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WORKSHOP

Modular Invariance Approach to the
Lepton and Quark Flavour Problems:
from Bottom-up to Top-down

May 13 – 17, 2024

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Theoretical Physics

Flavour Models with Multiple Modular Symmetries

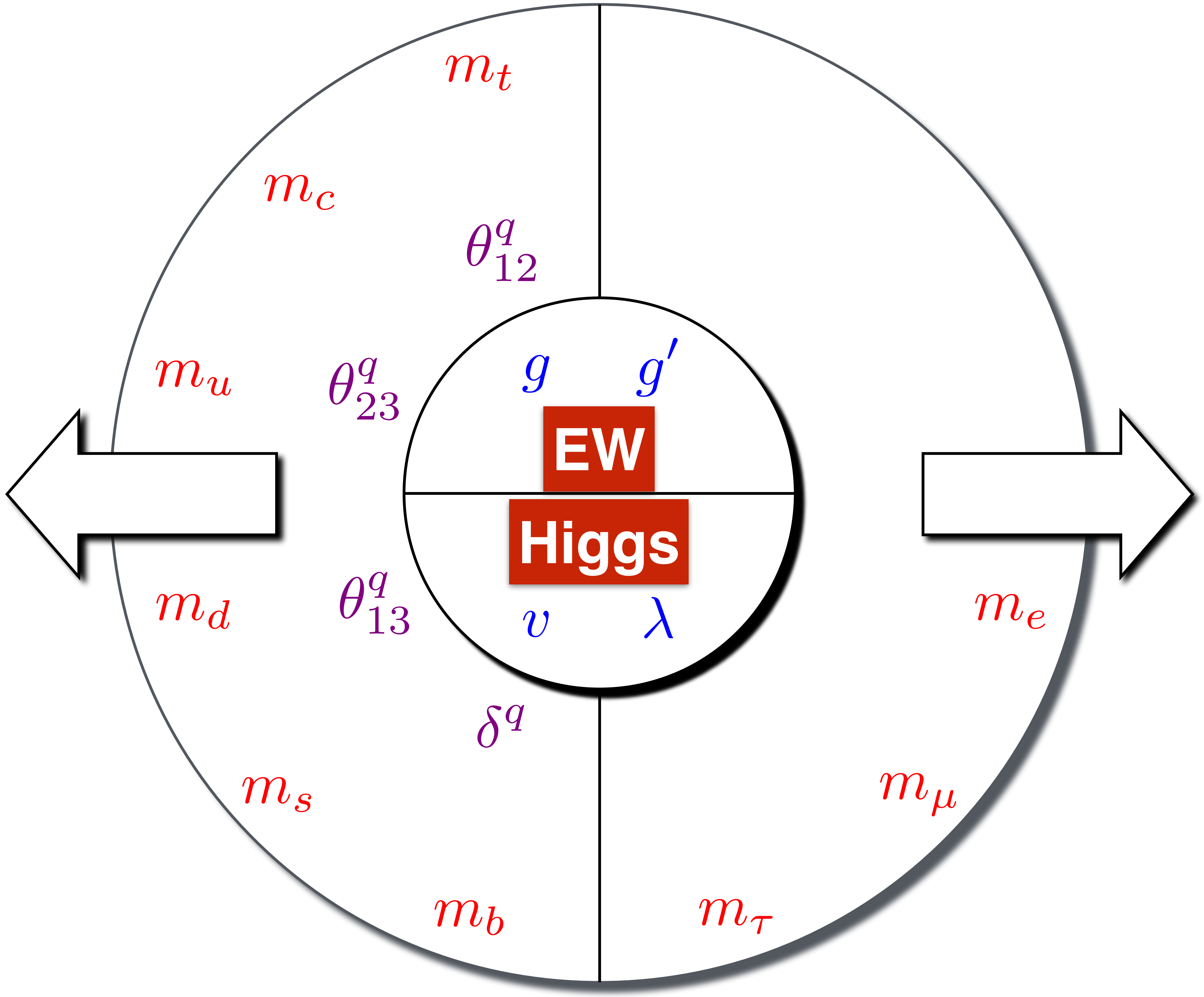
Ye-Ling Zhou, HIAS-UCAS, 2024-5-13

Outline

- *Favour symmetry and residual symmetries in flavour models*
- *Modular symmetry as the direct origin of lepton mixing*
- *From a modular symmetry to multiple modular symmetries*
- *TM_1 and TM_2 mixing achieved in multiple modular symmetries*

The Stardard Model (SM)

17+2 free parameters



Quark masses and mixing

Charged Lepton masses

QCD g_s

$\bar{\theta}$ **Strong CP**

Neutrino masses and lepton mixing

Flavour eigenstates

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}$$

Mass eigenstates

m_1

m_2

m_3

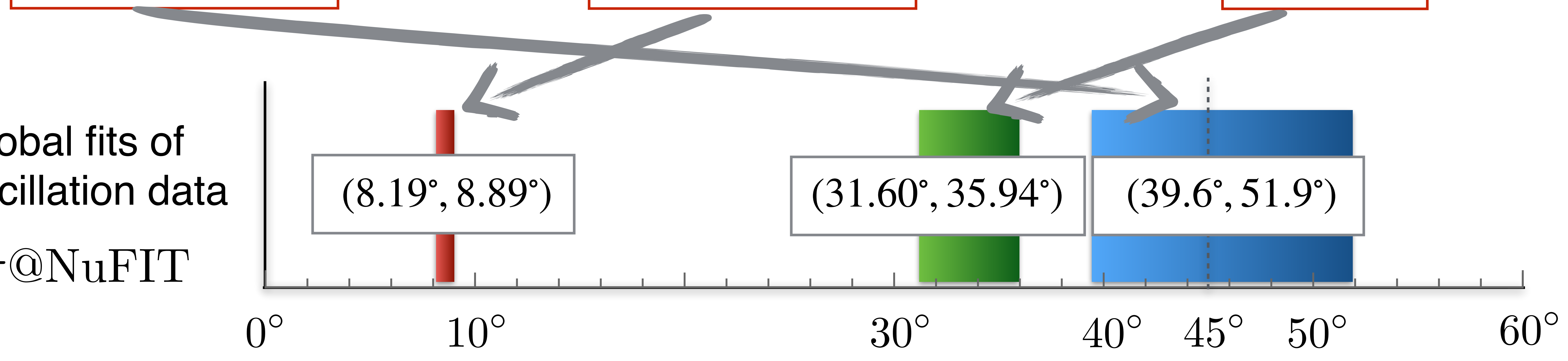
Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{bmatrix} \begin{bmatrix} \cos\theta_{13} & 0 & \sin\theta_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -\sin\theta_{13}e^{i\delta} & 0 & \cos\theta_{13} \end{bmatrix} \begin{bmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{bmatrix}$$

Atmospheric **Reactor** **Solar** **(Majorana)**

Global fits of oscillation data

3σ@NuFIT



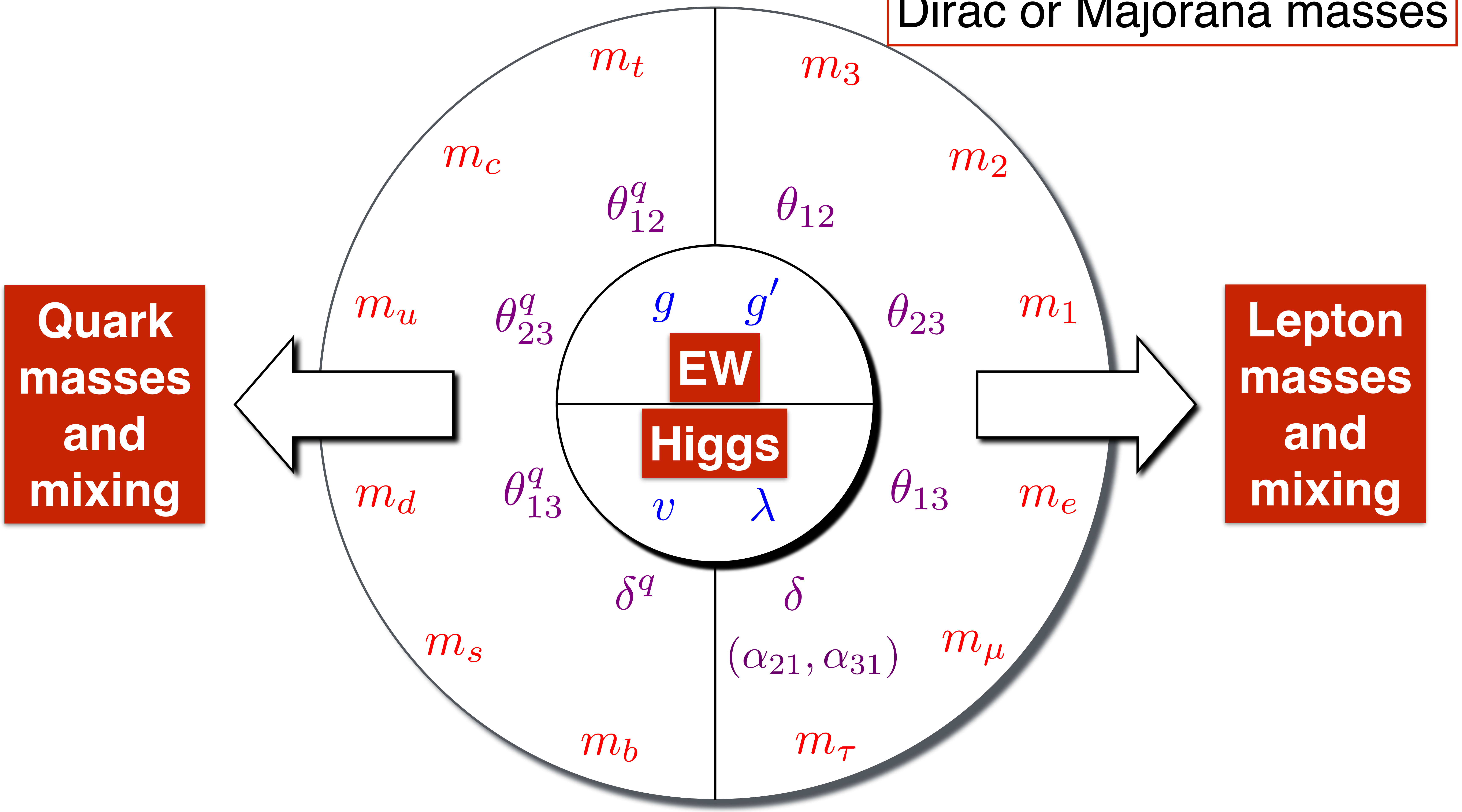
$\delta \in (108°, 404°)$

α_{21}, α_{31} undetermined

SM + massive neutrinos

24(26)+2 free parameters

Neutrino may take
Dirac or Majorana masses



Quark masses and mixing

m_t m_c m_u m_d m_s m_b

θ^q_{12} θ^q_{23} θ^q_{13} δ^q

EW

g g'

Higgs

v λ

θ_{12} θ_{23} θ_{13} δ $(\alpha_{21}, \alpha_{31})$

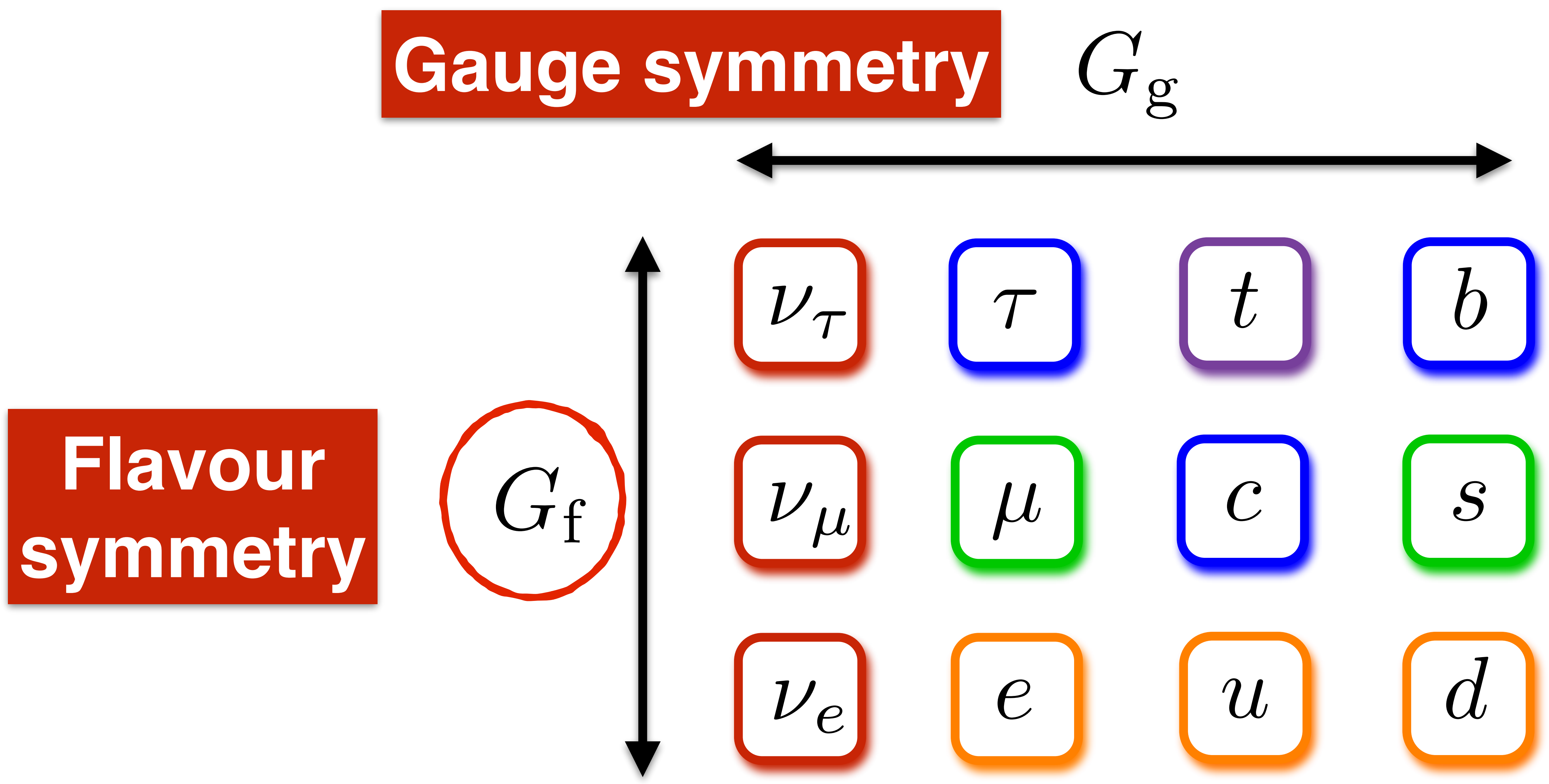
m_3 m_2 m_1 m_e m_μ m_τ

Lepton masses and mixing

QCD g_s

$\overline{\theta}$ **Strong CP**

What is the flavour symmetry?



