)$$

$$a_\mu^{\mathrm{HVP;NLO(IV)}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^{2+1} \int\limits_{s_0}^{\infty} \frac{ds}{s} \left(K^{(\mathrm{NLO})}(s) - \tilde{K}_1^{(\mathrm{NLO})}(s)\right) R(s)$$

$$K_1^{(\mathrm{NLO})}(s) \approx c_0^{(\mathrm{NLO})} s + \frac{c_1^{(\mathrm{NLO})}}{s} + \frac{c_2^{(\mathrm{NLO})}}{s^2} + \frac{c_3^{(\mathrm{NLO})}}{s^3}\,, \qquad s_{\mathrm{th}} \le s \le s_0$$

$$\tilde{K}_1^{(\mathrm{NLO})}(s) \approx + \frac{\tilde{c}_1^{(\mathrm{NLO})}}{s} + \frac{\tilde{c}_2^{(\mathrm{NLO})}}{s^2} + \frac{\tilde{c}_3^{(\mathrm{NLO})}}{s^3}\,, \qquad s \ge s_0$$

Minimization I (*least square fit*)

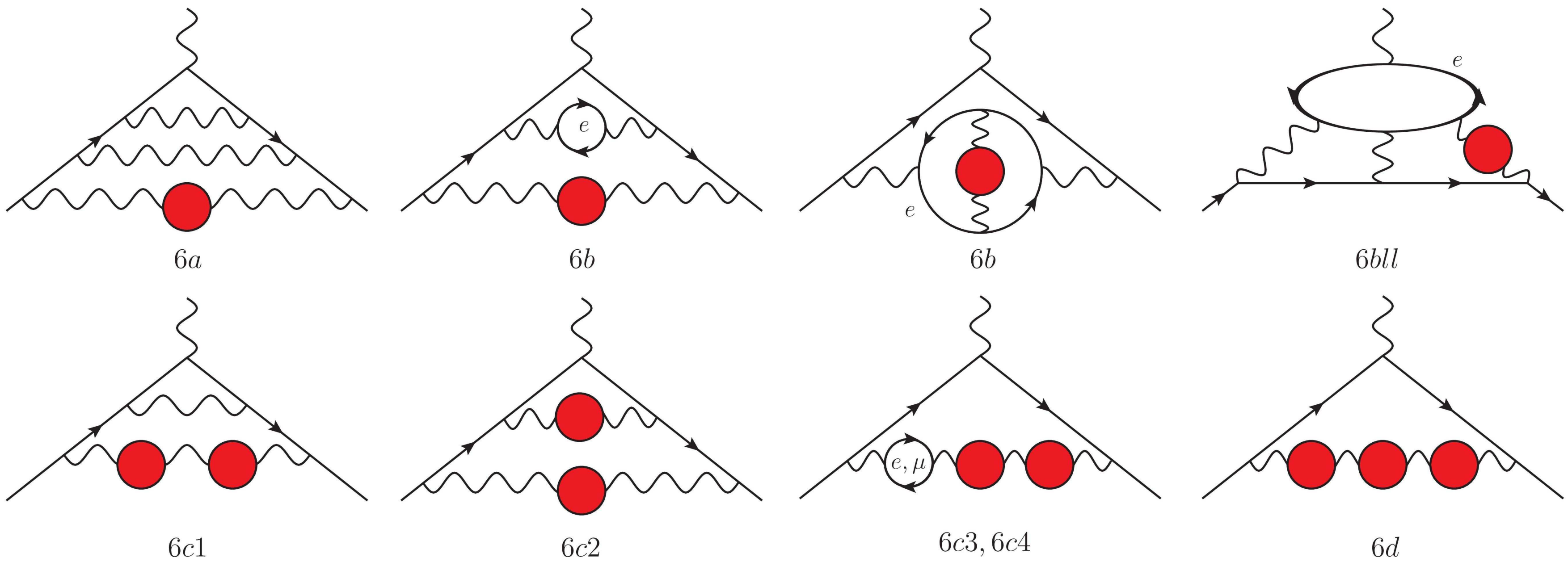
s_0	$(1.8\text{GeV})^2$	$(2.5\text{GeV})^2$	$(12\text{GeV})^2$
$a_\mu^{\text{HVP;LO(I)}} \cdot 10^{11}$	6868.0	6899.2	6944.7
$a_\mu^{\text{HVP;LO(II)}} \cdot 10^{11}$	58.8	36.2	2.9
$a_\mu^{\text{HVP;LO(III)}} \cdot 10^{11}$	4.1	-4.5	-16.7
$a_\mu^{\text{HVP;LO(IV)}} \cdot 10^{11}$	-0.011	0.005	$-1.3 \cdot 10^{-7}$
total	6930.9	6930.9	6930.9

$a_\mu^{\text{HVP;LO(II)}} \sim 1\%\ a_\mu^{\text{HVP;LO}}$ at $s_0 = (1.8\text{GeV})^2$

s_0	$(1.8\text{GeV})^2$	$(2.5\text{GeV})^2$	$(12\text{GeV})^2$
$a_\mu^{\text{HVP;NLO(4a)(I)}} \cdot 10^{11}$	-187.5	-194.8	-211.4
$a_\mu^{\text{HVP;NLO(4a)(II)}} \cdot 10^{11}$	-20.2	-14.8	-2.3
$a_\mu^{\text{HVP;NLO(4a)(III)}} \cdot 10^{11}$	-0.05	1.98	6.07
$a_\mu^{\text{HVP;NLO(4a)(IV)}} \cdot 10^{11}$	0.074	-0.082	$2.3 \cdot 10^{-4}$
total	-207.7	-207.7	-207.7

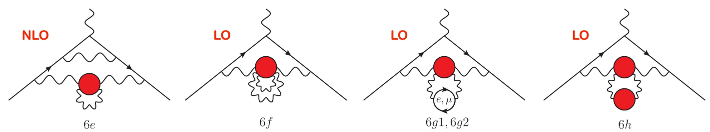
$a_\mu^{\text{HVP;NLO(4a)(II)}} \sim 10\%\ a_\mu^{\text{HVP(4a);NLO}}$ at $s_0 = (1.8\text{GeV})^2$

$a_\mu^{\text{HVP}}(\text{NLO; }4a) : 41\% \to 90\%$ *(preliminary)*

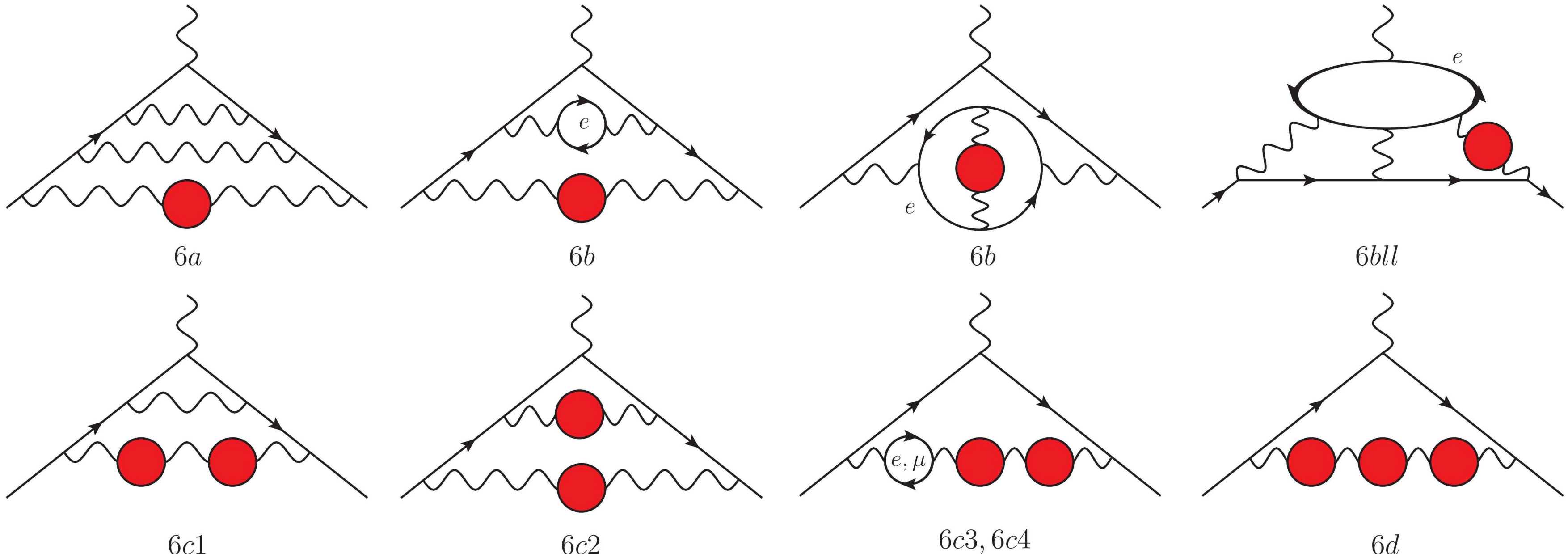


sample NNLO diagrams

- set 6a contains also diagrams with muon loops (like 6b 6bll)



- HVP with internal corrections already incorporated in NLO and LO
- $a_\mu^{\rm HVP}({\rm NNLO}; \mathit{total}) = +12.4(1) \times 10^{-11}$ Kurz Liu Marquard Steinhauser 2014



$$a_\mu^{\rm HVP}({\rm NNLO};6a) = \frac{\alpha^3}{\pi^4}\int_{m_\mu^2}^{\infty} \frac{ds}{s}\, K^{(6)}(s/m_\mu^2)\, {\rm Im}\Pi(s) = -\frac{\alpha^3}{\pi^4}\int_{-\infty}^{0} \frac{dt}{t}\Pi(t)\, {\rm Im}K^{(6)}(t/m_\mu^2)$$

- NLO: $K^{(4)}(s/m_\mu^2)$ is known analytically
- NNLO: $K^{(6)}(s/m_\mu^2)$ is NOT known analytically.
- Only a few terms of the *asymptotic expansions* for large s are known.
- We need to find *approximated* spacelike kernels from the asymptotic expansions

$K^{(6a)}(s/m_\mu^2)$: Only the first 4 terms of the expansion in power series of $r = m_\mu^2/s$ are known $\rightarrow$n=4

