NEW METHODS FOR FEYNMAN INTEGRALS AND THEIR APPLICATIONS TO MU-E SCATTERING

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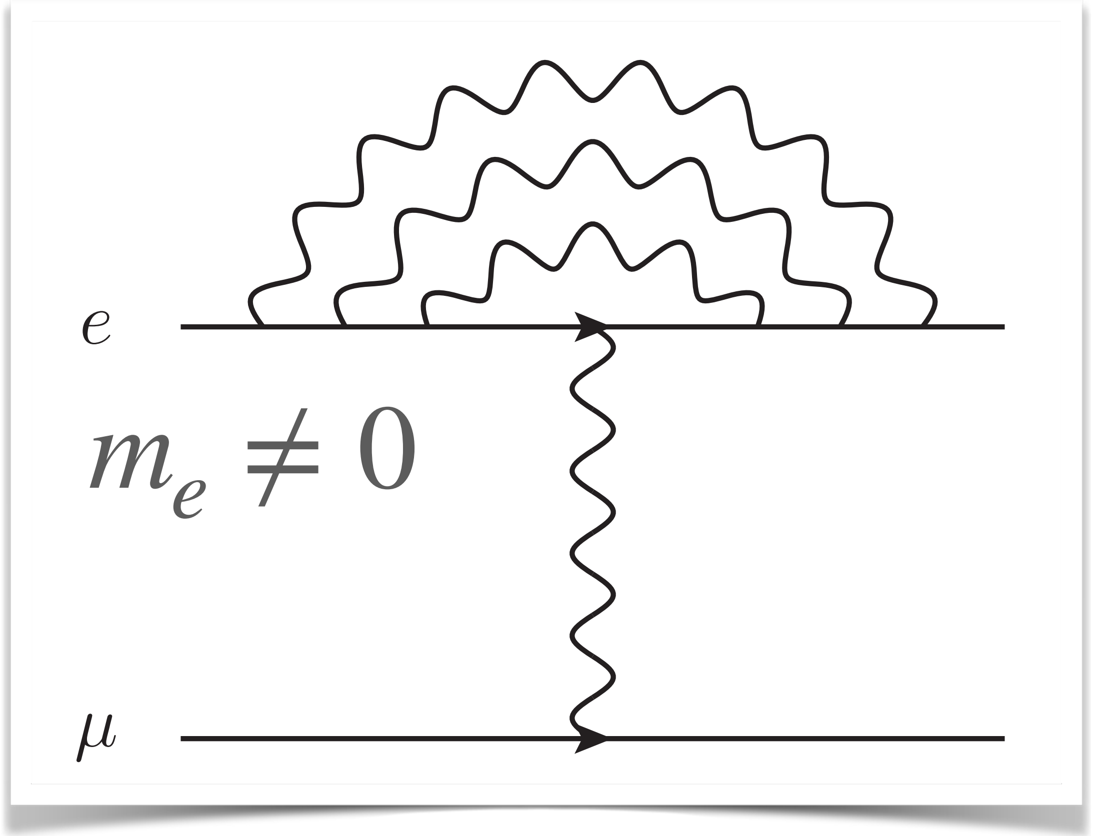
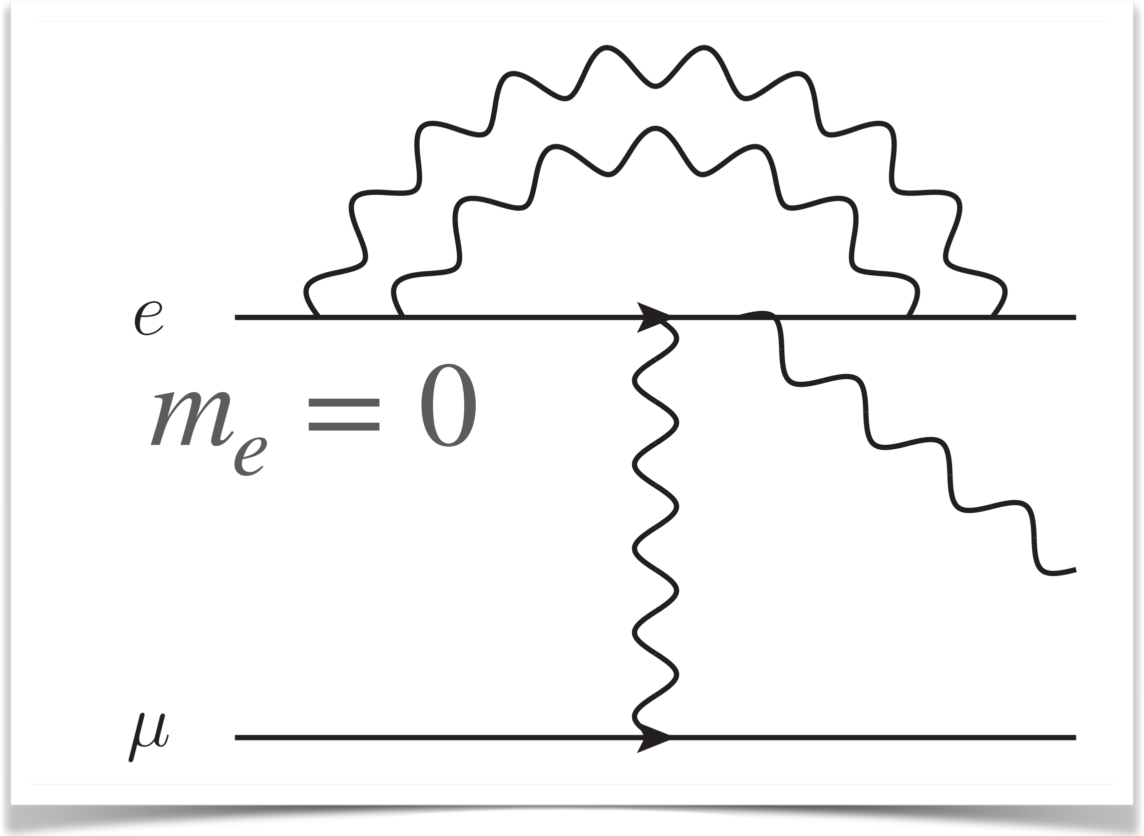
Funded by the European Union

IMPRESSIVE PROGRESS TOWARDS MU-E SCATTERING AT N3LO



Three-loop amplitude $\gamma^\star \to e^+e^-$

MF, Lange, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022), Phys.Rev.D 106 (2023), Phys.Rev.D 107 (2023)



Two-loop amplitude $\gamma^\star \to e^+e^-\gamma$

Badger, Krys, Moodie, Zoia,JHEP 11 (2023) 041.
Fadin, Lee, JHEP 11 (2023) 148

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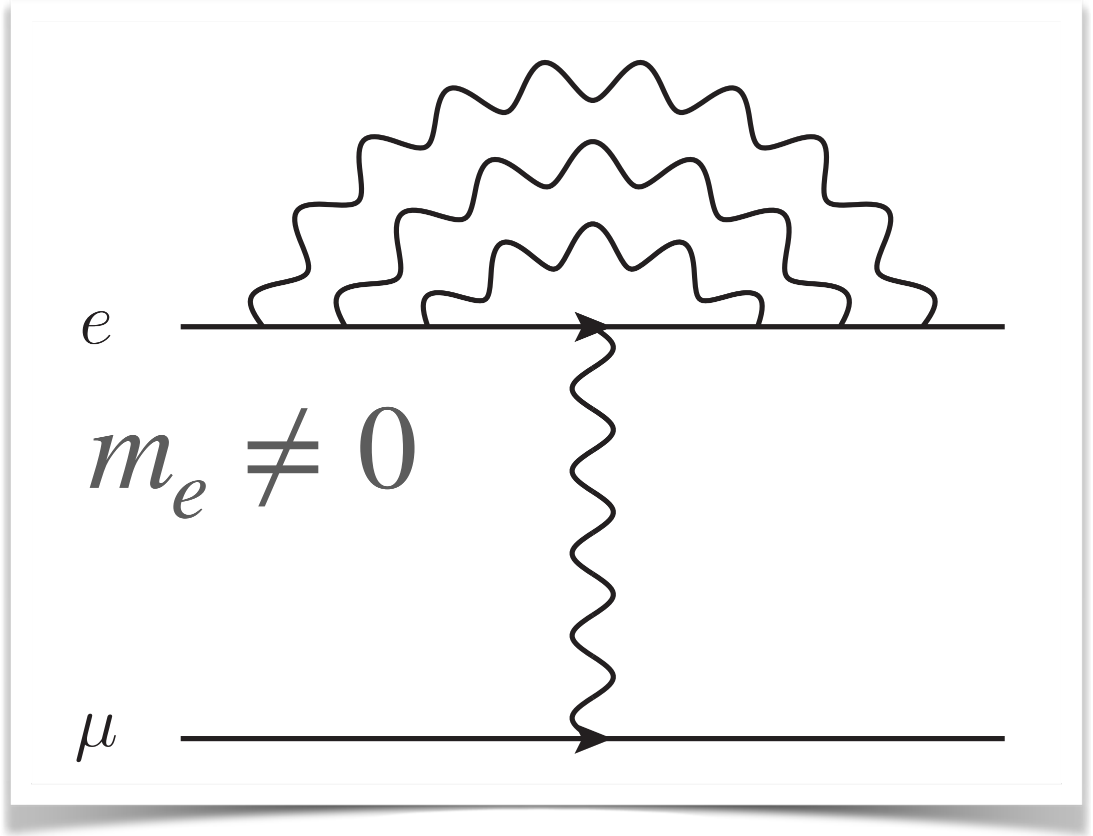

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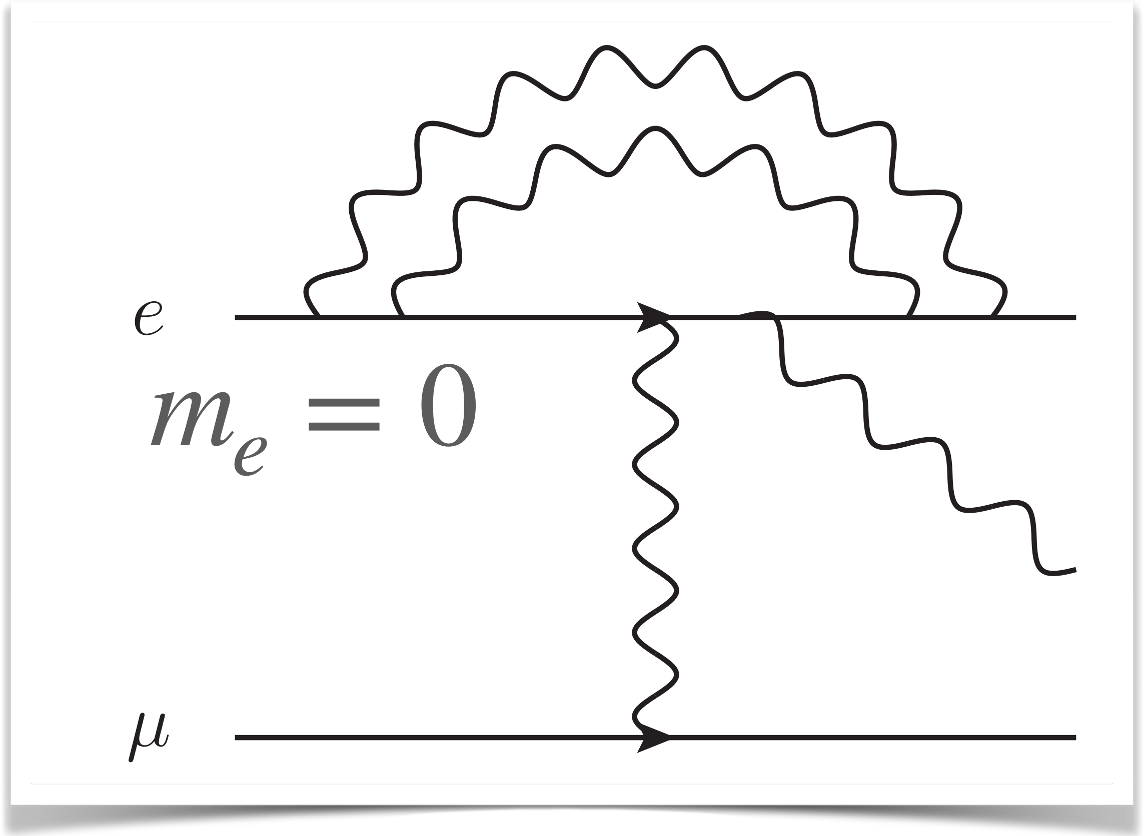

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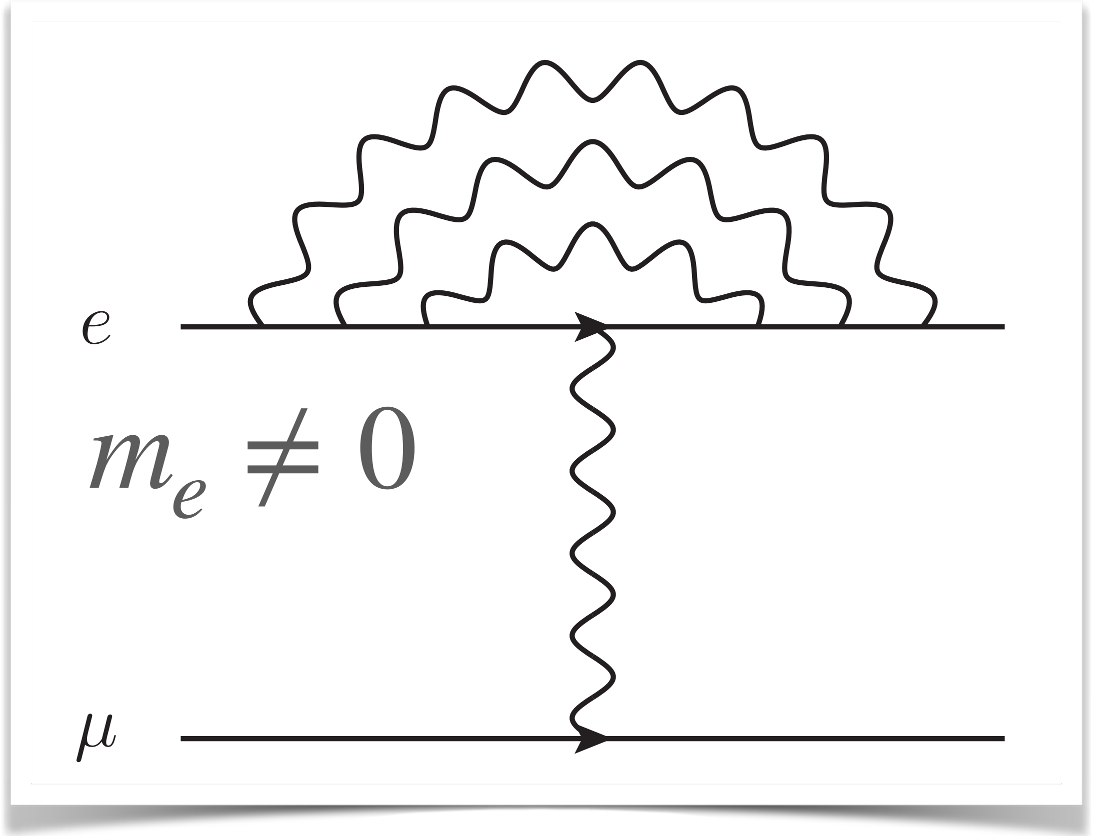
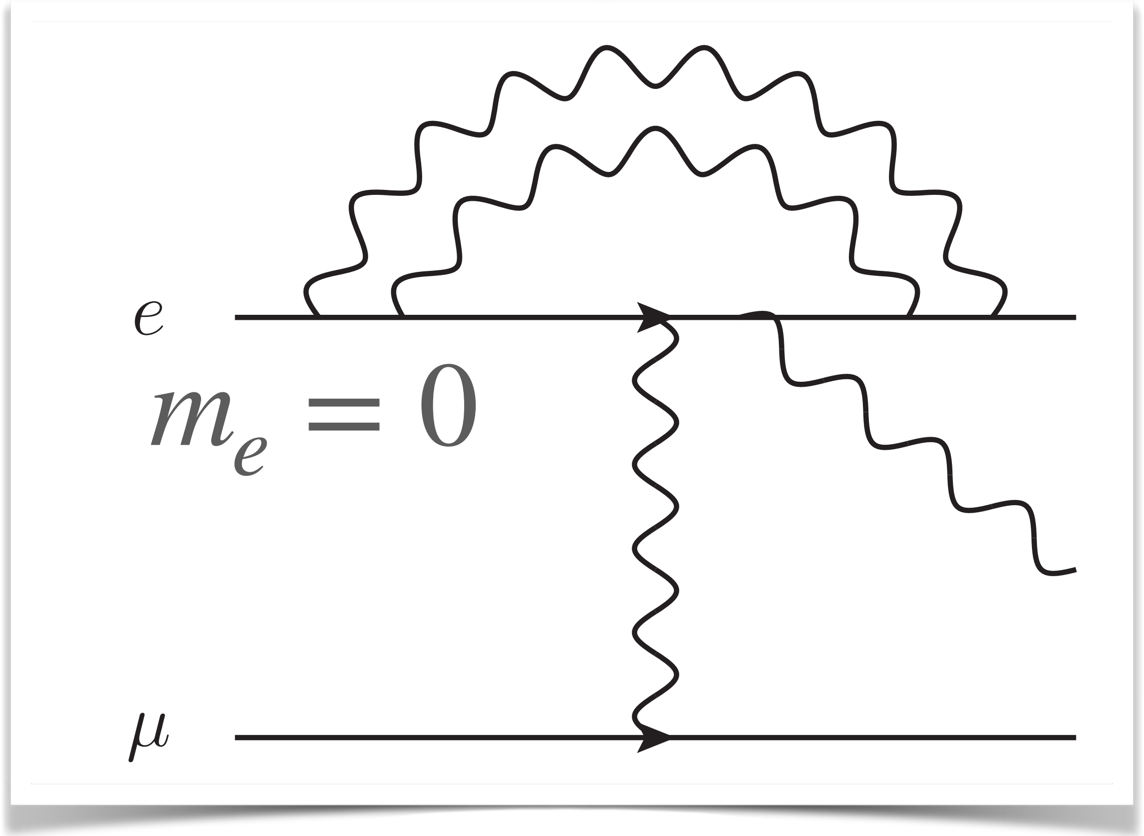
What is it still to be done?