Gauge symmetry G_{g}

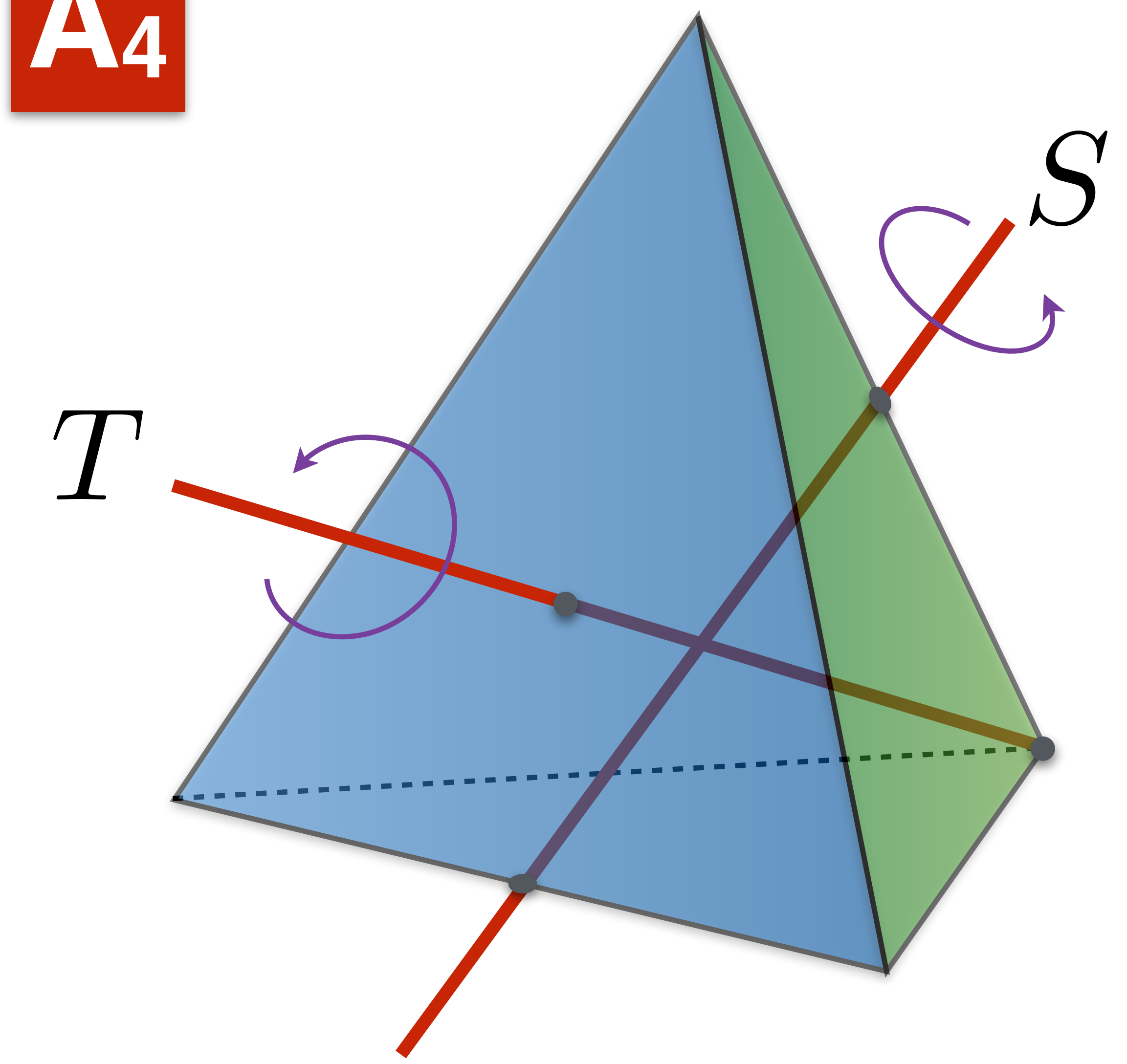
Flavour symmetry G_{f}

ν_τ	τ	t	b
ν_μ	μ	c	s
ν_e	e	u	d

	Continuous	**Discrete**
Abelian	**U(1)**	**Z_n**
Non-Abelian	**SU(3), SO(3),…**	**S_3, A_4, S_4, A_5, …**

Common-used non-Abelian discrete symmetries

A4

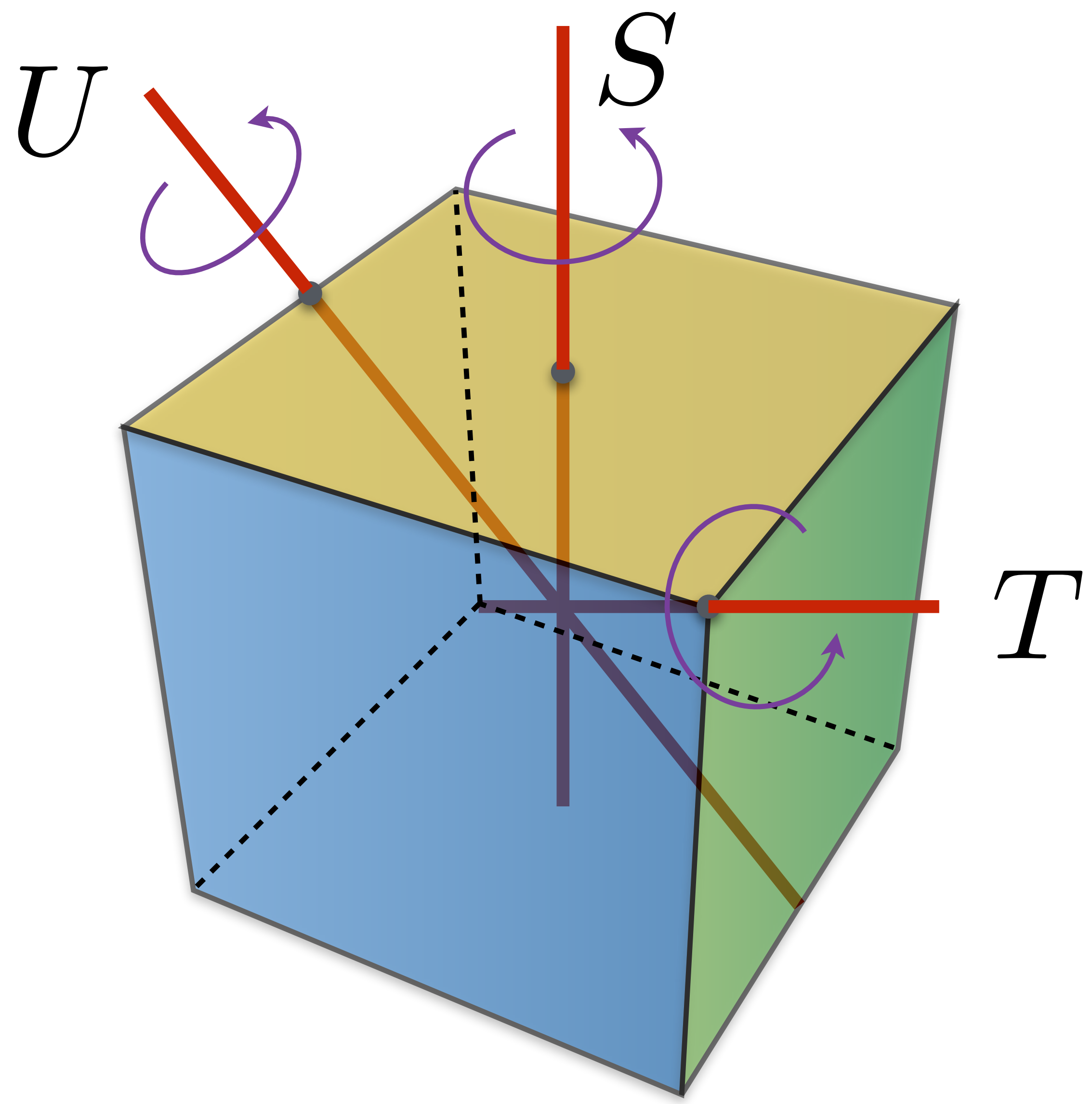


Ma, Rajasekaran, 01

$$T^3 = S^2 = (ST)^3 = \mathbf{1}$$

irrep: $\mathbf{1}, \mathbf{1}', \mathbf{1}'', \mathbf{3}$

S4



Lee, Mohapatra, 94

$$T^3 = S^2 = (ST)^3 = \mathbf{1}$$

$$U^2 = (SU)^2 = (TU)^2 = (STU)^4 = \mathbf{1}$$

irrep: $\mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{3}, \mathbf{3}'$

In this talk, 3d irrep will always be presented in the Altarelli-Feruglio basis

Typical mixing patterns and achievement in flavour symmetries

- Tri-bimaximal (TBM)

$$|U| = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$3\sin^2\theta_{12} = 1$$

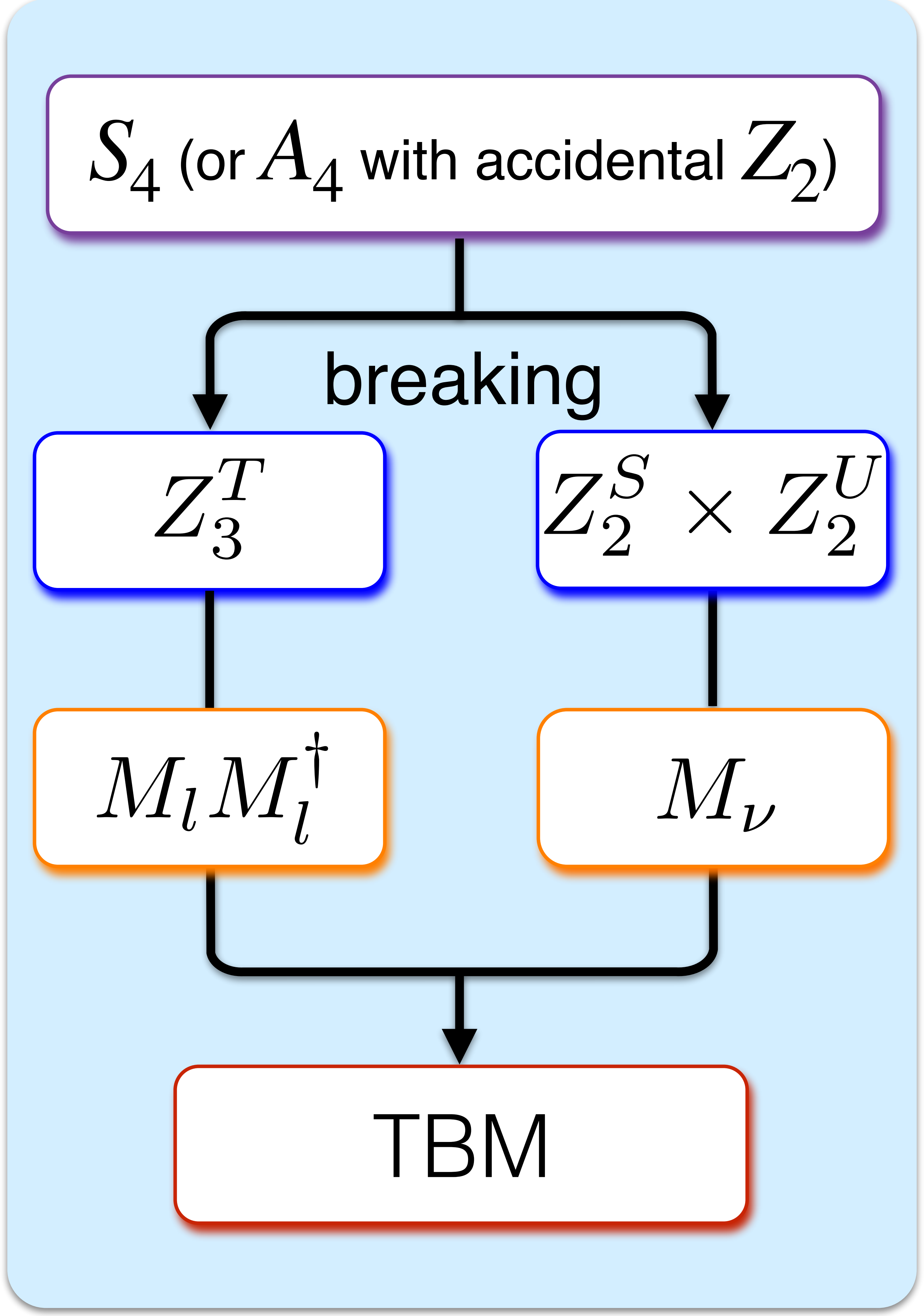
$$2\sin^2\theta_{23} = 1$$

$$\sin^2\theta_{13} = 0$$

Harrison, Perkins, Scott, 02

Xing, 02

- Tri-bimaximal (TBM)



S_4 (or A_4 with accidental Z_2)

breaking

Z_3^T

$Z_2^S \times Z_2^U$

$M_l M_l^\dagger$

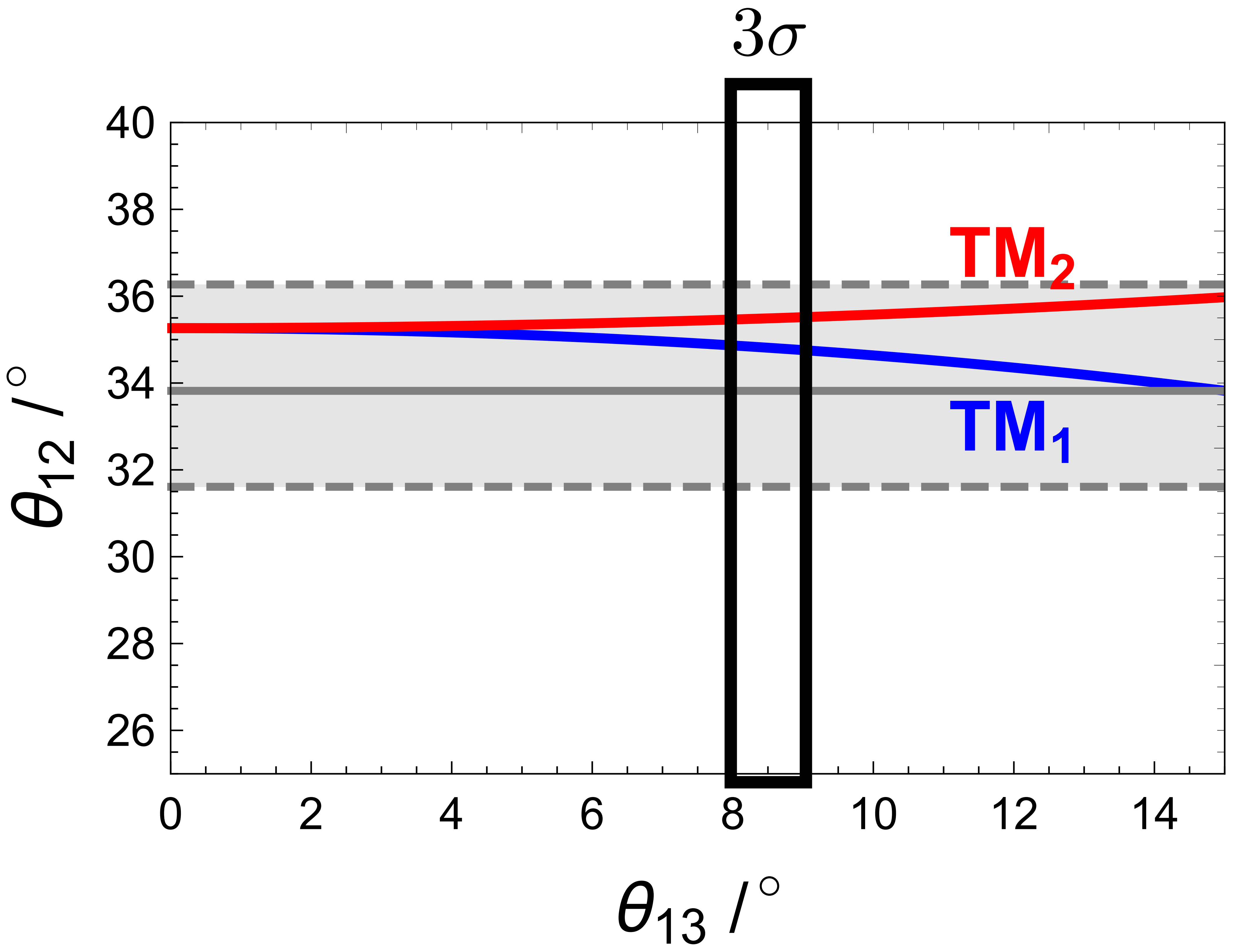
M_ν

TBM

From Tri-bimaximal to Trimaximal mixing

- Trimaximal (TM) mixing

$$|U| = \begin{array}{c} \\ \\ \end{array}\left(\begin{array}{ccc} \mathrm{TM}_1 & \mathrm{TM}_2 & \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{array}\right)$$



Xing, Zhou, 0607302; Lam, 0611017; Albright, Rodejohann, 0812.0436

Bjorken, Harrison, Scott, 0511201; He, Zee, 0607163; Grimus, Lavoura, 0809.0226; 0810.4516