Kurz, Liu, Marquard, Steinhauser, PLB734 (2014) 144

The expansion in small r contain terms with $r^n \ln r$, $r^n \ln^2 r$ and $r^n \ln^3 r$. We use an integral ansatz:

$$K^{(6a)}(s/m_\mu^2) = r \int_0^1 d\xi \left[\frac{L^{(6a)}(\xi)}{\xi + r} + \frac{P^{(6a)}(\xi)}{1 + r\xi} \right] \qquad L^{(6a)}(\xi) = G^{(6a)}(\xi) + H^{(6a)}(\xi) \ln \xi + J^{(6a)}(\xi) \ln^2 \xi \quad \text{new@NNLO}$$

$G^{(6a)}$, $H^{(6a)}$,$J^{(6a)}$, $P^{(6a)}$ polynomials of degree 3

$$G^{(6a)}(\xi) = \sum_{i=0}^{3} g_i^{(6a)} \xi^i, \quad H^{(6a)}(\xi) = \sum_{i=0}^{3} h_i^{(6a)} \xi^i, \quad J^{(6a)}(\xi) = \sum_{i=0}^{3} j_i^{(6a)} \xi^i, \quad P^{(6a)}(\xi) = \sum_{i=0}^{3} p_i^{(6a)} \xi^i$$

We integrate in ξ, expand in r, and we fit the coefficients $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$ and $p_i^{(6a)}$, $i = 0, 1, 2, 3$, in order to match the coefficients of the asymptotic expansion in r of $K^{(6a)}(s/m_\mu^2)$. The approximated kernel $\bar{\kappa}^{(6a)}(x)$ is

$$a_\mu^{HVP}(\text{NNLO}; 6a) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^1 dx\, \bar{\kappa}^{(6a)}(x)\, \Delta\alpha_{\text{had}}(t(x)),$$

$$\bar{\kappa}^{(6a)}(x) = \begin{cases} \frac{2-x}{x(1-x)} P^{(6a)}\left(\frac{x^2}{1-x}\right), & 0 < x < x_\mu = (\sqrt{5} - 1)/2 = 0.618\ldots \\ \frac{2-x}{x^3} L^{(6a)}\left(\frac{1-x}{x^2}\right), & x_\mu < x < 1 \quad \text{discontinuous in } x_\mu \end{cases}$$

- The contributions of classes $(6b)$ and $(6bll)$ can be calculated similarly to class $(6a)$.
- $a_\mu^{\text{HVP}}(\text{NNLO}; 6a) = +8.0 \times 10^{-11}$ $\quad a_\mu^{\text{HVP}}(\text{NNLO}; 6b) = -4.1 \times 10^{-11}$ $\quad a_\mu^{\text{HVP}}(\text{NNLO}; 6bll) = +9.1 \times 10^{-11}$
- The uncertainty due to the series approximations of $K^{(6a)}$, $K^{(6b)}$, $K^{(6bll)}$ are estimated to be less than $O(10^{-12})$

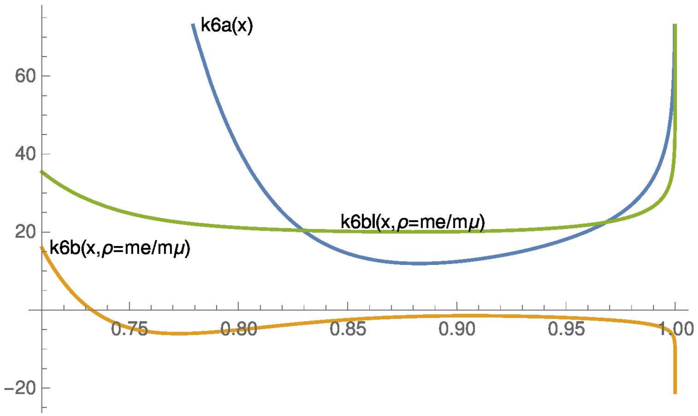
NNLO class *6a 6b 6bll 6c1 6c2 6c3 6c4 6d*

(6a)	
$j_0 = 0;$	$h_0 = -\frac{359}{36};$
$j_1 = -\frac{3793}{864};$	$h_1 = \frac{122293}{5184};$
$j_2 = \frac{35087}{21600};$	$h_2 = -\frac{43879427}{648000};$
$j_3 = \frac{1592093}{43200};$	$h_3 = \frac{14388407}{48000};$
$g_0 = \frac{1301}{144} - \frac{19\pi^2}{9};$	
$g_1 = \frac{441277}{10368} + \pi^2\left(-\frac{355}{648} + \ln 4\right) + \frac{25\ \zeta(3)}{2};$	
$g_2 = -\frac{5051645167}{38880000} + \pi^2\left(\frac{221411}{32400} - 18\ln 2\right) - \frac{3919\ \zeta(3)}{60};$	
$g_3 = \frac{14588342017}{38880000} + \pi^2\left(-\frac{2479681}{64800} + 112\ln 2\right) + \frac{3113\ \zeta(3)}{10};$	
$p_0 = -\frac{1808080780513}{14580000} + \frac{41851\pi^4}{15} + \frac{8432\ln^4 2}{3} + 67456\ a_4 + \frac{2085448\ \zeta(3)}{15} + +\pi^2\left(-\frac{11944163099}{194400} + \frac{272}{3}(180 - 31\ln 2)\ln 2 + \frac{115072\ \zeta(3)}{3}\right) - \frac{575360\ \zeta(5)}{3};$	
$p_1 = \frac{134017456919}{96000} - \frac{4481182\pi^4}{135} - \frac{98420\ln^4 2}{3} - 787360\ a_4 + 2255200\ \zeta(5) + +\pi^2\left(\frac{23549054249}{32400} - 201122\ln 2 + \frac{98420\ln^2 2}{3} - 451040\ \zeta(3)\right) - \frac{57189259\ \zeta(3)}{36};$	
$p_2 = -\frac{13069081405453}{3888000} + \frac{330073\pi^4}{4} + 80790\ln^4 2 + 1938960\ a_4 + \frac{77371609\ \zeta(3)}{20} + +\pi^2\left(-\frac{729995599}{405} + 6(85313 - 13465\ln 2)\ln 2 + 1114360\ \zeta(3)\right) - 5571800\ \zeta(5);$	
$p_3 = \frac{1274611832039}{583200} - \frac{986377\pi^4}{18} - 53340\ln^4 2 - 1280160\ a_4 + \frac{11057200\ \zeta(5)}{3} + +\pi^2\left(\frac{5809659289}{4860} + 420\ln 2(-823 + 127\ln 2) - \frac{2211440\ \zeta(3)}{3}\right) - \frac{22833188\ \zeta(3)}{9};$	

Table 1: The coefficients $g_i^{(6a)}$, $h_i^{(6a)}$, $j_i^{(6a)}$, $p_i^{(6a)}$ $(i = 0,1,2,3)$. The superscript $(6a)$ has been dropped for simplicity. In the above coefficients, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$ and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.

(6b)	
$j_0 = 0;$	$h_0 = \frac{65}{54};$
$j_1 = \frac{11}{27};$	$h_1 = -\frac{3559}{1296} + \rho^2 + \frac{5}{18}\ln\rho;$
$j_2 = \frac{41}{120};$	$h_2 = \frac{3917}{432} - \frac{82\rho^2}{3} + \frac{61}{10}\ln\rho;$
$j_3 = -\frac{507}{40};$	$h_3 = -\frac{4109}{80} + \frac{2211\rho^2}{10} - \frac{1763}{30}\ln\rho;$
$g_0 = \frac{1}{108}\left(259 - 72\rho^2 + 276\ln\rho\right);$	
$g_1 = -\frac{9215}{1296} + \frac{65\pi^2}{162} - \frac{3\pi^2\rho}{4} + \frac{49\rho^2}{36} + \left(-\frac{301}{54} + 8\rho^2\right)\ln\rho + \frac{4}{3}\ln^2\rho + 2\ \zeta(3);$	
$g_2 = \frac{501971}{40500} - \frac{113\pi^2}{36} + \frac{270\pi^2\rho}{36} - \frac{8417\rho^2}{180} + \left(\frac{3479}{900} - 44\rho^2\right)\ln\rho - 8\ln^2\rho - 12\ \zeta(3);$	
$g_3 = -\frac{2523823}{324000} + \frac{625\pi^2}{36} - 49\pi^2\rho + \frac{84946\rho^2}{225} + \left(\frac{987}{50} + 200\rho^2\right)\ln\rho + \frac{112}{3}\ln^2\rho + 56\ \zeta(3);$	
$p_0 = -\frac{95519053063}{486000} - 7275\pi^2\rho + \left(-\frac{587150693}{5400} + \frac{75272\rho^2}{3} + \frac{120800\pi^2}{9}\right)\ln\rho + \left(\frac{1135508}{9} + 96\rho^2\right)\zeta(3) + +4720\ln^2\rho + \frac{1067115409\rho^2}{5400} + \pi^2\left(\frac{24382331}{810} - \frac{285184}{9}\ln 2\right) - 32\pi^2\rho^2(687 + \ln 4);$	
$p_1 = \frac{279489728279}{121500} + \frac{179283\pi^2\rho}{2} + \left(\frac{2280933773}{1800} - 309540\rho^2 - \frac{1419328\pi^2}{9}\right)\ln\rho - \frac{10}{3}\left(446023 + 216\rho^2\right)\zeta(3) + -\frac{174712}{3}\ln^2\rho - \frac{174350167\rho^2}{75} + \pi^2\left(-\frac{143574463}{405} + \frac{3352256\ln 2}{9}\right) + \frac{16}{3}\pi^2\rho^2(48481 + 90\ln 2);$	
$p_2 = -\frac{229560199193}{40500} - \frac{912495\pi^2\rho}{4} + \left(-\frac{1867939691}{600} + 788488\rho^2 + \frac{1168336\pi^2}{3}\right)\ln\rho + \left(\frac{11034553}{3} + 1440\rho^2\right)\zeta(3) + +148348\ln^2\rho + \frac{258653648\rho^2}{45} + \frac{4}{135}\pi^2(29597029 - 31048560\ln 2) - \frac{320}{3}\pi^2\rho^2(5989 + \ln 512);$	
$p_3 = \frac{72762177677}{19440} + 154035\pi^2\rho - \frac{7}{108}\left(-31650719 + 3973440\pi^2 + 8220240\rho^2\right)\ln\rho - \frac{280}{9}\left(78283 + 27\rho^2\right)\zeta(3) + -100240\ln^2\rho - \frac{513692207\rho^2}{135} + \frac{35}{162}\pi^2(-2687659 + 2816064\ln 2) + \frac{140}{3}\pi^2\rho^2(9055 + \ln 4096);$	