IMPRESSIVE PROGRESS TOWARDS MU-E SCATTERING AT N3LO





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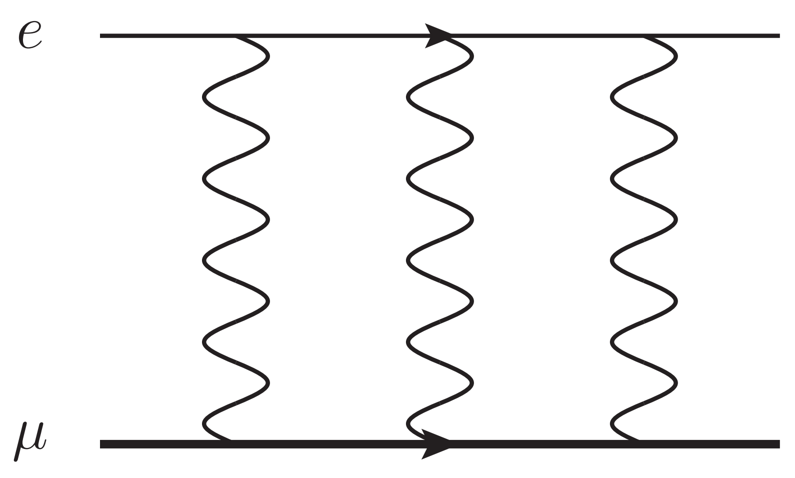
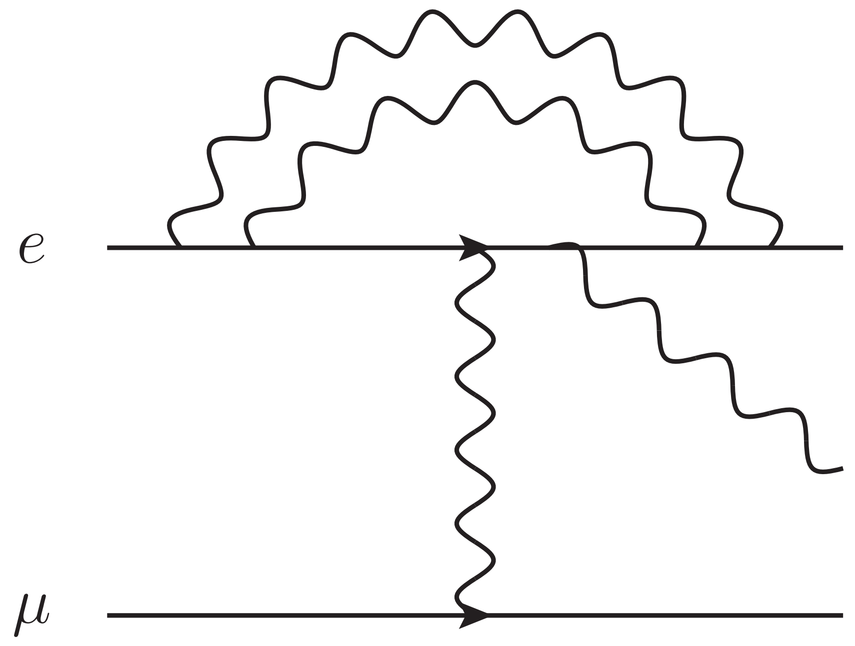
Badger, Krys, Moodie, Zoia,JHEP 11 (2023) 041.
Fadin, Lee, JHEP 11 (2023) 148

What is it still to be done? What can be done?

MU-E SCATTERING: WHATS NEXT?

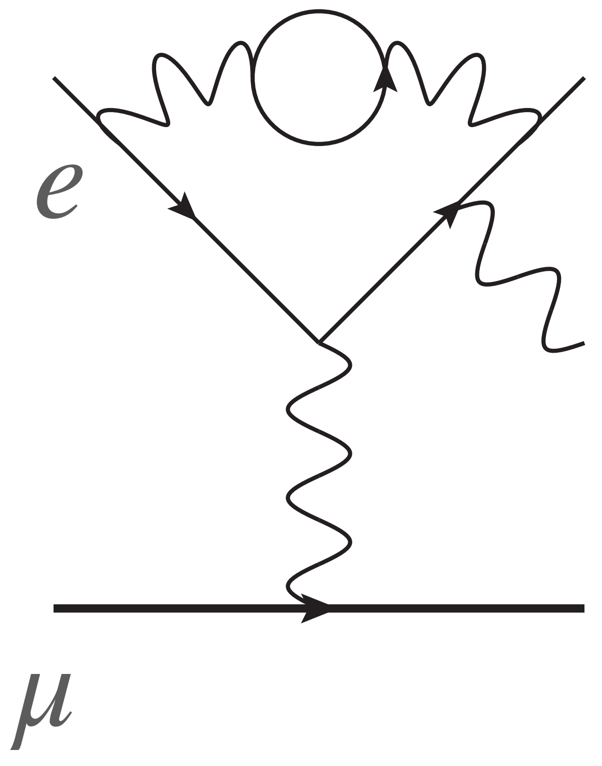
NNLO and N3LO amplitudes

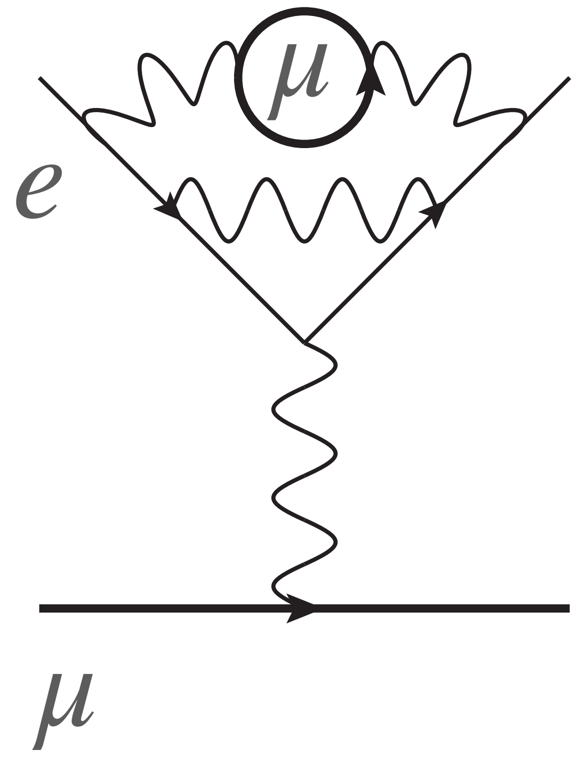




- Estimate finite m_e effects?
  - fix m_e/M_μ to some (few) values
  - grids in external kinematics?

n_f contributions at N3LO





- n_f corrections to $\gamma^\star \to e^+e^-\gamma$ with $m_e \neq 0$
  - dispersive or hypershperical method
- n_f corrections to VVV with unequal masses

OUTLOOK

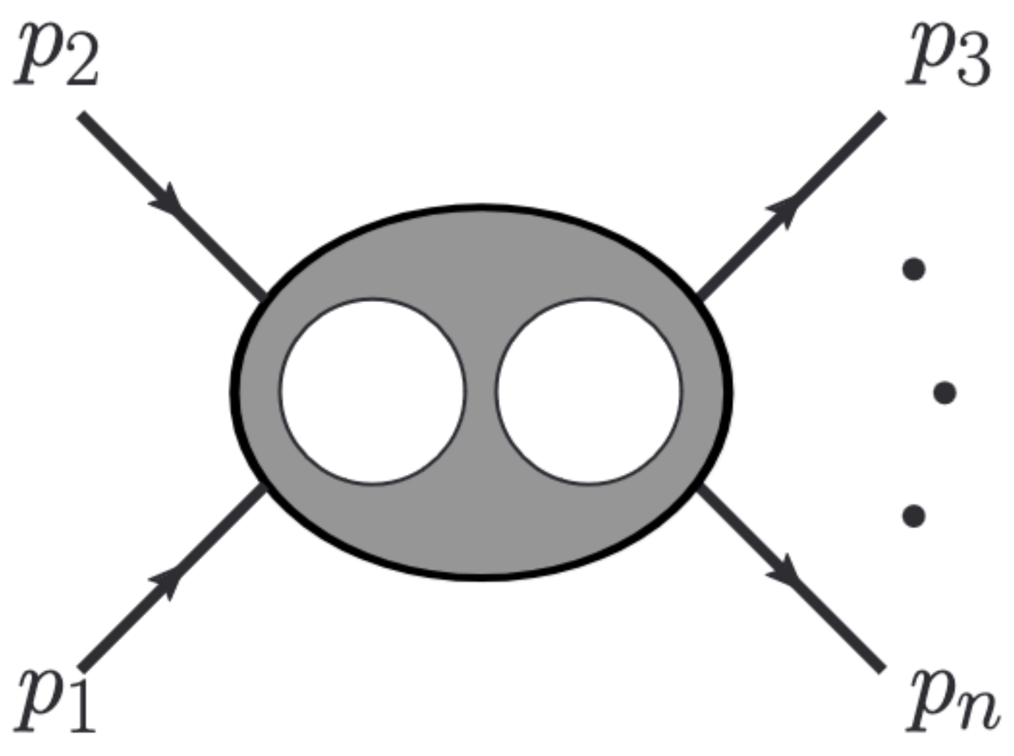
- Numerical evaluation of master integrals
- Improvements for generation of IBP system and solutions

Disclaimer: I will present some interesting recent developments.

Personal selection based on my experience and tests that I've done

SCATTERING AMPLITUDES AND FEYNMAN INTEGRALS

$$\mathscr{M} =$$



$$= \sum_i c_i I_i$$

RATIONAL FUNCTIONS

- Integration-by-part relations
- Analytic or numerical methods

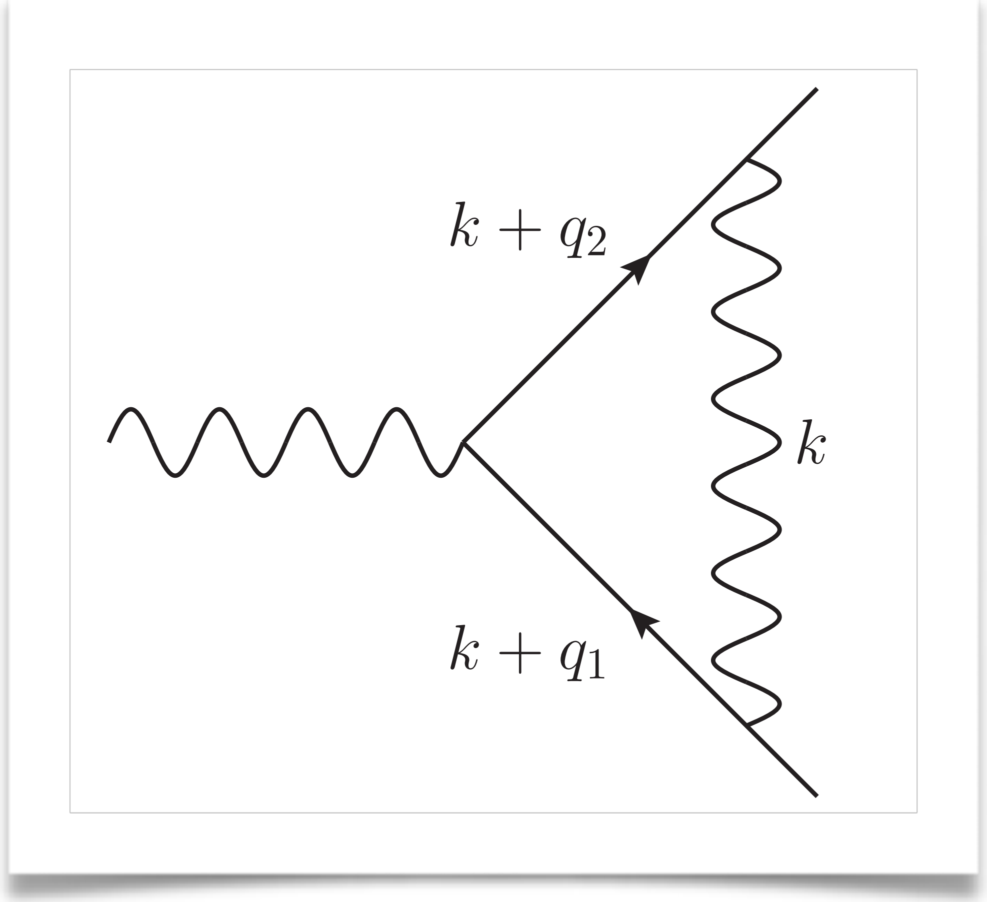
FEYNMAN INTEGRALS

- Complicated loop integrations
- Polylogarithms and Elliptic functions
  - Analytic/numerical method

