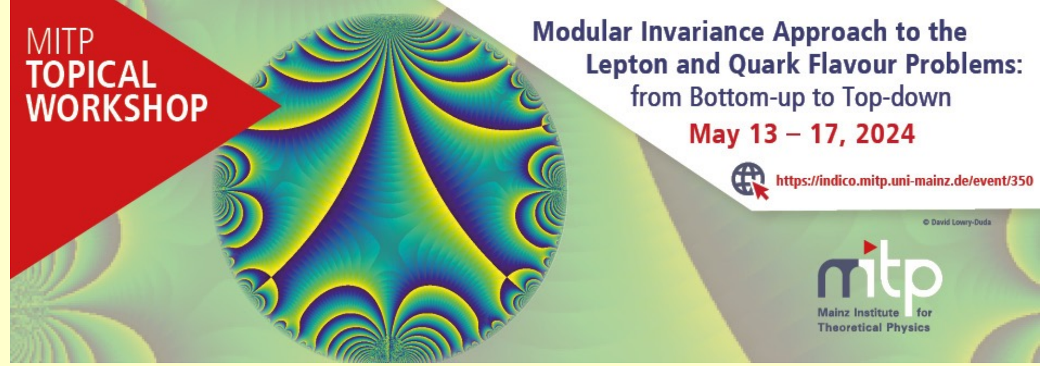

Quark mass hierarchy and CP violation in modular symmetry

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1 Introduction

We can approach the flavor problem
based on the modular symmetry

Mass hierarchy

Flavor mixing

CP violation

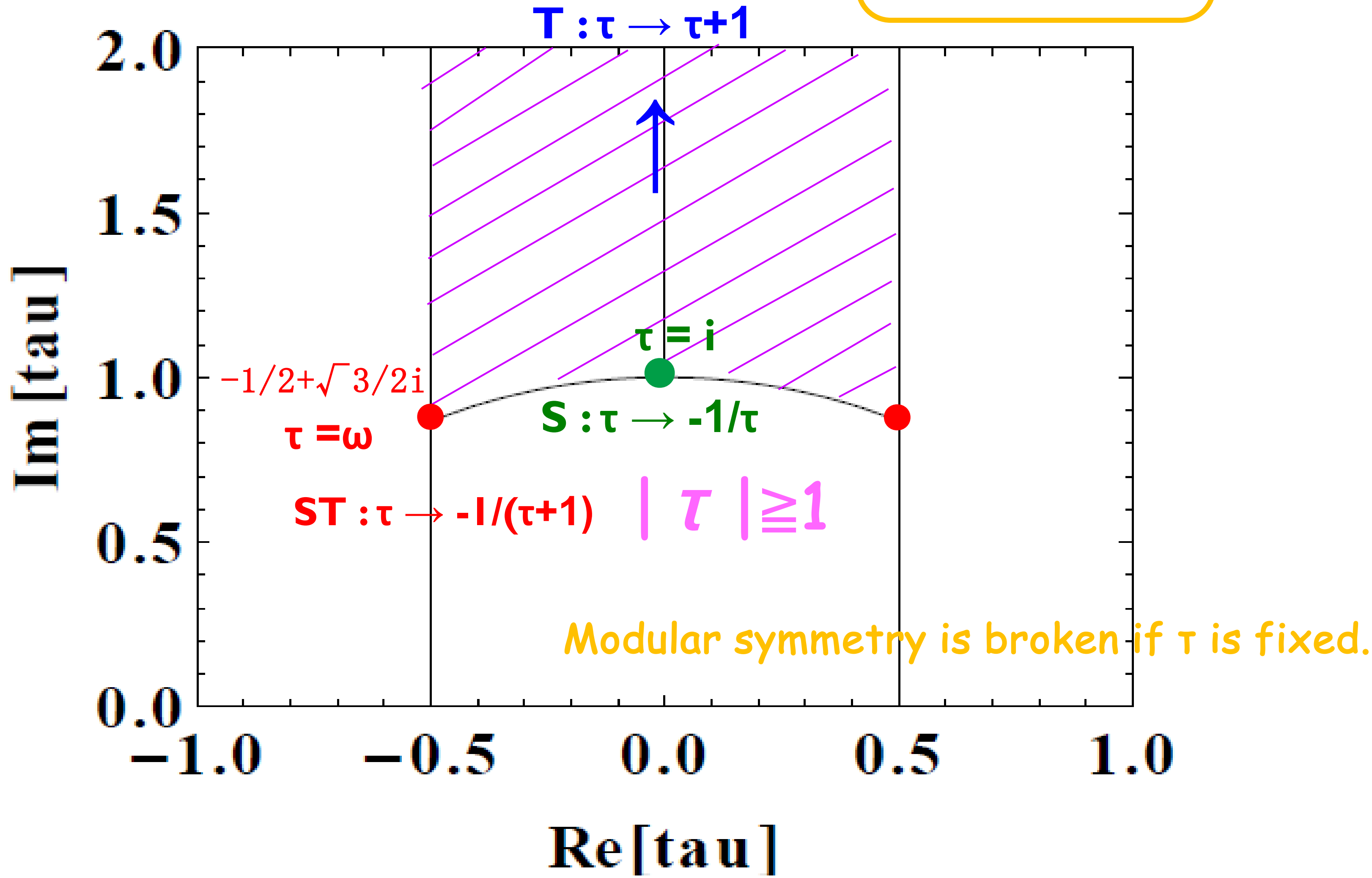
of quarks/leptons

Modular forms meet the flavor problem !

