Atomic Binding Corrections For Fixed Target $\mu e \to \mu e$ Scattering

RYAN PLESTID | NTN FELLOW, CALTECH

COLLABORATORS | MARK WISE (CALTECH)

MITP TOPICAL WORKSHOP | MAINZ, DE | JUNE 2024

Caltech

Neutrino Theory Network

Theory for muon-electron scattering @ 10 ppm

A report of the MUonE theory initiative

factories [80]. Finally, one should not forget that electrons are bound inside the target and the impact of bound-state effects should be evaluated. This will require considering the possibility of scattering of the incident muons off core valence electrons, for which off-shell effects due to the finite binding energy and momentum distribution must be considered.

PART 1

HYDROGEN

- Setup of problem on hydrogen.
- Kinematics and expansion of amplitude.

PART 2

HELIUM++

- Summing over final states.
- The virial theorem and sum rules.

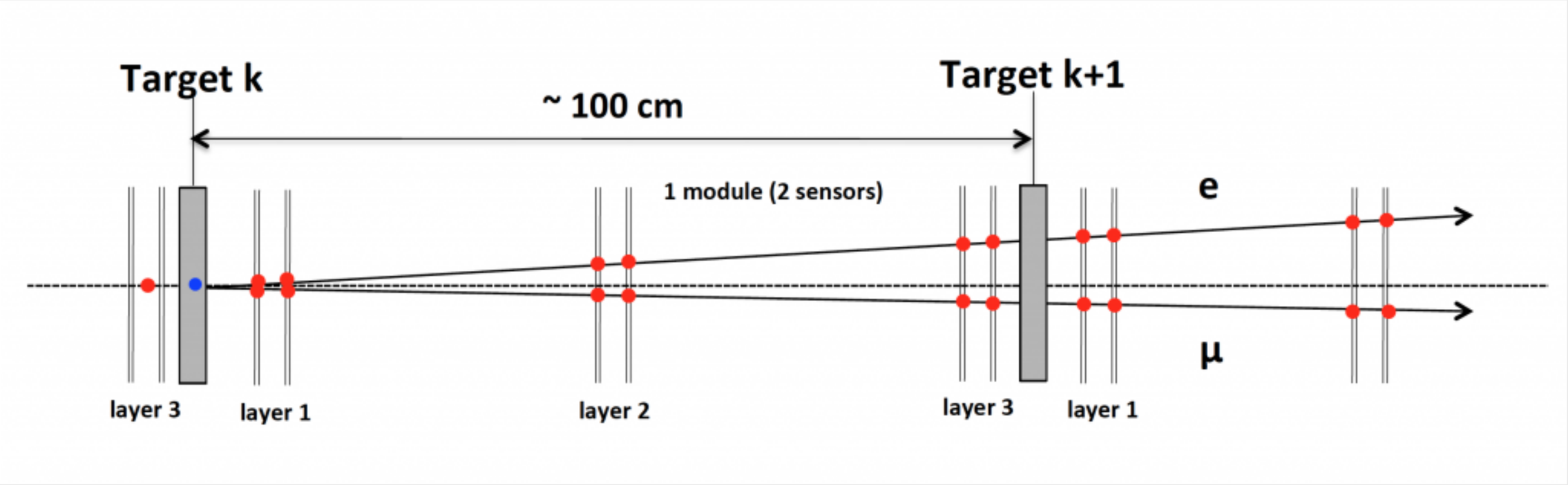
PART 3

PHENO.

- Implications for MuonE.
- Ongoing work on additional radiative corrections.

High Energy Fixed Target Scattering

$$E_\mu \sim E_e \sim 100\ \text{GeV} \quad s \sim 2m_e E_\mu + m_\mu^2 \sim (0.3\ \text{GeV})^2$$