Integral family

$$I(a_1, a_2, a_3) = \int d^d k \frac{1}{[k^2]^{a_1} [(k+q_1) - m^2]^{a_2} [(k+q_2) - m^2]^{a_3}}$$

with $s = (q_1 - q_2)^2$



Integration-by-part reduction

Master integrals

$$I(2,1,1) = \frac{(d-2)\left(4dm^2 + ds - 20m^2 - 4s\right)}{2(d-6)(d-5)m^4 s^2} I(0,0,1) + \frac{4(d-3)}{(d-6)s^2} I(0,1,1)$$

Differential Equations

Kotikov, Phys. Lett. B 254 (1991) 158;
Gehrmann, Remiddi, Nucl. Phys. B 580 (2000) 485

$$\frac{\mathrm{d}}{\mathrm{ds}} I(0,1,1) = \frac{\mathrm{d}}{\mathrm{ds}} \int d^d k \frac{1}{k^2[(k+q_1-q_2)^2 - m^2]}$$

$$= \frac{I(-1,2,1)}{s-4} - \frac{I(0,1,1)}{s-4} + \frac{2I(0,1,1)}{(s-4)s} - \frac{2I(0,2,0)}{(s-4)s}$$

$$\overset{IBP}{=} \frac{(d-2)}{s\left(4m^2-s\right)} I(0,0,1) + \frac{\left(-4dm^2+ds+12m^2-4s\right)}{2s\left(s-4m^2\right)} I(0,1,1)$$

Differential Equations

Kotikov, Phys. Lett. B 254 (1991) 158;
Gehrmann, Remiddi, Nucl. Phys. B 580 (2000) 485

$$\frac{d}{ds}\begin{pmatrix} I(0,0,1) \\ I(0,1,1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{d-2}{s\left(4m^2-s\right)} & \frac{-4dm^2+ds+12m^2-4s}{2s\left(s-4m^2\right)} \end{pmatrix} \begin{pmatrix} I(0,0,1) \\ I(0,1,1) \end{pmatrix}$$

Boundary conditions

$$I(0,0,1)\,|_{s=0} = (m^2)^{1-\epsilon}\Gamma(\epsilon-1)$$
$$I(0,1,1)\,|_{s=0} = \ldots$$

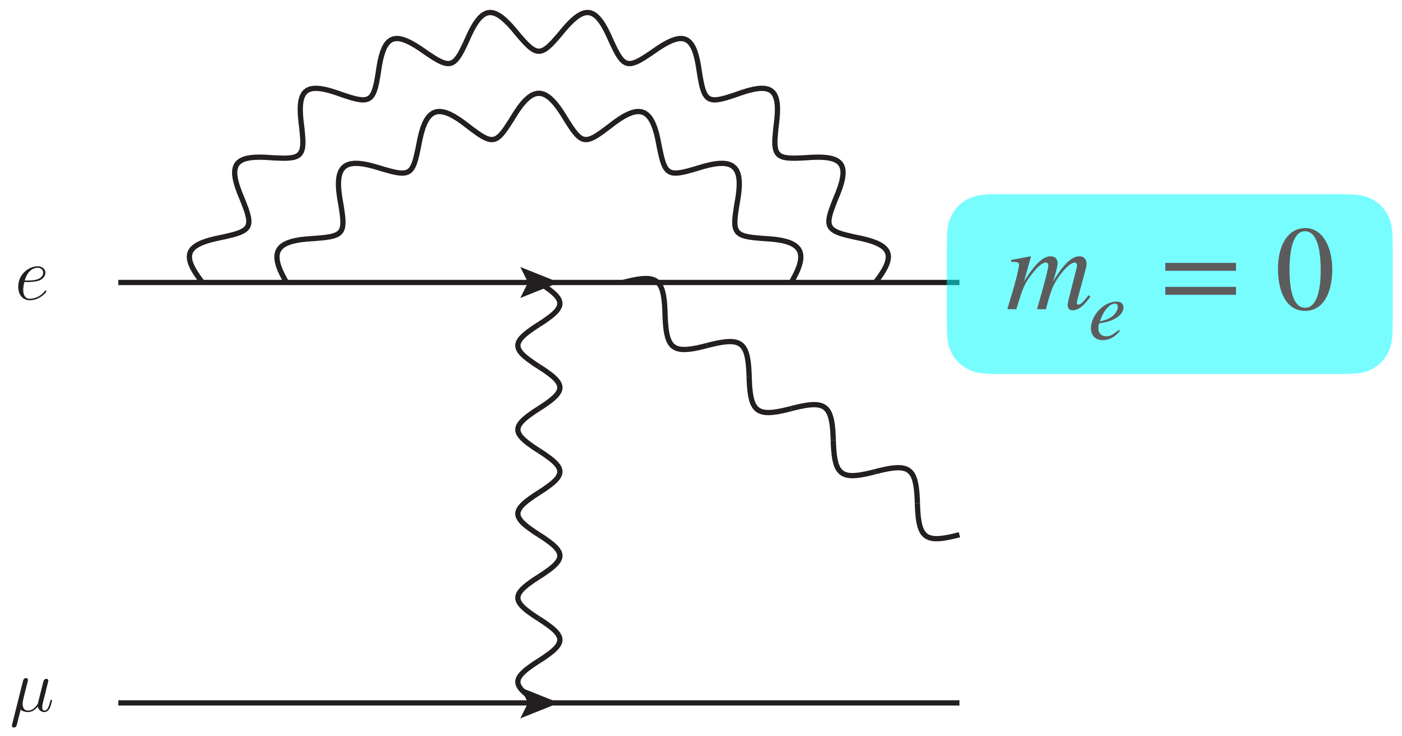
Analytic solution

- Solve in terms of known constants/functions
- Function properties well understood
- Known analytic structures and series expansions
- Fast and generic numerical evaluation tools

Numerical solution

- Oriented to phenomenological studies
- Applicable to larger class of problems
- Finite numerical accuracy

TWO LOOP CORRECTIONS TO $\gamma^\star \to e^+e^-\gamma$



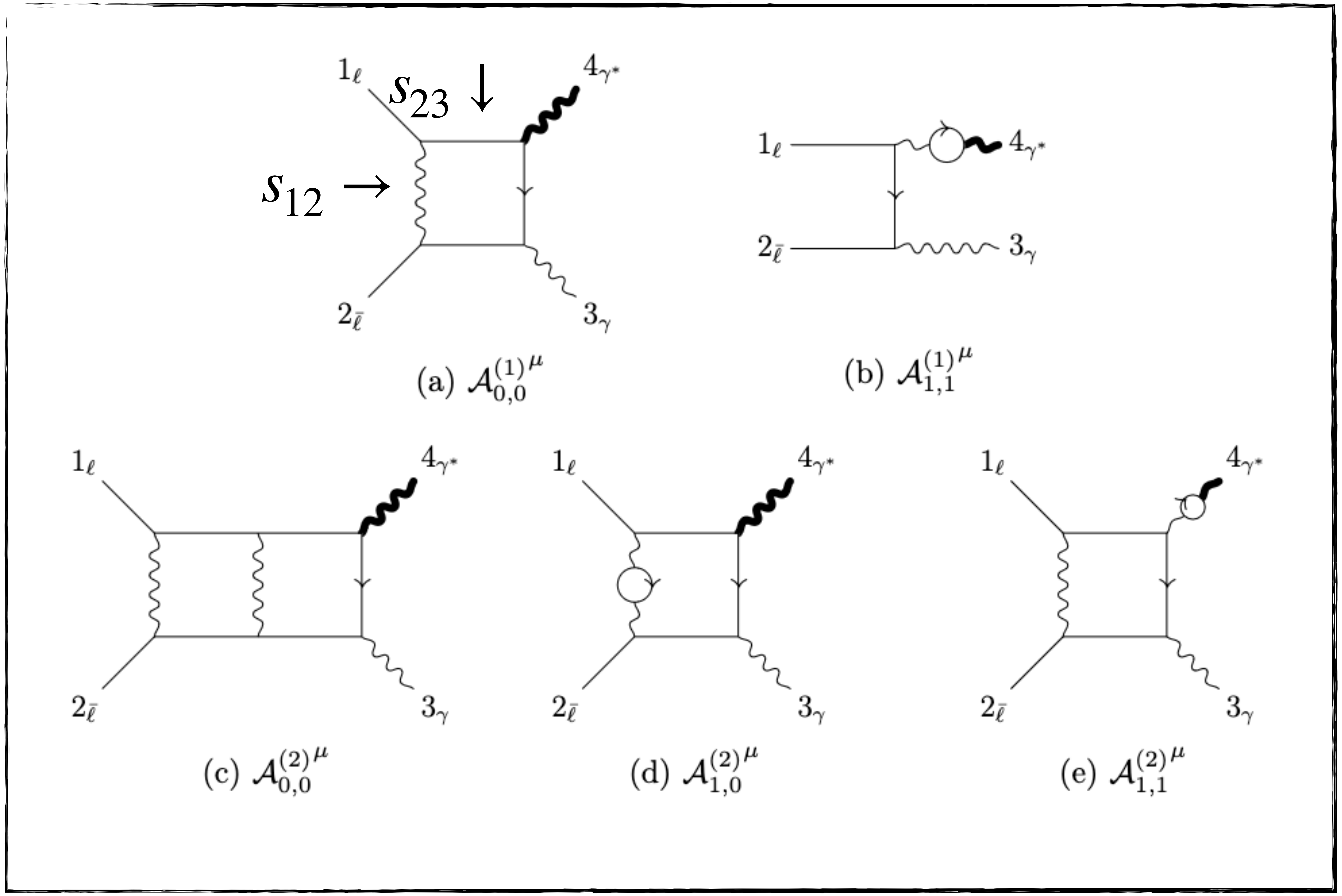
Badger, Krys, Moodie, Zoia JHEP 11 (2023) 041.
Fadin, Lee, JHEP 11 (2023) 148

$p_e^2 = p_{\bar{e}}^2 = 0$

$p_\gamma^2 = 0$

$p_{\gamma^\star}^2 = s_4$

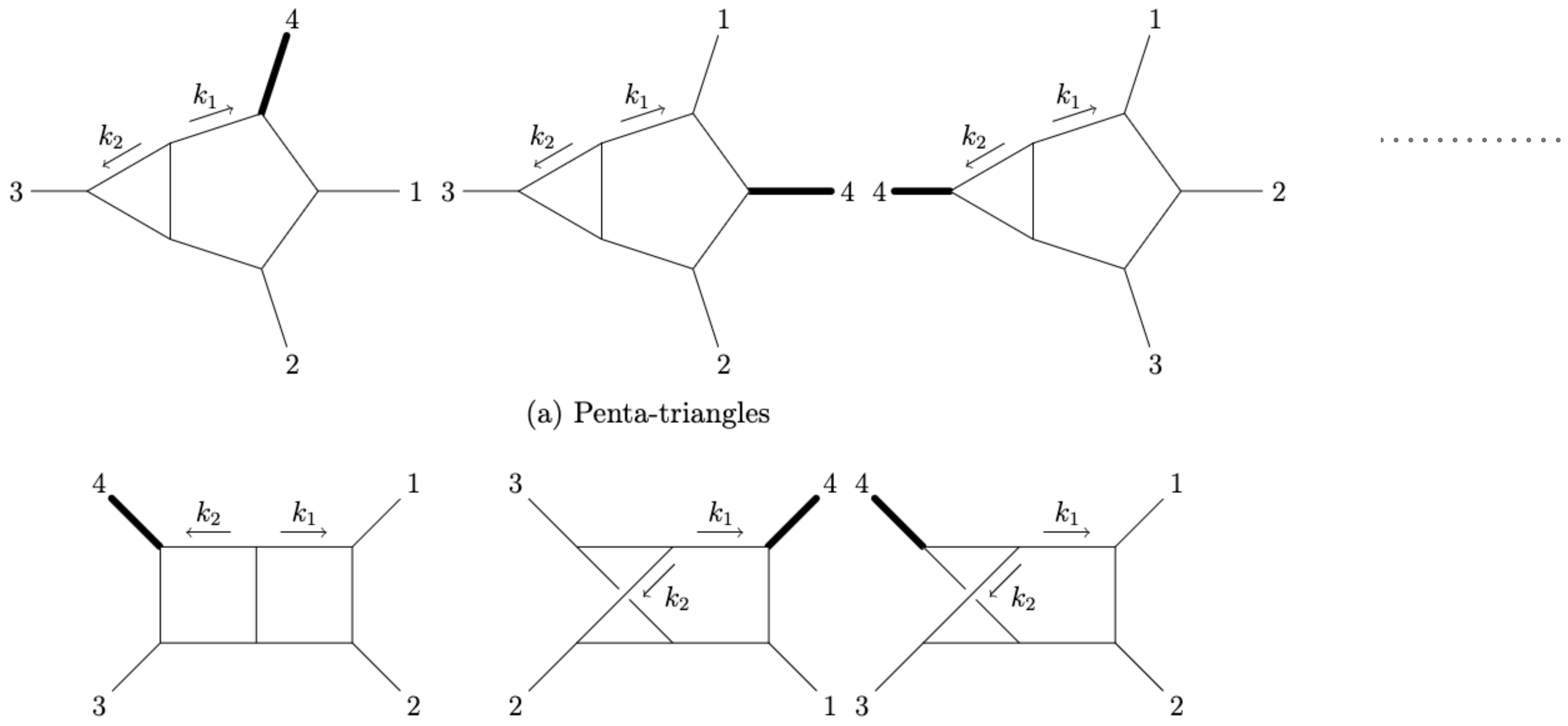
$s_{12} = (p_e + p_{\bar{e}})^2, \quad s_{23} = (p_{\bar{e}} + p_\gamma)^2$



Canonical differential equations

$$\frac{\partial \mathrm{MI}_i}{\partial s_j} = \epsilon A^{(j)}(\vec{s})_{ik}\mathrm{MI}_k$$

Henn, Phys.Rev.Lett. 110 (2013) 251601



(a) Penta-triangles

(b) Double-box

(c) Crossed double-boxes

Badger, Krys, Moodie, Zoia, *JHEP* 11 (2023) 041

Previous calculations

Gehrmann, Remiddi, NPB 601 (2001) 248;
Gehrmann, Remiddi, NPB 601 (2001) 287;
Gehrman Jakubcik, Mella, Syrrakos, Tancredi, JHEP 04 (2023) 016

Express all MIs in terms of a set of algebraically independent special functions $\{F_i^{(w)}(s)\}$

$$\mathrm{MI}(\vec{s};\epsilon) = \sum_{k=-4} \epsilon^k\,\mathrm{MI}^{(k)}(\vec{s})$$

$$\mathrm{MI}^{(2)}(\vec{s}) = \sum_i \alpha_i F_i^{(2)}(\vec{s}) + \sum_{i\le j} \beta_{ij} F_i^{(1)}(\vec{s}) F_j^{(1)}(\vec{s}) + \gamma\zeta_2$$

Multiple Polylogarithms $G(a_1, \ldots, a_n; 1)$

STANDARD AMPLITUDE WORKFLOW

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \text{Feynman diagram}_i$$

FORM + Mathematica

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i c_i\left(\{p\},\epsilon\right)\, \mathrm{I}_i\left(\{p\},\epsilon\right)$$

Integration-by-parts (IBPs)

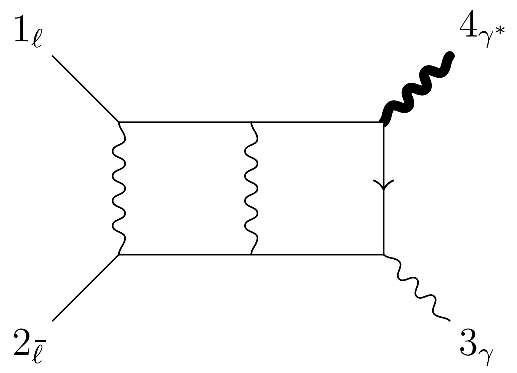
Chetyrkin, Tkachov '81; Laporta 2000

$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i d_i\left(\{p\},\epsilon\right)\, \mathrm{MI}_i\left(\{p\},\epsilon\right)$$

Expansion around $\epsilon = 0$

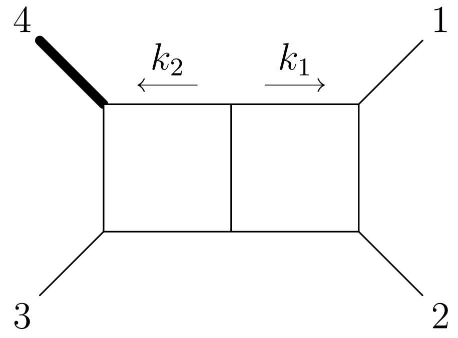
$$A^{(2)}\left(\{p\},\epsilon\right) = \sum_i \sum_{k=-4}^{0} \epsilon^k\, r_{ki}\left(\{p\}\right)\, \mathrm{F}_i\left(\{p\}\right)$$

5



with QGRAF

Nogueira



Scalar Feynman integrals

E.g. $\mathrm{I} = \int \frac{d^D k_1\, d^D k_2}{k_1^2 (k_1+p_1)^2 (k_1+p_1+p_2)^2 \ldots}$

"Master integrals"

Special functions, e.g.