Relax residual symmetries

$$(Z_3^T, Z_2^{SU}) \Rightarrow \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, (Z_3^T, Z_2^S) \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, (Z_3^T, Z_2^U) \Rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

A toy model in A_4

Simplified based on Altarelli, Feruglio, hep-ph/0504165, 0512103

- Field contents

Flavons $\varphi = (\varphi_1, \varphi_2, \varphi_3) \sim \mathbf{3}, \chi = (\chi_1, \chi_2, \chi_3) \sim \mathbf{3}$ $\eta \sim \mathbf{1}$

Matter fields $L = (L_e, L_\mu, L_\tau) \sim \mathbf{3}, (e^c, \mu^c, \tau^c) \sim (\mathbf{1}, \mathbf{1}', \mathbf{1}''), \nu^c = (N_1, N_2, N_3) \sim \mathbf{3}$ $H \sim \mathbf{1}$

- Lagrangian

$$W \supset \left[y_e (L\varphi)_1 e^c + \frac{y_\mu}{\Lambda} (L\varphi)_{1'} \mu^c + \frac{y_\tau}{\Lambda} (L\varphi)_{1''} \tau^c \right] \frac{H_d}{\Lambda} \Rightarrow \boxed{M_l}$$

$$+ y_D L \nu^c H_u + \boxed{\frac{y_1}{2} (\nu^c \nu^c)_3 \chi + \frac{y_2}{2} (\nu^c \nu^c)_1 \eta +} \mathrm{h.c.} \Rightarrow \boxed{M_\nu}$$

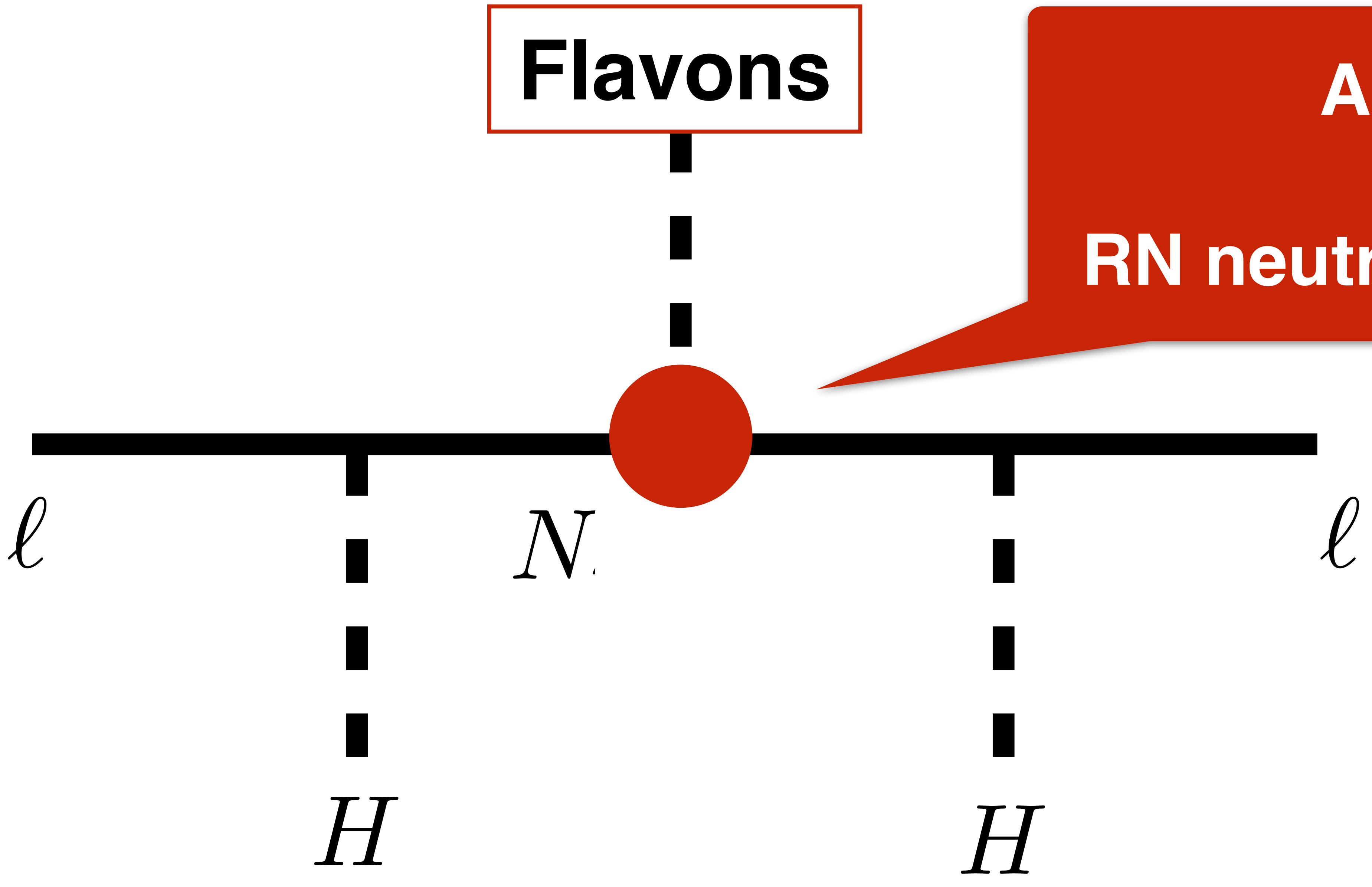
- Vacuum alignment and flavour mixing

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{\begin{array}{l} \langle \varphi \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} v_\varphi \\ \langle \chi \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \frac{v_\chi}{\sqrt{3}} \\ \langle \eta \rangle = v_\eta \end{array}} \Rightarrow \boxed{\begin{array}{l} M_l = \mathrm{diag}\{y_e, y_\mu, y_\tau\} \frac{v_\phi v_d}{\Lambda} \\ M_D = y_D P_{23} v_u \\ \hline M_\nu = \begin{pmatrix} a + 2b & -b & -b \\ -b & 2b & a - b \\ -b & a - b & 2b \end{pmatrix} \\ \hline a = y_2 v_\eta \,, \quad b = \frac{1}{2\sqrt{3}} y_1 v_\chi \end{array}} \Rightarrow \boxed{\text{TBM mixing}}$$

Two approaches for model building

In the framework of type-I seesaw

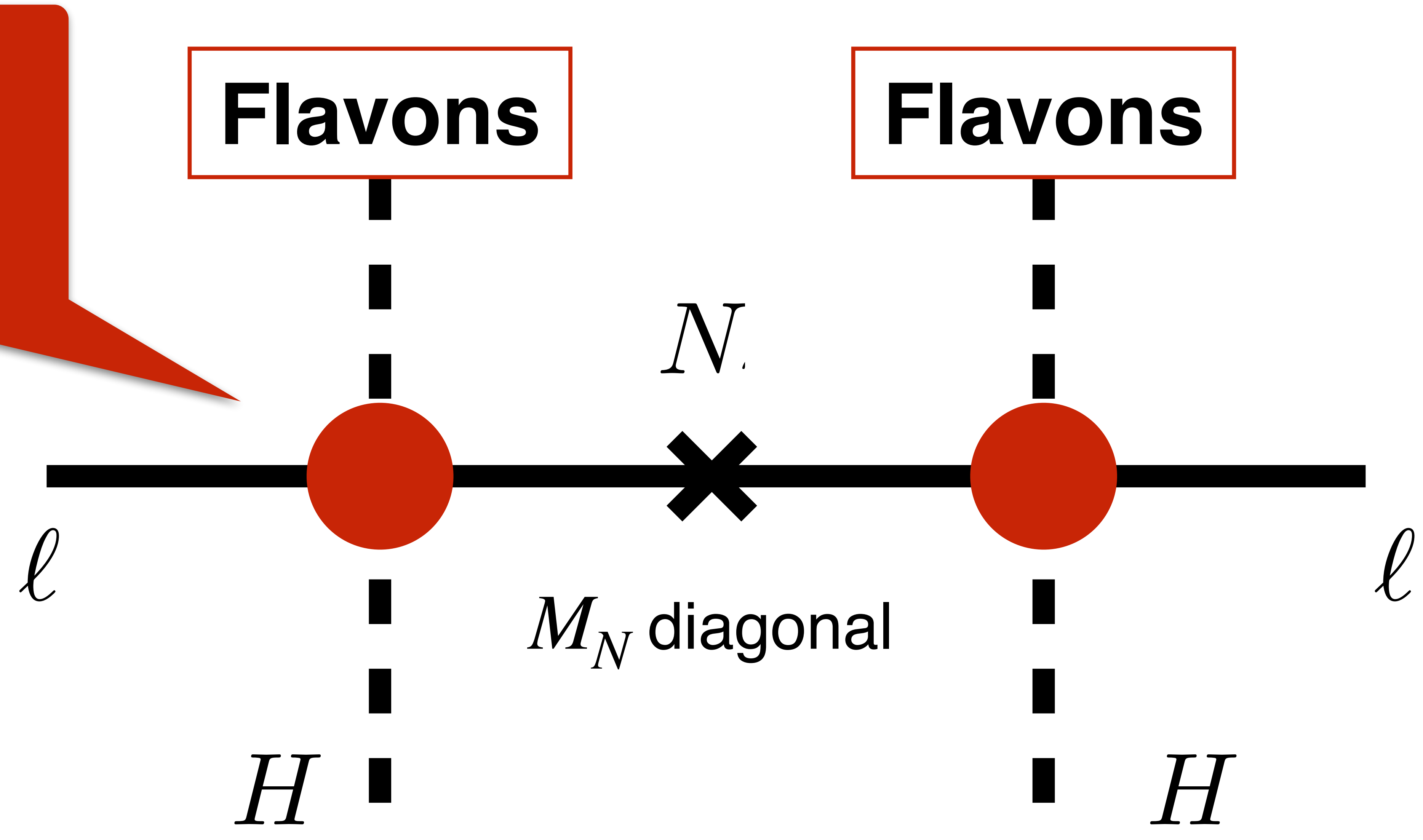


Appoach I:
via
RN neutrino mass matrix

e.g., the toy model in the last slide
+ hundreds of models …

For some reviews, see
Alterelli, Feruglio, 1002.0211; Ishimori, Kobayashi, et al, 1003.3552; King, Luhn, 1301.1340; King, Merle, Morisi, Shimizu, Tanimoto, 1402.4271; Xing, 1909.09610; Feruglio, Romanino, 1912.06028;

Approach II:
via
Yukawa couplings



e.g., the sequential dominance series

Flavour symmetries from modular symmetry

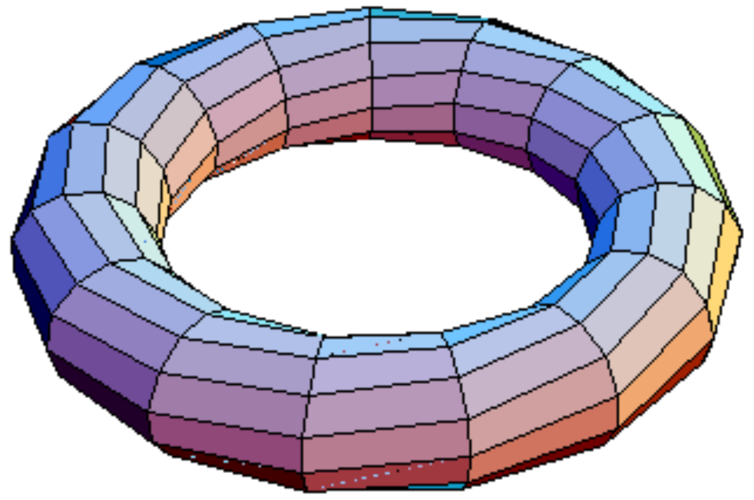
- Modular symmetry predicted in string orbifold compactifications

Ferrara, Lust, Theisen, 89

$$\mathbb{T}^2/\Lambda$$

$$\tau = 2B + i\sqrt{3}R^2$$

$$\mathrm{Im}\tau > 0$$



$$S_\tau : \tau \to \frac{-1}{\tau} \qquad T_\tau : \tau \to \tau + 1 \qquad S_\tau^2 = (S_\tau T_\tau)^3 = 1$$

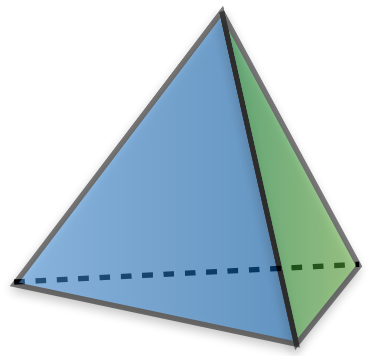
$$\overline{\Gamma} = \left\{ \gamma \middle| \gamma\tau = \frac{a\tau + b}{c\tau + d}, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

- Finite modular symmetries

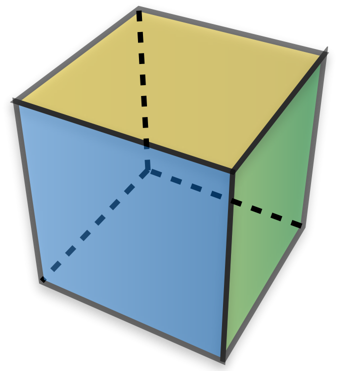
$$T_\tau^3 = 1 \qquad T_\tau^4 = 1 \qquad T_\tau^5 = 1$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

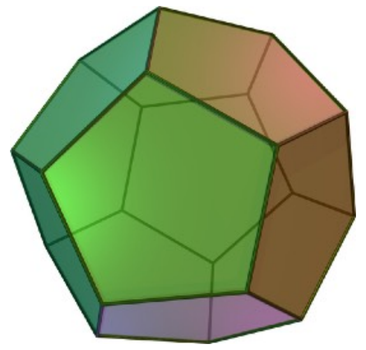
$\Gamma_3 \simeq A_4$



$\Gamma_4 \simeq S_4$



$\Gamma_5 \simeq A_5$



$$\boxed{S = S_\tau, T = T_\tau}$$

$$\boxed{S = T_\tau^2, T = S_\tau T_\tau, U = T_\tau S_\tau T_\tau^2 S_\tau}$$

de Adelhart Toorop, Feruglio
and Hagedorn, 1112.1340

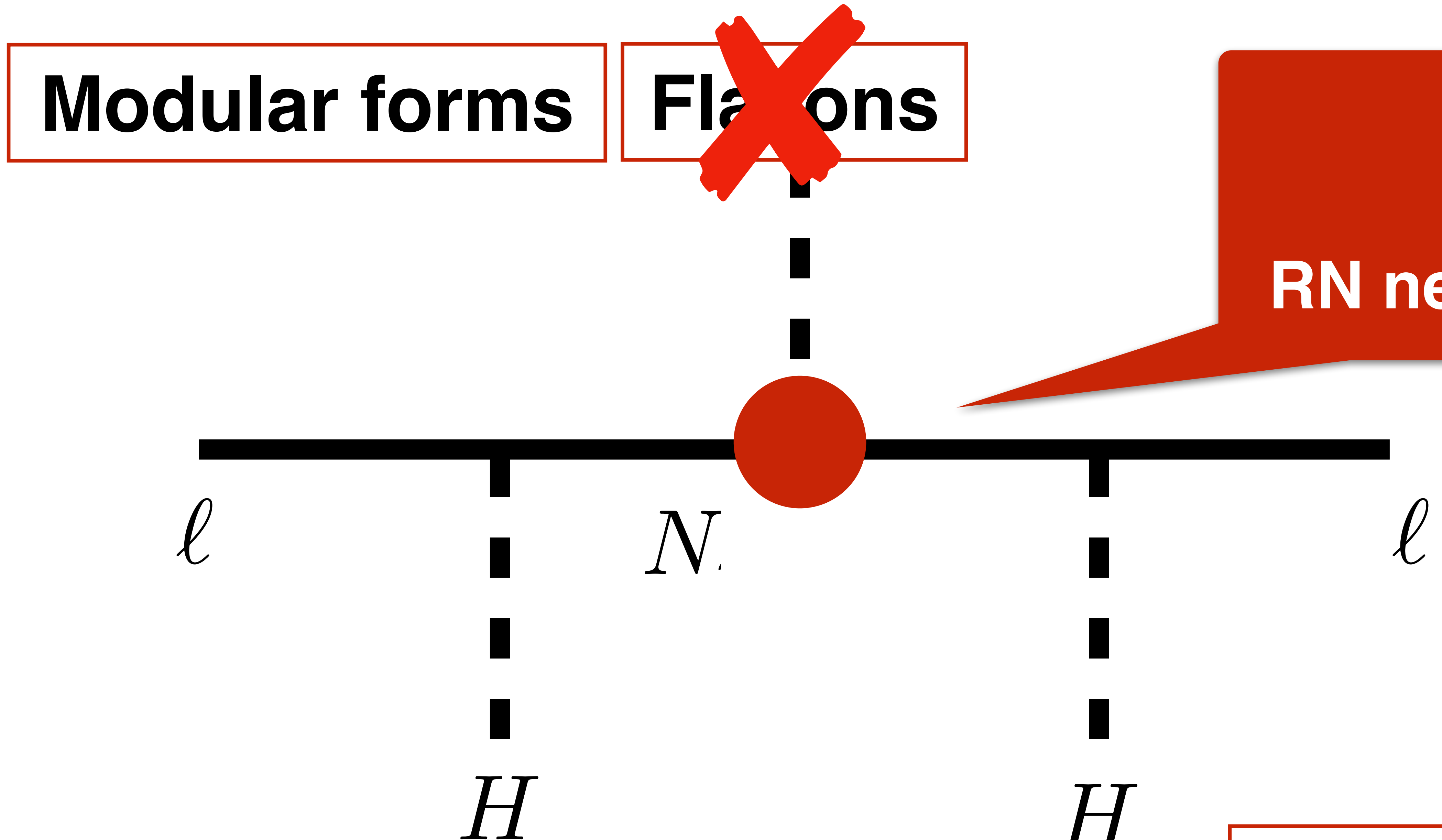
Modular symmetry as direct origin of flavour mixing