Is modulus τ origin of mass hierarchy ?
origin of CP violation ?

Fundamental Domain of τ

$$S:\tau \longrightarrow -\frac{1}{\tau},$$
$$T:\tau \longrightarrow \tau+1.$$



● ● **Symmetric point of τ** **(Residual symmetry)**

2 Modular forms at nearby fixed points

Consider A_4 triplet modular forms with weigh k=2. (N=3)

$$
\begin{aligned}
Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\
Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\
Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots).
\end{aligned}
$$

$$q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}$$

$\varepsilon = 6\,|q|^{1/3}$

$\tau \to \infty i$ $\quad (Y_1, Y_2, Y_3)^T \to (1, -\varepsilon, -1/2\,\varepsilon^2)^T \to (1, 0, 0)^T$

A_4 triplet $\qquad |\varepsilon| << 1$

k=4 $\quad \mathbf{Y}_{\mathbf{3}}^{(4)} = Y_0^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_{\mathbf{1}}^{(4)} = Y_0^2, \quad \mathbf{Y}_{\mathbf{1'}}^{(4)} = 0,$

$$\rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

Z_3 symmetry

k=6 $\quad \mathbf{Y}_{\mathbf{3}}^{(6)} = Y_0^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_{\mathbf{3'}}^{(6)} = 0, \quad \mathbf{Y}_{\mathbf{1}}^{(6)} = Y_0^3,$

k=8 $\quad \mathbf{Y}_{\mathbf{3}}^{(8)} = Y_0^4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_{\mathbf{3'}}^{(8)} = 0, \quad \mathbf{Y}_{\mathbf{1}}^{(8)} = Y_0^4, \quad \mathbf{Y}_{\mathbf{1'}}^{(8)} = 0, \quad \mathbf{Y}_{\mathbf{1''}}^{(8)} = 0$

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \cdots, \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \cdots), \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \cdots). \end{aligned}$$

$$q = e^{2\pi i\tau} = e^{2\pi i \mathrm{Re}\tau} e^{-2\pi \mathrm{Im}\tau}$$

$$\varepsilon = 6\,|q|^{1/3}$$

Modular forms are also hierarchical at $\tau=\omega$

$$\rho(ST) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

Unitary transformation

Z_3 symmetry

$\tau = \omega$

$$\mathbf{Y_3^{(2)}} = Y_0 \begin{pmatrix} 1 \\ \omega \\ -\frac{1}{2}\omega^2 \end{pmatrix} \longrightarrow \mathbf{Y_3^{(2)}} = \frac{3}{2}\omega Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

k=2

$$\mathbf{Y_3^{(4)}} = \frac{9}{4} Y_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{Y_3^{(6)}} = 0\,, \qquad \mathbf{Y_{3'}^{(6)}} = \frac{27}{8}\omega^2 Y_0^3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

k=4

k=6

Modular forms at τ=i

Z_2 symmetry

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Unitary transformation

τ = i

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} 1 \\ 1-\sqrt{3} \\ -2+\sqrt{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \sqrt{6}-3/\sqrt{2} \\ \sqrt{3/2} \end{pmatrix}$$

k=4

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \end{pmatrix}$$

F. Feruglio, V. Gherardi, A. Romanino and A. Titov, JHEP 05 (2021), 242; arXiv:2101.08718

We can construct the quark mass matrix with hierarchical masses by using the modular forms towards : $\tau = \infty i$ and ω

$$\mathcal{M}_q \sim v_q \begin{pmatrix} \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}_{RL}$$

$\varepsilon = 6\,|q|^{1/3}$

This hierarchical structure is not accidental.

Thanks to Residual symmetry Z_3 (N=3)

P.P.Novichkov, J.T.Penedo, S.T.Petcov, JHEP 04(2021)206, arXiv:2102.07488

Hierarchical fermion mass matrices arise due to the proximity of the modulus τ to a symmetric point, in which a residual symmetery remainds.