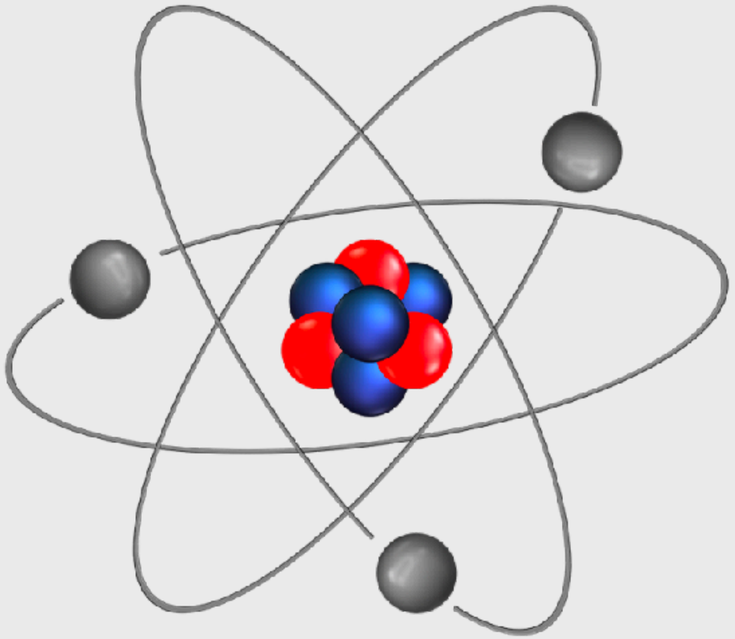
Idealized Limit: Electron At Rest

$$\mu^{\pm}(\mathbf{k}) + e^{-}(\mathbf{0}) \to \mu^{\pm}(\mathbf{k}') + e^{-}(\mathbf{p}')$$

- Electron is approximately ``at rest'' inside of atom.
- Can imagine fixed target as N_e electrons at rest.
- Compute radiative corrections to high order to predict $\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$

Reality: Electron's In Atoms

$$\mu^{\pm}(\mathbf{k}) + A(\mathbf{0}) \rightarrow \mu^{\pm}(\mathbf{k}') + e^{-}(\mathbf{p}') + B^{+}(\mathbf{p})$$

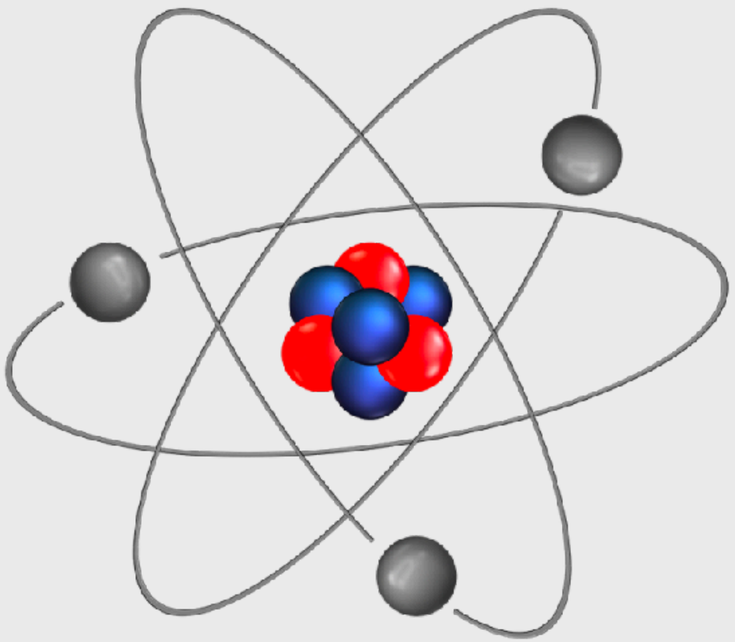


- How are these pictures related?
- What controls atomic binding effects?
- Is this relevant for MuonE?

Motivation: Kinematics

$$s_0 = 2m_e E_\mu + m_e^2 + m_\mu^2$$

CONSIDER AN ELECTRON IN MOTION



$$s = \underbrace{2(E_e E_\mu - \mathbf{p}_e \cdot \mathbf{p}_\mu)}_{\simeq 2m_e E_\mu \left(1 - \frac{p_e}{m_e}\cos\theta\right)} + m_e^2 + m_\mu^2$$

CORRECTIONS OF ORDER $p_e/m_e \sim \alpha$!