- “classical” flavour model		- Flavour model with modular symmetry
$\gamma \in G_{\mathrm{f}}$	**Symmetry**	$\gamma \in \Gamma_N$
		$\tau \to \gamma\tau = \dfrac{a\tau + b}{c\tau + d}$
$\boxed{\psi \to \rho_I(\gamma)\psi}$	**Matter field**	$\boxed{\psi \to (c\tau + d)^{2k}\rho_I(\gamma)\psi}$
$\boxed{Y(\{\phi_i\}) \to \rho_{I_Y}(\gamma)Y(\{\phi_i\})}$	**Yukawa couplings**	$\boxed{Y(\tau) \to (c\tau + d)^{2k_Y}\rho_{I_Y}(\gamma)Y(\tau)}$

- If the lepton has non-vanishing modular weight, the modular symmetry can be directly used to explain flavour mixing

Feruglio, 1706.08749

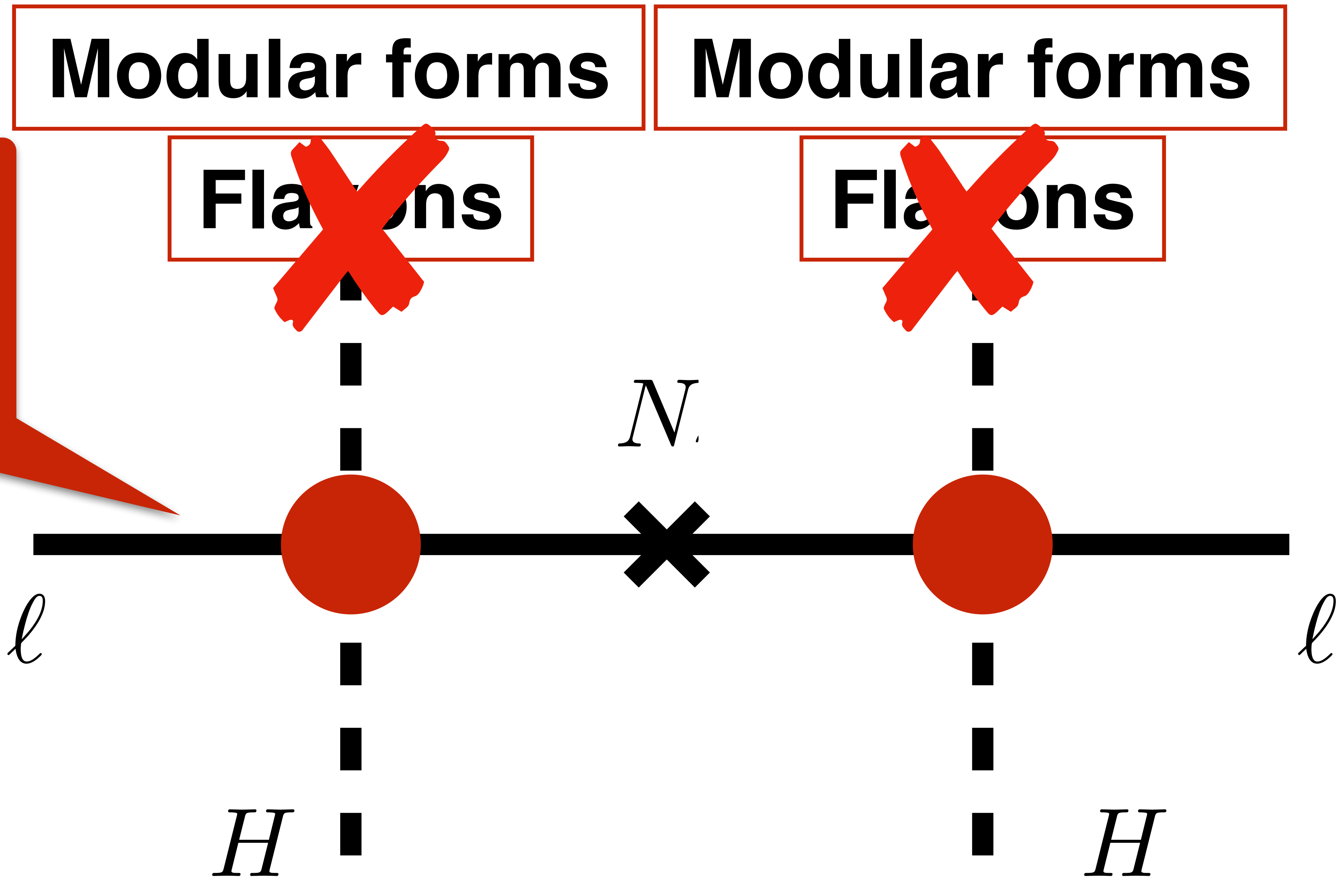
Flavour models with a modular symmetry



Appoach I:
via
RN neutrino mass matrix

Replacing flavons by modular forms reduces free parameters and new d.o.f.

Approach II:
via
Yukawa couplings



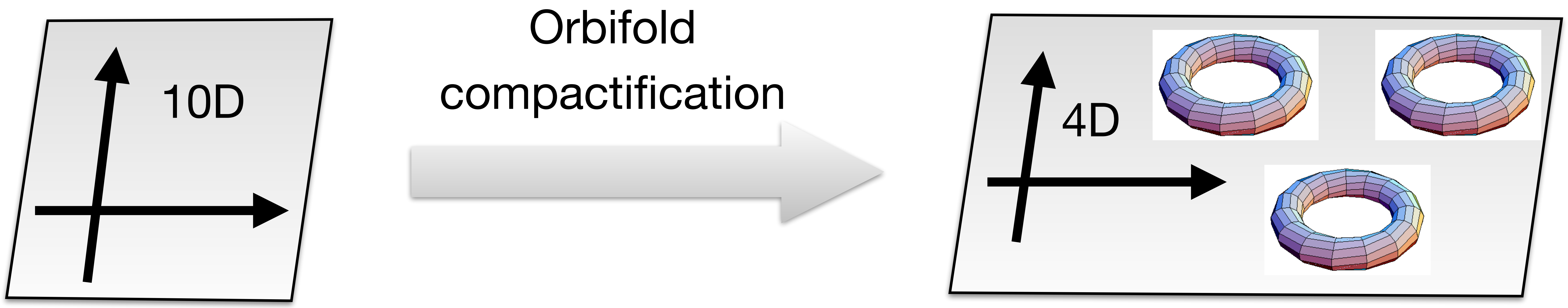
see Ding, King, 2311.09282 for a recent review

From a single modular symmetry to multiple modular symmetries

- Motivation for multiple modular symmetries

$$\tau_i = 2B_i + i\sqrt{3}R_i^2$$

Top-down



Bottom-up

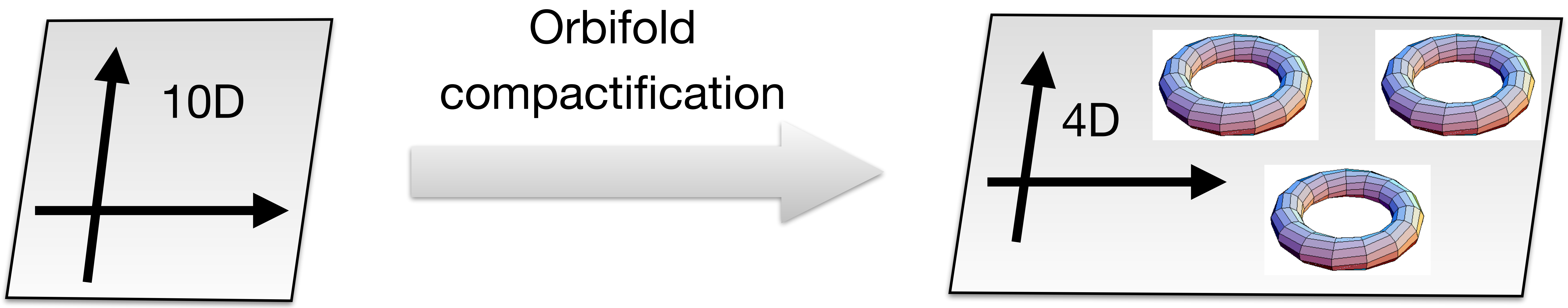
1. Multiple moduli fields can be introduced. They take different VEVs, enlarging the parameter space.
2. In particular, VEVs may be fixed at some special values, called stabilisers, leading to residual symmetries unbroken.
3. The flavour mixing, following the classical flavour model building approach, arises from the misalignment of different breaking directions.

From a single modular symmetry to multiple modular symmetries

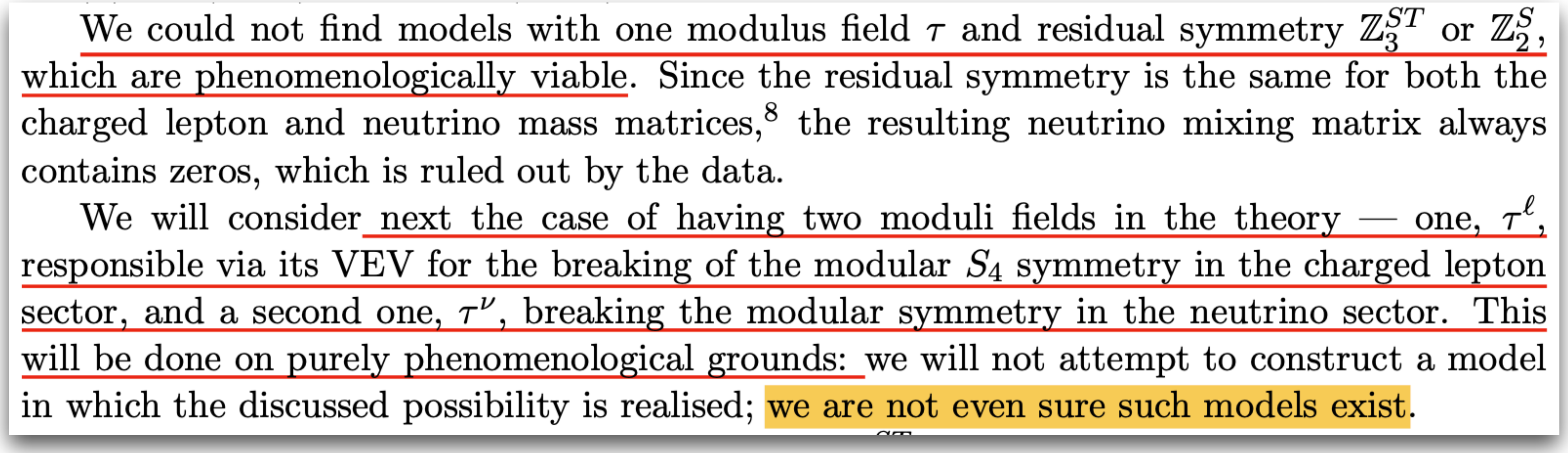
- Motivation for multiple modular symmetries

$$\tau_i = 2B_i + i\sqrt{3}R_i^2$$

Top-down



Bottom-up



We could not find models with one modulus field τ and residual symmetry $\mathbb{Z}_3^{ST}$ or $\mathbb{Z}_2^{S}$, which are phenomenologically viable. Since the residual symmetry is the same for both the charged lepton and neutrino mass matrices,[8] the resulting neutrino mixing matrix always contains zeros, which is ruled out by the data.

We will consider next the case of having two moduli fields in the theory — one, τ^ℓ, responsible via its VEV for the breaking of the modular S_4 symmetry in the charged lepton sector, and a second one, τ^ν, breaking the modular symmetry in the neutrino sector. This will be done on purely phenomenological grounds: we will not attempt to construct a model in which the discussed possibility is realised; we are not even sure such models exist.

in Sec 5 in [Novichkov, Penedo, Petcov, Titov, 1811.04933]

Framework of multiple modular symmetries

- Multiple finite modular symmetries (simplest case, direct product)

$$\Gamma^1_{N_1} \times \Gamma^2_{N_2} \times \ldots$$

- Multiple moduli fields $\tau_1, \tau_2, \ldots$ as target space

de Medeiros Varzielas,
King, **YLZ**, 1906.02208

- $\gamma_J \in \Gamma^J_{N_J}$ acts on τ_J as

$$\gamma_J : \tau_J \rightarrow \gamma_J \tau_J = \frac{a_J \tau_J + b_J}{c_J \tau_J + d_J}$$

- γ_J acts on superfields ϕ_i and Yukawa forms as

$$\begin{aligned} \phi_i(\tau_1, ..., \tau_M) \quad \rightarrow \quad & \phi_i(\gamma_1 \tau_1, ..., \gamma_M \tau_M) \\ &= \prod_{J=1,...,M} (c_J \tau_J + d_J)^{-2k_{i,J}} \bigotimes_{J=1,...,M} \rho_{I_{i,J}}(\gamma_J) \phi_i(\tau_1, \tau_2, ..., \tau_M) \end{aligned}$$

$$\begin{aligned} Y_{(I_{Y,1},...,I_{Y,M})}(\tau_1, ..., \tau_M) \rightarrow & Y_{(I_{Y,1},...,I_{Y,M})}(\gamma_1 \tau_1, ..., \gamma_M \tau_M) \\ &= \prod_{J=1,...,M} (c_J \tau_J + d_J)^{2k_{Y,J}} \bigotimes_{J=1,...,M} \rho_{I_{Y,J}}(\gamma_J) Y_{(I_{Y,1},...,I_{Y,M})}(\tau_1, ..., \tau_M) \; . \end{aligned}$$

Framework of multiple modular symmetries

- $\mathcal{N} = 1$ SUSY

$$\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta}\, K + \left[\int d^4x d^2\theta\, W + \mathrm{h.c.}\right]$$

- Kahler potential (the simplest case)

$$K = -\sum_J h_J \log(-i\tau_J + i\bar{\tau}_J) + \sum_i \frac{\bar{\phi}_i \phi_i}{\prod_J (-i\tau_J + i\bar{\tau}_J)^{2k_{i,J}}}$$