Under the modular transformation γ, modular invariant mass matrix M(τ) satisfies

$$(\phi^{(I)})_i(x) \longrightarrow (c\tau+d)^{-k_I}\rho(\gamma)_{ij}(\phi^{(I)})_j(x)$$

Automorphy factor

$$M(\gamma\tau) = (c\tau+d)^K \rho^c(\gamma)^* M(\tau)\rho(\gamma)^\dagger \qquad K = k^c + k$$

At τ=i ∞, mass matrix is invariant under T transformation (Z_N symmetry)

$$M_{ij}(T\tau) = (\rho_i^c\,\rho_j)^*\, M_{ij}(\tau)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$\tau \to \tau + 1$

redefine $q \equiv exp(i2\pi\tau/N)$ $\quad q \xrightarrow{T} \xi q$, with $\xi = exp(i\,2\pi/N)$

$$M_{ij}(\zeta q) = (\rho_i^c\rho_j)^* M_{ij}(q) \;\rightarrow\; \xi^n M_{ij}^{(n)}(0) = (\rho_i^c\,\rho_j)^* M_{ij}^{(n)}(0) \;\rightarrow\; (\rho_i^c\,\rho_j)^* = \xi^n$$

$$M_{ij}(q) = a_0\, q^{\ell} + a_1\, q^{\ell+N} + a_2\, q^{\ell+2N} + \ldots\,,$$

$$q \equiv exp(i2\pi\tau/N) \qquad \ell = 0, 1, 2, ..., N-1,$$

in the vicinity of symmetric point $\tau = i\infty$

For N=3 $\quad M(\tau) \sim \mathcal{O}(\epsilon^{\ell}) \quad \ell = 0, 1, 2$

due to residual symmetry Z_3 $\qquad \varepsilon = |q|$

Mass hierarchy is also realized close to τ=ω

$$M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau)\rho(\gamma)^\dagger \qquad K = k^c + k$$

mass matrix is invariant under ST transformation (Z_3 symmetry)

Near τ=ω $\quad u = \frac{\tau-\omega}{\tau-\omega^2} \ (u = 0 \ @ \ \tau = \omega) \qquad |u| = \epsilon$

ST transformation : $u \to \omega^2 u$

$$M(ST\tau)_{ij} = M(\omega^2 u)_{ij} = (-(\tau+1))^K [\rho^c(\gamma)_i \rho(\gamma)_j]^* M(u)_{ij}$$

$$M(\tau) \sim \mathcal{O}(\epsilon^\ell) \quad \ell = 0, 1, 2$$

due to residual symmetry Z_3

3 Examples in A_4 modular symmetry

@ $\tau=\omega$ S.T.Petcov, M.Tanimoto, Eur. Phys. J. C 83(2023)579 [arXiv:2212.13336]

	Q	(u^c, c^c, t^c), (d^c, s^c, b^c)	H_q	$\mathbf{Y_3^{(6)}}$, $\mathbf{Y_{3'}^{(6)}}$	$\mathbf{Y_3^{(4)}}$	$\mathbf{Y_3^{(2)}}$
$SU(2)$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
A_4	$\mathbf{3}$	$(\mathbf{1}, \mathbf{1''}, \mathbf{1'})$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$
k_I	2	(4, 2, 0)	0	$k=6$	$k=4$	$k=2$

$$W_d = \left[\alpha_d(\mathbf{Y_3^{(6)}}Q)_1 d_1^c + \alpha'_d(\mathbf{Y_{3'}^{(6)}}Q)_1 d_1^c + \beta_d(\mathbf{Y_3^{(4)}}Q)_{1'} s_{1'}^c + \gamma_d(\mathbf{Y_3^{(2)}}Q)_{1''} b_{1'}^c\right] H_d$$

Suppose all coefficients are same order.

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1'^{(6)} & Y_3^{(6)} + g_q Y_3'^{(6)} & Y_2^{(6)} + g_q Y_2'^{(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}_{RL}$$

$g_q = \alpha'_q / \alpha_q$ arXiv 2212.13336 S. Petcov and M.T

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1'^{(6)} & Y_3^{(6)} + g_q Y_3'^{(6)} & Y_2^{(6)} + g_q Y_2'^{(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}_{RL}$$

At τ = ω **in the diagonal base of ST**

$$\mathbf{Y}_3^{(2)} = \frac{3}{2}\omega Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{Y}_3^{(4)} = \frac{9}{4} Y_0^2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{Y}_3^{(6)} = 0\,, \quad \mathbf{Y}_{3'}^{(6)} = \frac{27}{8}\omega^2 Y_0^3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathcal{M}_q^{(0)} = M_q V_{\mathrm{ST}}^\dagger = v_q \begin{pmatrix} 0 & 0 & \frac{27}{8}\hat{\alpha}_q g_q \omega \\ 0 & 0 & \frac{9}{4}\hat{\beta}_q \omega^2 \\ 0 & 0 & \frac{3}{2}\hat{\gamma}_q \end{pmatrix}$$