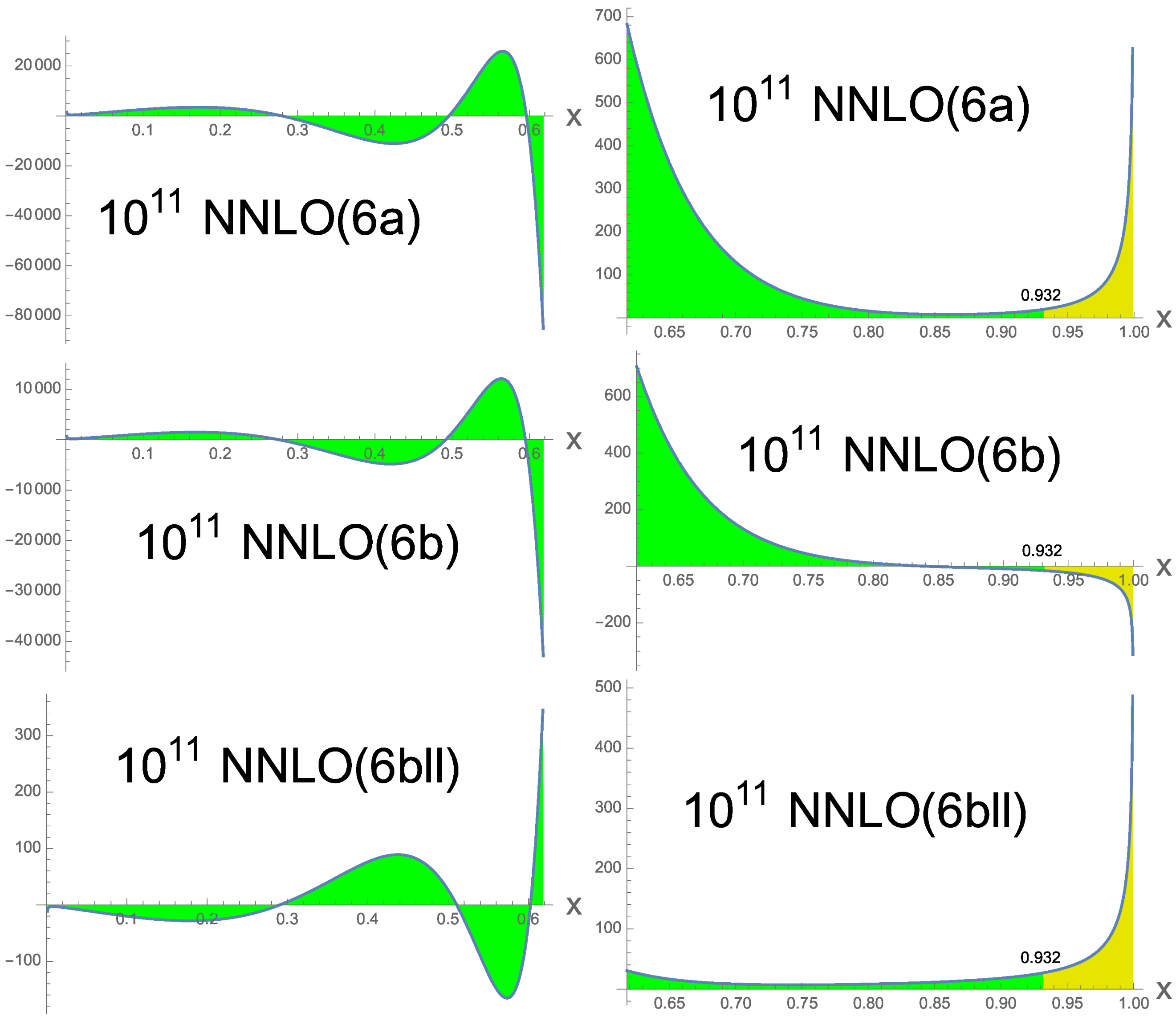
Table 2: The coefficients $g_i^{(6b)}$, $h_i^{(6b)}$, $j_i^{(6b)}$, $p_i^{(6b)}$ $(i = 0,1,2,3)$. The superscript $(6b)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.



(6bll)	
$j_0 = 0;$	$h_0 = -\frac{9}{2};$
$j_1 = \frac{4}{27} - \frac{9\rho^2}{2};$	$h_1 = \frac{59}{9} - \frac{275\rho^2}{36} - 18\rho^2\ln\rho;$
$j_2 = -\frac{41}{48} + \frac{2201\rho^2}{216};$	$h_2 = -\frac{485}{32} + \frac{1351\rho^2}{48} + \frac{659\rho^2}{18}\ln\rho;$
$j_3 = \frac{3037}{900} - \frac{5909\rho^2}{216};$	$h_3 = \frac{282617}{6750} - \frac{10481\rho^2}{108} - \frac{851\rho^2}{9}\ln\rho;$
$g_0 = \frac{43}{8} - 4\pi^2\rho + 15\rho^2 + \pi^2\rho^2 - 18\rho^2\ln\rho + 6\rho^2\ln^2\rho;$	
$g_1 = -\frac{73}{81} + \frac{8\pi^2}{81} + \frac{40\pi^2\rho}{9} + \frac{2437\rho^2}{108} + \frac{17\pi^2\rho^2}{9} + \frac{607\rho^2}{18}\ln\rho - \frac{20\rho^2}{3}\ln^2\rho + \frac{2}{3}\zeta(3) + 2\rho^2\zeta(3);$	
$g_2 = -\frac{385}{162} - \frac{41\pi^2}{72} - \frac{28\pi^2\rho}{3} - \frac{89873\rho^2}{5184} - \frac{997\pi^2\rho^2}{324} - \frac{1961\rho^2}{72}\ln\rho + 14\rho^2\ln^2\rho - \frac{5}{2}\zeta(3) - \frac{16\rho^2}{3}\zeta(3);$	
$g_3 = \frac{2691761}{202500} + \frac{3037\pi^2}{1350} + 24\pi^2\rho + \frac{655429\rho^2}{97200} + \frac{2359\pi^2\rho^2}{324} + \frac{6943\rho^2}{360}\ln\rho - 36\rho^2\ln^2\rho + \frac{42}{5}\zeta(3) + 15\rho^2\zeta(3);$	
$p_0 = -\frac{343277101}{45000} - \frac{33156604927\rho^2}{583200} + \pi^2\left(-\frac{615427}{4050} + \frac{6776\rho}{3} + \frac{763121\rho^2}{972}\right) - \frac{4\pi^4}{135}(7817 + 3212\rho^2) + +\left(-\frac{7290521}{3240} + \frac{49622\pi^2}{27} - \frac{128\pi^4}{9}\right)\rho^2\ln\rho + \left(-3388 - \frac{80\pi^2}{3}\right)\rho^2\ln^2\rho + +\left(25642 + \frac{1515724\rho^2}{27} - 128\pi^2\rho^2 - 160\rho^2\ln\rho\right)\zeta(3) - \frac{1280}{3}\rho^2\zeta(5);$	
$p_1 = \frac{89280434843}{972000} + \frac{248834878697\rho^2}{388800} - \frac{1}{324}\pi^2\left(-533001 + 9110736\rho + 3110417\rho^2\right) + \frac{2}{135}\pi^4\left(180247 + 73530\rho^2\right) + +\left(\frac{11101973}{1080} - \frac{193400\pi^2}{9} + \frac{320\pi^4}{3}\right)\rho^2\ln\rho + \frac{2}{3}\left(63269 + 300\pi^2\right)\rho^2\ln^2\rho + +\frac{1}{45}\left(-13410977 + 100\left(-292301 + 432\pi^2\right)\rho^2 + 54000\rho^2\ln\rho\right)\zeta(3) + 3200\rho^2\zeta(5);$	
$p_2 = -\frac{6209532853}{27000} - \frac{29997466847\rho^2}{19440} + \pi^2\left(-\frac{114521}{30} + 71840\rho + \frac{1970140\rho^2}{81}\right) - \frac{4}{9}\pi^4\left(14685 + 6032\rho^2\right) + -\frac{1}{54}\left(190613 - 2847360\pi^2 + 11520\pi^4\right)\rho^2\ln\rho - 80\left(1347 + 5\pi^2\right)\rho^2\ln^2\rho + -\frac{10}{9}\left(-658509 + \left(-1431463 + 1728\pi^2\right)\rho^2 + 2160\rho^2\ln\rho\right)\zeta(3) - 6400\rho^2\zeta(5);$	
$p_3 = \frac{49726331179}{324000} + \frac{7324831423\rho^2}{7290} + \pi^2\left(\frac{3897971}{1620} - \frac{145880\rho}{3} - \frac{3977785\rho^2}{243}\right) + \frac{14}{27}\pi^4\left(8269 + 3419\rho^2\right) + +\frac{7}{81}\left(-81551 - 401520\pi^2 + 1440\pi^4\right)\rho^2\ln\rho + \frac{140}{3}\left(1563 + 5\pi^2\right)\rho^2\ln^2\rho + +\frac{35}{27}\left(-371889 + 16\left(-50437 + 54\pi^2\right)\rho^2 + 1080\rho^2\ln\rho\right)\zeta(3) + \frac{11200}{3}\rho^2\zeta(5);$	

Table 3: The coefficients $g_i^{(6bll)}$, $h_i^{(6bll)}$, $j_i^{(6bll)}$, $p_i^{(6bll)}$ $(i = 0,1,2,3)$. The superscript $(6bll)$ has been dropped for simplicity. In the above coefficients, $\rho = m_e/m$, the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} 1/n^k$, and $a_4 = \sum_{n=1}^{\infty} 1/(2^n n^4) = \mathrm{Li}_4(1/2)$.