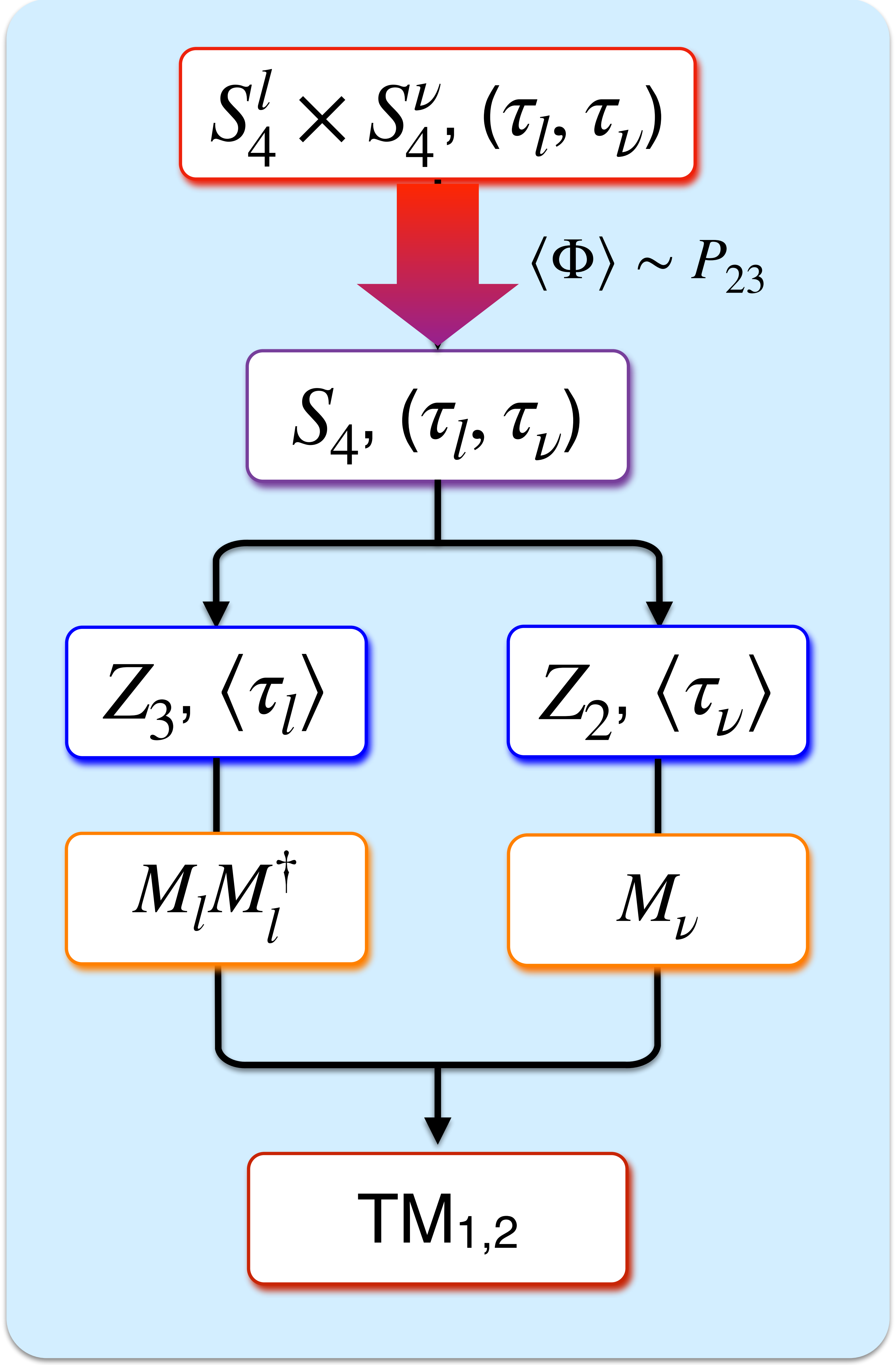
$$\Rightarrow \sum_J \frac{h_J}{\langle -i\tau_J + i\bar{\tau}_J \rangle^2} \partial_\mu \bar{\tau}_J \partial^\mu \tau_J + \sum_i \frac{\partial_\mu \bar{\phi}_i \partial^\mu \phi_i}{\prod_J \langle -i\tau_J + i\bar{\tau}_J \rangle^{2k_{i,J}}}$$

- Superpotential (should by invariant under any modular transformation)

$$W = \sum_n \prod_{i_1,\ldots,i_n} (Y_{I_{Y,1},I_{Y,2},\ldots}\, \phi_{i_1}, \ldots, \phi_{i_n})_{\mathbf{1}}$$

Framework of models with multiple modular symmetries

Twin S_4 as example



$S_4^l \times S_4^\nu, (\tau_l, \tau_\nu)$

$\langle\Phi\rangle \sim P_{23}$

$S_4, (\tau_l, \tau_\nu)$

$Z_3, \langle\tau_l\rangle$

$Z_2, \langle\tau_\nu\rangle$

$M_l M_l^\dagger$

M_ν

TM1,2

- Two S_4 are broken to a single S_4 along flat direction of bi-triplet
    - Bi-triplet scalar $\Phi \sim (\mathbf{3},\mathbf{3})$ of $S_4^l \times S_4^\nu$
    - Driving fields $\chi^d \sim (\mathbf{3},\mathbf{3})$, $\tilde{\chi}^d \sim (\mathbf{1},\mathbf{3})$

$$W_d = [\underline{(\Phi\Phi)_{(3,3)} + \mathrm{M}\,\Phi]}\,\chi^d + \underline{(\Phi\Phi)_{(1,3)}}\,\tilde{\chi}^d$$

- There are 24 solutions

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\langle\Phi\rangle \sim \rho(\gamma) P_{23}$ for γ spanning in S_4

All equivalent to P_{23} after basis transformation

ρ: 3D irrep matrix in <u>Altarelli-Feruglio</u> basis

- Same method applies to $A_4^l \times A_4^\nu \to A_4$

Also 24 solutions, $\langle\Phi\rangle \sim \rho(\gamma) P_{23}, \rho(\gamma)$

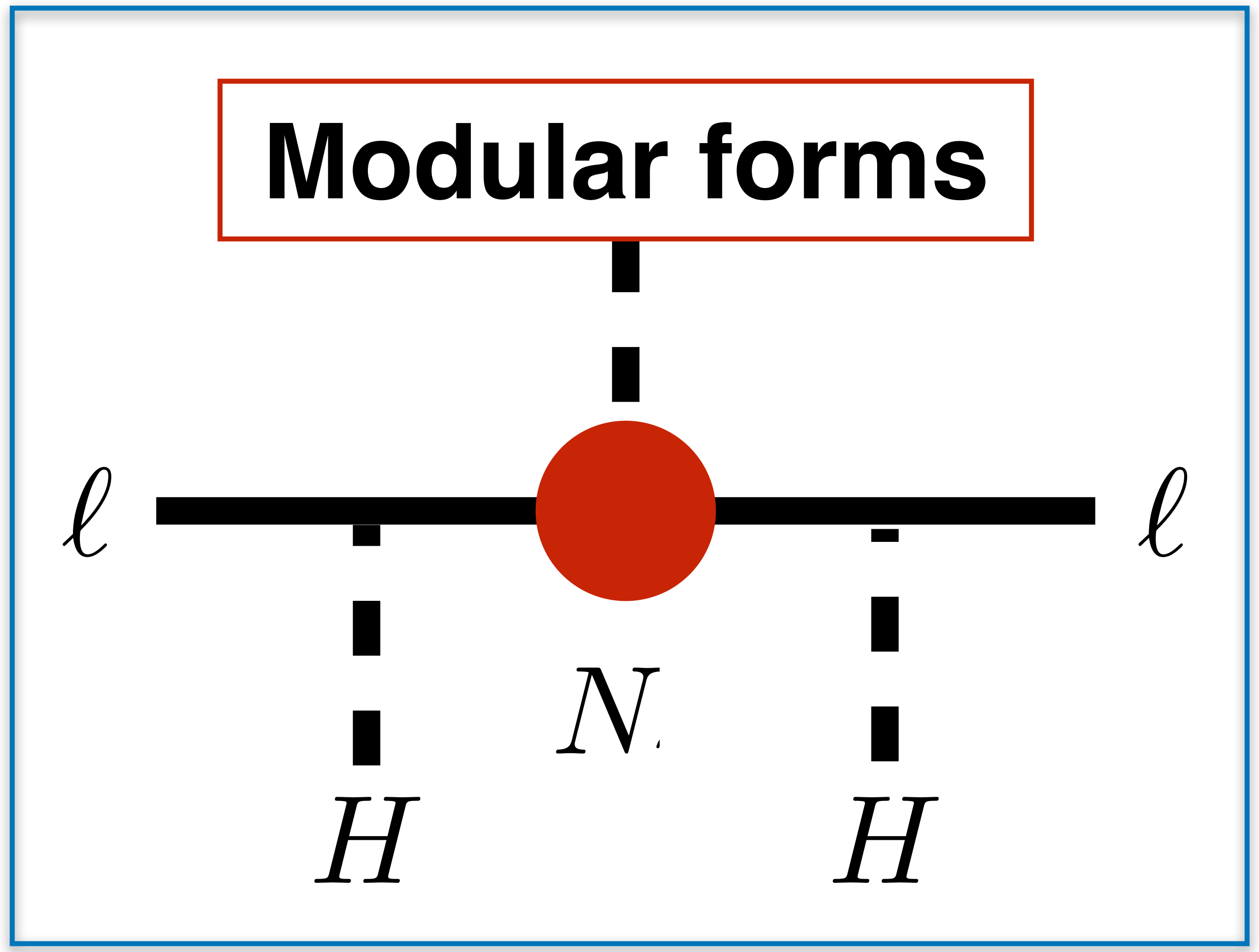
de Medeiros Varzielas, Lourenco, 2107.04042

But not all equivalent

Zhang, YLZ, 2401.17810

TM_1 in approach I

King, **YLZ**, 1908.02770



LH lepton $L \sim \mathbf{3}$

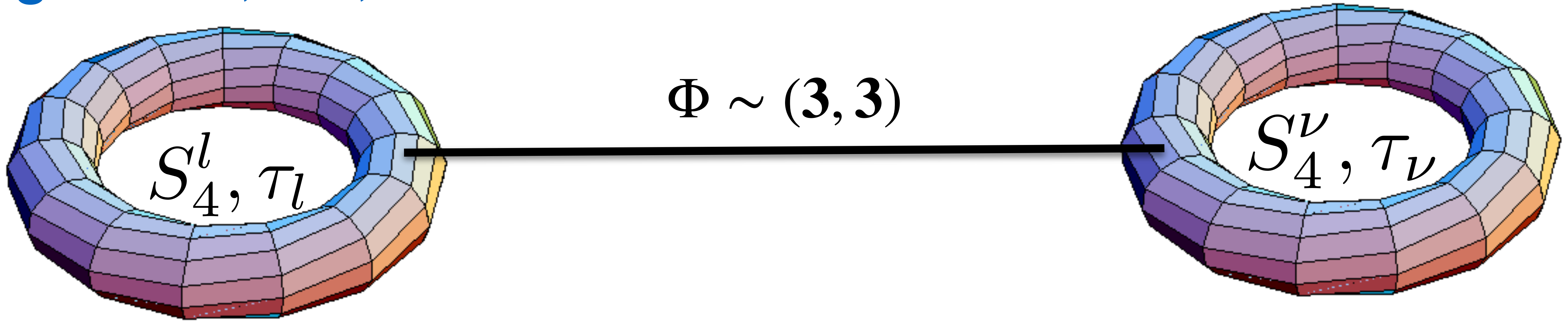
$S_4^l \times S_4^\nu,\ (\tau_l, \tau_\nu)$

RH neutrino

$\nu^c = (N_1, N_2, N_3) \sim \mathbf{3}$

RH lepton $e^c, \mu^c, \tau^c \sim \mathbf{1}'$

weights: $-6, -4, -2$

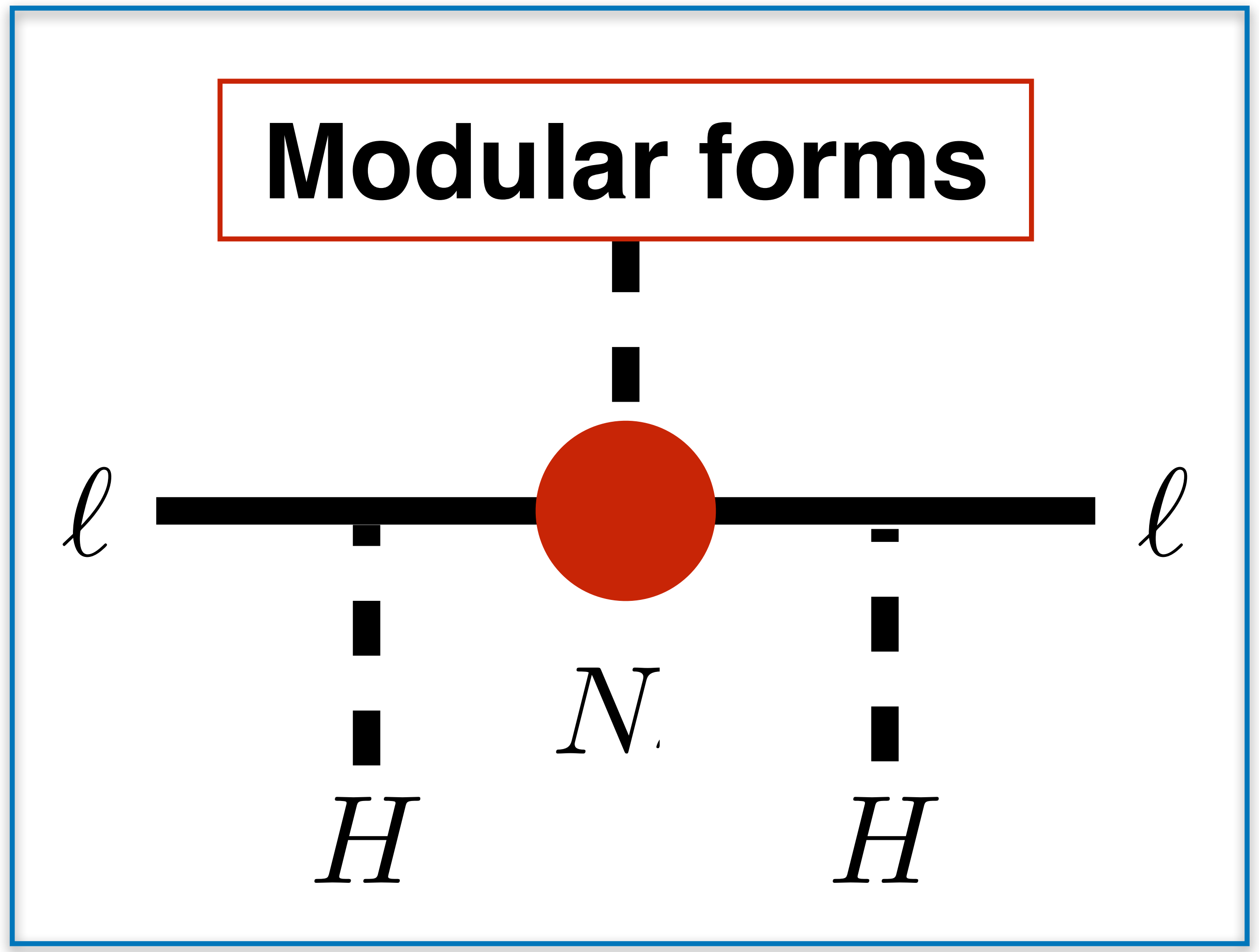


Fields	S_4^l	S_4^ν	$2k_l$	$2k_\nu$
e^c	$\mathbf{1}'$	$\mathbf{1}$	-6	-2
μ^c	$\mathbf{1}'$	$\mathbf{1}$	-4	-2
τ^c	$\mathbf{1}'$	$\mathbf{1}$	-2	-2
L	$\mathbf{3}$	$\mathbf{1}$	0	$+2$
ν^c	$\mathbf{1}$	$\mathbf{3}$	0	-2
Φ	$\mathbf{3}$	$\mathbf{3}$	0	0
$H_{u,d}$	$\mathbf{1}$	$\mathbf{1}$	0	0

$$\begin{aligned} w &= \left[L Y_e(\tau_l) e^c + L Y_\mu(\tau_l) \mu^c + L Y_\tau(\tau_l) \tau^c\right] H_d \\ &\quad + \frac{y_\nu}{\Lambda} L \Phi \nu^c H_u + \frac{1}{2} M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2} M_{\mathbf{2}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{2}} + \frac{1}{2} M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}} \,. \end{aligned}$$