rank one matrix

$$\rho(ST) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2 & -\omega & 2\omega^2 \\ 2 & 2\omega & -\omega^2 \end{pmatrix} \qquad \tau = \omega + \epsilon$$

very small

$$Y_2^2 + 2Y_1Y_3 = 0$$

$$\frac{Y_2(\tau)}{Y_1(\tau)} \simeq \omega\,(1+\epsilon_1)\,, \qquad \frac{Y_3(\tau)}{Y_1(\tau)} \simeq -\frac{1}{2}\omega^2\,(1+\epsilon_2)\,,$$

$$\epsilon_1 = \frac{1}{2}\epsilon_2 \simeq 2.1\, i\,\epsilon\,.$$

Move to diagonal base of ST

$$\mathcal{M}_q \sim v_q \begin{pmatrix} \hat{\alpha}_q\omega Y_1^3 & 0 & 0 \\ 0 & \hat{\beta}_q\omega^2 Y_1^2 & 0 \\ 0 & 0 & \hat{\gamma}_q Y_1 \end{pmatrix} \begin{pmatrix} (-3+\frac{3}{4}g_q)\epsilon_1^2 & -\frac{9}{2}\epsilon_1(1+\frac{g_q}{2}) & g_q\frac{27}{8} \\ -\frac{3}{2}\epsilon_1^2 & \frac{3}{2}\epsilon_1 & \frac{9}{4} \\ \frac{1}{3}\epsilon_1^2 & -\epsilon_1 & \frac{3}{2} \end{pmatrix}$$

$g_q \sim 1$ **(down-quarks)**

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : |\epsilon_1| : |\epsilon_1|^2 \simeq 1 : |\epsilon| : |\epsilon|^2$$

$g_q >> 1$ **(up-quarks)**

$$m_{q3} : m_{q2} : m_{q1} \simeq \quad 1 : \frac{|\epsilon_1|}{|g_q|} : \left(\frac{|\epsilon_1|}{|g_q|}\right)^2$$

Observed Yukawa ratios at GUT scale with tanβ=10

S. Antusch, V. Maurer, JHEP 1311 (2013) 115 [arXiv:1306.6879].

$$\frac{y_d}{y_b} = 9.21 \times 10^{-4}\,(1 \pm 0.111)\,, \qquad \frac{y_s}{y_b} = 1.82 \times 10^{-2}\,(1 \pm 0.055)$$

$$\frac{y_u}{y_t} = 5.39 \times 10^{-6}\,(1 \pm 0.311)\,, \qquad \frac{y_c}{y_t} = 2.80 \times 10^{-3}\,(1 \pm 0.043)$$

$$m_{b\,(t)} : m_{s\,(c)} : m_{d\,(u)} \sim 1 : |\epsilon| : |\epsilon|^2$$

For down quark sector ε_d = 0.02～0.03

For up quark sector ε_u= 0.002～0.003

$$\tau = \omega + \epsilon$$

one mudulus

How is the CP violation ？

CP phase structure of mass matrix

$$\tau = \omega + \epsilon \quad \frac{Y_2(\tau)}{Y_1(\tau)} \simeq \omega\,(1+\,\epsilon_1)\,, \quad \frac{Y_3(\tau)}{Y_1(\tau)} \simeq -\frac{1}{2}\omega^2\,(1+\,\epsilon_2)\,, \quad \epsilon_1 = \frac{1}{2}\epsilon_2 \simeq 2.1\, i\,\epsilon\,.$$

$$\mathcal{M}_q^{gen} = v_q \begin{pmatrix} i^2\,\epsilon^2 & i\,\epsilon & 1 \\ i^2\,\epsilon^2 & i\,\epsilon & 1 \\ i^2\,\epsilon^2 & i\,\epsilon & 1 \end{pmatrix}\,, \quad q = d, u$$

$$(\mathcal{M}_q^{gen})^\dagger \mathcal{M}_q^{gen} = v_q^2 \begin{pmatrix} -\,i\,e^{-i\,\kappa_q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i\,e^{i\,\kappa_q} \end{pmatrix} \begin{pmatrix} |\epsilon_q|^4 & |\epsilon_q|^3 & |\epsilon_q|^2 \\ |\epsilon_q|^3 & |\epsilon_q|^2 & |\epsilon_q| \\ |\epsilon_q|^2 & |\epsilon_q| & 1 \end{pmatrix} \begin{pmatrix} i\,e^{i\,\kappa_q} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\,i\,e^{-i\,\kappa_q} \end{pmatrix}$$