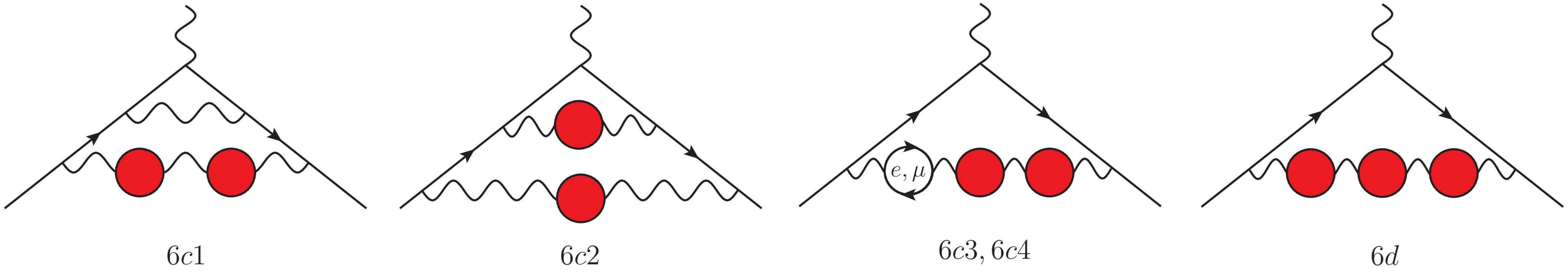
NNLO integrands $6a$ $6b$ $6bll$



Plot of spacelike NNLO integrands $6a$ $6b$ $6bll$

Huge, almost complete cancellations between positive and negative parts of integrands

Part of the integral directly scanned by MUonE: $6a$: 15%, $6b$: 16%, $6bll$: 38%.



$$a_\mu^{HVP}(\text{NNLO}; 6c) = a_\mu^{HVP}(\text{NNLO}; 6c1) \; + a_\mu^{HVP}(\text{NNLO}; 6c2) \; + a_\mu^{HVP}(\text{NNLO}; 6c3) \; + a_\mu^{HVP}(\text{NNLO}; 6c4)$$

$$a_\mu^{HVP}(\text{NNLO}; 6c1) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \mathrm{d}x \left[\kappa^{(4)}(x) \; - \; \frac{2\pi}{\alpha}\kappa^{(2)}(x)\, \Delta\alpha_\mu^{(2)}(t(x))\right] \left[\Delta\alpha_{\text{had}}(t(x))\right]^2$$

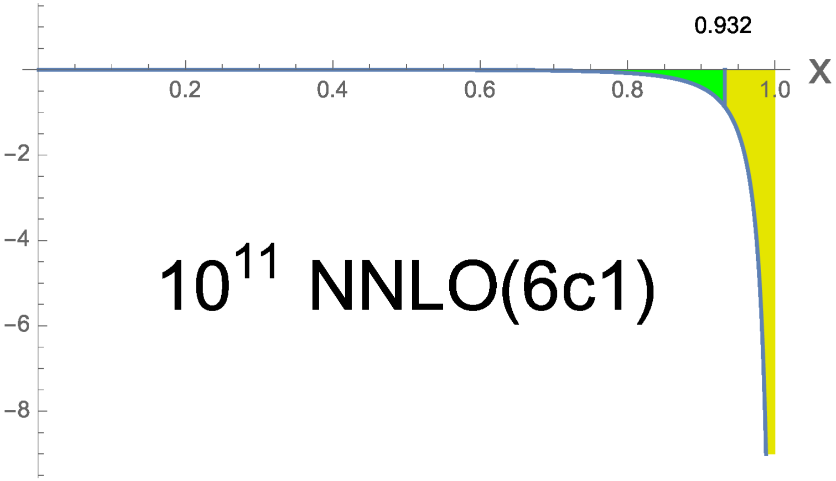
6c4 separated
multiplicity=3

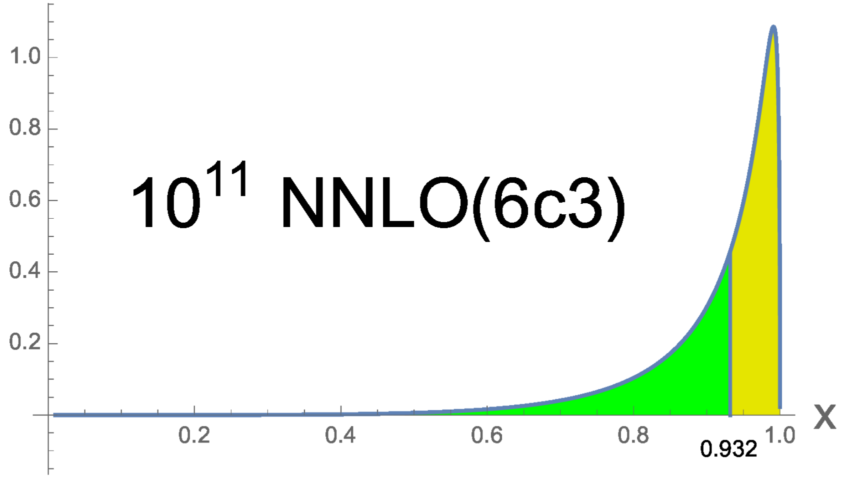
$$a_\mu^{HVP}(\text{NNLO}; 6c3) = \frac{3\alpha}{\pi} \int_0^1 \mathrm{d}x\, \kappa^{(2)}(x) \left[\Delta\alpha_{\text{had}}(t(x))\right]^2 \Delta\alpha_e^{(2)}(t(x))$$

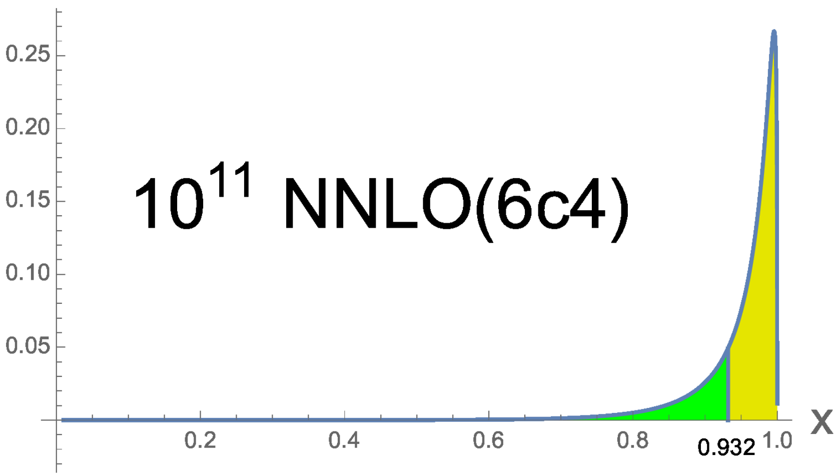
$$a_\mu^{HVP}(\text{NNLO}; 6c4) = \frac{3\alpha}{\pi} \int_0^1 \mathrm{d}x\, \kappa^{(2)}(x) \left[\Delta\alpha_{\text{had}}(t(x))\right]^2 \Delta\alpha_\mu^{(2)}(t(x))$$

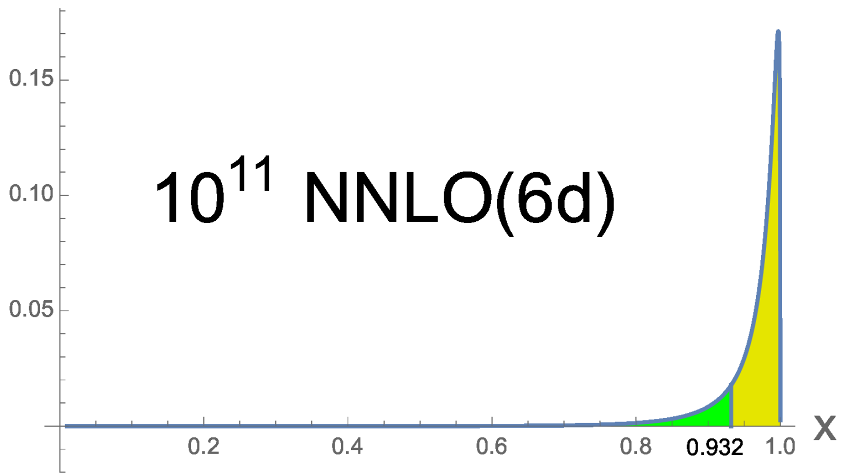
$$a_\mu^{HVP}(\text{NNLO}; 6d) \; = \; \frac{\alpha}{\pi} \int_0^1 \mathrm{d}x\, \kappa^{(2)}(x) \; \left[\Delta\alpha_{\text{had}}(t(x))\right]^3$$

$a_\mu^{HVP}(6c1) = -5 \times 10^{-12}$, $a_\mu^{HVP}(6c3) = 0.9 \times 10^{-12}$, $a_\mu^{HVP}(6c4) = 0.1 \times 10^{-12}$, $a_\mu^{HVP}(6d) = 0.05 \times 10^{-12}$ 6c2 ?



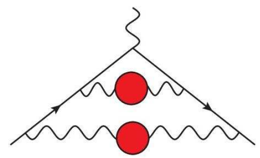






peak at $x = 0.997$

Part of the integral directly scanned by MUonE: $6c1 : 9\%$, $6c3 : 44\%$, $6c4 : 23\%$, $6d : 16\%$.



$6c2$

This class requires *double* integrals

$$a_\mu^{HVP}(\text{NNLO}; 6c2) = \frac{\alpha^2}{\pi^4}\int_{s_0}^{\infty}\frac{\mathrm{d}s}{s}\int_{s_0}^{\infty}\frac{\mathrm{d}s'}{s'}\,K^{(6c2)}(s/m_\mu^2, s'/m_\mu^2)\mathrm{Im}\Pi_{\text{had}}(s)\,\mathrm{Im}\Pi_{\text{had}}(s').$$

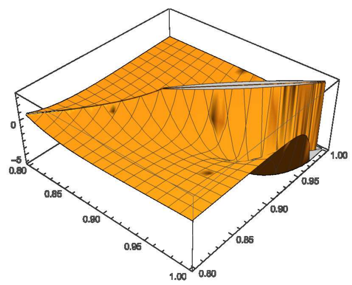
$$a_\mu^{HVP}(\text{NNLO}; 6c2) \;=\; \left(\frac{\alpha}{\pi}\right)^2\int_{x_\mu}^{1}\mathrm{d}x\int_{x_\mu}^{1}\mathrm{d}x'\,\bar{\kappa}^{(6c2)}(x,x')\Delta\alpha_{\text{had}}(t(x))\Delta\alpha_{\text{had}}(t(x')),$$

$\bar{\kappa}^{(6c2)}(x,x')$ space-like bidimensional kernel, $\quad x_\mu < \{x, x'\} < 1$

$$\bar{\kappa}^{(6c2)}(x,x') \;=\; \frac{2-x}{x^3}\frac{2-x'}{x'^3}\,G^{(6c2)}\!\left(\frac{1-x}{x^2}, \frac{1-x'}{x'^2}\right)$$