TM$_1$ in approach I

King, **YLZ**, 1908.02770



LH lepton $L \sim \mathbf{3}$

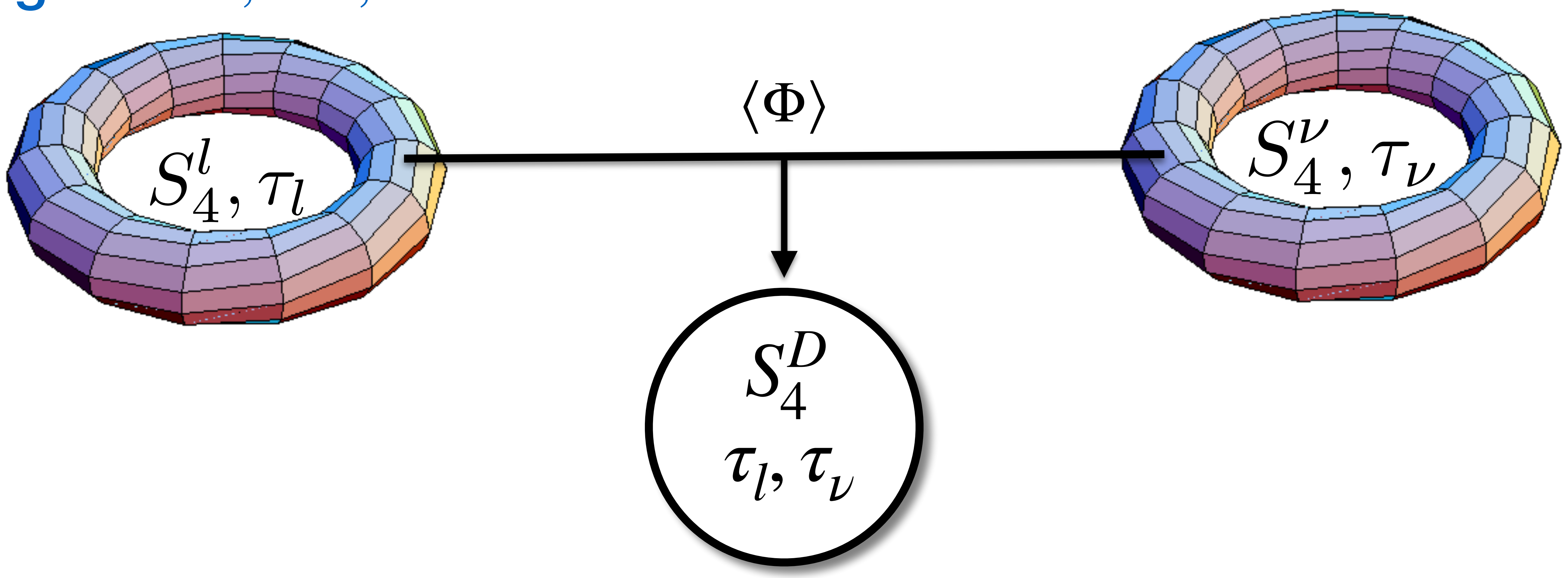
$S_4^l \times S_4^\nu$, (τ_l, τ_ν)

RH neutrino

$\nu^c = (N_1, N_2, N_3) \sim \mathbf{3}$

RH lepton $e^c, \mu^c, \tau^c \sim \mathbf{1}'$

weights: $-6, -4, -2$

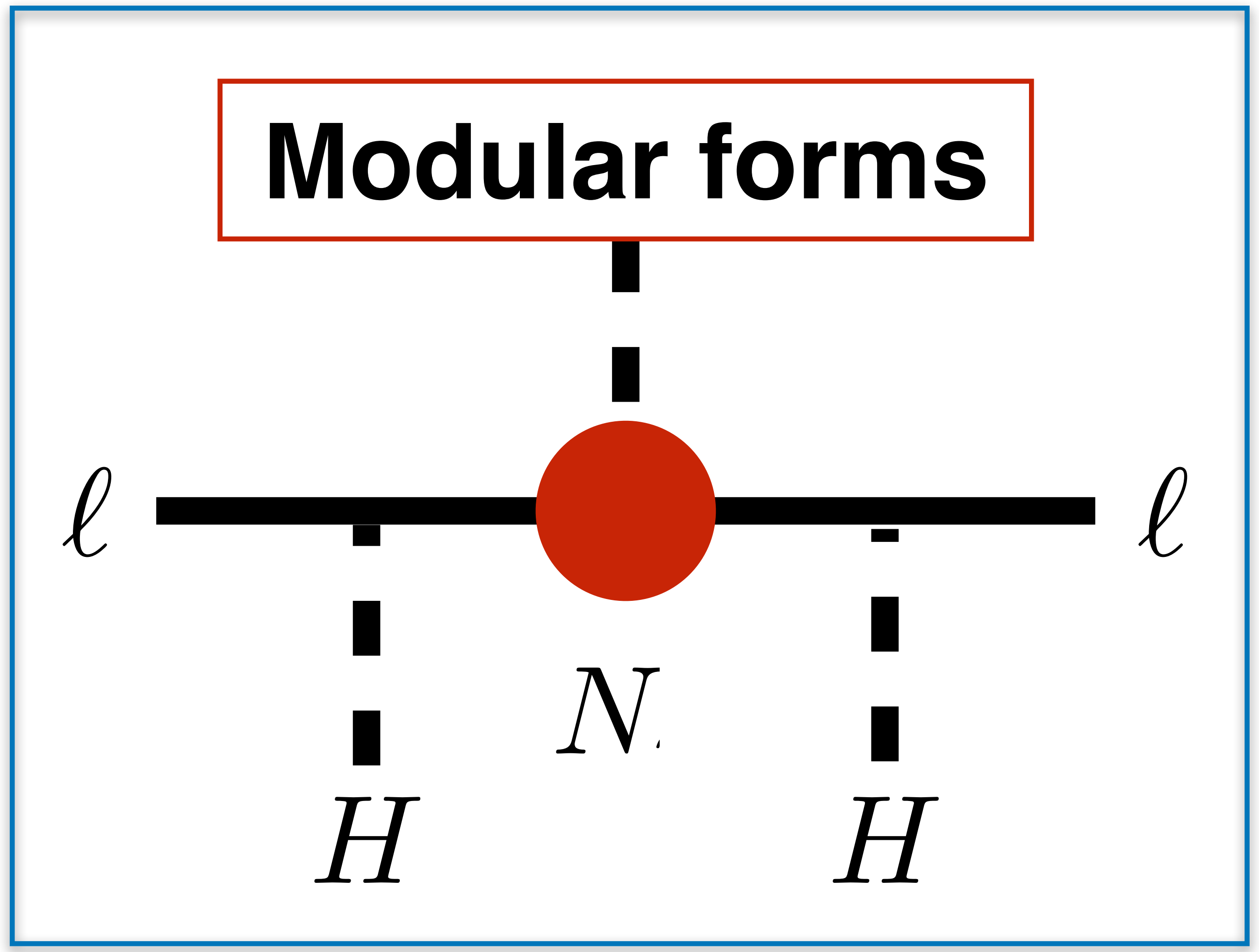


Fields	S_4^l	S_4^ν	$2k_l$	$2k_\nu$
e^c	$\mathbf{1}'$	$\mathbf{1}$	-6	-2
μ^c	$\mathbf{1}'$	$\mathbf{1}$	-4	-2
τ^c	$\mathbf{1}'$	$\mathbf{1}$	-2	-2
L	$\mathbf{3}$	$\mathbf{1}$	0	$+2$
ν^c	$\mathbf{1}$	$\mathbf{3}$	0	-2
Φ	$\mathbf{3}$	$\mathbf{3}$	0	0
$H_{u,d}$	$\mathbf{1}$	$\mathbf{1}$	0	0

$$\begin{aligned} w_{\rm eff} &= \left[L Y_e(\tau_l) e^c + L Y_\mu(\tau_l)\mu^c + L Y_\tau(\tau_l)\tau^c\right] H_d \\ &+ y_D L \nu^c H_u + \frac{1}{2} M_{\mathbf{1}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{1}} + \frac{1}{2} M_{\mathbf{2}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{2}} + \frac{1}{2} M_{\mathbf{3}}(\tau_\nu)(\nu^c\nu^c)_{\mathbf{3}} \,. \end{aligned}$$

TM$_1$ in approach I

King, **YLZ**, 1908.02770



Fields	S_4^l	S_4^ν	$2k_l$	$2k_\nu$
e^c	$\mathbf{1}'$	$\mathbf{1}$	-6	-2
μ^c	$\mathbf{1}'$	$\mathbf{1}$	-4	-2
τ^c	$\mathbf{1}'$	$\mathbf{1}$	-2	-2
L	$\mathbf{3}$	$\mathbf{1}$	0	$+2$
ν^c	$\mathbf{1}$	$\mathbf{3}$	0	-2
Φ	$\mathbf{3}$	$\mathbf{3}$	0	0
$H_{u,d}$	$\mathbf{1}$	$\mathbf{1}$	0	0

LH lepton $L \sim \mathbf{3}$

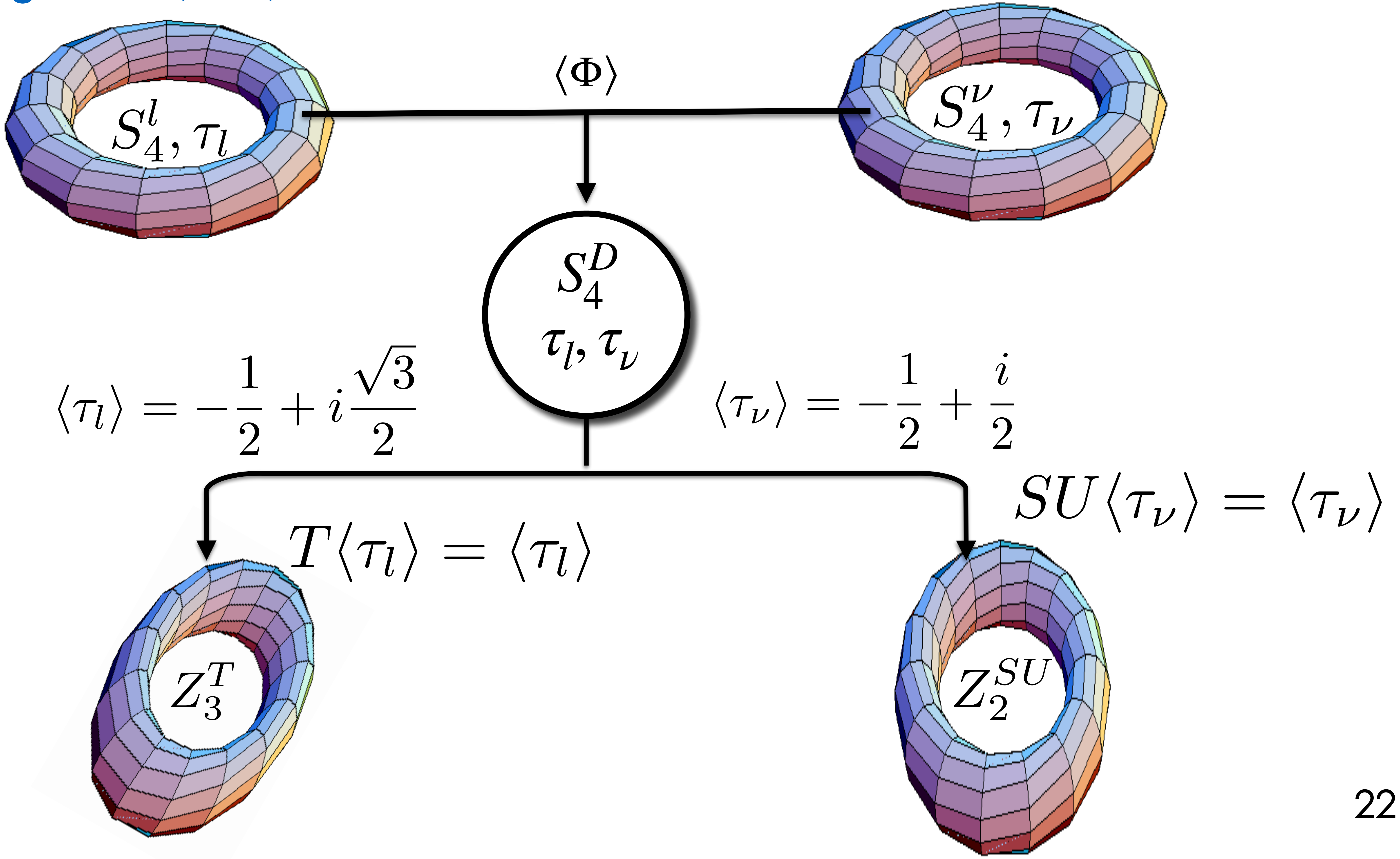
$S_4^l \times S_4^\nu, (\tau_l, \tau_\nu)$

RH neutrino

$\nu^c = (N_1, N_2, N_3) \sim \mathbf{3}$

RH lepton $e^c, \mu^c, \tau^c \sim \mathbf{1}'$

weights: $-6, -4, -2$



$\langle\Phi\rangle$

S_4^l, τ_l

S_4^ν, τ_ν

S_4^D
τ_l, τ_ν

$$\langle\tau_l\rangle = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\langle\tau_\nu\rangle = -\frac{1}{2} + \frac{i}{2}$$

$$T\langle\tau_l\rangle = \langle\tau_l\rangle$$

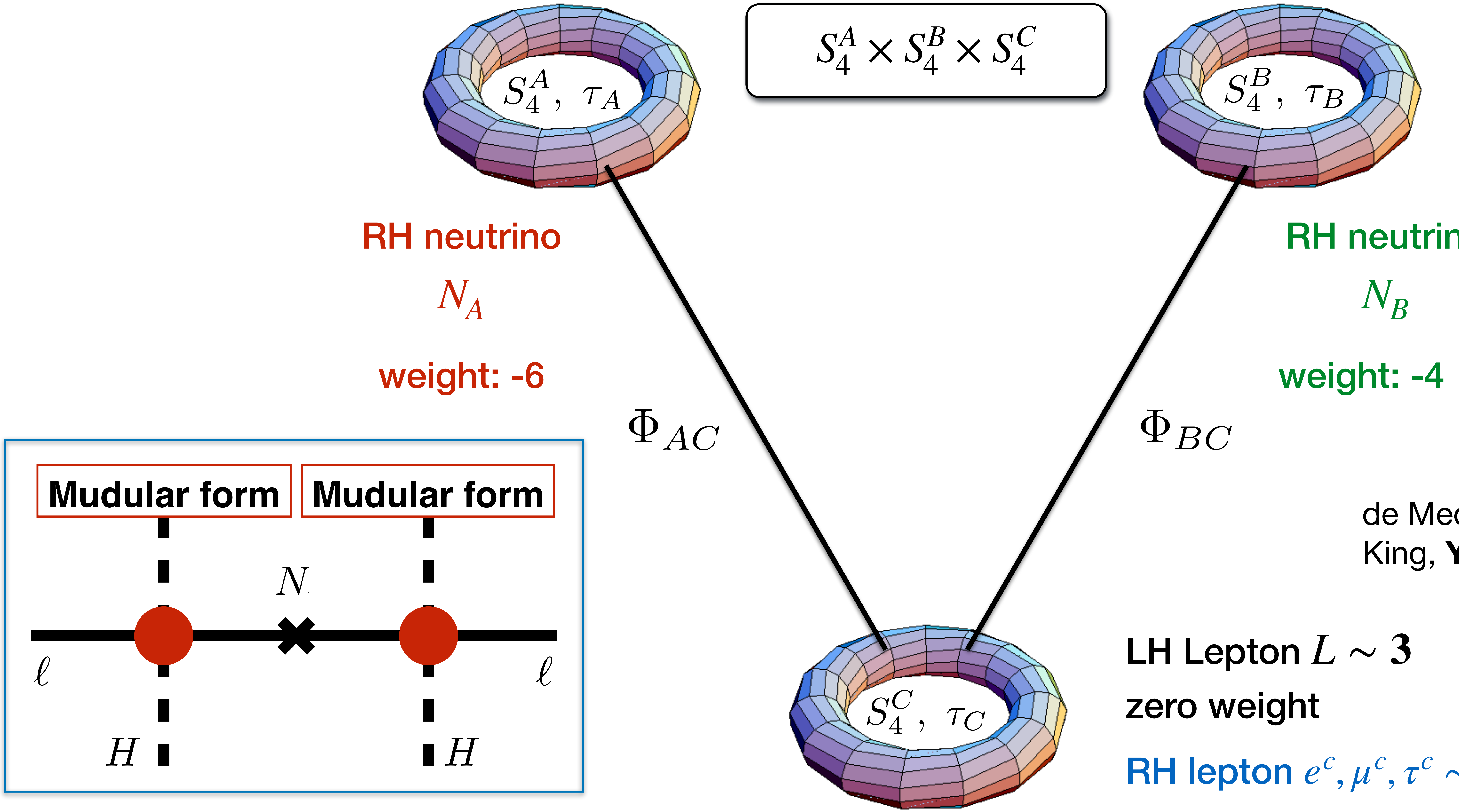
$$SU\langle\tau_\nu\rangle = \langle\tau_\nu\rangle$$