$P(\kappa_q)$ $\qquad$ $\epsilon_q = |\epsilon_q|e^{i\,\kappa_q}$ $\qquad$ $P(\kappa_q)^*$

$$\mathrm{U}^{\mathrm{gen}}_{\mathrm{CKM}} = \mathrm{O}^{\mathrm{T}}_{\mathrm{u}}\, \mathrm{P}^*(\kappa_{\mathrm{u}})\mathrm{P}(\kappa_{\mathrm{d}})\mathrm{O}_{\mathrm{d}}$$

$$\mathrm{P}(\kappa_{\mathrm{q}}) = \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\,(\kappa_{\mathrm{q}}+\pi/2)}, 1, \mathrm{e}^{\mathrm{i}\,(\kappa_{\mathrm{q}}+\pi/2)})$$

Common τ $\epsilon_{1d} = \epsilon_{1u} \quad \kappa_d = \kappa_u$ $\mathrm{P}^*(\kappa_{\mathrm{u}})\mathrm{P}(\kappa_{\mathrm{d}}) = 1$

CP conserving if other parameters are real

Two different τ $\epsilon_d \neq \epsilon_u$ $\mathrm{P}^*(\kappa_{\mathrm{u}})\mathrm{P}(\kappa_{\mathrm{d}}) \neq 1$

CP violation even if other parameters are real

Spontaneous CP violation

$$\tau = \omega + \epsilon$$

Real parameters

ϵ	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	g_d	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	g_u
$0.01779 + i\,0.02926$	3.26	0.43	-1.40	1.05	0.80	-16.1

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$J_{\rm CP}$
Fit	1.52	8.62	2.50	5.43	0.2230	0.0786	0.00368	-2.9×10^{-8}
Exp $1\,\sigma$	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times 10^{-5}$

Towards τ=i∞

$2\mathrm{Im}\,\tau$ is large

S.T.Petcov, M.Tanimoto, JHEP 08 (2023)086 [arXiv:2306.05730]

	Q	(d^c, s^c, b^c) , (u^c, c^c, t^c)		H_u	H_d
$SU(2)$	2	1		2	2
A_4	3	$(1', 1', 1')$	$(1', 1', 1')$	1	1
k	2	$(4, 2, 0)$	$(6, 2, 0)$	0	0

Irreducible representations

A_4 : 1, 1', 1'', 3

Weight k is set to vanish

automorphy factor $(c\tau + d)^k$

$$W_d = \left[\alpha_d(\mathbf{Y}_3^{(6)}Q)_1 d_1^c + \alpha'_d(\mathbf{Y}_{3'}^{(6)}Q)_1 d_1^c + \beta_d(\mathbf{Y}_3^{(4)}Q)_{1'} s_{1'}^c + \gamma_d(\mathbf{Y}_3^{(2)}Q)_{1''} b_{1'}^c\right] H_d$$

$$M_d = v_d \begin{pmatrix} \hat{\alpha}'_d & 0 & 0 \\ 0 & \hat{\beta}_d & 0 \\ 0 & 0 & \hat{\gamma}_d \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(6)} & \tilde{Y}_2^{(6)} & \tilde{Y}_1^{(6)} \\ \tilde{Y}_3^{(4)} & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}, \quad M_u = v_u \begin{pmatrix} \hat{\alpha}'_u & 0 & 0 \\ 0 & \hat{\beta}_u & 0 \\ 0 & 0 & \hat{\gamma}_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(8)} & \tilde{Y}_2^{(8)} & \tilde{Y}_1^{(8)} \\ \tilde{Y}_3^{(4)} & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

$$\tilde{Y}_i^{(6)} = g_d Y_i^{(6)} + Y_i'^{(6)}, \qquad \tilde{Y}_i^{(8)} = f_u Y_i^{(8)} + Y_i'^{(8)}, \qquad g_d \equiv \alpha_d/\alpha'_d \qquad f_u \equiv \alpha_u/\alpha'_u$$

$$\mathrm{Det}\,[\mathcal{M}_u^2] = 0$$

due to

$$\mathbf{Y}^{(8)} = (Y_1^2 + 2Y_2Y_3)\mathbf{Y}^{(4)}$$

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$q \equiv \exp\left(2i\pi\tau\right) = \left(p\,\epsilon\right)^3$$

$$\epsilon = \exp\left(-\frac{2}{3}\pi\,\mathrm{Im}\left[\tau\right]\right)\,, \qquad p = \exp\left(\frac{2}{3}\pi\,i\,\mathrm{Re}\left[\tau\right]\right)$$

τ=i∞

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = Y_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{Y}_{\mathbf{3}}^{(4)} = Y_0^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} = Y_0^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} = 0 \qquad \mathbf{Y}_{\mathbf{3}}^{(8)} = Y_0^4 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{Y}_{\mathbf{3'}}^{(8)} = 0$$

Superpotential

$$W_d = \left[\alpha_d(\mathbf{Y}_3^{(6)}Q)_1 d_1^c + \alpha'_d(\mathbf{Y}_{3'}^{(6)}Q)_1 d_1^c + \beta_d(\mathbf{Y}_3^{(4)}Q)_{1'} s_{1'}^c + \gamma_d(\mathbf{Y}_3^{(2)}Q)_{1''} b_{1'}^c\right] H_d$$