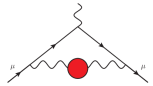
From the leading terms of the known asymptotic expansion of $K^{(6c2)}(s/m_\mu^2, s'/m_\mu^2)$:
$s/s' \ll 1$ or $s/s' \approx 1$ or $s/s' \ggg 1$ and $s, s' \gg m_\mu^2$ we get the approximated space-like kernel

$$G^{(6c2)}(\xi,\xi') = \frac{1855-188\pi^2}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')}{\max(\xi,\xi')^2} + \frac{988\pi^2-9765}{4\,(32\pi^2-315)}\frac{\min(\xi,\xi')^2}{\max(\xi,\xi')^3} + \frac{6\,(435-44\pi^2)}{32\pi^2-315}\frac{\min(\xi,\xi')^3}{\max(\xi,\xi')^4}$$



plot of $\bar{\kappa}^{(6c2)}(x,x')$

Contribution of 6c2 class is $a_\mu^{HVP}(6c2) = -1.8\times10^{-12}$
The uncertainty of this leading order approximation is estimated to be $\sim 10^{-13}$
NNLO(6c2): part of the integral directly scanned by MUonE= 6% of the diagram contribution



$$a_\mu^{\rm HVP}({\rm LO}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\, \tilde{K}_2(t, m_\mu)$$

- $G(t)$ correlator of e.m.currents ← lattice
- $\tilde{K}_2(t, m_\mu)$ LO time-kernel
- t Euclidean time

(Bernecker Meyer 2011)

$$\tilde{K}_2(t, m_\mu) = \tilde{f}_2(t) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega}\, f_2(\omega^2) \left[\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right]$$

$$f_2(\omega^2) = \frac{1}{\pi} \frac{{\rm Im}K^{(2)}(-\omega^2/m_\mu^2)}{-\omega^2} = \frac{1}{m_\mu^2} \frac{1}{y(-\hat{\omega}^2)(1 - y^2(-\hat{\omega}^2))}$$

${\rm Im}K^{(2)}(q^2)$ LO space-like kernel

$y(z) \equiv \frac{z-\sqrt{z(z-4)}}{z+\sqrt{z(z-4)}}$ $\hat{\omega} = \omega/m_\mu$

Analytical integration possible!: $\hat{t} = m_\mu t$ $(\hat{t} = 1 \;\rightarrow\; t = 1.86{\rm fm})$

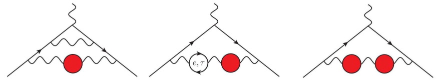
$$\frac{m_\mu^2}{8\pi^2}\tilde{f}_2(t) = \frac{1}{4} \overbrace{G^{2,1}_{1,3}\left(\begin{smallmatrix} \frac{3}{2} \\ 0,1,\frac{1}{2} \end{smallmatrix} \middle| \hat{t}^2\right)}^{\text{Meijer G-function}} + \frac{\hat{t}^2}{4} + \frac{1}{\hat{t}^2} + 2(\ln\hat{t} + \gamma) - \frac{2}{\hat{t}} K_1(2\hat{t}) - \frac{1}{2}$$

(Della Morte et al 2017)

$$= -\pi t^2 \overbrace{\left(\mathbf{L}_{-1}(2\hat{t})K_0(2\hat{t}) + \mathbf{L}_0(2\hat{t})K_1(2\hat{t})\right)}^{\text{Struve Bessel functions}} + \frac{\hat{t}^2}{4} + \frac{1}{\hat{t}^2} - \left(\frac{2}{\hat{t}} + \hat{t}\right) K_1(2\hat{t}) + 2(\ln\hat{t} + \gamma) - \frac{1}{2}$$

(E.Balzani, S.L, M.Passera 2023)

$$\frac{m_\mu^2}{8\pi^2}\tilde{f}_2(t) = \begin{cases} \frac{\hat{t}^4}{72} + \frac{(120(\ln\hat{t}+\gamma)-169}{43200}\hat{t}^6 + \ldots & \hat{t} \ll 1 \\ \frac{t^2}{4} - \frac{\pi\hat{t}}{2} + 2(\ln\hat{t} + \gamma) - \frac{1}{2} + \frac{1}{\hat{t}^2} \underbrace{+\sqrt{\frac{\pi}{\hat{t}}}e^{-2\hat{t}}\left[-\frac{1}{4} - \frac{55}{64\hat{t}} + \ldots\right]}_{\text{exponentially suppressed}} & \hat{t} \gg 1 \end{cases}$$



$$a_\mu^{\rm HVP}(\text{NLO;4a}) = \left(\frac{\alpha}{\pi}\right)^3 \int_0^\infty dt\; G(t)\; \tilde{K}_4(t, m_\mu)$$

- $G(t)$ correlator of e.m.currents $\leftarrow$ lattice
- $\tilde{K}_4(t, m_\mu)$ NLO(4a) time-kernel
- t Euclidean time

$$\tilde{K}_4(t, m_\mu) = \tilde{f}_4(t) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega}\; f_4(\omega^2) \left[\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right]$$

$$\hat{f}_4(\hat{\omega}^2) = m_\mu^2 f_4(\hat{\omega}^2) = \frac{2\; F_4(1/y(-\hat{\omega}^2))}{-\omega^2}$$

F_4 NLO(4a) space-like kernel

- integral with $(\omega t)^2$ analytically easy; integral with $\sin(\omega t)$ difficult.
- $F_4(1/y)$ contains logarithms and dilogarithms of $\pm y(\omega)$:
- analytical integration in ω of some dilogarithms and products of logarithms complicated but feasible $\rightarrow$ Bessel functions, exponential integrals, generalized Meijer G-functions,
  example: $\int_0^\infty d\hat{\omega} \frac{\ln^2\left(\frac{1}{2}\left(\sqrt{\hat{\omega}^2+4}+\hat{\omega}\right)\right)}{\sqrt{\hat{\omega}^2+4}} \cos(\hat{\omega}\hat{t}) = \frac{\partial^2}{\partial n^2} K_n(2\hat{t})|_{n=0} - \frac{1}{4}\pi^2 K_0(2\hat{t})$
- but still not able to integrate analytically all the integral with $\mathrm{Li}_2(\pm y)$

Alternative: Series expansions

We split the interval of integration in a intermediate point $\hat{\omega}_0(\hat{t})$:

$$\int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\omega^2)\left[(\hat{\omega}t)^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right] = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\hat{\omega}^2) g(\hat{\omega}\hat{t}) = \int_0^{\hat{\omega}_0(\hat{t})} \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\hat{\omega}^2) g(\hat{\omega}\hat{t}) \quad \leftarrow \begin{array}{l}\text{expand } g \text{ for } \hat{t} \ll 1 \\ \text{change } \hat{\omega} \to y(-\hat{\omega}^2)\end{array}$$

$$+ \int_{\hat{\omega}_0(\hat{t})}^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\hat{\omega}^2) g(\hat{\omega}\hat{t}) \quad \leftarrow \text{ expand } \hat{f}_4 \text{ for } \hat{\omega} \gg 1$$

integral independent of $\hat{\omega}_0$: convenient choice for calculation: $\hat{\omega}_0 = \frac{1-\hat{t}}{\sqrt{\hat{t}}} \gg 1 \quad \Rightarrow y\left(-\hat{\omega}_0^2\right) = -\hat{t}$.

The final result of expansion:

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \sum_{\substack{n\geq 4 \\ n \text{ even}}} \frac{\hat{t}^n}{n!}\left(a_n + b_n\pi^2 + c_n\left(\ln(\hat{t}) + \gamma\right) + d_n\left(\ln(\hat{t}) + \gamma)^2\right)\right)$$

- π^2 and $(\ln \hat{t} + \gamma)^2$ appear at NLO
- Coefficients a_n, b_n, c_n, d_n up to $\hat{t}^{30}$ were calculated
- series converges for every $\hat{t}$, but for $\hat{t} \gtrsim 5$ terms grow fast, then change sign and start decreasing: huge cancellations!
- It needs other kind of expansions to cover the large-$\hat{t}$ region

NLO time-kernel: Expansion for small t

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \sum_{\substack{n\geq 4\\ n\ \text{even}}} \frac{\hat{t}^n}{n!}\left(a_n + b_n\pi^2 + c_n\left(\ln(\hat{t})+\gamma\right) + d_n\left(\ln(\hat{t})+\gamma\right)^2\right)$$

n	$\mathbf{a_n}$	$\mathbf{b_n}$	$\mathbf{c_n}$	$\mathbf{d_n}$
4	$\frac{317}{216}$	$-\frac{1}{3}$	$\frac{23}{18}$	0
6	$\frac{843829}{259200}$	$-\frac{371}{432}$	$\frac{877}{1080}$	$\frac{19}{36}$
8	$\frac{412181237}{5292000}$	$-\frac{233}{48}$	$-\frac{824603}{25200}$	$\frac{141}{20}$
10	$\frac{6272504689}{10584000}$	$-\frac{1165}{48}$	$-\frac{460711}{1680}$	$\frac{961}{20}$
12	$\frac{404220031035193}{121022748000}$	$-\frac{42443}{378}$	$-\frac{1359283213}{873180}$	$\frac{79342}{315}$
14	$\frac{14790819716039431}{890463974400}$	$-\frac{142931}{288}$	$-\frac{4138386457}{540540}$	$\frac{28243}{24}$
16	$\frac{38888413518277699}{503454631680}$	$-\frac{12895145}{6048}$	$-\frac{489120278261}{13970880}$	$\frac{2605993}{504}$
18	$\frac{395063308536506 7019}{11462583132000}$	$-\frac{116506871}{12960}$	$-\frac{4589675124823}{29937600}$	$\frac{23642359}{1080}$
20	$\frac{3647218698026344775775 71}{243865691961091200}$	$-\frac{55559731}{1485}$	$-\frac{37593205363634911}{57616158600}$	$\frac{44767436}{495}$
22	$\frac{773922392827939458822 49}{12165635426630400}$	$-\frac{610873921}{3960}$	$-\frac{26135521670035411}{9602693100}$	$\frac{121188929}{330}$
24	$\frac{27318770927965379913670522297}{10248726666544814448 00}$	$-\frac{19509636989}{30888}$	$-\frac{5138081420797732289}{459392837904}$	$\frac{3789385597}{2574}$
26	$\frac{4499684907681688287146651006 63}{40761981060121421100 00}$	$-\frac{5618399257}{2184}$	$-\frac{15810911801773817669}{348024877200}$	$\frac{151912159}{26}$
28	$\frac{2511462939294980551566835497 73}{55458477632818260 0000}$	$-\frac{678234361}{65}$	$-\frac{3787066553671821473}{20715766500}$	$\frac{1495034796}{65}$
30	$\frac{10079211746301768464355522417826916 8501}{546805545707624630499072 00000}$	$-\frac{2551294690547}{60480}$	$-\frac{30599625762869165887 5533}{419236121304000}$	$\frac{64743309493}{720}$