$\mathrm{Li}_n(x)$, $G(a_1, \ldots; x)\ldots$

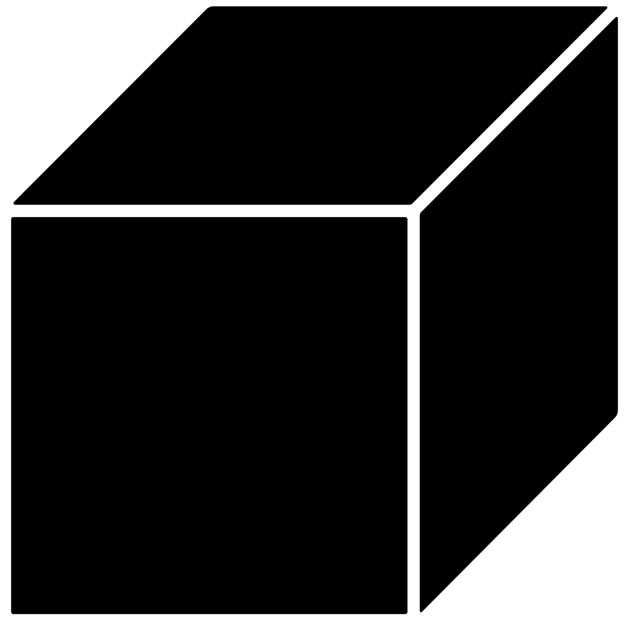
Large intermediate expressions which eventually simplify!

slide by S. Zoia @ Moriond QCD 2024

NUMERICAL SAMPLING

Sample $\{p\}, \epsilon$

Perform all operations modulus a large prime P



$r_{ki}(\{p\}) \bmod P$

Reconstruction of analytic from numerical samples

Analytic solution

- Solve in terms of known constants/functions
- Function properties well understood
- Known analytic structures and series expansions
- Fast and generic numerical evaluation tools

Numerical solution

- Oriented to phenomenological studies
- Applicable to larger class of problems
- Finite numerical accuracy

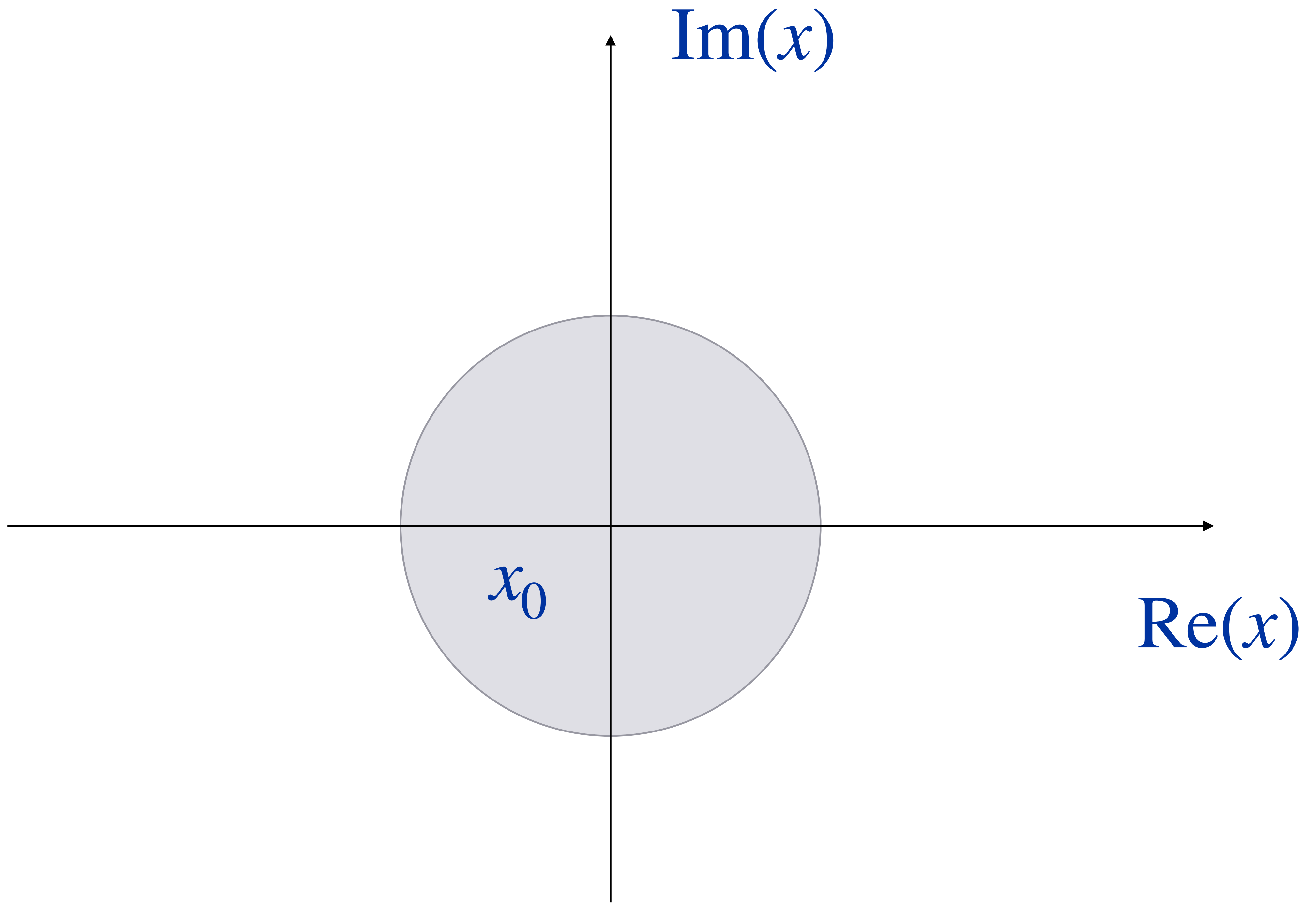
SOLVING DIFFERENTIAL EQUATIONS NUMERICALLY

$$I_a(x,\epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} c_{a,mn}\,\epsilon^m (x-x_0)^n$$

Construct a series expansion around some point x_0

[and $\epsilon = (d-4)/2$]

$$\frac{\partial \vec{I}}{\partial x} = M(x,\epsilon)\vec{I}$$

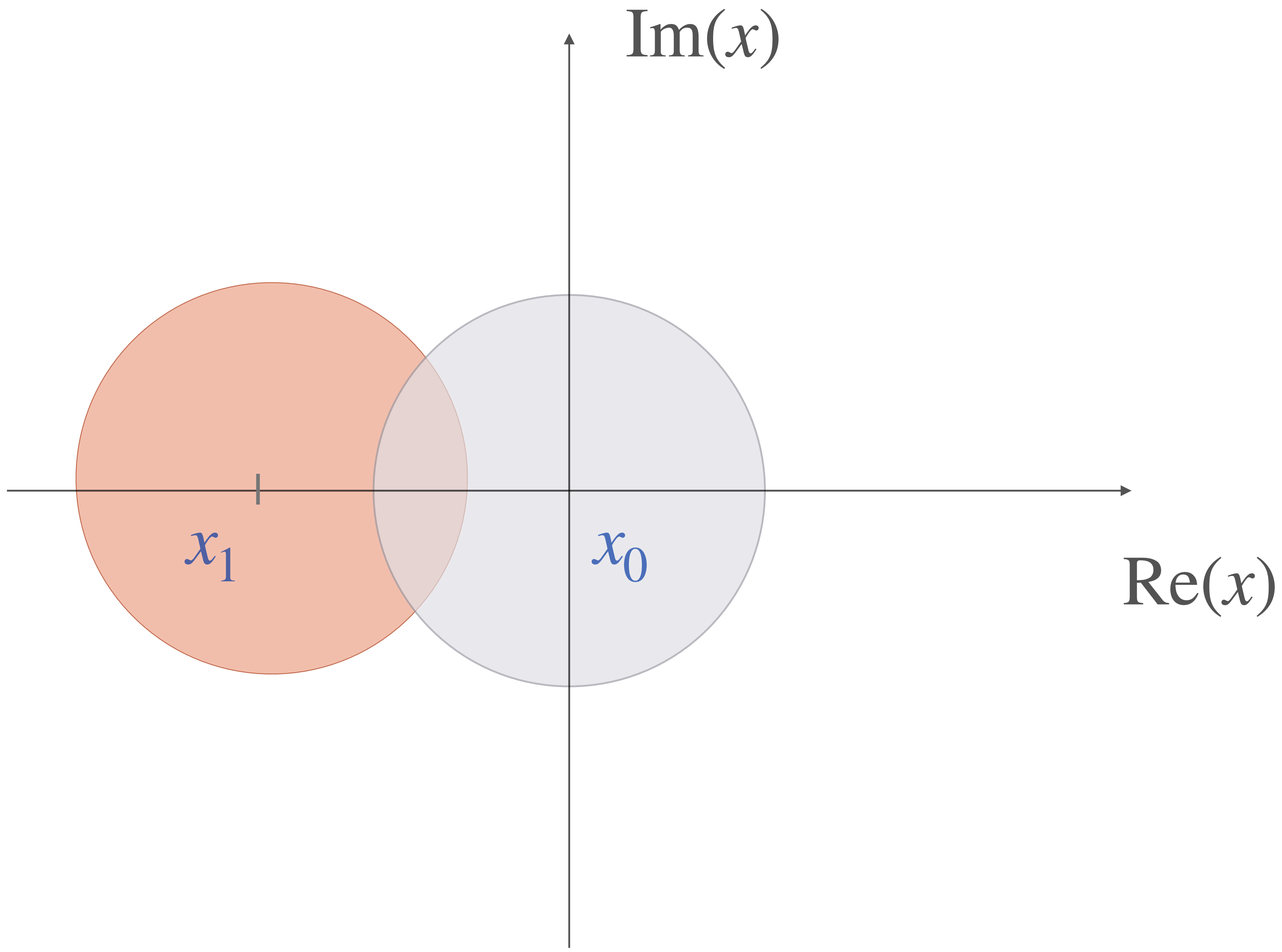


S. Pozzorini and E. Remiddi, Comput. Phys. Commun. 175, 381 (2006), arXiv:hep-ph/0505041.
X. Liu, Y.-Q. Ma, and C.-Y. Wang, Phys. Lett. B 779, 353 (2018), arXiv:1711.09572 [hep-ph].
R. N. Lee, A. V. Smirnov, and V. A. Smirnov, JHEP 03, 008 (2018), arXiv:1709.07525 [hep-ph].
M. K. Mandal and X. Zhao, JHEP 03, 190 (2019), arXiv:1812.03060 [hep-ph].
M. L. Czakon and M. Niggetiedt, JHEP 05, 149 (2020), arXiv:2001.03008 [hep-ph].
F. Moriello, JHEP 01, 150 (2020), arXiv:1907.13234, [hep-ph]
MF, Lange, Schönwald, Steinhauser JHEP 09 (2021) 152
Hidding, Comput.Phys.Commun. 269 (2021) 108125
Armadillo, Bonciani, Devoto, Rana, Vicini, Comput.Phys.Commun. 282 (2023) 108545

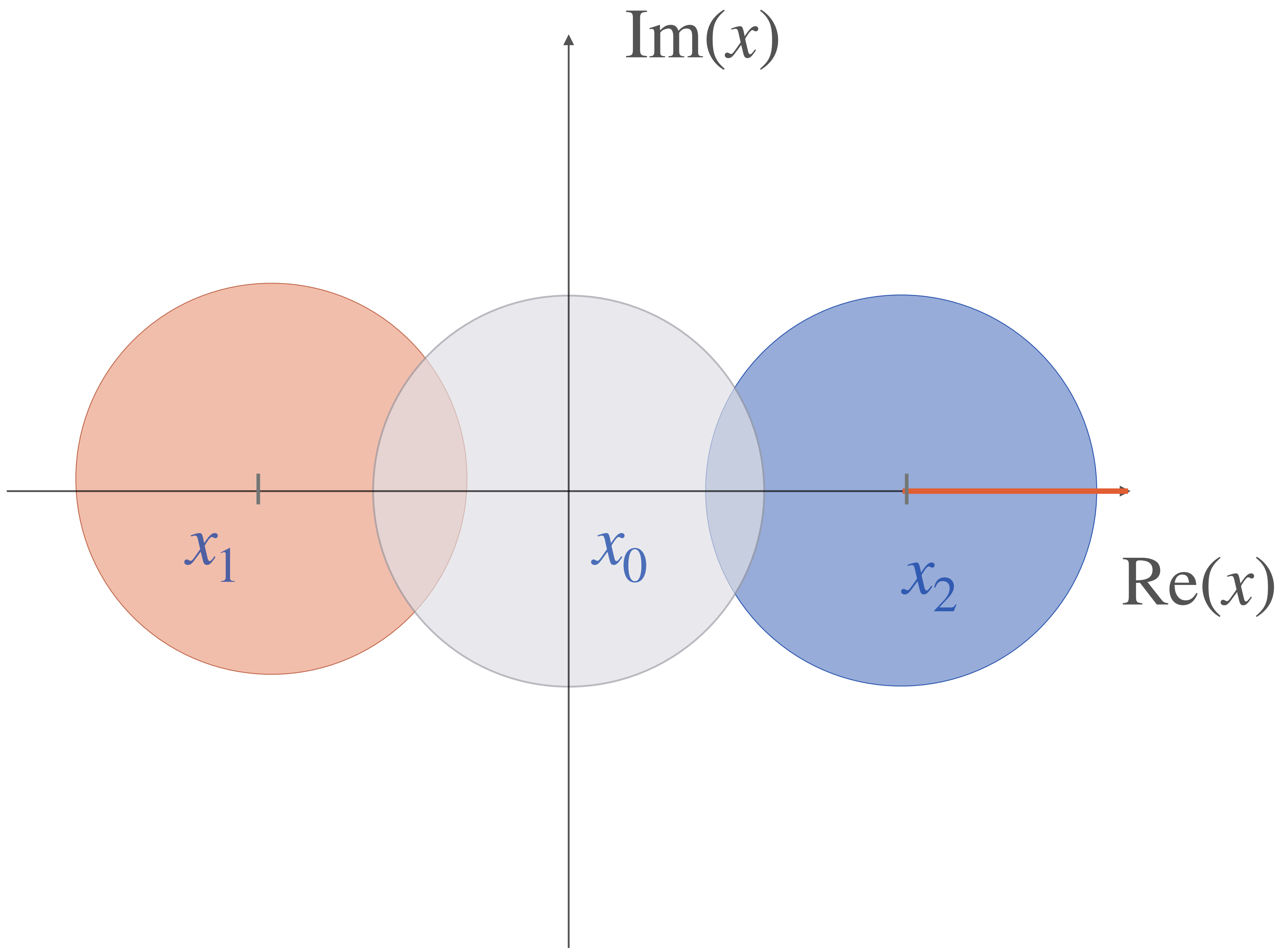
Compare order by order in x_0 and ϵ

$$\underbrace{\sum_m \sum_{n=1} n c_{a,mn} \epsilon^m (x - x_0)^{n-1}}_{\partial I_a / \partial x} = \sum_b M_{ab}(x, \epsilon) \underbrace{\sum_m \sum_{n=0} c_{b,mn} \epsilon^m (x - x_0)^n}_{I_b}$$