Kinetic terms

$$\sum_I \frac{|\partial_\mu \psi^{(I)}|^2}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}$$

We renormalize superfields to get canonical kinetic terms

$$\psi^{(I)} \to \sqrt{(2\mathrm{Im}\tau_q)^{k_I}}\, \psi^{(I)}$$

$$\alpha_u \to \hat{\alpha}_u = \alpha_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha_u\,(2\mathrm{Im}\tau)^4, \qquad \alpha'_u \to \hat{\alpha}'_u = \alpha'_u \sqrt{(2\mathrm{Im}\tau)^8} = \alpha'_u\,(2\mathrm{Im}\tau)^4\,,$$
$$\beta_u \to \hat{\beta}_u = \beta_u \sqrt{(2\mathrm{Im}\tau)^4} = \beta_u\,(2\mathrm{Im}\tau)^2\,, \qquad \gamma_u \to \hat{\gamma}_u = \gamma_u \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_u\,(2\mathrm{Im}\tau)\,,$$
$$\alpha_d \to \hat{\alpha}_d = \alpha_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha_d\,(2\mathrm{Im}\tau)^3, \qquad \alpha'_d \to \hat{\alpha}'_d = \alpha'_d \sqrt{(2\mathrm{Im}\tau)^6} = \alpha'_d\,(2\mathrm{Im}\tau)^3\,,$$
$$\beta_d \to \hat{\beta}_d = \beta_d \sqrt{(2\mathrm{Im}\tau)^4} = \beta_d\,(2\mathrm{Im}\tau)^2\,, \qquad \gamma_d \to \hat{\gamma}_d = \gamma_d \sqrt{(2\mathrm{Im}\tau)^2} = \gamma_d\,(2\mathrm{Im}\tau)\,.$$

$2\mathrm{Im}\,\tau$ is large

Down type quark mass matrix

At τ=i∞

$$M_q = v_q \begin{pmatrix} g_q\,\hat\alpha'_q & 0 & 0 \\ 0 & \hat\beta_q & 0 \\ 0 & 0 & \hat\gamma_q \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}_{RL}$$

rank one

$$\mathcal{M}_q^{2(0)} \equiv M_q^\dagger M_q = v_q^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |g_q|^2\,\hat\alpha'^2_q + \hat\beta_q^2 + \hat\gamma_q^2 \end{pmatrix}$$

In the vicinity of τ=i∞ $|\alpha'_q| \sim |\beta_q| \sim |\gamma_q| \quad \hat\alpha'_q = \alpha'_q\,(2\mathrm{Im}\tau_q)^3$

$$\mathcal{M}_q = v_q \begin{pmatrix} \hat\alpha'_q & 0 & 0 \\ 0 & \hat\beta_q & 0 \\ 0 & 0 & \hat\gamma_q \end{pmatrix} \begin{pmatrix} 18\,(\epsilon\,p)^2(4-g_q) & -6\,(\epsilon\,p)(2+g_q) & g_q \\ 54\,(\epsilon\,p)^2 & 6\,(\epsilon\,p) & 1 \\ -18\,(\epsilon\,p)^2 & -6\,(\epsilon\,p) & 1 \end{pmatrix}$$

$$\mathcal{M}_q^2 \sim \begin{pmatrix} \epsilon^4 & \epsilon^3 p^* & \epsilon^2 p^{*2} \\ \epsilon^3 p & \epsilon^2 & \epsilon p^* \\ \epsilon^2 p^2 & \epsilon p & 1 \end{pmatrix}$$

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : \left|\frac{12\epsilon}{I_\tau g_q}\right| : \left|\frac{12\epsilon}{I_\tau g_q}\right|^2$$

$I_\tau = 2\mathrm{Im}\,\tau$

$g_q > \mathcal{O}(1)$

Up type quark mass matrix

In order to protect a massless quark, we can consider dimesuion 6 mass operator

$(u^c Q H_u)(H_u H_d)/\Lambda^2$ with $k_Q = 2 - k_{Hd}\,, \quad k_{u^c} = 6 + k_{Hd} - k_{Hu}$

or SUSY breaking by F term F/Λ^2

F. Feruglio, V. Gherardi, A. Romanino and A. Titov, JHEP 05 (2021), 242; arXiv:2101.08718

$$M_u = v_u \begin{pmatrix} \hat{\alpha}'_u & 0 & 0 \\ 0 & \hat{\beta}_u & 0 \\ 0 & 0 & \hat{\gamma}_u \end{pmatrix} \begin{pmatrix} \tilde{Y}_3^{(8)}(1 + C_{u1}) & \tilde{Y}_2^{(8)} & \tilde{Y}_1^{(8)} \\ \tilde{Y}_3^{(4)}(1 + C_{u2}) & \tilde{Y}_2^{(4)} & \tilde{Y}_1^{(4)} \\ Y_3^{(2)}(1 + C_{u3}) & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}$$

$$m_t : m_c : m_u \simeq \left[1 : \left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right) : \frac{3}{2}\left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right)^2 f_u^3 I_\tau |C_u|\right] I_\tau^4 f_u$$

$$C_u = 3 f_u\,(C_{u1} - C_{u2}) + (-4C_{u1} + 3C_{u2} + C_{u3}) \qquad I_\tau = 2\,\mathrm{Im}\,\tau$$