Start With Hydrogen

$$T = \quad {}_{\text{out}}\langle e'\mu' p' \,|\, \mu \mathrm{H} \rangle_{\text{in}}$$

$$|\mathrm{H}\rangle = \int [\mathrm{d}p]\ \psi(\mathbf{p})\,|\,e^{-}(\mathbf{p})\rangle\,|\,p^{+}(-\mathbf{p})\rangle + \ldots$$

- Valence electron and proton with wavefunction.
- Can now compute matrix elements

Tree-Level With Bound State

$$T \simeq \quad \langle e'p' | J^{\nu} | \mathrm{H} \rangle \; D_{\mu\nu}(k - k') \; \bar{u}(k') \gamma^{\mu} u(k)$$

SINGLE PHOTON EXCHANGE

$$\langle e'p' | J^{\nu} | \mathrm{H} \rangle \simeq \int [\mathrm{d}p] \; \psi(\mathbf{p}) \; \langle e'p' | \bar{\psi}_e \gamma^{\nu} \psi_e | ep \rangle$$

Strategy For Approximations

- Relate momentum and binding energy via virial theorem.
- Ignore all terms that are suppressed by large kinematic energies, i.e. $\epsilon_H/E_\mu \sim 10\ \text{eV}/100\ \text{GeV} \sim 10^{-10}$.
- Systematically track terms suppressed by electron mass, p_e^2/m_e^2 or ϵ_H/m_e.
- Taylor expand amplitude and fold against wave function.

Kinematics

MOMENTUM COMES FROM FINAL STATE

$$\mathbf{k}_\mu = \mathbf{p}'_e + \mathbf{k}'_\mu + \mathbf{P}_B$$

PROTON

$$\mathbf{k}_\mu - \mathbf{P}_B = \mathbf{p}'_e + \mathbf{k}'_\mu \longleftrightarrow \mathbf{k}_\mu + \mathbf{p}_e = \mathbf{p}'_e + \mathbf{k}'_\mu$$

ENTANGLED

$$|\mathrm{H}\rangle = \int [\mathrm{d}p]\ \psi(\mathbf{p})\,|e^-(\mathbf{p})\rangle\,|p^+(-\mathbf{p})\rangle + \ldots$$

Kinematics

BINDING ENERGY CHANGES ENERGY CONSERVATION

$$\omega_\mu + m_e - \epsilon_H = \omega'_\mu + E'_e$$

ELECTRON IN HYDROGEN

BINDING ENERGY

IN MOMENTUM SPACE

$$\omega_\mu + E_e + \langle V \rangle = \omega'_\mu + E'_e \qquad \langle V \rangle = -\epsilon_H - T_e$$

- Six (rather than three) mandelstam variables.

Modified Four-Momentum Relations

$$(p' \cdot k) = p \cdot k' + \langle V \rangle (k' - p)_0 - \frac{1}{2}\langle V \rangle^2$$

$$(p' \cdot k') = p \cdot k + \langle V \rangle (p + k)_0 + \frac{1}{2}\langle V \rangle^2$$

$$(k \cdot k') = p \cdot p' + m_\mu^2 + \langle V \rangle (p' - p)_0 - \frac{1}{2}\langle V \rangle^2$$

Modified Four-Momentum Relations

$$(p' \cdot k) \simeq p \cdot k' + \langle V \rangle \omega'_\mu$$

$$(p' \cdot k') \simeq p \cdot k + \langle V \rangle \omega_\mu$$

$$(k \cdot k') \simeq p \cdot p' + m_\mu^2 + \langle V \rangle E'_e$$

MANIFESTLY

$O(m_e \omega_\mu)$

- Convenient to work in terms of invariants with manifest scaling.
- Trading Lorentz invariants introduces binding energy.

Expand The Matrix Element

$$\mathsf{M} = \mathscr{M}_{\text{free}}(e\mu \to e\mu)$$

$$\sum_{\text{spins}} |\mathsf{M}|^2 \simeq 32e^4 \frac{(k\cdot p')(k'\cdot p) + (k\cdot p)(k'\cdot p') - m_\mu^2(p\cdot p')}{[(k-k')^2]^2}$$

- Write in terms of $(k\cdot p)$, $(p'\cdot p)$, $(k'\cdot p)$.
- Expand in p_e and $\langle V \rangle$.