- Linear system of equations for the expansion coefficients $c_{k,mn}$
- Solve the liner system in term of a minimal set of coefficients
- The minimal set of undetermined coefficients are fixed from boundary conditions



- Proceeds with a new expansion around
- Match new expansion to the previous one (with finite accuracy)
- Iterate until all range of is covered



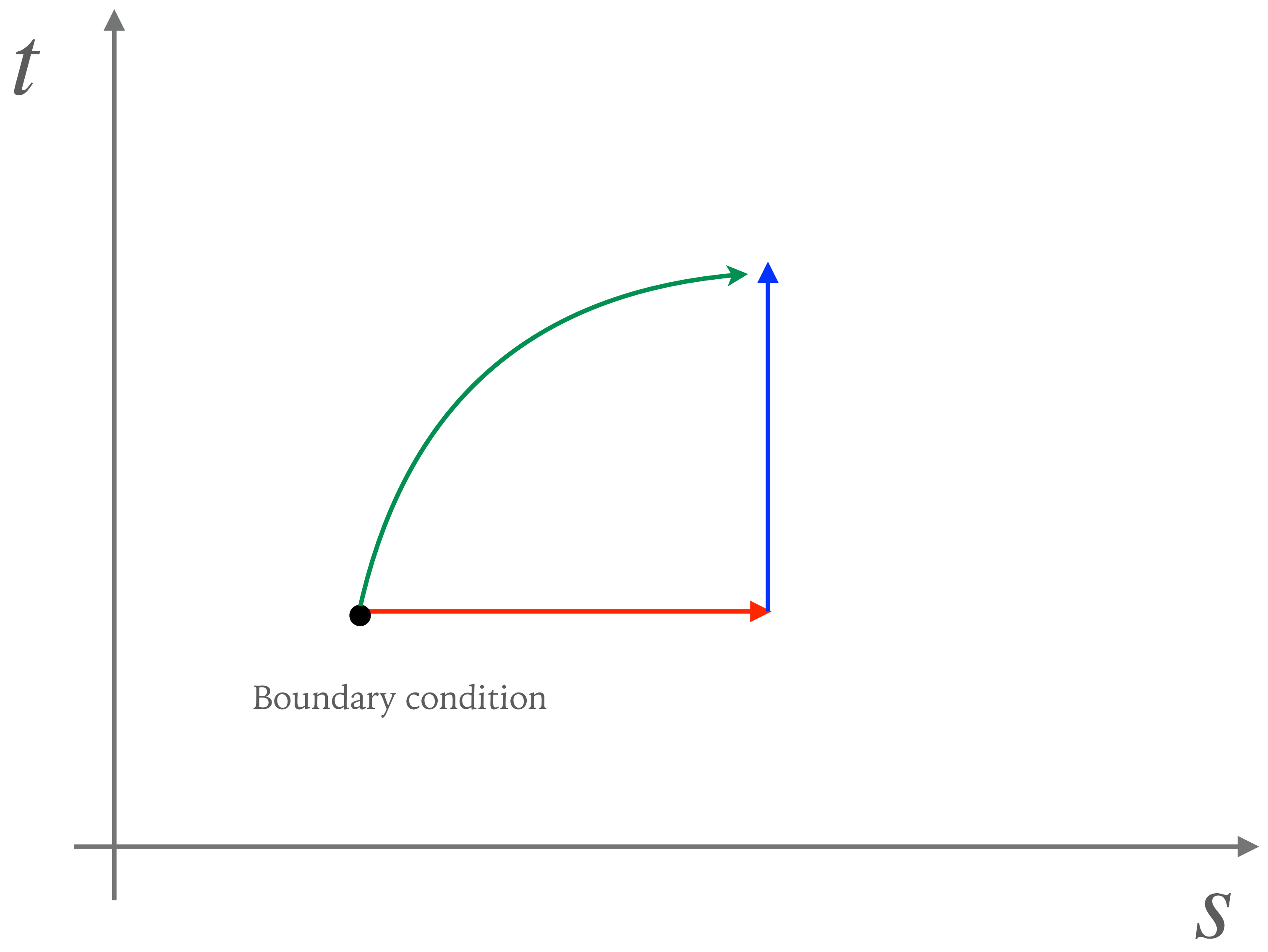
- Proceeds with a new expansion around
- Match new expansion to the previous one (with finite accuracy)
- Iterate until all range of is covered

- Power-log expansion around singular points (thresholds)

$$I_a(x,\epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} \sum_{l\geq 0} c_{a,mnl}\, \epsilon^m\, (x-x_2)^{\alpha n-\beta} \log^l(x-x_2)$$

MANY VARIATIONS

- The solution is not in close form
- Solution with arbitrary number of digits
- Multivariate case can be approached by considering one variable at a time
- Final amplitude can be parametrised via grids

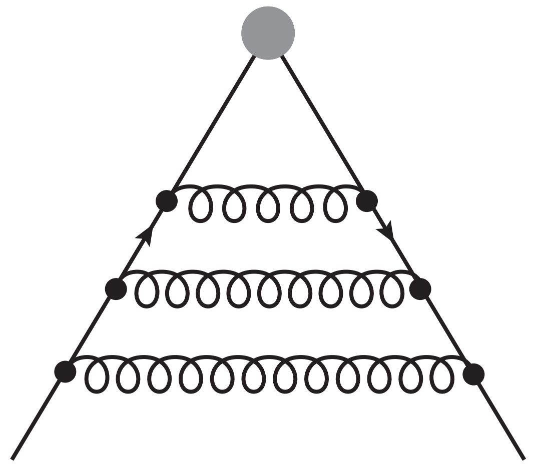


Several approaches

- DESS
  Lee, Smirnov, Smirnov, JHEP 03 (2018) 008
- DiffExp
  Hidding, Comput.Phys.Commun. 269 (2021) 108125
- SeaSide
  Armadillo, Bonciani, Devoto, Rana, Vicini, Comput.Phys.Commun. 282 (2023) 108545
- AMFlow
  Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565
- Expand and match
  MF, Lange, Schönwald, Steinhauser JHEP 09 (2021) 152

Heavy quark form factors at $O(\alpha_s^3)$

MF, Lange, Schönwald, Steinhauser Phys.Rev.Lett. 128 (2022) 17; Phys.Rev.D 106 (2022) 3, 034029; Phys.Rev.D 107 (2023), 094017



AUXILIARY MASS METHOD

Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565

$$I(\vec{n}) = \int \prod_{i=1}^{L} d^D\ell_i \frac{1}{D_1^{n_1}\dots D_N^{n_N}}$$

$$= \lim_{\eta \to i0^-} I_{\text{aux}}(\vec{n}, \eta)$$

Fix all external kinematics to numerical values

$s = 2, t = 1/10, m = 1, \text{etc}$

Integrals with auxiliary mass parameter η

$$I_{\text{aux}}(\vec{n}, \eta) = \int \prod_{i=1}^{L} d^D\ell_i \frac{1}{(D_1-\eta)^{n_1}\dots(D_K-\eta)^{n_K}\dots D_N^{n_N}}$$

Method of regions

$$\frac{1}{(\ell+p)^2-m^2-\eta}=\frac{1}{\ell^2-\eta}\sum_i\left(-\frac{2p\cdot\ell+p^2-m^2}{\ell^2-\eta}\right)^i$$

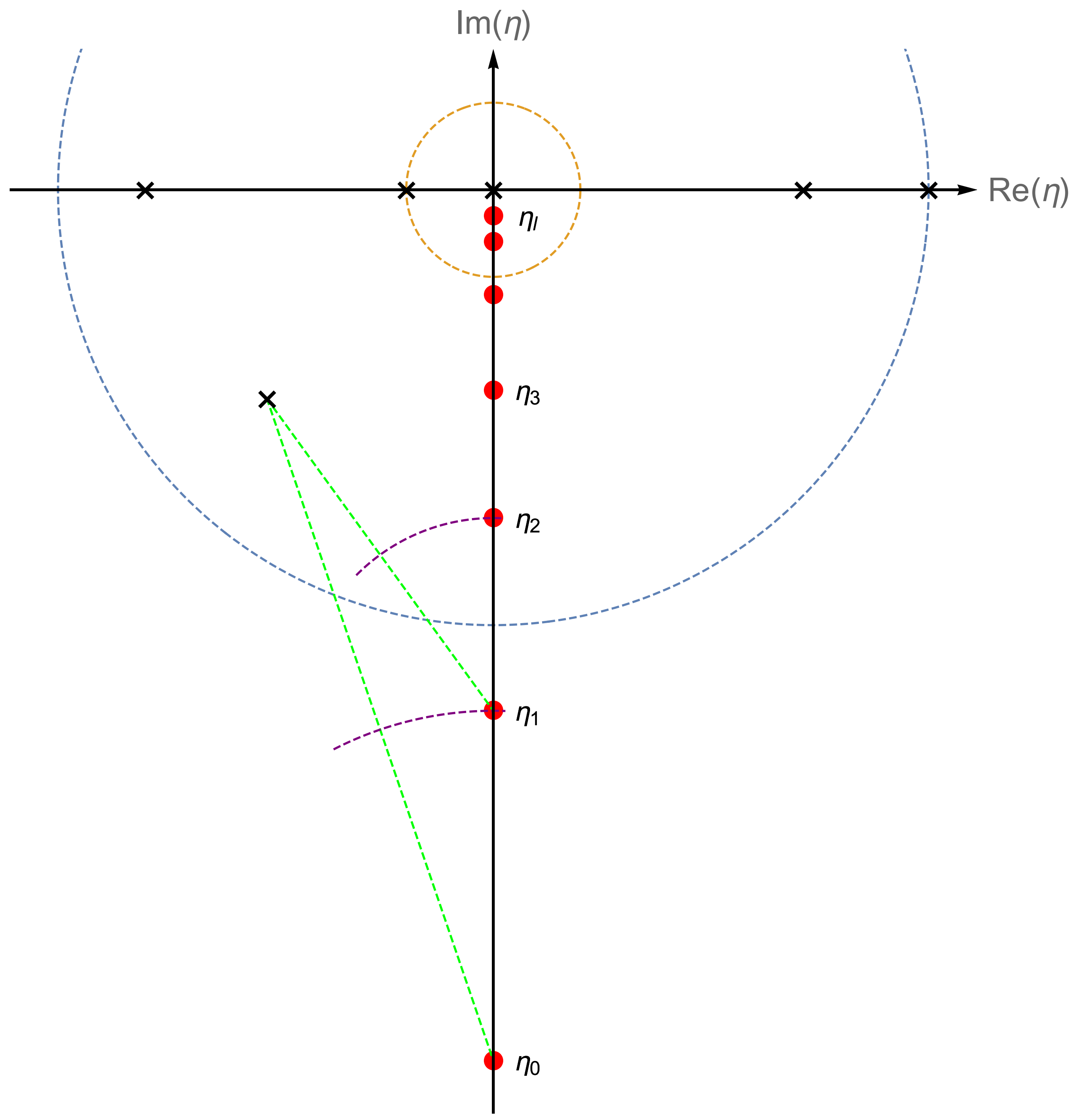
Differential equations

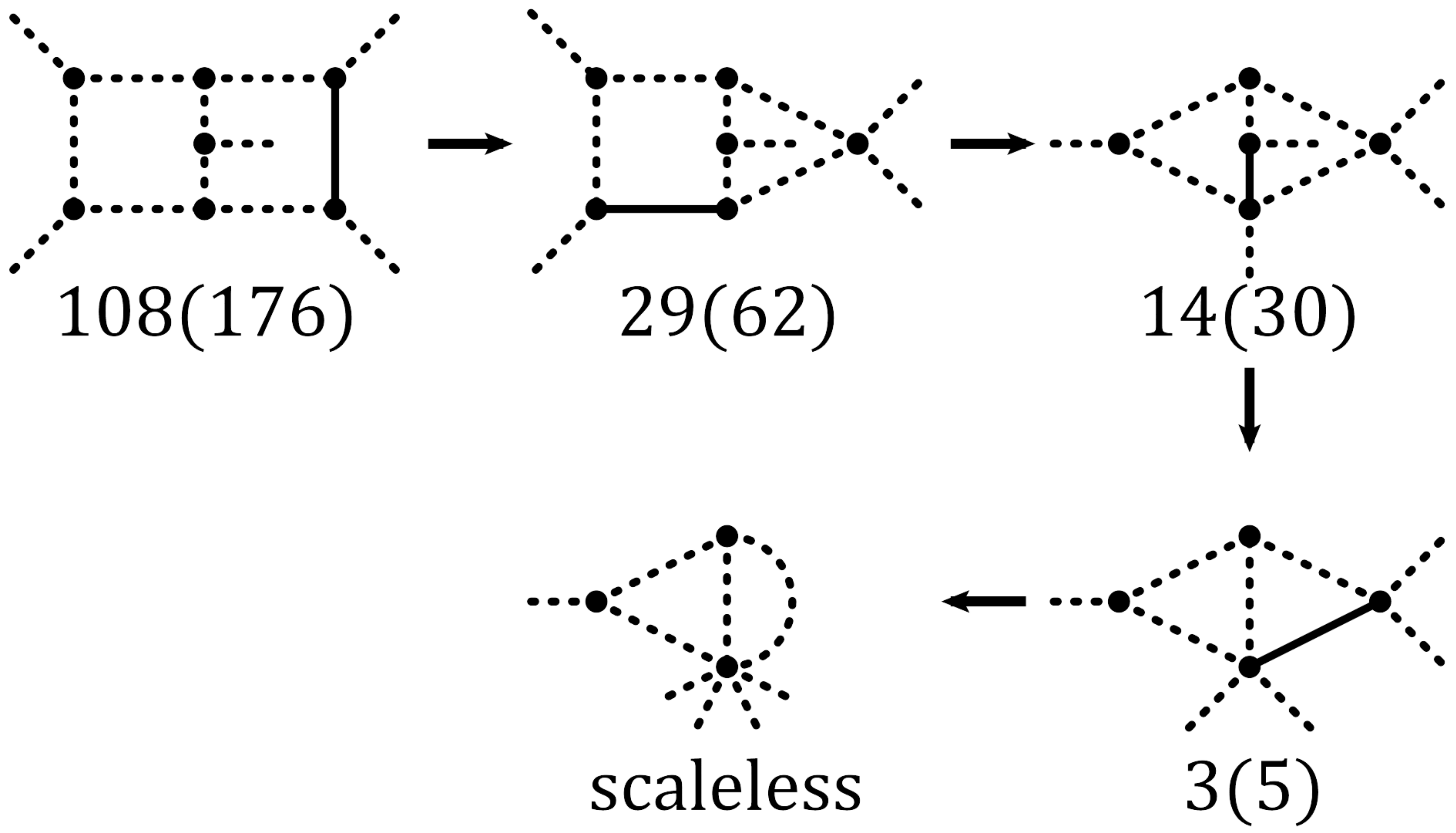
$$\frac{\partial I_{\text{aux}}(\eta)}{\partial\eta}=A(\eta)I_{\text{aux}}(\eta)$$