I_τ is a overall normalization factor for canonical kinetic terms

Down type quark masses k=2,4,6 modular forms

$$m_{q3} : m_{q2} : m_{q1} \simeq 1 : \left|\frac{12\epsilon}{I_\tau g_q}\right| : \left|\frac{12\epsilon}{I_\tau g_q}\right|^2$$

Up type quark masses k=2, 4, 8 modular forms

$$m_t : m_c : m_u \simeq \left[1 : \left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right) : \frac{3}{2}\left(\frac{12\epsilon}{I_\tau f_u}\frac{1}{I_\tau f_u}\right)^2 f_u^3 I_\tau |C_u|\right] I_\tau^4 f_u$$

$$I_\tau = 2\mathrm{Im}\,\tau$$

$$\tilde{Y}_i^{(6)} = g_d Y_i^{(6)} + Y_i^{\prime(6)}\,, \qquad \tilde{Y}_i^{(8)} = f_u Y_i^{(8)} + Y_i^{\prime(8)}\,, \qquad g_d \equiv \alpha_d/\alpha'_d \qquad f_u \equiv \alpha_u/\alpha'_u$$

Fitting parameters

τ	$\frac{\beta_d}{\alpha'_d}$	$\frac{\gamma_d}{\alpha'_d}$	g_d	$\frac{\beta_u}{\alpha'_u}$	$\frac{\gamma_u}{\alpha'_u}$	$\|f_u\|$	$\arg[f_u]$	C_{u1}
$-0.3952 + i\,2.4039$	3.82	1.17	-0.677	1.72	3.21	1.68	127.3°	-0.07147

$$q = e^{2\pi i\tau}$$

8 real parameters + 2 phase

!! Order 1 parameters, β_q/α_q , γ_q/α_q , g_d , f_u

$C_{u1} \sim (F/\Lambda^2) / \varepsilon^2$

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\|V_{us}\|$	$\|V_{cb}\|$	$\|V_{ub}\|$	$\|J_{CP}\|$	δ_{CP}
Fit	1.89	8.78	2.81	5.52	0.2251	0.0390	0.00364	2.94×10^{-5}	70.7°
Exp $1\,\sigma$	1.82 ± 0.10	9.21 ± 1.02	2.80 ± 0.12	5.39 ± 1.68	0.2250 ± 0.0007	0.0400 ± 0.0008	0.00353 ± 0.00013	2.8×10^{-5} ${}^{+0.14}_{-0.12}\times 10^{-5}$	66.2° ${}^{+3.4°}_{-3.6°}$

8 output $N\sigma$=2.0

4 Summary

- Quark mass hierarchy is obtained at nearby symmetric points $\tau = i\infty$ and ω thanks to the residual symmetry.

  Im τ is important for $\tau = i\infty$.

- One modulus or multi-modulei ?
  Especially, for CP violation ?

- Spontaneous CP violation is challenging
  Is VEV of τ origin of CP violation ?

$$\mathbf{Y}_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \dots) \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(6)} \equiv \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = (Y_1^2 + 2Y_2Y_3) \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} , \qquad \mathbf{Y}_{\mathbf{3'}}^{(6)} \equiv \begin{pmatrix} Y_1'^{(6)} \\ Y_2'^{(6)} \\ Y_3'^{(6)} \end{pmatrix} = (Y_3^2 + 2Y_1Y_2) \begin{pmatrix} Y_3 \\ Y_1 \\ Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} \equiv \begin{pmatrix} Y_1^{(8)} \\ Y_2^{(8)} \\ Y_3^{(8)} \end{pmatrix} = (Y_1^2 + 2Y_2Y_3) \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix} , \quad \mathbf{Y}_{\mathbf{3'}}^{(8)} \equiv \begin{pmatrix} Y_1'^{(8)} \\ Y_2'^{(8)} \\ Y_3'^{(8)} \end{pmatrix} = (Y_3^2 + 2Y_1Y_2) \begin{pmatrix} Y_2^2 - Y_1Y_3 \\ Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \end{pmatrix}$$

$$\mathbf{Y}_{\mathbf{3}}^{(8)} = (Y_1^2 + 2Y_2Y_3)\mathbf{Y}_{\mathbf{3}}^{(4)}$$

How is Mixing angles ?