Table 1: Coefficients of the expansion of $\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t)$ up to $\hat{t}^{30}$,

$$\tilde{f}_4(t) = 8\pi^2 \int_0^\infty \frac{d\omega}{\omega}\, f_4(\omega^2) \left[\omega^2 t^2 - 4\sin^2\left(\frac{\omega t}{2}\right)\right] \qquad \tilde{f}_4(t) = \tilde{f}_4^{(a)}(t) + \tilde{f}_4^{(b)}(t)$$

$$\frac{m_\mu^2}{8\pi^2}\tilde{f}_4^{(a)}(t) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\, \hat{f}_4(\hat{\omega}^2)\left(\hat{\omega}^2\hat{t}^2\right) = \frac{\hat{t}^2}{2}\int_{-\infty}^0 \frac{dz}{z}\,\frac{1}{\pi}\mathrm{Im}K_4(z) = \frac{\hat{t}^2}{2}K_4(0) = \frac{\hat{t}^2}{2}\left(\frac{197}{144} + \frac{\pi^2}{12} - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta(3)\right) \text{ easy}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b)}(t) = \int_0^\infty \frac{d\hat{\omega}}{\hat{\omega}}\hat{f}_4(\hat{\omega}^2)\left(-4\sin^2\left(\frac{\hat{\omega}\hat{t}}{2}\right)\right) \qquad \text{adimensionalized } \hat{f}_4(\hat{\omega}^2) \equiv m_\mu^2 f_4(\hat{\omega}^2)$$

Decomposition of $\tilde{f}_4^{(b)}(t)$ according to the different behaviour for $t \to \infty$.

$$\tilde{f}_4^{(b)}(t) = \tilde{f}_4^{(b;1)}(t) \qquad \to \text{ dominant no exponential prefactors new to NLO}$$
$$+ \tilde{f}_4^{(b;2)}(t) \qquad \to \text{ exponentially suppressed } e^{-2\hat{t}}\text{prefactor see LO expansion}$$

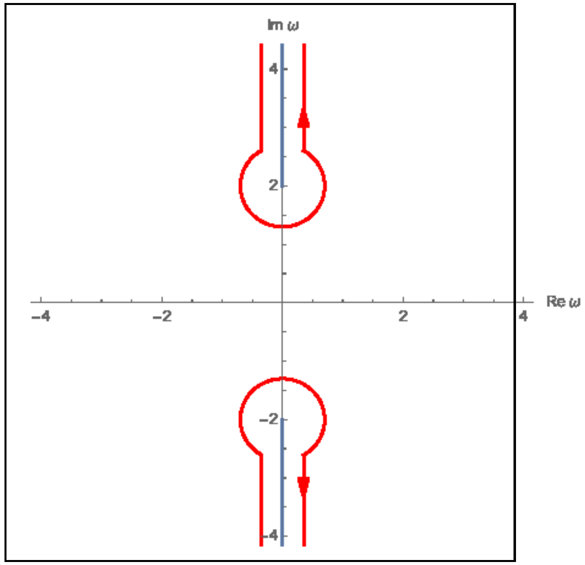
Expanding in series the dominant part (integrate formal expansion of $\hat{f}_4(\hat{\omega}^2)$ in $\hat{\omega} = 0$)

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1)}(t) = -\frac{\pi\hat{t}}{8} + \ln\hat{t} + \gamma - \frac{7\zeta(3)}{4} + \frac{7}{6}\pi^2\ln(2) - \frac{127\pi^2}{144} + \frac{653}{216} - \frac{5\left(\ln\hat{t} + \gamma\right)}{12\hat{t}^2} - \frac{\pi}{2\hat{t}} + \frac{209}{180\hat{t}^2} + \frac{277\pi}{360\hat{t}^3} + O\left(\frac{1}{\hat{t}^4}\right)$$

- series expansion is asymptotic, factorial growth of coefficients, example: $-\frac{12510892800}{19\hat{t}^{18}}$
- asymptotic series needs truncation, almost useless numerically, error $\sim e^{-2\hat{t}}$

$\tilde{f}_4^{(b;2)}(t)$: contour integral over $\mathcal{C}$

$$\frac{\tilde{f}_4^{(b;2)}(t)}{8\pi^2} = \int_{\mathcal{C}} \frac{d\hat{\omega}}{\hat{\omega}} \hat{f}_4(\hat{\omega}^2) 2\cos(\hat{\omega}\hat{t})$$



Its asymptotic expansion contains the factor $e^{-2\hat{t}}$:

$$\tilde{f}_4^{(b;2)}(t) = e^{-2\hat{t}} \sum_{n=0}^{\infty} \left(D_n + \frac{E_n \ln \hat{t} + F_n}{\sqrt{\hat{t}}} \right) \frac{1}{\hat{t}^n}$$

where D_n, E_n and F_n are constants.

- The exponential factor is due to the singularities of the integrand in $\hat{\omega} = \pm 2i$, which come from the terms containing $\sqrt{\hat{\omega}^2 + 4}$ in $\hat{f}_4(\hat{\omega})$
- coefficient of these series not useful, the truncation error of the dominant series ($\sim e^{-2\hat{t}}$) is of the same order of the exponentially suppressed series
- the $\mathcal{C}$ contour is around the imaginary axis: Fourier integrals $\to$ Laplace integrals
- We need expansions around *finite* points $\hat{t} = \hat{t}_0$ *converging* for $\hat{t} \to \infty$.

- In order to obtain numerically efficient expansions around finite $\hat{t}$, we have to introduce further splitting, separating according the prefactors: even and odd powers in $f_4^{(b;1)}(t)$ and integer and half-integer powers, and logarithms in $f_4^{(b;2)}(t)$.

$$\tilde{f}_4^{(b;1)}(t) = \tilde{f}_4^{(b;1;1)}(t) + \tilde{f}_4^{(b;1;2)}(t) + \tilde{f}_4^{(b;1;3)}(t)$$
$$\tilde{f}_4^{(b;2)}(t) = \tilde{f}_4^{(b;2;1)}(t) + \tilde{f}_4^{(b;2;2)}(t) + \tilde{f}_4^{(b;2;3)}(t)$$

where

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;1)}(t) \sim \frac{1}{\hat{t}} + O\left(\frac{1}{\hat{t}^3}\right), \qquad \text{only odd powers (which have a factor } \pi)$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;2)}(t) \sim \frac{1}{\hat{t}^2} + O\left(\frac{1}{\hat{t}^4}\right), \qquad \text{only even powers}$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;1)}(t) \sim e^{-2\hat{t}}\left[1 + O\left(\frac{1}{\hat{t}^2}\right)\right],$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;2)}(t) \sim e^{-2\hat{t}}\frac{\ln(\hat{t})}{\sqrt{\hat{t}}}\left[1 + O\left(\frac{1}{\hat{t}}\right)\right],$$
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;3)}(t) \sim e^{-2\hat{t}}\frac{1}{\sqrt{\hat{t}}}\left[1 + O\left(\frac{1}{\hat{t}}\right)\right],$$

$\tilde{f}_4^{(b;1;3)}(t)$ contains the part not included in the above asymptotic expansions:

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;3)}(t) = -\frac{\pi\hat{t}}{8} + \left(\ln\hat{t} + \gamma\right)\left(1 - \frac{5}{12\hat{t}^2}\right) + \frac{653}{216} - \frac{127\pi^2}{144} - \frac{7\zeta(3)}{4} + \frac{7}{6}\pi^2\ln(2)$$

Fourier→Laplace: We decompose the cosine in exponentials and rotate

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b)}(t) = c_0 + \tilde{h}_0(\hat{t}) + \tilde{h}_3(\hat{t}) + \int_0^2 dw\, 2\overbrace{\left(F_{02}(w) + \frac{1}{2w}\right)}^{\text{finite } w\to 0} e^{-w\hat{t}} + \int_2^\infty dw\, 2F_{2\infty}(w)e^{-w\hat{t}}$$

$$F_{02}(w) = \tfrac{4}{3w^3} + \tfrac{w}{16(w^2-4)} + \pi\sqrt{4-w^2}\left(\tfrac{w}{16(w^2-4)^2} - \tfrac{1}{8w^2} + \tfrac{7}{48}\right) + \left[\sqrt{4-w^2}\left(-\tfrac{4}{3w^4} - \tfrac{17}{48w^2} - \tfrac{5}{16(w^2-4)}\right.\right.$$
$$\left.\left. - \tfrac{1}{4(w^2-4)^2} + \tfrac{1}{8}\right) + \pi\left(\tfrac{1}{2w^3} + \tfrac{w}{2} - \tfrac{7}{6w}\right)\right]\arcsin\left(\tfrac{w}{2}\right) + \tfrac{23w}{144} - \tfrac{37}{144w} + \tfrac{5}{24}w\ln(w)$$

$$F_{2\infty}(w) = \tfrac{4}{3w^3} + \tfrac{w}{16(w^2-4)} + \left(\tfrac{7}{24} - \tfrac{1}{4w^2}\right)\sqrt{w^2-4}\ln\left(w\left(w^2-4\right)\right) + \sqrt{w^2-4}\left(-\tfrac{1}{3w^4} + \tfrac{115}{144w^2} + \tfrac{23}{144(w^2-4)} - \tfrac{23}{144}\right)$$
$$+\left[-\tfrac{4}{3w^5} + \tfrac{7}{6w^3} + \tfrac{w}{2(w^2-4)} - \tfrac{29w}{24} + \tfrac{47}{12w} - \sqrt{w^2-4}\left(-\tfrac{4}{3w^4} - \tfrac{17}{48w^2} - \tfrac{5}{16(w^2-4)} - \tfrac{1}{4(w^2-4)^2} + \tfrac{1}{8}\right)\right]\tfrac{\ln(y(w))}{2}$$
$$+ \tfrac{23w}{144} - \tfrac{37}{144w} + \tfrac{5}{24}w\ln(w) - \left(\tfrac{1}{w^3} + w - \tfrac{7}{3w}\right)L(y(w))$$