Boundary conditions at $\eta = i\infty$: Equal mass vacuum integrals

Davydychev and Tausk, Nucl, Phys. B, 1993, Broadhurst, Eur. Phys. J. C, 1999, Schroder and Vuorinen, JHEP, 2005, Kniehl, Pikelner and Veretin, JHEP, 2017, Luthe, phdthesis, 2015, Luthe, Maier, Marquard et al, JHEP, 2017

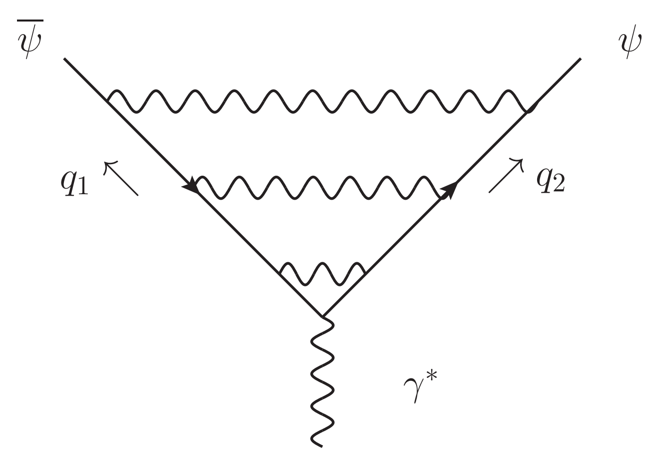
https://gitlab.com/multiloop-pku/amflow



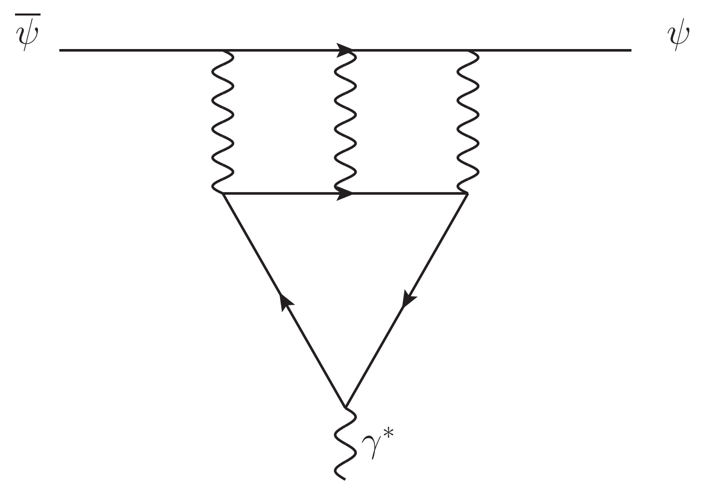


MASSIVE FORM FACTORS

	non singlet	n_h singlet	n_l singlet
diagrams	271	66	66
families	34	17	13
masters	422	316	158



Non-singlet



Singlet

- We obtain deep series expansion for the master integrals around singular and regular points in $s = (q_1 + q_2)^2$
- Boundary conditions with a mixture of analytic method and numerical evaluation with AMFlow

	Current	Form factors
vector	$j^v_\mu = \bar{\psi}\gamma_\mu\psi$	$\Gamma^v_\mu(s) = F^v_1(s)\gamma_\mu - \frac{i}{2m}F^v_2(s)\sigma_{\mu\nu}q^\nu$
axial-vector	$j^a_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$	$\Gamma^a_\mu(s) = F^a_1(s)\gamma_\mu\gamma_5 - \frac{1}{2m}F^a_2(s)\gamma_5 q_\mu$
scalar	$j_s = m\bar{\psi}\psi$	$\Gamma^s(s) = mF^s(s)$
pseudo-scalar	$j_p = im\bar{\psi}\gamma_5\psi$	$\Gamma^p(s) = imF^p(s)$

BOUNDARY CONDITIONS

- ➤ Analytic boundary conditions can be reconstructed from AMFlow evaluation

  see e.g. MF, Herren,, hep-ph/2403.03976
- ➤ No need to study specific kinematic limit
- ➤ Generic analytic solution must be also evaluated with many digits
- ➤ Basis of transcendental constants must be guessed in advance

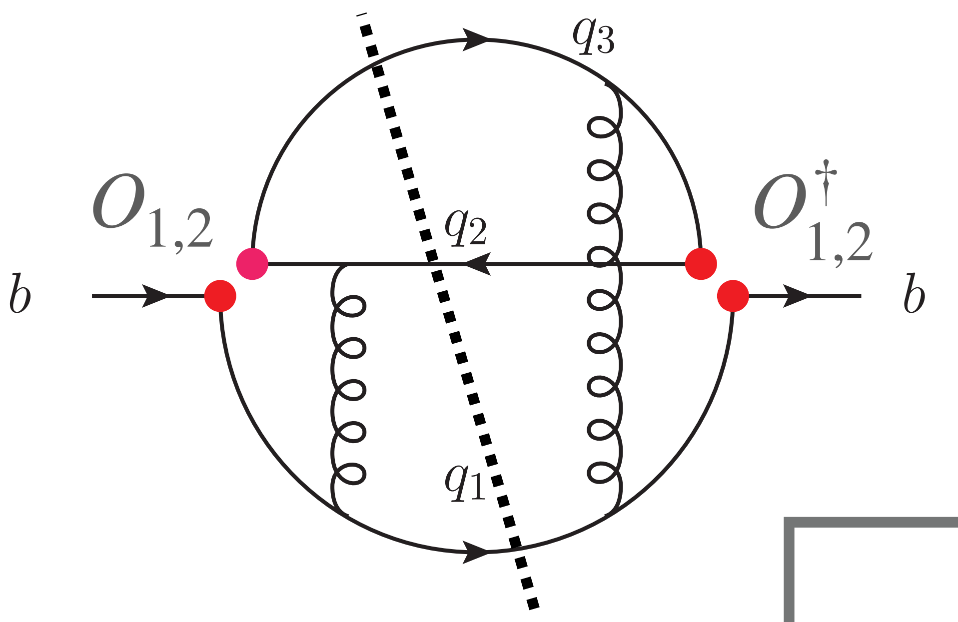
PolyLogTools + GiNaC: Duhr, Dulat, JHEP 08 (2019) 135; Bauer, Frink, Kreckel, , J. Symb. Comput. 33 (2002) 1

Ferguson, Bailey, Arno, Mathematics of Computation 68 (1999) 351.

$$2.18269754013877673 46\ldots = \frac{13\pi^2}{72} + \frac{\zeta_3}{3}$$

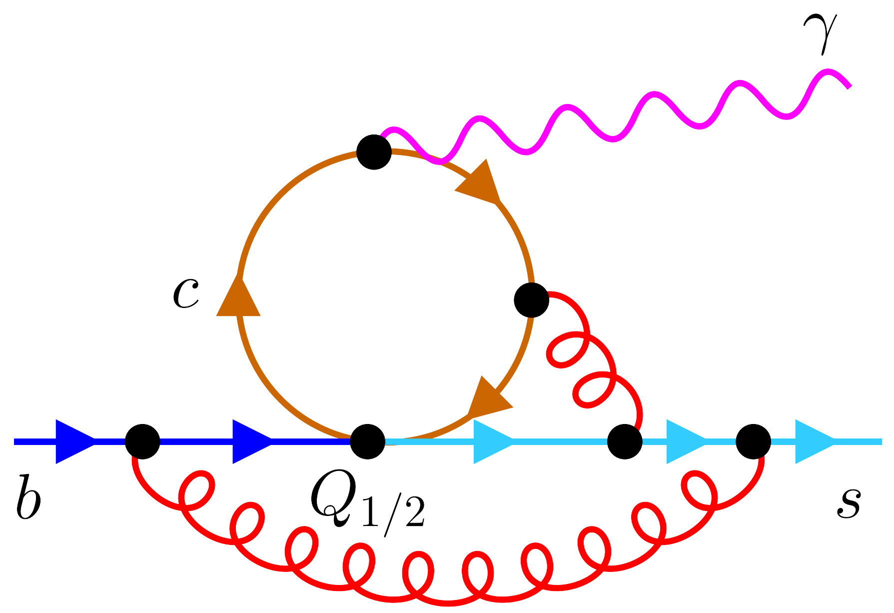
Suddenly, new applications are enabled…

$\Gamma(B) \simeq$



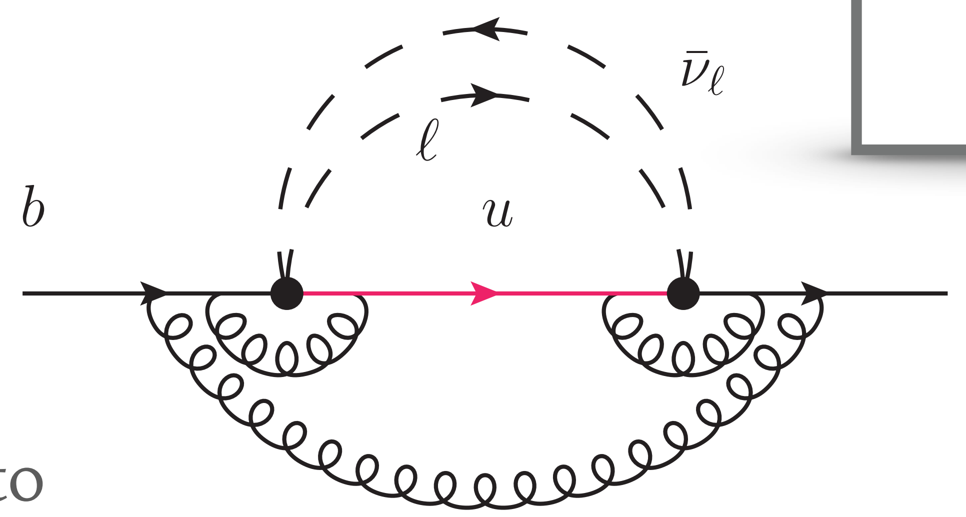
NNLO corrections to B lifetime

Egner, **MF** Schönwald, Steinhauser, *JHEP* 09 (2023) 112, in preparation



Three-loop corrections to $b \to s\gamma$ vertex

MF, Lange, Schönwald, Steinhauser, 2309.14706
Misiak et al, 2309.14707



N3LO corrections to
$b \to ul\bar{\nu}_l$ and $t \to bW$ decays

MF, Usovitsch, Phys.Rev.D 108 (2023) 11, 11
Chen, Li, Li, Wang, Wand, Wu, hep-ph/2309.00762
Long Chen, Xiang Chen, Xin Guan, Yan-Qing Ma, hep-ph/2309.01937

$B-\overline{B}$ mixing:
Reeck, Shtabovenko, Steinhauser, 2405.14698 [hep-ph]

$gg \to HH$:
Davies, Schönwald, Steinhauser, Phys.Lett.B 845 (2023) 138146

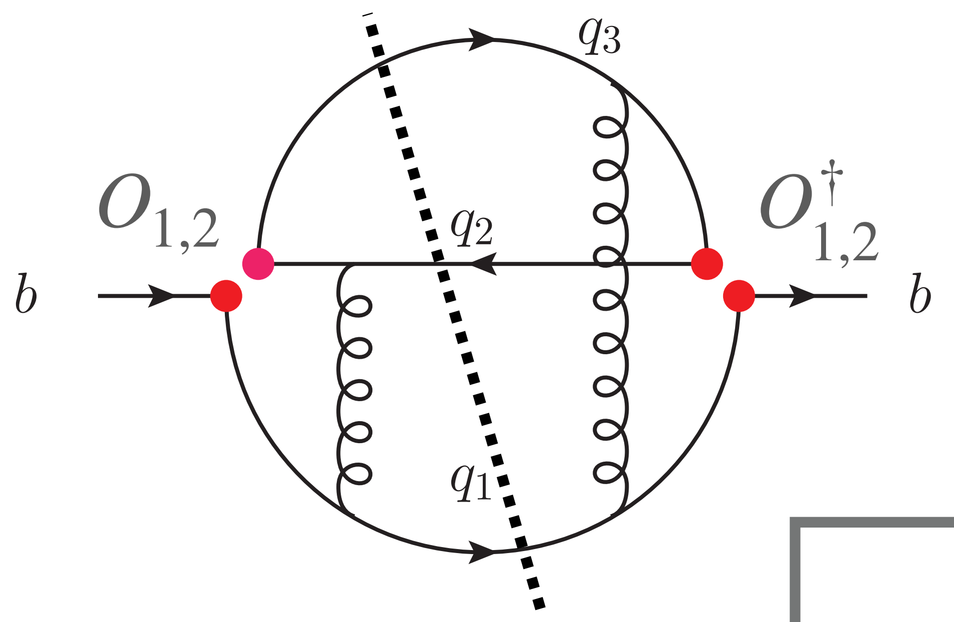
$gg \to H$:
Niggetiedt, Usovitsch, JHEP 02 (2024) 087

Drell-Yan:
Armadillo et al, 2405.00612 [hep-ph], JHEP 05 (2022) 072

$\sigma_{\rm tot}(e^+e^- \to q\bar{q})$:
Xiang Chen, Xin Guan, Chuan-Qi He, Xiao Liu, Yan-Qing Ma, *Phys.Rev.Lett.* 132 (2024) 10, 10