Z_3^T

Z_2^{SU}

TM$_1$ in approach II

$S_4^A \times S_4^B \times S_4^C$



RH neutrino

N_A

weight: -6

RH neutrino

N_B

weight: -4

de Medeiros Varzielas,
King, **YLZ**, 1906.02208

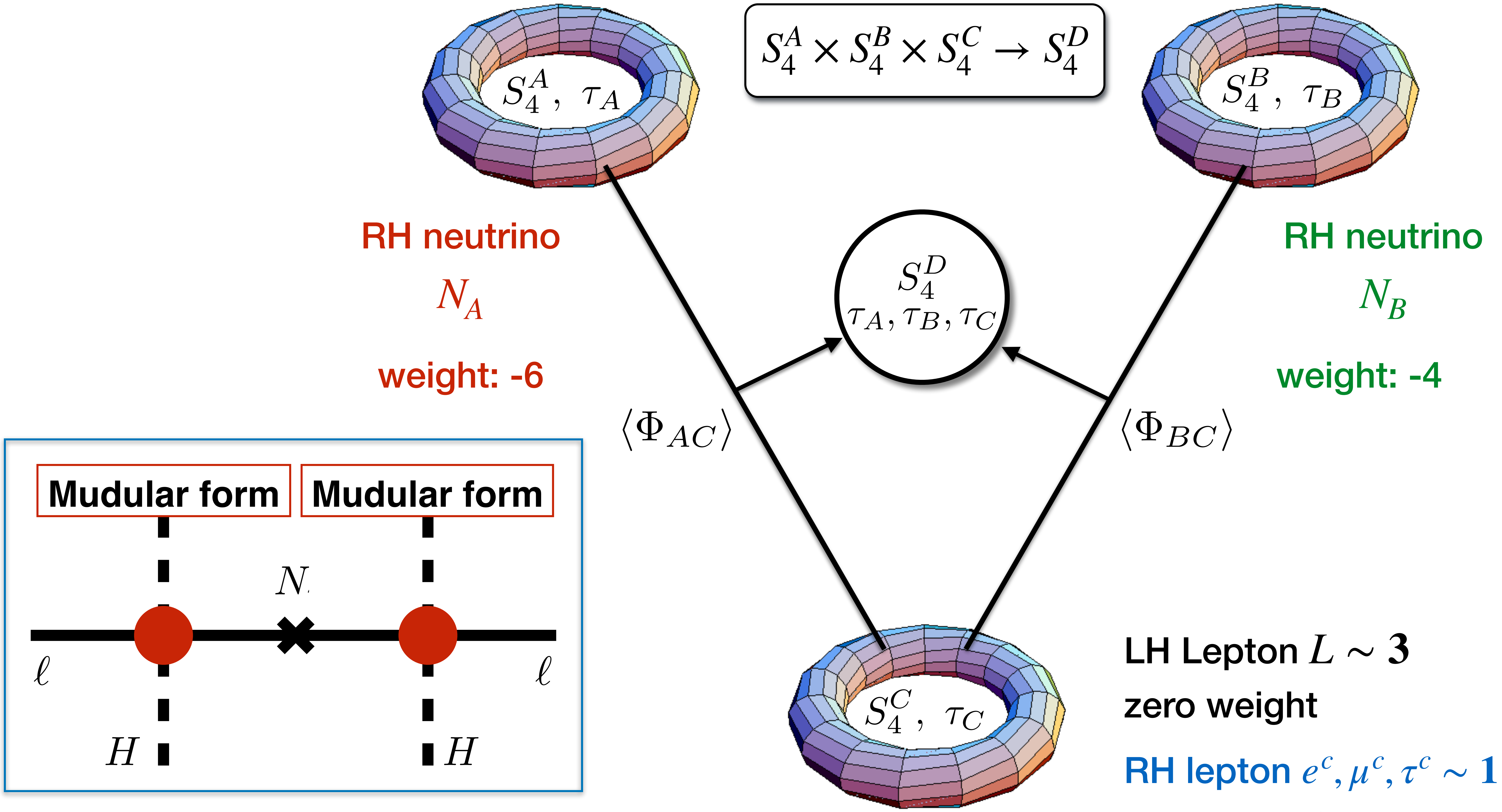
LH Lepton $L \sim \mathbf{3}$

zero weight

RH lepton $e^c, \mu^c, \tau^c \sim \mathbf{1}$

weights: $-6, -4, -2$

TM_1 in approach II



$$S_4^A \times S_4^B \times S_4^C \rightarrow S_4^D$$

RH neutrino

N_A

weight: -6

RH neutrino

N_B

weight: -4

LH Lepton $L \sim \mathbf{3}$

zero weight

RH lepton $e^c, \mu^c, \tau^c \sim \mathbf{1}$

weights: $-6, -4, -2$

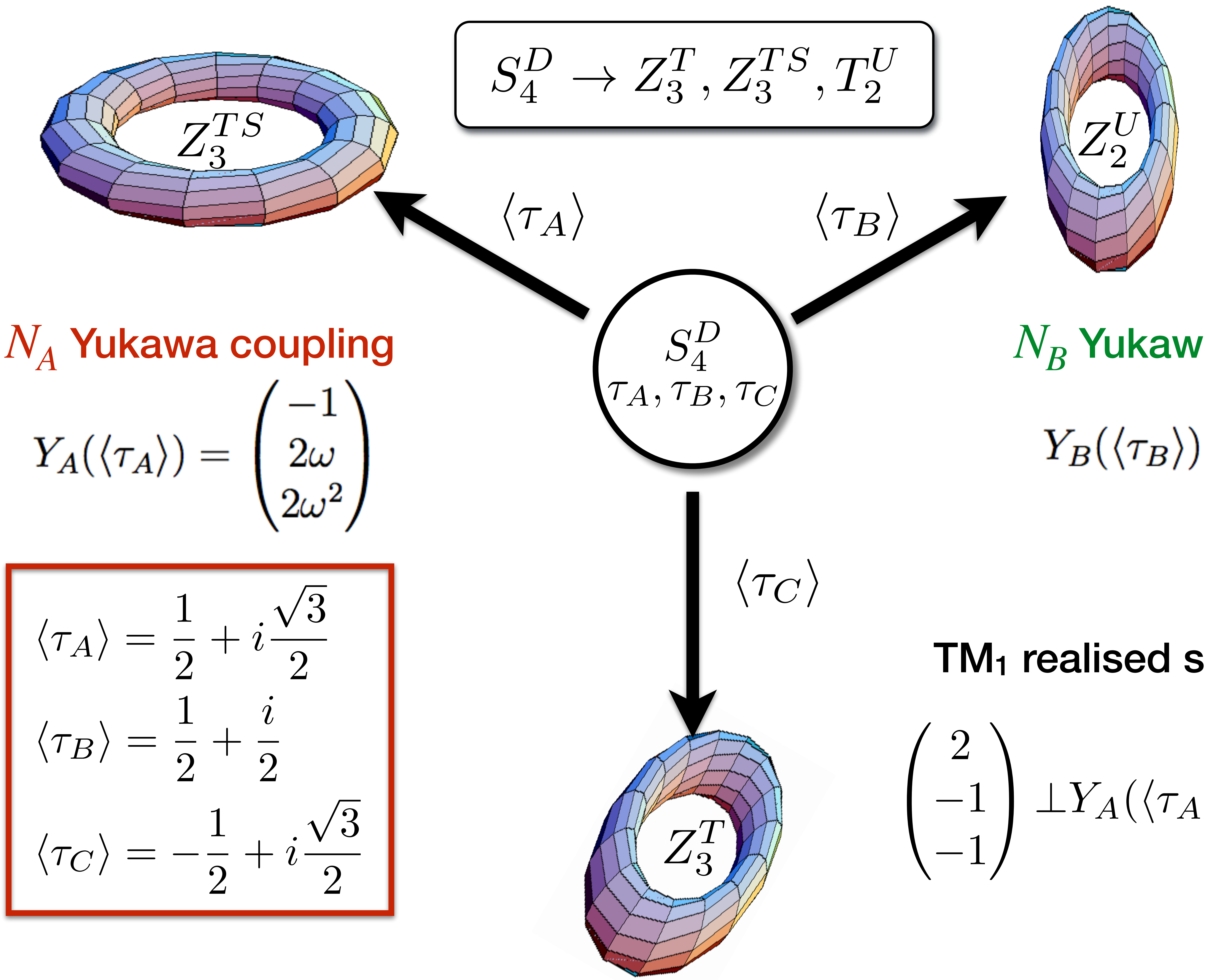
TM_1 in approach II

- Two-triplet scalars Φ_{AC} and Φ_{BC} : bridges to connect different modular symmetries
- VEVs of both Φ_{AC} and Φ_{BC} are achieved via the flat directions from the bi-triplet contraction and triplet contraction
- Superpotential before and after $S_4^A \times S_4^B \times S_4^C$ breaking

$$\begin{aligned} w_\ell \;=\; & \frac{1}{\Lambda}\left[L\Phi_{AC}Y_A(\tau_A)N_A^c + L\Phi_{BC}Y_B(\tau_B)N_B^c\right]H_u \\ & +\left[LY_e(\tau_C)e^c + LY_\mu(\tau_C)\mu^c + LY_\tau(\tau_C)\tau^c\right]H_d \\ & +\frac{1}{2}M_A(\tau_A)N_A^cN_A^c + \frac{1}{2}M_B(\tau_B)N_B^cN_B^c + M_{AB}(\tau_A,\tau_B)N_A^cN_B^c \end{aligned}$$

$$\begin{aligned} w_\ell^{\rm eff} \;=\; & \left[\frac{v_{AC}}{\Lambda}LY_A(\tau_A)N_A^c + \frac{v_{BC}}{\Lambda}LY_B(\tau_B)N_B^c\right]H_u \\ & +\left[LY_e(\tau_C)e^c + LY_\mu(\tau_C)\mu^c + LY_\tau(\tau_C)\tau^c\right]H_d \\ & +\frac{1}{2}M_A(\tau_A)N_A^cN_A^c + \frac{1}{2}M_B(\tau_B)N_B^cN_B^c + M_{AB}(\tau_A,\tau_B)N_A^cN_B^c \end{aligned}$$

TM_1 in approach II



$$\boxed{S_4^D \to Z_3^T, Z_3^{TS}, T_2^U}$$

N_A **Yukawa coupling**

$$Y_A(\langle\tau_A\rangle) = \begin{pmatrix} -1 \\ 2\omega \\ 2\omega^2 \end{pmatrix}$$

N_B **Yukawa coupling**

$$Y_B(\langle\tau_B\rangle) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \langle\tau_A\rangle &= \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ \langle\tau_B\rangle &= \frac{1}{2} + \frac{i}{2} \\ \langle\tau_C\rangle &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{aligned}}$$

TM_1 realised since

$$\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \perp Y_A(\langle\tau_A\rangle), Y_B(\langle\tau_B\rangle)$$

Diagonal charged lepton Yukawa coupling

Multiple modular symmetries in GUTs

King, YLZ, 2103.02633

- SU(5) GUT with $S_4^F \times S_4^N$

Fields	$SU(5)$	S_4^F	S_4^N	$2k_F$	$2k_N$
T_1	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$+4$	$+2$
T_2	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$+3$	$+1$
T_3	$\mathbf{10}$	$\mathbf{1'}$	$\mathbf{1}$	0	0
F	$\overline{\mathbf{5}}$	$\mathbf{3}$	$\mathbf{1}$	0	$+2$
N	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	-2
H_5	$\mathbf{5}$	$\mathbf{1}$	$\mathbf{1}$	0	0
$H_{\overline{5}}$	$\overline{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{1}$	0	0
$H_{\overline{45}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	0	0
Φ	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0
ϕ_1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	-1	-1
ϕ_2	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	-3	-1

$$Y_{\mathbf{3}}^{(2)}(\tau_{SU}) = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$Y_{\mathbf{3}}^{(4)}(\tau_{SU}) = \sqrt{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \sqrt{3}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

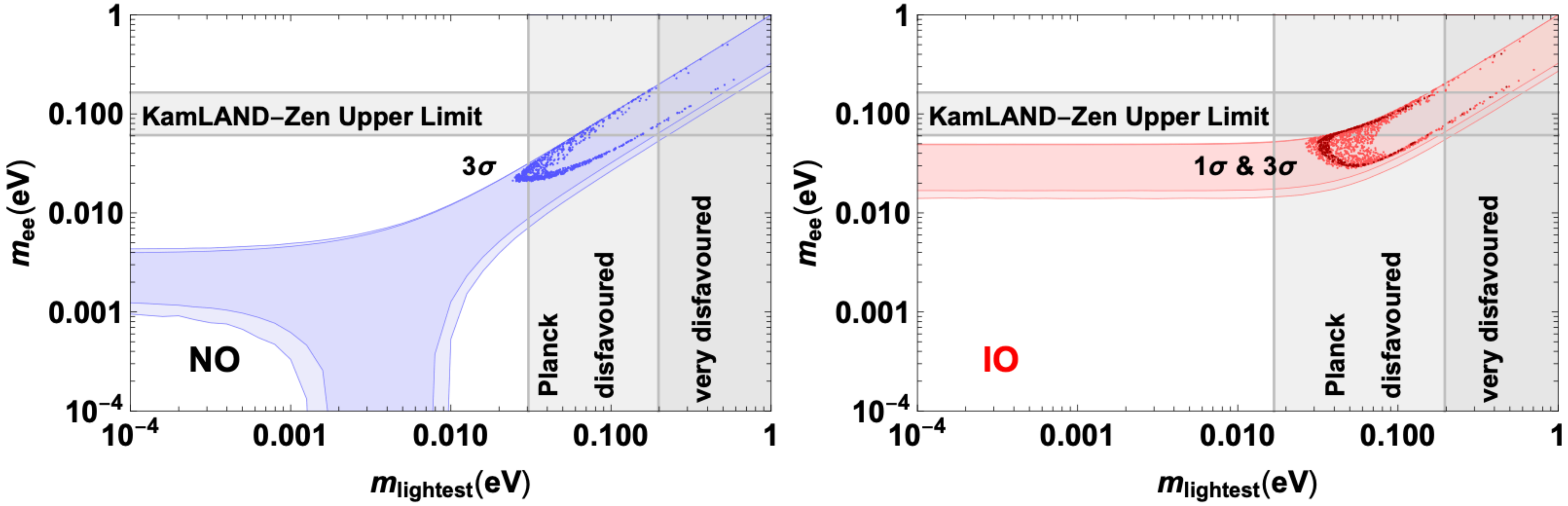
$$Y_d = \begin{pmatrix} y_{dd}\epsilon_1\epsilon_2^3 & y_{ds}\epsilon_1^2\epsilon_2^2 & y_{db}\epsilon_2^4 \\ 0 & y_{ss}\epsilon_1\epsilon_2^2 & 0 \\ 0 & 0 & y_{bb}\epsilon_1^2 \end{pmatrix}^* \qquad Y_u = \begin{pmatrix} y_{uu}\epsilon_1^2\epsilon_2^2 & y_{uc}\epsilon_1\epsilon_2^2 & y_{ut}\epsilon_1\epsilon_2 \\ y_{uc}\epsilon_1\epsilon_2^2 & y_{cc}\epsilon_2^2 & y_{ct}\epsilon_2 \\ y_{ut}\epsilon_1\epsilon_2 & y_{ct}\epsilon_2 & y_{tt} \end{pmatrix}^*$$

$$Y_e = \begin{pmatrix} y_{ee}\epsilon_1\epsilon_2^3 & 0 & 0 \\ y_{\mu e}\epsilon_1^2\epsilon_2^2 & y_{\mu\mu}\epsilon_1\epsilon_2^2 & 0 \\ y_{\tau e}\epsilon_2^4 & 0 & y_{\tau\tau}\epsilon_1^2 \end{pmatrix}^*$$

$$\epsilon_1 = \frac{\langle\phi_1\rangle}{\Lambda} \qquad \epsilon_2 = \frac{\langle\phi_2\rangle}{\Lambda}$$