$$\mathcal{M}_q^2 \sim \begin{pmatrix} |\epsilon^4| & |\epsilon|^2\epsilon^* & \epsilon^{2*} \\ |\epsilon|^2\epsilon & |\epsilon^2| & \epsilon^* \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}$$

Vcb=0.04 Vub/Vcb=0.08

Vus=0.22

θ_{23} $|M^2_q(2,3)/M^2_q(3,3)| \fallingdotseq \varepsilon_1$

$$\epsilon_1 \simeq 2.235\, i\, \epsilon$$

θ_{13} $|M^2_q(1,3)/M^2_q(3,3)| \fallingdotseq \varepsilon_1^{\,2}$

ε=0.02

θ_{12} Enhancement factor due to $g_q \sim -1$

$|M^2_q(1,2)/M^2_q(2,2)| \sim (4-g)/(2+g)\varepsilon_1$

Rough approximation $5 \times \varepsilon_1$

θ_{12}

Go to diagonal base of (2-3) submatrix

$$\mathcal{M}_q^2 \sim \begin{pmatrix} |\epsilon^4| & |\epsilon|^2\epsilon^* & \epsilon^{2*} \\ |\epsilon|^2\epsilon & |\epsilon^2| & \epsilon^* \\ \epsilon^2 & \epsilon & 1 \end{pmatrix} \rightarrow \mathcal{M}_q^{2'} \sim v_q^2 \begin{pmatrix} \mathcal{O}|(\epsilon|^4) & \mathcal{O}(|\epsilon|^2\epsilon^*) & \mathcal{O}(\epsilon^{2*}) \\ \mathcal{O}(|\epsilon|^2\epsilon) & \mathcal{O}(|\epsilon|^2) & 0 \\ \mathcal{O}(\epsilon^2) & 0 & 1 \end{pmatrix}$$

$$\theta_{12} \sim |M^{2\prime}_q(1,2)/M^{2\prime}_q(2,2)| \sim k\,\varepsilon_1$$

k=4-5 for $g_q \sim -1$

Fitting parameters

12 parameters

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1^{(6)} + g_q Y_1'^{(6)} & Y_3^{(6)} + g_q Y_3'^{(6)} & Y_2^{(6)} + g_q Y_2'^{(6)} \\ Y_2^{(4)} & Y_1^{(4)} & Y_3^{(4)} \\ Y_3^{(2)} & Y_2^{(2)} & Y_1^{(2)} \end{pmatrix}_{RL}$$

$$\tau = \omega + \epsilon$$

ϵ	$\frac{\beta_d}{\alpha_d}$	$\frac{\gamma_d}{\alpha_d}$	$\lvert g_d\rvert$	$\arg[g_d]$	$\frac{\beta_u}{\alpha_u}$	$\frac{\gamma_u}{\alpha_u}$	$\lvert g_u\rvert$	$\arg[g_u]$
$0.00048 + i\,0.02670$	2.30	0.39	0.88	161°	1.69	0.49	16.2	205°

	$\frac{m_s}{m_b}\times 10^2$	$\frac{m_d}{m_b}\times 10^4$	$\frac{m_c}{m_t}\times 10^3$	$\frac{m_u}{m_t}\times 10^6$	$\lvert V_{us}\rvert$	$\lvert V_{cb}\rvert$	$\lvert V_{ub}\rvert$	$\lvert J_{CP}\rvert$	δ_{CP}
Fit	1.53	8.88	3.13	2.02	0.2229	0.0777	0.00333	5.2×10^{-5}	67.0°
Exp $1\,\sigma$	1.82 ±0.10	9.21 ±1.02	2.80 ±0.12	5.39 ±1.68	0.2250 ±0.0007	0.0400 ±0.0008	0.00353 ±0.00013	2.8×10^{-5} $^{+0.14}_{-0.12}\times 10^{-5}$	66.2° $^{+3.4°}_{-3.6°}$

How to obtain the mass hierarchy

First, construct a model of mass matrix with rank one at symmetric point , $\tau = \omega$ or $i\infty$, in which some couplings vanishes.

$$M_f^2 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \tau = \tau_{\rm sym}$$

In the vicinity of symmetric point,

tiny yukawa couplings $\varepsilon^{\#}$ **(#=1,2)** appear !

$$M_f^2 \sim \begin{pmatrix} \epsilon^{\#} & \epsilon^{\#} & \epsilon^{\#} \\ \epsilon^{\#} & \epsilon^{\#} & \epsilon^{\#} \\ \epsilon^{\#} & \epsilon^{\#} & 1 \end{pmatrix} \qquad \epsilon \sim |\tau - \tau_{\rm sym}| > 0$$

$$M(\tau) \sim \mathcal{O}(\epsilon^{\#}) \qquad \# = 0, 1, 2$$

$$m_3 : m_2 : m_1 \sim 1 : \epsilon : \epsilon^2$$