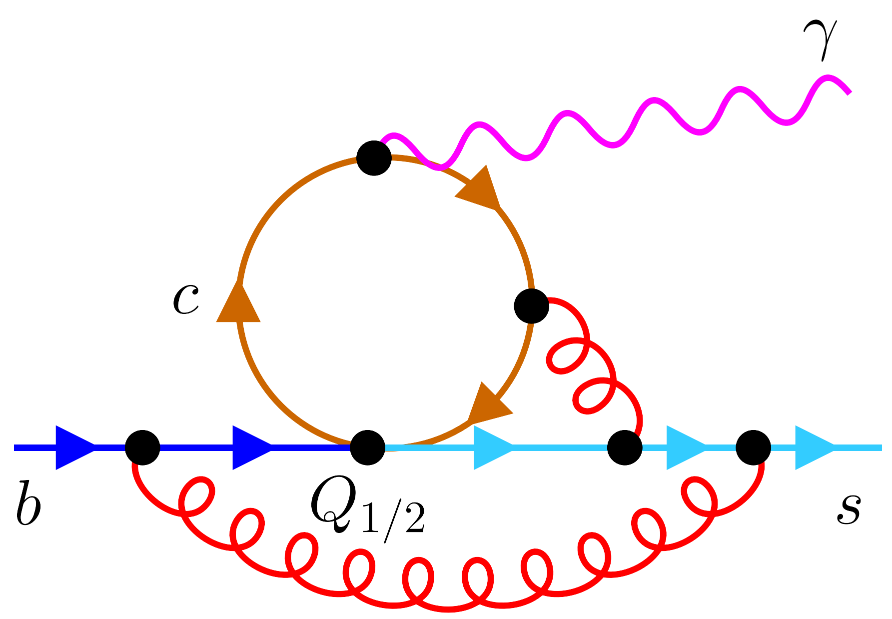
Suddenly, new applications are enabled…

$\Gamma(B) \simeq$



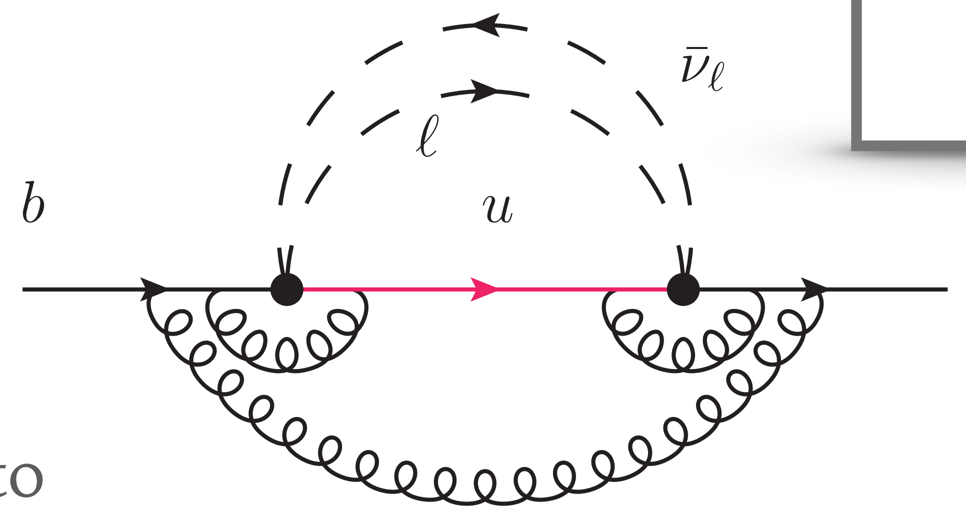
NNLO corrections to B lifetime

Egner, **MF** Schönwald, Steinhauser, *JHEP* 09 (2023) 112, in preparation



Three-loop corrections to $b \to s\gamma$ vertex

MF, Lange, Schönwald, Steinhauser, 2309.14706
Misiak et al, 2309.14707



N3LO corrections to
$b \to u l \bar{\nu}_l$ and $t \to bW$ decays

MF, Usovitsch, Phys.Rev.D 108 (2023) 11, 11
Chen, Li, Li, Wang, Wand, Wu, hep-ph/2309.00762
Long Chen, Xiang Chen, Xin Guan, Yan-Qing Ma, hep-ph/2309.01937

$B - \overline{B}$ mixing:
Reeck, Shtabovenko, Steinhauser, 2405.14698 [hep-ph]

$gg \to HH$:
Davies, Schönwald, Steinhauser, Phys.Lett.B 845 (2023) 138146

$gg \to H$:
Niggetiedt, Usovitsch, JHEP 02 (2024) 087

Drell-Yan:
Armadillo et al, 2405.00612 [hep-ph], JHEP 05 (2022) 072

$\sigma_{\text{tot}}(e^+e^- \to q\bar{q})$:
Xiang Chen, Xin Guan, Chuan-Qi He, Xiao Liu, Yan-Qing Ma, *Phys.Rev.Lett.* 132 (2024) 10, 10

Can we perform IBP reduction for the amplitude?

Can we generate the DEQs?

LINEAR RELATIONS AMONG FEYNMAN INTEGRALS

Integration-by-parts relations

Chetyrkin, Tkachov, Nucl. Phys. B 192 (1981) 159.
Tkachov, Phys. Lett. B 100 (1981) 65.

$$\int d^dk_1 \ldots d^dk_L \frac{\partial}{\partial (k_i)_\mu} \left((q_j)_\mu \frac{1}{[D_1]^{a_1} \ldots [D_N]^{a_N}} \right) = 0$$

Lorentz-invariance relations

Gehrmann, Remiddi,Nucl.Phys.B580,485

$$q_i^\mu q_j^\nu \left(\sum_{r=1}^{E} q_{r[\nu} \frac{\partial}{\partial q_r^{\mu]}} \right) I(a_1, \ldots, a_n) = 0$$

Symmetries

$$I(\ldots, a_m, \ldots, a_n, \ldots) = I(\ldots, a_n, \ldots, a_m, \ldots)$$

LAPORTA ALGORITHM

Laporta, Int. J. Mod. Phys. A 15 (2000) 5087

IBP relation templates

$$c_1(\{a_f\},\vec{s},d)I(a_1,\ldots,a_N-1)+\ldots+c_m(\{a_f\},\vec{s},d)I(a_1+1,\ldots,a_N)=0$$

Laporta algorithm

- Insert seeds into IBP relations:

This stage can blow up the memory for complicated problems

$$a_1=1,a_2=0,a_3=-1,\ \ldots$$
$$a_1=2,a_2=0,a_3=-1,\ \ldots$$

$\rightarrow$

$$0=c_1I(1,-1,-1,\ldots)+c_2I(2,0,-1,\ldots)+c_3I(1,0,-2,\ldots)+\ldots$$
$$0=c_1I(2,-1,-1,\ldots)+c_2I(3,0,-1,\ldots)+c_3I(2,0,-2,\ldots)+\ldots$$

- Solve highly redundant and sparse linear system (Gaussian elimination)

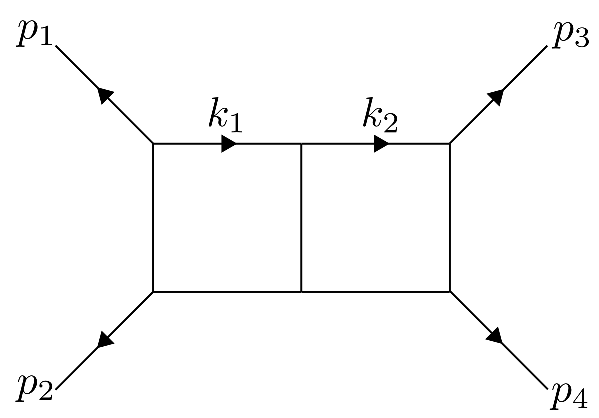
Complicated operations over multivariate polynomials

BETTER COMPUTER ALGEBRA SYSTEMS

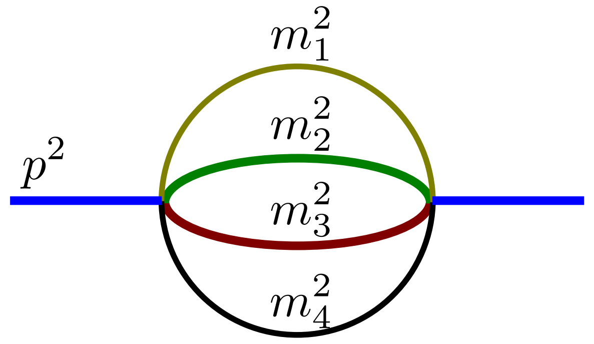
- Kira, FIRE and Reduze in C++
  https://gitlab.com/kira-pyred/kira
  https://gitlab.com/feynmanintegrals/fire
  https://reduze.hepforge.org/
- FERMAT has beed standard for many years
  http://home.bway.net/lewis/
- FLINT
  https://flintlib.org
- Symbolica by Ben Ruijl
  https://symbolica.io/

Comparisons with FIRE

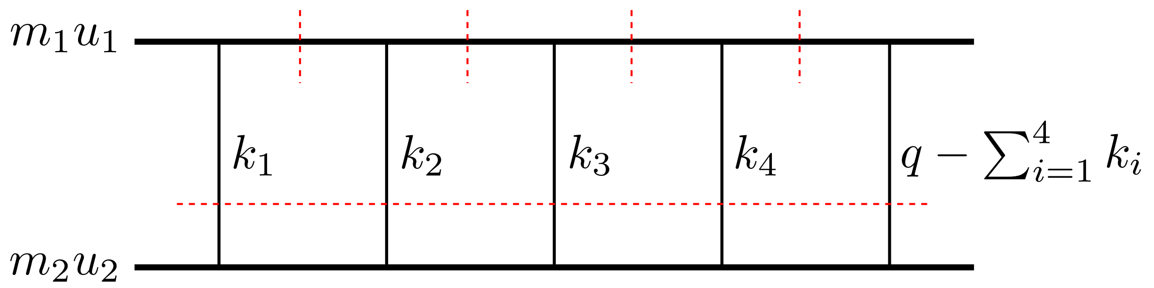
Smirnov, Zeng, Comput.Phys.Commun. 302 (2024) 109261



Simplifier	rank-2 time (s)	rank-8 time (1000 s)
FLINT	7.5, 10.8	0.13, 0.30
Symbolica	6.7, 9.2	0.13, 0.26
Fermat	7.9, 14.9	0.25, 0.48



Simplifier	Time (1000s)
FLINT	0.62, 1.01
Symbolica	1.02, 1.66
Fermat	5.36, 9.23

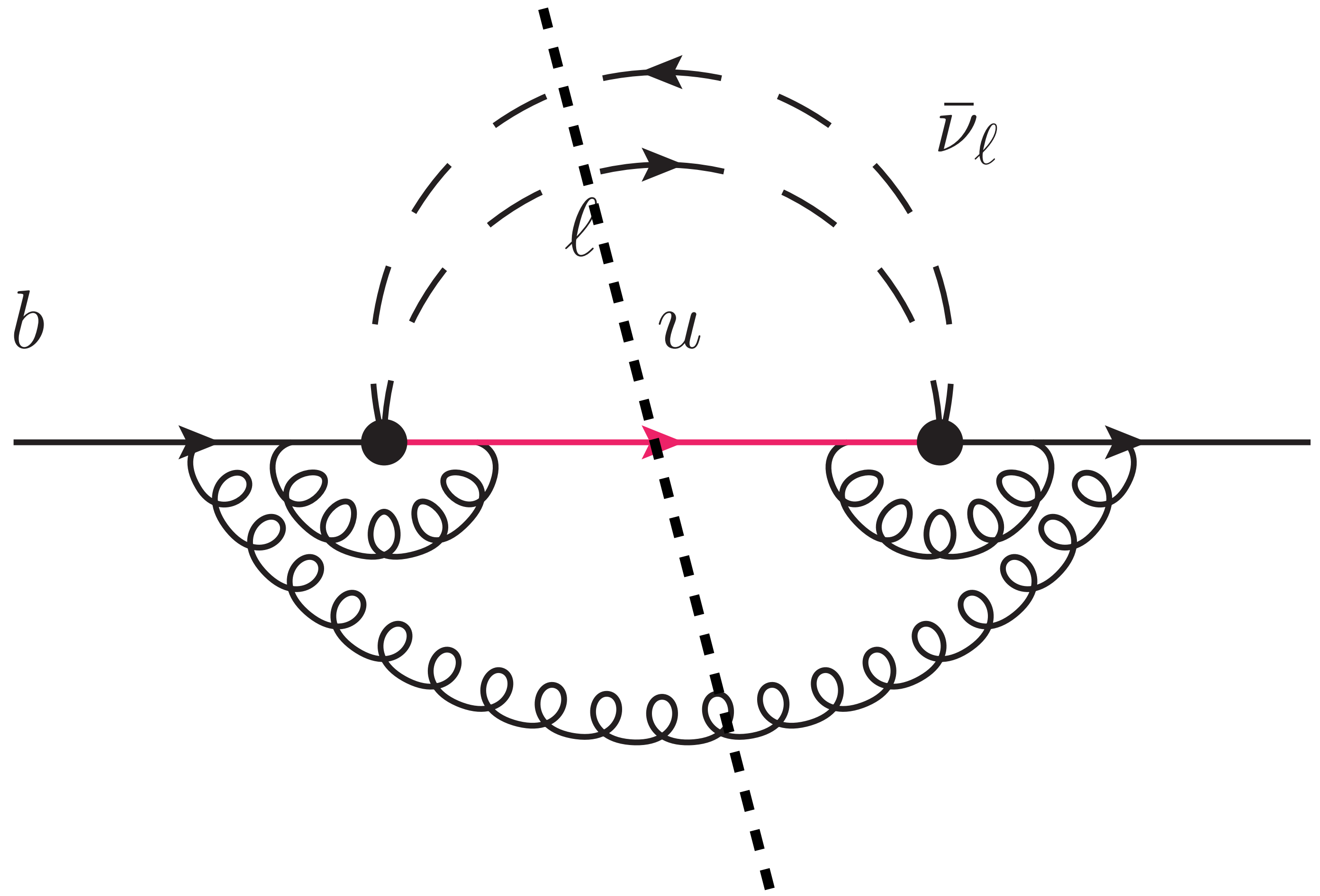


Simplifier	Time (1000s)
FLINT	0.45, 0.97
Symbolica	0.45, 0.94
Fermat	0.82, 1.60

PROBLEMS WITH SEEDING

Challenging 5loop families:

12 propagators + 8 numerators



MF, Usovitsch, Phys.Rev.D 108 (2023) 11, 11

- N3LO corrections to $\Gamma(b \to u l \bar{\nu}_l)$
- Set $m_q = 0$. Integrals depend only on $\epsilon = (4-d)/2$
- Integrals up to 5 irreducible scalar products (sum of negative indices)

```
- reduce_sectors:
    reduce:
      - {topologies: [SLTOP51542], sectors: [4095], r: 12, s: 5}
```

- Insane combinatorics when seeding the IBP vectors
- We do not even generate the system with Kira 2.3

PROBLEMS WITH SEEDING

- Fix sum positive indices r, negative indices s, and dots d

$$I(1,1,1,1,1,1,0,-5) \qquad r=6, s=5, d=0$$

- Example $I(1,1,1,1,1,1,0,-5)$
  - Top sector $b111\,111\,00$ seed with $r=6, s=5, d=0$
  - Subsector $b111\,110\,00$ seed with $r=6, s=5, d=1$
  - …
  - Subsector $b111\,100\,00$ seed with $r=6, s=5, d=2$
- High values of s lead to huge combinatorics in lower sectors

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Kira developers are improving the seeding strategy …

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For n missing propagators

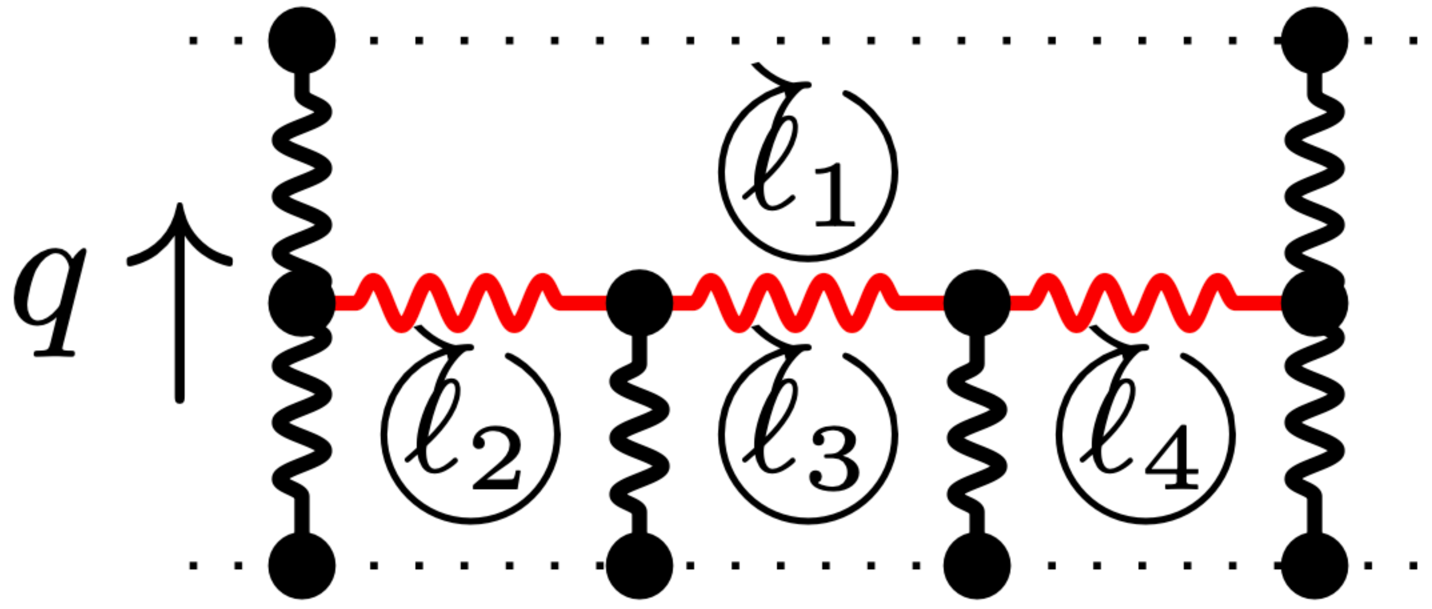
$\sim$ reduce total negative indices by n