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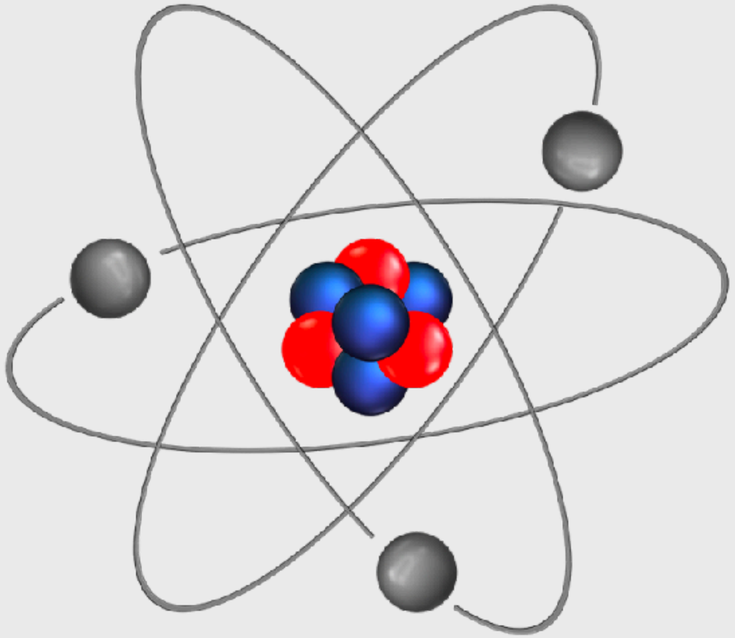
Reality: Carbon Is Not Hydrogen

$$\sum_{B} (\mu^{\pm}(\mathbf{k}) + A(\mathbf{0}) \rightarrow \mu^{\pm}(\mathbf{k}') + e^{-}(\mathbf{p}') + B^{+}(\mathbf{p})$$



- Considering helium and beyond.
- Final state is no longer a proton.
- Need to account for ***all*** final states.
- Complicated atomic wave function.

Energy Conservation

BINDING ENERGY CHANGES ENERGY CONSERVATION

$$\omega_\mu + m_e - \epsilon_A = \omega'_\mu + E'_e - \epsilon_B$$

$$\epsilon_H \to \epsilon_A - \epsilon_B$$

Summing Over Final States

$$\sigma = \int \mathrm{d}\epsilon \int \frac{1}{2\omega} \int \frac{\mathrm{d}\Pi_e}{2\omega'} (2\pi)\delta(\mathcal{E} - \epsilon)$$
$$\sum_B \int \mathrm{d}\Pi_B \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{2E_q}} \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_p}} \delta(\epsilon_A - \epsilon_B - \epsilon) \ \langle A|\hat{a}^\dagger_{\mathbf{q}}|B^+\rangle \ \langle B^+|\hat{a}_{\mathbf{p}}|A\rangle \ \frac{1}{2} \sum_{\text{spins}} |\mathsf{M}(\mathbf{p}, \mathbf{k}, \mathbf{p}', \mathbf{k}')|^2$$

Spectral Function

$$S_A(\epsilon, \mathbf{p}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \langle A|\hat{a}^\dagger_{\mathbf{p}} \delta(\epsilon_A - \hat{H} - \epsilon)\hat{a}_{\mathbf{q}}|A\rangle \ .$$

Sum Rules

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \mathrm{d}\epsilon \quad (-\epsilon) \sum_B \langle A \,|\, \hat{a}_p^\dagger \,|\, B \rangle \langle B \,|\, \delta(\epsilon - \epsilon_A + \epsilon_B) a_p \,|\, A \rangle$$

Sum Rules

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \mathrm{d}\epsilon \; (\epsilon_B - \epsilon_A) \sum_B \langle A \,|\, \hat{a}_p^\dagger \,|\, B \rangle \langle B \,|\, \delta(\epsilon - \epsilon_A + \hat{H}) a_p \,|\, A \rangle$$

Sum Rules

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int \mathrm{d}\epsilon \sum_B \langle A \,|\, \hat{a}_p^\dagger \,|\, B \rangle \langle B \,|\, \delta(\epsilon - \epsilon_A + \hat{H})[\hat{H}, a_p] \,|\, A \rangle$$