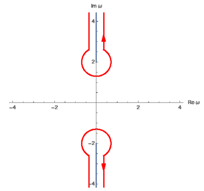
$$L(x) = \mathrm{Li}_2(-x) + 2\mathrm{Li}_2(x) + \tfrac{1}{2}\ln x\left(\ln(1+x) + 2\ln(1-x)\right)$$

$$c_0 = -2\int_0^2 dw\,\left(F_{02}(w) + \frac{1}{2w}\right) - 2\int_2^\infty dw\, F_{2\infty}(w) = \frac{653}{216} + \frac{\pi}{16} - \ln(2) - \frac{163}{144}\pi^2 + \frac{7}{6}\pi^2\ln(2) - \frac{7\zeta(3)}{4}$$

$$\tilde{h}_3(\hat{t}) = \int_0^2 dw\,\frac{1-e^{-w\hat{t}}}{w} = -\mathrm{Ei}(-2\hat{t}) + \ln(2\hat{t}) + \gamma$$

$$\tilde{h}_0(\hat{t}) = \int_0^\infty 2\left(\cos(\hat{\omega}\hat{t}) - 1\right)h_0(\hat{\omega})\,d\hat{\omega} = \frac{\pi\hat{t}}{16} + \frac{\pi^2}{8}\left(e^{-2\hat{t}} - 1\right) + \frac{1}{32}\pi^2\hat{t}\left(K_0(2\hat{t}) - \mathbf{L}_0(2\hat{t})\right)$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2)}(t) = \tilde{h}_2(\hat{t}) + \int_{\mathcal{C}} d\hat{\omega}\; 2\cos(\hat{\omega}\hat{t}) \overbrace{\left[\frac{\hat{f}_4(\hat{\omega}^2)}{\hat{\omega}} - h_2(\hat{\omega})\right]}^{g_5(\hat{\omega})}$$



where we added and subtracted the pole term $h_2(\hat{\omega}) = -\frac{\pi}{2(4+\hat{\omega}^2)}$

$$\tilde{h}_2(\hat{t}) = \int_0^\infty d\hat{\omega}\; 2\cos(\hat{\omega}\hat{t}) h_2(\hat{\omega}) = -\frac{\pi^2}{4}e^{-2\hat{t}}$$

We decompose the cosine and make the suitable change of variables

We take the difference between the values of g_5 between the two cuts, and on the left and the right of each cut:

$$F_5(w) = \frac{\mathrm{i}}{2}\left[\lim_{\epsilon\to 0^+} g_5(\epsilon+\mathrm{i}w) - \lim_{\epsilon\to 0^-} g_5(\epsilon+\mathrm{i}w) - \lim_{\epsilon\to 0^+} g_5(\epsilon-\mathrm{i}w) + \lim_{\epsilon\to 0^-} g_5(\epsilon-\mathrm{i}w)\right]$$

Finally

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2)}(t) = \tilde{h}_2(\hat{t}) + \int_2^\infty dw\; F_5(w) 2e^{-w\hat{t}}\;,$$

$$F_5(w) = \frac{-23w^6+230w^4-508w^2+192}{144w^4\sqrt{w^2-4}} - \frac{-29w^8+222w^6-348w^4-144w^2+128}{48w^5(w^2-4)}\ln(y(w))$$
$$-\left(\frac{1}{w^3}+w-\frac{7}{3w}\right)\left(L(y(w))+\frac{\pi^2}{4}\right) + \left(\frac{7}{24}-\frac{1}{4w^2}\right)\sqrt{w^2-4}\ln\left(w(w^2-4)\right)$$

Perusing the asymptotic expansions due to each term of the integrands, we can isolate and regroup the terms with same asymptotic behaviour. We found

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;1)}(t) = \tilde{h}_2(\hat{t}) + \int_2^\infty dw\; 2F_5^{(1)}(w)e^{-w\hat{t}}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;2)}(t) = \ln(\hat{t})\int_2^\infty dw\; 2F_5^{(2)}(w)e^{-w\hat{t}}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;2;3)}(t) = \int_2^\infty dw\; 2F_5^{(3)}(w)e^{-w\hat{t}}$$

$$F_5^{(1)}(w) = \frac{\pi^2}{4}\left(\frac{7}{3w} - w - \frac{1}{w^3}\right)$$

$$F_5^{(2)}(w) = \frac{1}{2}\left(\sqrt{w^2-4}\left(\frac{1}{4w^2} - \frac{7}{24}\right) - \left(\frac{1}{w^3} + w - \frac{7}{3w}\right)\frac{\ln(y(w^2))}{2}\right)$$

$$F_5^{(3)}(w) = F_5(w) - F_5^{(1)}(w) - F_5^{(2)}(w)\ln\hat{t}$$

$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;1)}(t) = \int_0^2 dw\; 2F_{02}^{\text{odd}}(w)e^{-w\hat{t}} + \int_2^\infty dw\; 2F_{2\infty}^{\text{odd}}(w)e^{-w\hat{t}}$$

$$\begin{aligned}\frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;2)}(t) =& c_0 - \hat{f}_4^{(b;1;3)}(t) - \tilde{h}_2(\hat{t}) + \tilde{h}_0(\hat{t}) + \tilde{h}_3(\hat{t}) \\ &+ \int_0^2 dw\; 2\left(F_{02}(w) + \frac{1}{2w} - F_{02}^{\text{odd}}(w)\right)e^{-w\hat{t}} \\ &+ \int_2^\infty dw\; 2\left(F_{2\infty}(w) - F_5(w) - F_{2\infty}^{\text{odd}}(w)\right)e^{-w\hat{t}}\end{aligned}$$

$$\begin{aligned}F_{02}^{\text{odd}}(w) =& \frac{\pi}{2}\left(\sqrt{4-w^2}\left(\frac{7}{24} - \frac{1}{4w^2}\right)\right. \\ &\left.+\left(\frac{1}{w^3} + w - \frac{7}{3w}\right)\arcsin\left(\frac{w}{2}\right)\right)\end{aligned}$$

$$F_{2\infty}^{\text{odd}}(w) = \frac{\pi^2}{4}\left(\frac{1}{w^3} + w - \frac{7}{3w}\right)$$

We define the series removing any leading factor

$$\bar{f}_4^{(b;2;1)}(t) = \tilde{f}_4^{(b;2;1)}(t)\; e^{2\hat{t}}$$
$$\bar{f}_4^{(b;2;2)}(t) = \tilde{f}_4^{(b;2;2)}(t)\; e^{2\hat{t}}\sqrt{\hat{t}}/\ln\hat{t}$$
$$\bar{f}_4^{(b;2;3)}(t) = \tilde{f}_4^{(b;2;3)}(t)\; e^{2\hat{t}}\sqrt{\hat{t}}$$
$$\bar{f}_4^{(b;1;1)}(t) = \tilde{f}_4^{(b;1;1)}(t)\; \hat{t}$$
$$\bar{f}_4^{(b;1;2)}(t) = \tilde{f}_4^{(b;1;2)}(t)\; \hat{t}^2$$

$\longrightarrow$

$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;1;1)}\left(\frac{\hat{t}_0}{\sqrt{1+v}}\right) = \sum_{n=0}^{\infty} a_n^{(b;1;1)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;1;2)}\left(\frac{\hat{t}_0}{\sqrt{1+v}}\right) = \sum_{n=0}^{\infty} a_n^{(b;1;2)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;2;1)}\left(\frac{\hat{t}_0}{1+v}\right) = \sum_{n=0}^{\infty} a_n^{(b;2;1)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;2;2)}\left(\frac{\hat{t}_0}{1+v}\right) = \sum_{n=0}^{\infty} a_n^{(b;2;2)} v^n$$
$$\frac{m_\mu^2}{16\pi^2}\bar{f}_4^{(b;2;3)}\left(\frac{\hat{t}_0}{1+v}\right) = \sum_{n=0}^{\infty} a_n^{(b;2;3)} v^n$$

- Convenient change of variable $t \to v$: $t = \hat{t}_0/(1+v)^n$ ($n = 1$ or $1/2$) and expand in v
- These particular substitutions improve the convergence of the series in v for $\hat{t} \to \infty$, corresponding to $v \to -1$.
- The series converge if $|v| \le 1$ corresponding to $\hat{t} \ge \hat{t}_0/2$
- The coefficients $a_n^{(b;x;y)}$ can be obtained from the w-integral representations by expanding the integrands in v and integrating *numerically* term by term in w.
- The whole timekernel $\tilde{f}_4(t)$ is worked out adding $\tilde{f}_4^{(a)}(t)$ and all the 6 contributions,

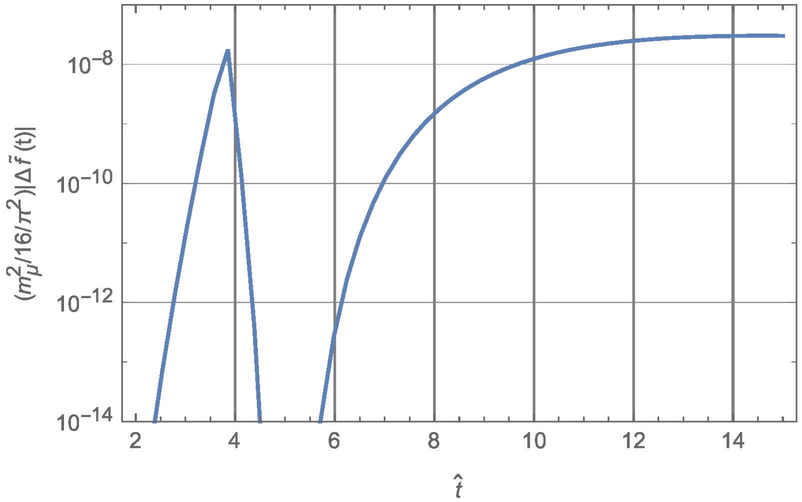
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \frac{\hat{t}^2}{2}\left(\frac{197}{144} + \frac{\pi^2}{12} - \frac{1}{2}\pi^2\ln 2 + \frac{3}{4}\zeta(3)\right) - \frac{\pi\hat{t}}{8} + (\ln\hat{t} + \gamma)\left(1 - \frac{5}{12\hat{t}^2}\right) + \frac{653}{216} - \frac{127\pi^2}{144} - \frac{7\zeta(3)}{4} + \frac{7}{6}\pi^2\ln(2)$$
$$+ \frac{1}{\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;1;1)}\left(\frac{\hat{t}_0^2}{\hat{t}^2} - 1\right)^n + \frac{1}{\hat{t}^2}\sum_{n=0}^{\infty} a_n^{(b;1;2)}\left(\frac{\hat{t}_0^2}{\hat{t}^2} - 1\right)^n + e^{-2\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;2;1)}\left(\frac{\hat{t}_0}{\hat{t}} - 1\right)^n$$
$$+ \frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\ln(\hat{t})\sum_{n=0}^{\infty} a_n^{(b;2;2)}\left(\frac{\hat{t}_0}{\hat{t}} - 1\right)^n + \frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\sum_{n=0}^{\infty} a_n^{(b;2;3)}\left(\frac{\hat{t}_0}{\hat{t}} - 1\right)^n$$