Kira developers are improving the seeding strategy …

FIFTH POST-MINKOWSKIAN, FIRST SELF-FORCE ORDER CONTRIBUTIONS TO CONSERVATIVE BLACK HOLE SCATTERING

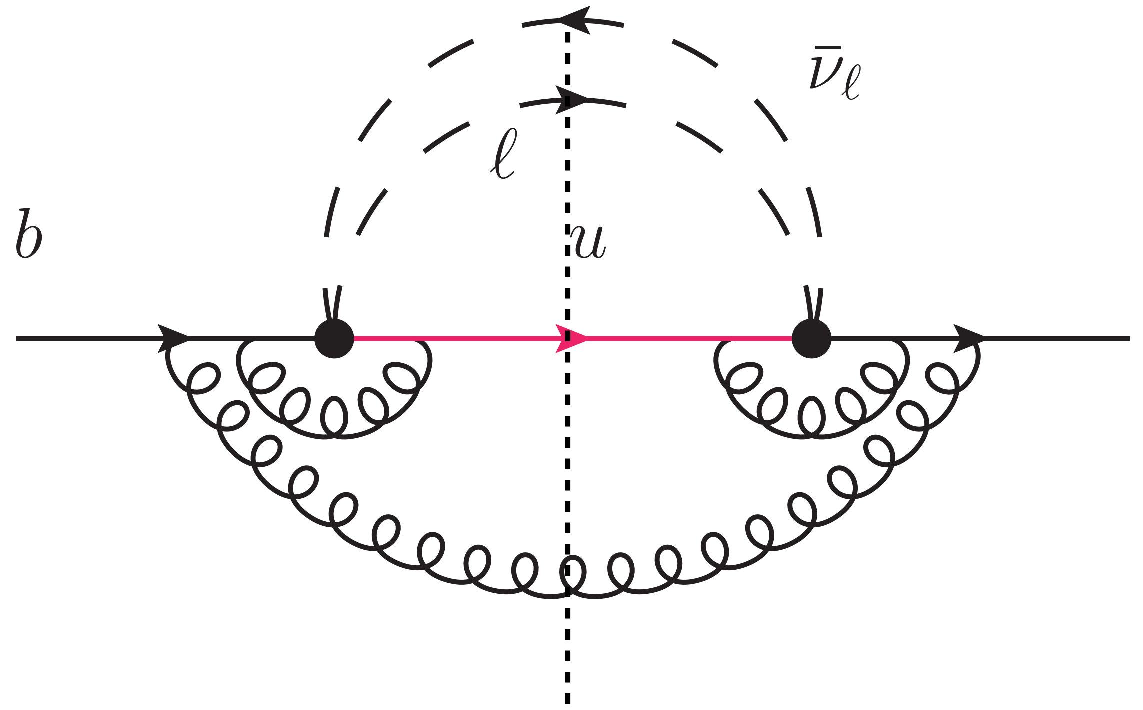
Driesse, Jakobsen, Mogull, Plefka, Sauer, Usovitsch, 2403.07781 [hep-th]

- 4 loop calculation
- 13 propagators + nine irreducible scalar propagators
- Perform reduction on a 1.5TiB RAM



$\Gamma(b \to u l \bar{\nu}_l)$: NUMERICAL EVALUATION WITH AMFLOW

- Example: DEQs generation for the auxiliary mass flow for a complicated 5 loop family from the bosonic diagrams:
  - 4×10^6 equations → 300 000 equations
  - FireFly: 500s per probe → 16s per probe



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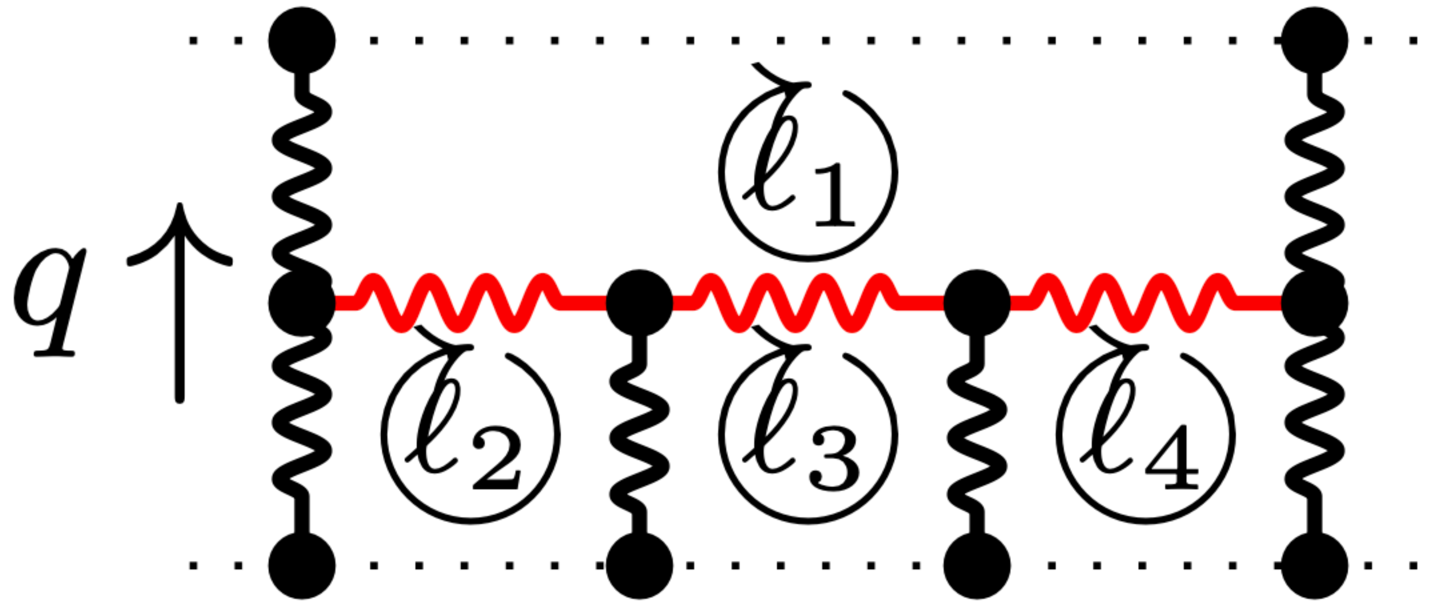

$\Gamma(b \to ul\bar{\nu}_l)$: NUMERICAL EVALUATION WITH AMFLOW

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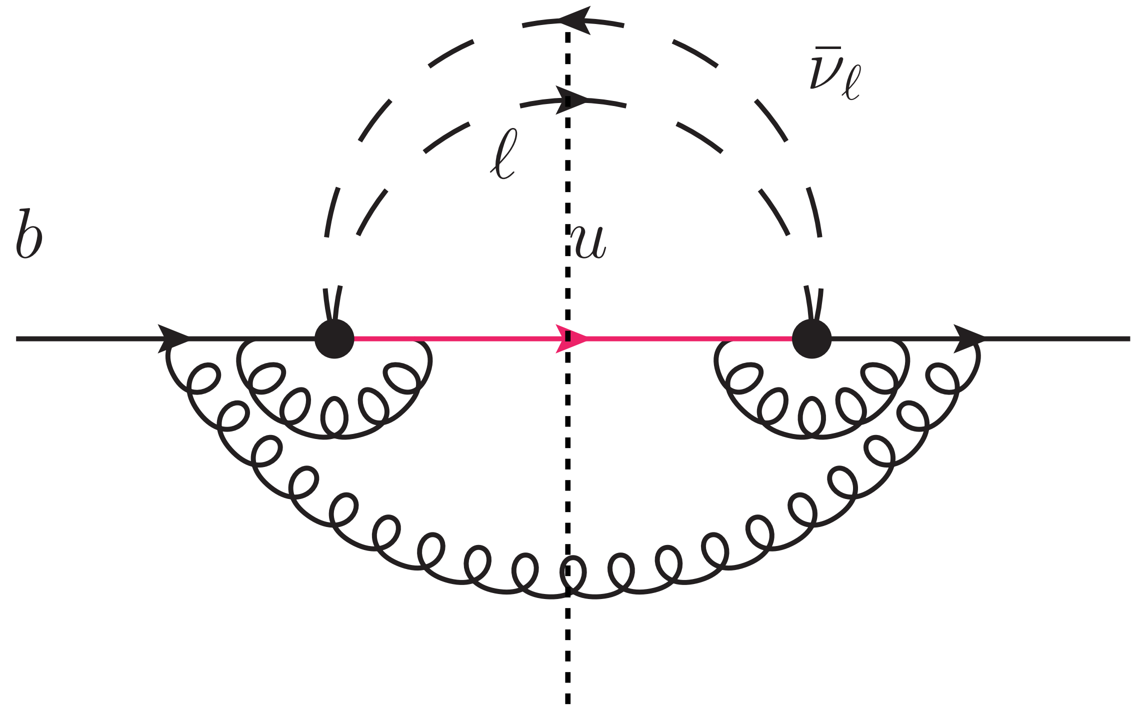
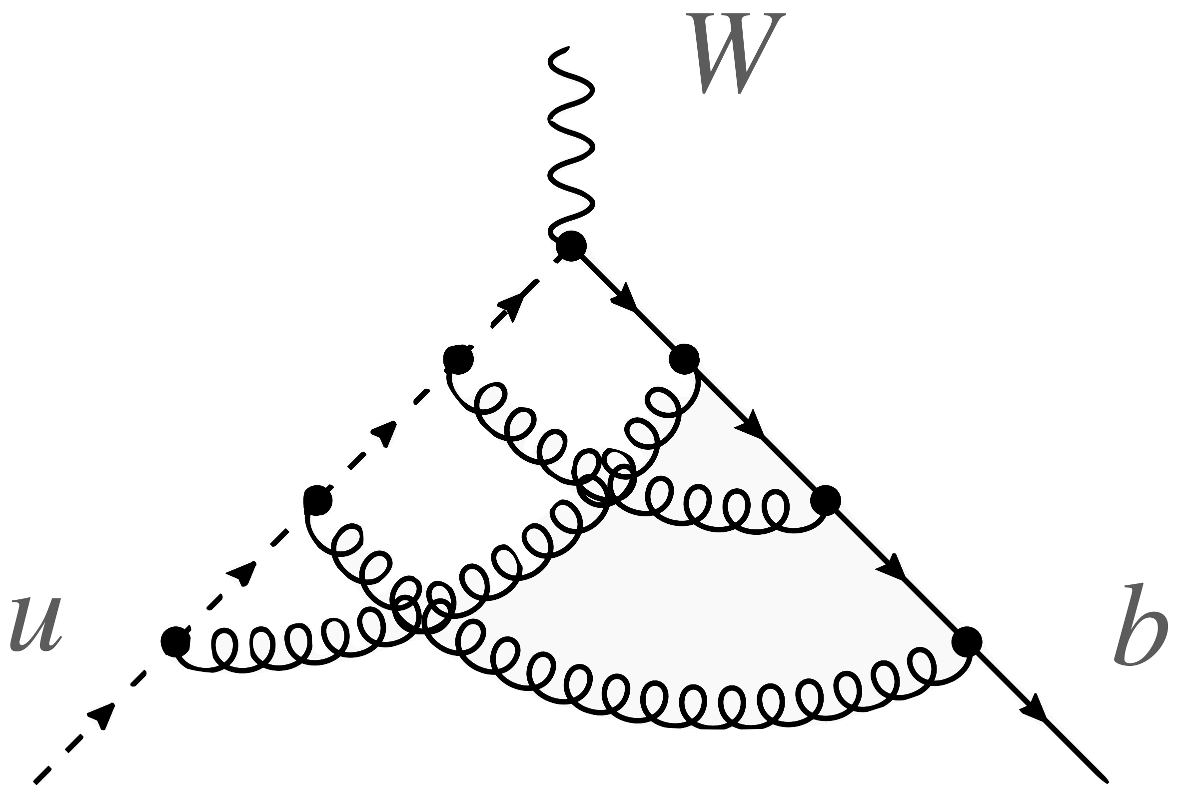


Approximate 10^{L-1} runtime improvement!



- Heavy-to-light form factors at three loops

  [Fael, Huber, **FL**, Müller, Schönwald, Steinhauser SOON]

- Reduce all integrals required for amplitude including gauge dependence

	Kira 2.3	Kira 2.3, tuned	Kira dev
# of generated equations	304 821 183	198 405 903	3 779 262
# of selected equations	21 666 745	24 708 949	828 010
# of terms	828 060 148	583 976 288	12 516 161
probe time	2255 s	3726 s	19 s
memory (generator)	194 GiB	129 GiB	3.2 GiB

Improvement: $\times 119$ in runtime and $\times 61$in memory usage

slide by F. Lange, Loops and Legs 2024

SYZYGY EQUATIONS

Gluza, Kajda, Kosower, Phys. Rev. D 83 (2011) 045012

- IBP relations without raising operators

$$\sum_i \phi_i^\mu \frac{\partial}{\partial \ell_\mu} \frac{1}{(\ell + p)^2 - m^2} = \gamma \frac{1}{(\ell + p)^2 - m^2}$$

- Contract an optimised system
- Solution of syzygy equations simpler
- NeatIBP

Wu, Boehm, Ma, Xu, Zhang,Comput.Phys.Commun. 295 (2024) 108999
https://github.com/yzhphy/NeatIBP

BLOCK-TRIANGULAR FORM

Xiao Liu and Yan-Qing Ma. Phys.Rev.D 99 (2019) 071501. Xin Guan, Xiao Liu and Yan-Qing Ma, Chin.Phys.C 44 (2020) 9, 093106
Xin Guan, Xiao Liu, Yan-Qing Ma and Wen-Hao Wu. hep-ph/2405.14621
https://gitlab.com/multiloop-pku/blade

- Many equations to solve irrelevant auxiliary integrals.
- Construct (guess) linear relations within much smaller sets

$$\sum_{i=1}^{N} Q_i(\epsilon, \vec{s}) I_i(\epsilon, \vec{s}) = 0$$

Q_i are simple polynomials

- $Q_i(\epsilon, \vec{s})$ numerically computable from IBP system over finite fields.
- Find relations to write difficult integrals in terms of simpler ones

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_n \end{pmatrix} = \begin{pmatrix} 0 & X & X & X & \dots & X \\ 0 & 0 & X & X & \dots & X \\ 0 & 0 & 0 & X & \dots & X \\ \vdots & & & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & X \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ M_m \end{pmatrix}$$

CONCLUSIONS

- Ongoing efforts to including dominant N3LO corrections to μ-e scattering.
  - VVV: massive form factors at three loops ✓
  - VVR: two-loop amplitude with $m_e = 0$ ✓
- Numerical evaluation of master integrals: new approach to higher-order calculations.
- New interesting methods to improve/speed up IBP reductions with many variables or many loops.
- MUonE has already profited and will profit a lot!