Sum Rules

INTER-ELECTRON INTERACTIONS

$$\hat{H} = \hat{T} + \hat{V}_1 + \hat{V}_2$$

$$\int \mathrm{d}\epsilon \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathbf{p}^2}{2m_e} S_A(\epsilon, \mathbf{p}) = \langle \hat{T} \rangle_A = \epsilon_A \ ,$$

VIRIAL THEOREM

$$\int \mathrm{d}\epsilon \int \frac{\mathrm{d}^3 p}{(2\pi)^3} (-\epsilon)\ S_A(\epsilon, \mathbf{p}) = \langle \hat{T} \rangle_A + \langle \hat{V}_1 \rangle_A + 2\langle \hat{V}_2 \rangle_A$$

$$= \left(-3\epsilon_A - \langle \hat{V}_1 \rangle_A\right) .$$

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Shape Only Measurement

- Assuming a shape only measurement relevant quantity is

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

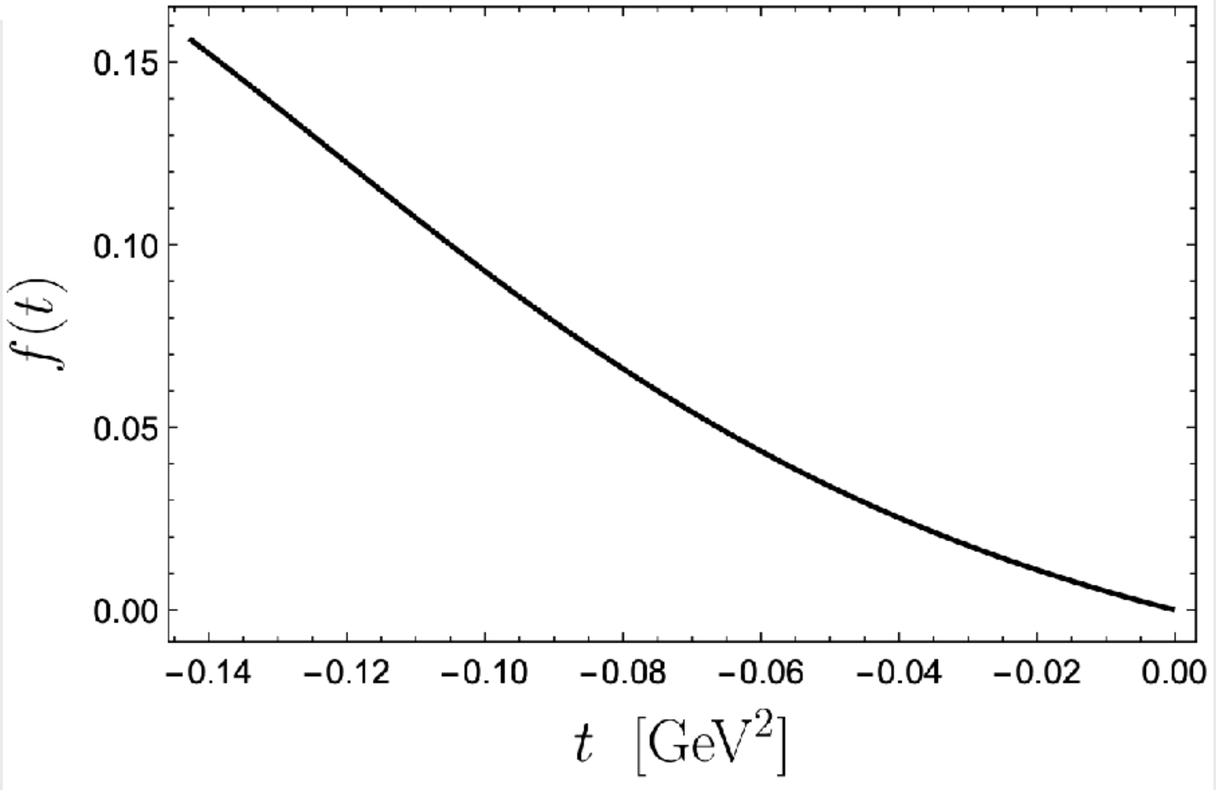
- Take matrix element and expand like with hydrogen.