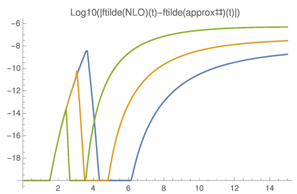
$$\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t) = \frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(a)}(t) + \frac{m_\mu^2}{16\pi^2}\tilde{f}_4^{(b;1;3)}(t) + \frac{1}{\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;1;1)}\left(\frac{\hat{t}_0^2}{\hat{t}^2}-1\right)^n + \frac{1}{\hat{t}^2}\sum_{n=0}^{\infty} a_n^{(b;1;2)}\left(\frac{\hat{t}_0^2}{\hat{t}^2}-1\right)^n + e^{-2\hat{t}}\sum_{n=0}^{\infty} a_n^{(b;2;1)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n + \frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\ln(\hat{t})\sum_{n=0}^{\infty} a_n^{(b;2;2)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n + \frac{e^{-2\hat{t}}}{\sqrt{\hat{t}}}\sum_{n=0}^{\infty} a_n^{(b;2;3)}\left(\frac{\hat{t}_0}{\hat{t}}-1\right)^n$$

- We can use the expansions for small and for large $\hat{t}$ to obtain the values of $\tilde{f}_4(t)$ for any value of $\hat{t}$.
- We choose a point of separation $\hat{t} = \hat{t}_c$
- In the region $\hat{t} \leq \hat{t}_c$ we compute $\tilde{f}_4(t)$ from the small-t expansion
- In the region $\hat{t} > \hat{t}_c$, we choose a suitable value of $\hat{t}_0$ and we use the expansion in $\hat{t} = \hat{t}_0$ to obtain $\tilde{f}_4(t)$
- The choice of the optimal $\hat{t}_c$, $\hat{t}_0$, and the numbers of terms of the expansions depend on the level of precision required. Using the small-$\hat{t}$ expansion up to $\hat{t}^{30}$ we choose $\hat{t}_c = 3.82$ and $\hat{t}_0 = 5$ We calculated the coefficients of the expansion up to $n = 12$ (see table)
- These values allow to obtain $\tilde{f}_4(t)$ with an error $\Delta\tilde{f}_4 < 3 \times 10^{-8}$ for any value of $\hat{t} \geq 0$. See figure →
- Checked with $G_e(t)$ from QED 1-loop VP, $\Delta a_\mu^{\text{QED VP}} \sim 10^{-11}$ ($G_e(t) \sim e^{-2m_\mu t}$) for large t

n	$a_n^{(b;1,1)}$	$a_n^{(b;1,2)}$	$a_n^{(b;2,1)}$	$a_n^{(b;2,2)}$	$a_n^{(b;2,3)}$
0	−1.4724671380	1.1589872337	−4.8942765691	0.2973718753	2.1170734478
1	0.1002442629	−0.0022459376	−2.9475017651	0.4127862149	1.0364595246
2	0.0021557710	0.0008279191	−0.5075497783	0.1109534688	0.1101698869
3	0.0001282655	0.0007999410	0.0115794503	−0.0040980259	0.0167667530
4	−0.0001467432	−0.0006094594	−0.0013940058	0.0003899989	−0.0035236970
5	9.35581×10^{-6}	7.37693×10^{-6}	0.0001421294	−0.0000133805	0.0008586372
6	0.0000260037	0.0002711371	7.67679×10^{-6}	−0.00001764961	−0.0002257379
7	−0.0000189910	−0.0002551246	−0.00001492424	.000011742325	0.0000612688
8	6.93309×10^{-6}	0.0001291619	8.61706×10^{-6}	-5.92454×10^{-6}	−0.0000164422
9	3.18779×10^{-7}	−0.0000121615	-4.20065×10^{-6}	2.78837×10^{-6}	4.04750×10^{-6}
10	-2.93399×10^{-6}	−0.0000553459	1.95419×10^{-6}	-1.29025×10^{-6}	-7.17744×10^{-7}
11	2.98580×10^{-6}	0.0000760414	-9.00478×10^{-7}	5.98351×10^{-7}	-7.67136×10^{-8}
12	-2.08433×10^{-6}	−0.0000669985	4.17032×10^{-7}	-2.80343×10^{-7}	1.94188×10^{-7}

Table 2: Coefficients of the expansions in v of $\frac{m_\mu^2}{16\pi^2}\tilde{f}_4(t)$ up to v^{12} with $\hat{t}_0 = 5$,

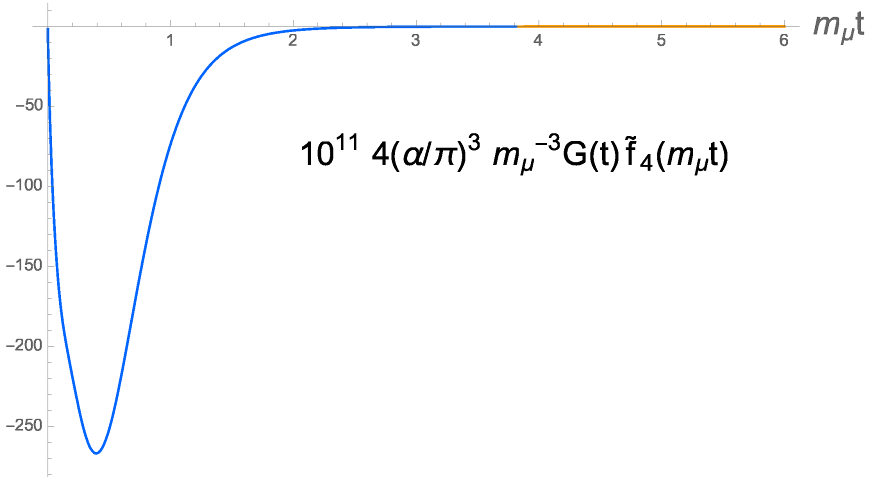




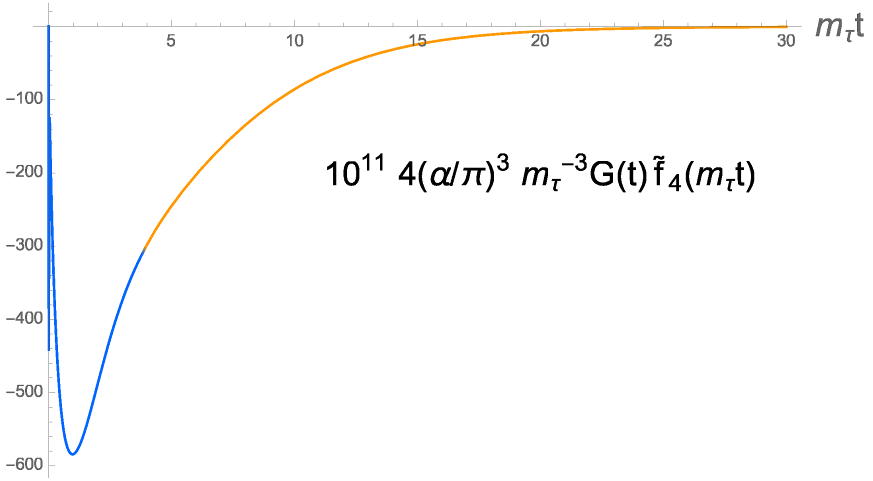
$\hat{t}_0 = 3, 4, 5$ ↑

Plots the integrands of $G(t)\tilde{f}_4(mt)$ for μ and τ

expansion around $t = 0$ expansion around $t = t_0$



$\rightarrow \quad a_\mu^{\rm HVP}(\rm NLO) = -207 \times 10^{-11}$



$\rightarrow \quad a_\tau^{\rm HVP}(\rm NLO) = -3150 \times 10^{-11}$

$$G(t) = \int_0^\infty d\omega \frac{1}{3} R(\omega)\omega^2 e^{-\omega t}$$

- NLO: Exact NLO space-like kernels are known.
- MUonE directly scans 41%, 82%, and 49% of the integrals of NLO ($4a$), ($4b$) and ($4c$), respectively.
- Using the alternative approach on the timelike integral the percentages of the contributions deduced from the MUonE data can be substantially improved *(work in progress)*.
- NNLO: Approximated space-like NNLO kernels were obtained from the first terms of the asymptotic expansions. Due to the peaking at high energies MUonE scans 15%, 15%, and 38% of the contributions ($6a$), ($6b$) and ($6bll$), respectively. For one set (6c2) containing two HVP insertions on *different* photon lines, we worked out a *bidimensional* approximated space-like kernel. The precision of the contributions of all the approximated space-like kernels obtained is at the level of 10^{-13}.
- We have obtained analytical coefficients of the series expansion of the NLO time-kernel valid for small $\hat{t}$
- We have found representations of all the components of the NLO time-kernel as Laplace integrals.
- From these representations we have worked out compact and fast numerical expansions of all the components of the NLO time-kernel, centered in finite values $\hat{t}_0$ of time $\hat{t}$, and converging for $\hat{t} > \hat{t}_0/2$.
- The combination of these expansions, with a suitable choice of numbers of terms, of the expansion point $\hat{t}_0$ and of the separation point $\hat{t}_s$ between regimes, allows to determine the NLO time-kernel with an error $\Delta\tilde{f} < 3 \times 10^{-8}$ for every value of $\hat{t}$.

Thank You

BACKUP SLIDES

SUMMARY

class	$10^{11} a_\mu$(diagram)	$10^{11} a_\mu$(MUonE)	% of the diagram contribution
LO(2)	6930	5828	84.1%
NLO($4a$)	-207	-85	41.3%
NLO($4b$)	$+106$	$+87$	82.0%
NLO($4c$)	$+3.4$	$+1.6$	48.6%
NNLO($6a$)	$+8.1$	$+1.2$	15%
NNLO($6b$)	-4.0	-0.6	15%
NNLO($6bll$)	$+9.1$	$+3.5$	38%
NNLO($6c1$)	-0.50	-0.04	9%
NNLO($6c2$)	-0.019	-0.001	6%
NNLO($6c3$)	$+0.09$	$+0.04$	44%
NNLO($6c4$)	$+0.011$	$+0.002$	23%
NNLO($6d$)	$+0.005$	$+0.0008$	16%