- Use sum rules to simplify relevant atomic matrix elements.

Expand The Matrix Element

$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}t} \simeq \frac{1}{\sigma^{(0)}}\frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}t}\left(1 - \frac{f(t)}{Z_A m_e}\left[\frac{11}{3}\epsilon_A + \langle \hat{V}_1 \rangle_A\right]\right)$$

- Distribution receives corrections.
- Numerical size set by binding energy per electron.



Size Of Effect

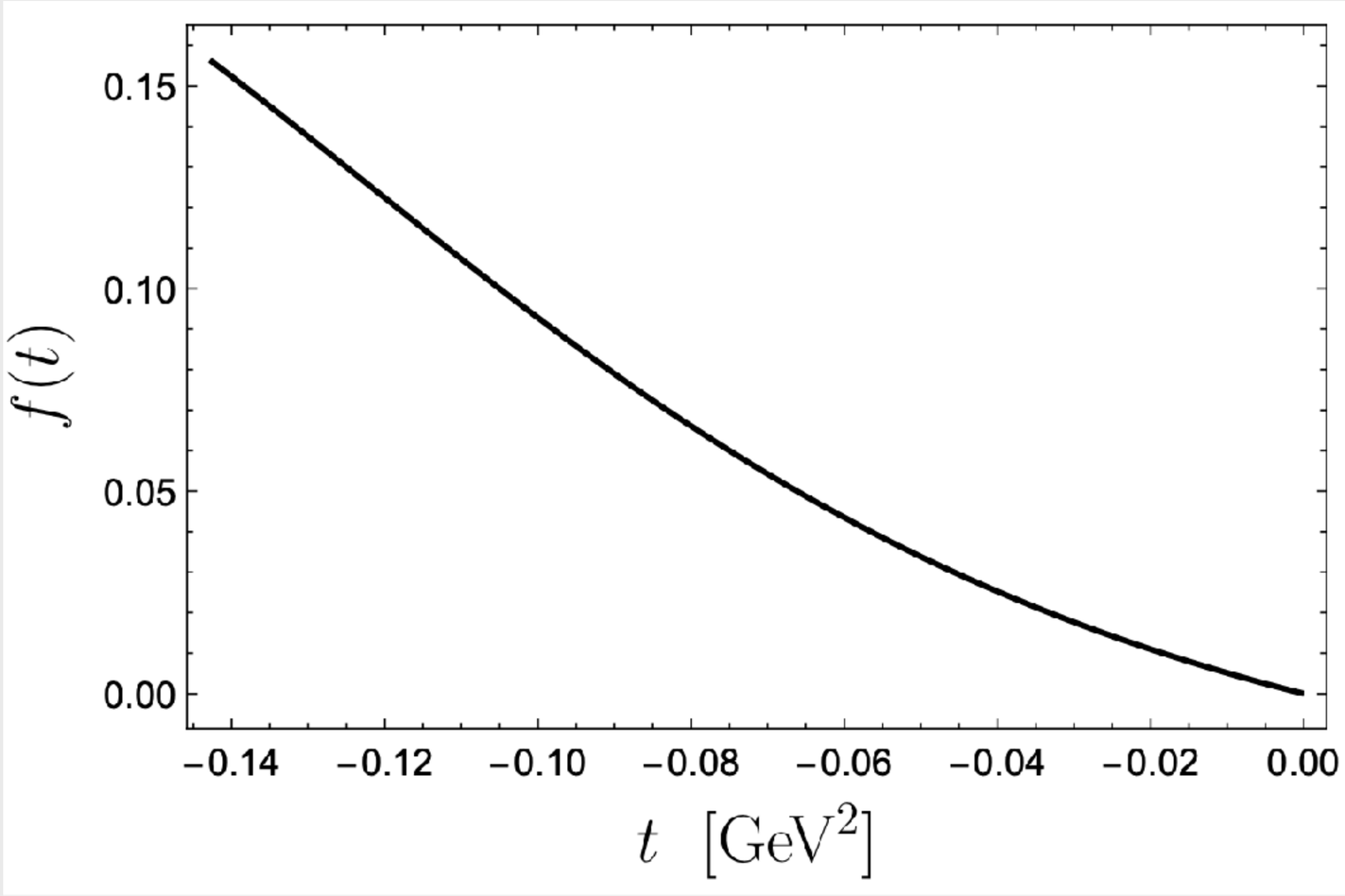
- For a carbon target we find that the distortion is

$$\sim 5 \times 10^{-5}.$$

- Inputs are:

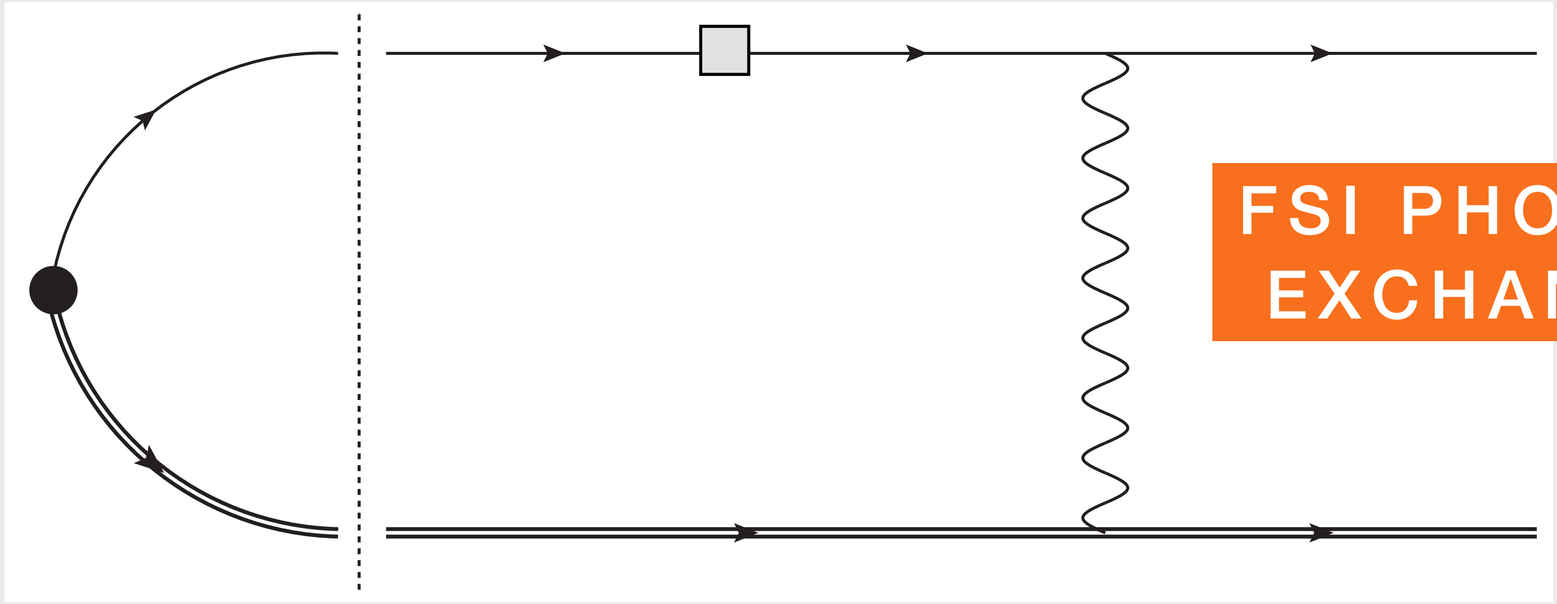
  1) Measured binding energy.

  2) Average of $\langle 1/\hat{r}_e \rangle$



Open Questions / Ongoing Work

$\tilde{\psi}(p)$



- We have ignored higher order photon exchanges with the proton/nucleus/atom (both from the muon and the "struck" electron). These are formally of the same order.

Work In Progress

Open Questions / Ongoing Work

- We have assumed that existing radiative correction calculations (free electron at rest) apply at zeroth order in the binding expansion.

  ** ***It would be interesting to see this derived.***

- We have ignored higher order photon exchanges with the proton/ nucleus/atom (both from the muon and the "struck" electron). These are formally of the same order. **Work In Progress**