Necessary for small quark mixing

$$M_R = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + c\sqrt{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - c\sqrt{3}\begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & -2 \end{pmatrix}$$



Stabilisers and residual symmetries

A stabiliser of $\gamma \in \Gamma_N$ refers to some value of τ satisfying $\gamma\tau_\gamma = \tau_\gamma$ $\Rightarrow Y(\tau_\gamma) = Y(\gamma\tau_\gamma) = (c\tau_\gamma + d)^{2k}\rho_I(\gamma)Y(\tau_\gamma)$

$$\Rightarrow \boxed{\rho_I(\gamma)\ Y(\tau_\gamma)\ = (c\tau_\gamma + d)^{-2k}\ Y(\tau_\gamma)}$$

A modular form at a stabiliser τ_γ is an eigenvector of the representation matrix $\rho_I(\gamma)$ with eigenvalue $(c\tau_\gamma + d)^{-2k}$

$\Gamma_4 \simeq S_4$

$$\boxed{S = T_\tau^2, T = S_\tau T_\tau, U = T_\tau S_\tau T_\tau^2 S_\tau}$$

Typical stabilisers (not complete)

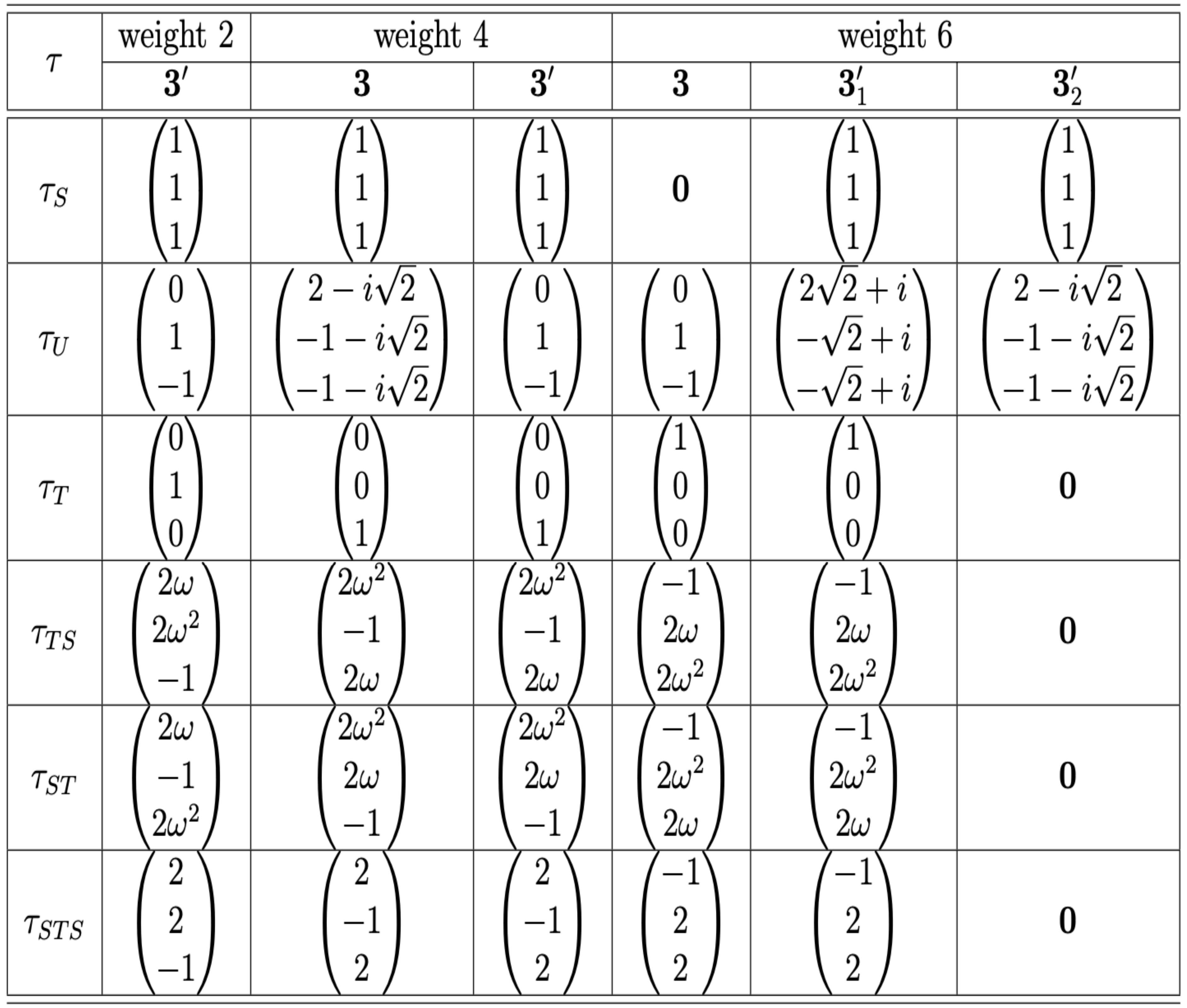
$$\tau_S = i\infty\,,\ \ \tau_T = \omega = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\,,\ \ \tau_U = \tfrac{1}{2} + \tfrac{i}{2}\,,$$
$$\tau_{TS} = -\omega^2 = \tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\,,\ \ \tau_{ST} = \tfrac{1}{2} + \tfrac{i}{2\sqrt{3}}\,,\ \ \tau_{STS} = -\tfrac{1}{2} + \tfrac{i}{2\sqrt{3}}\,.$$

$$\rho_3(S_\tau) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & 2\omega \\ 2\omega & 2 & -\omega^2 \\ 2\omega^2 & -\omega & 2 \end{pmatrix}$$

$$Y_3(\tau_{S_\tau}) \propto \begin{pmatrix}1\\1\\1\end{pmatrix},\ x\begin{pmatrix}2\\-1\\-1\end{pmatrix} + y\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\rho_3(T_\tau) = \frac{1}{3}\begin{pmatrix} -1 & 2\omega & 2\omega^2 \\ 2\omega & 2\omega^2 & -1 \\ 2\omega^2 & -1 & 2\omega \end{pmatrix}$$

$$Y_3(\tau_{T_\tau}) \propto \begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix}$$

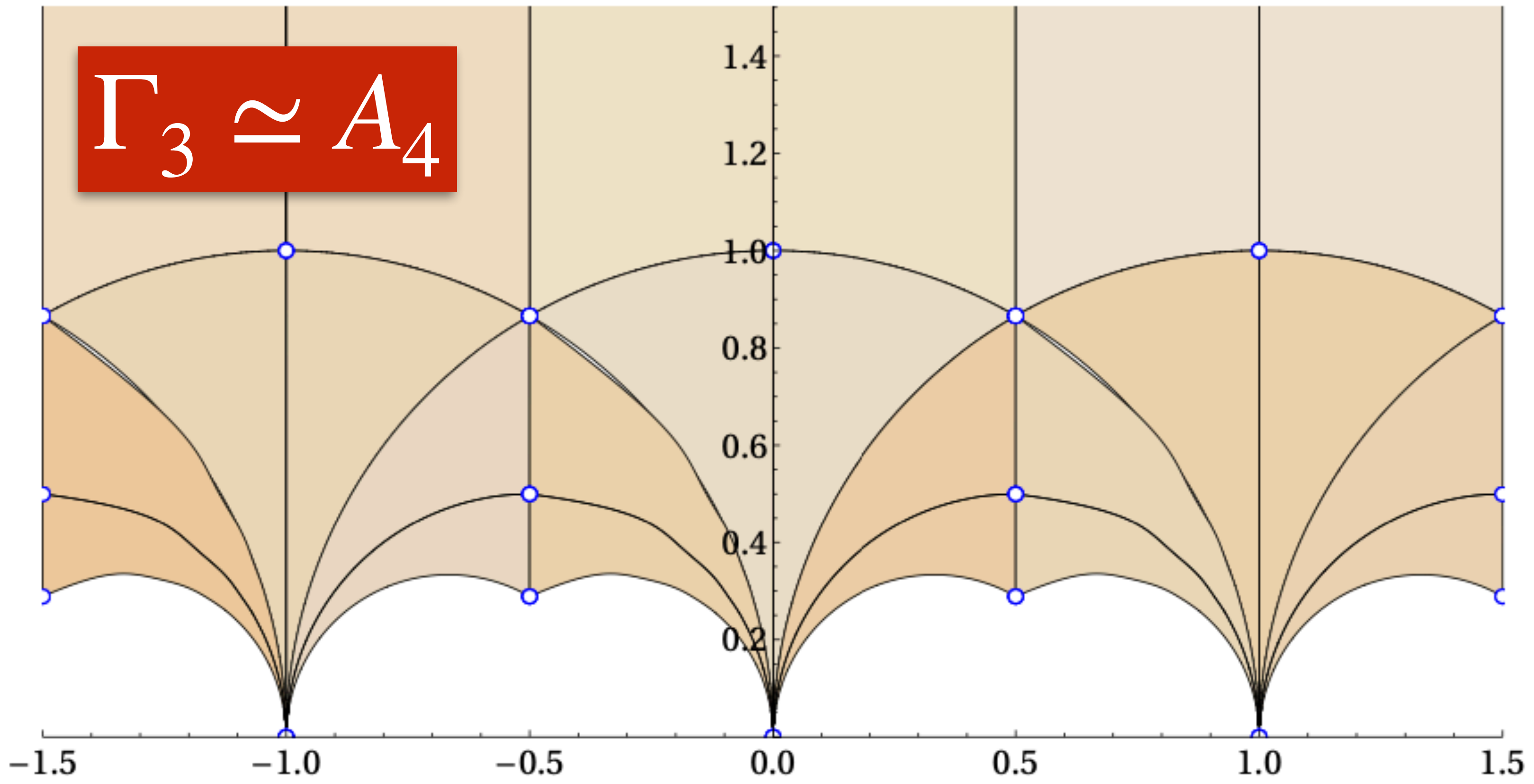


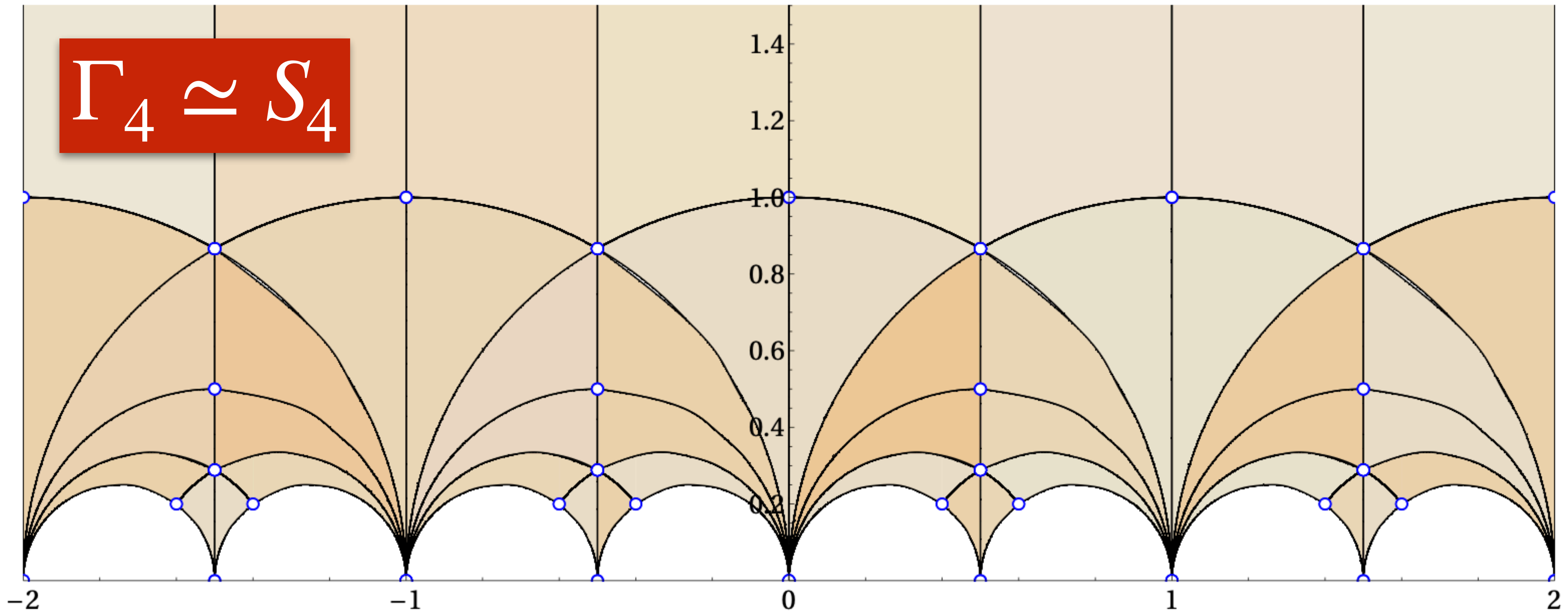
τ	weight 2	weight 4		weight 6		
	$\mathbf{3}'$	$\mathbf{3}$	$\mathbf{3}'$	$\mathbf{3}$	$\mathbf{3}'_1$	$\mathbf{3}'_2$
τ_S	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\mathbf{0}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$
τ_U	$\begin{pmatrix}0\\1\\-1\end{pmatrix}$	$\begin{pmatrix}2-i\sqrt{2}\\-1-i\sqrt{2}\\-1-i\sqrt{2}\end{pmatrix}$	$\begin{pmatrix}0\\1\\-1\end{pmatrix}$	$\begin{pmatrix}0\\1\\-1\end{pmatrix}$	$\begin{pmatrix}2\sqrt{2}+i\\-\sqrt{2}+i\\-\sqrt{2}+i\end{pmatrix}$	$\begin{pmatrix}2-i\sqrt{2}\\-1-i\sqrt{2}\\-1-i\sqrt{2}\end{pmatrix}$
τ_T	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\mathbf{0}$
τ_{TS}	$\begin{pmatrix}2\omega\\2\omega^2\\-1\end{pmatrix}$	$\begin{pmatrix}2\omega^2\\-1\\2\omega\end{pmatrix}$	$\begin{pmatrix}2\omega^2\\-1\\2\omega\end{pmatrix}$	$\begin{pmatrix}-1\\2\omega\\2\omega^2\end{pmatrix}$	$\begin{pmatrix}-1\\2\omega\\2\omega^2\end{pmatrix}$	$\mathbf{0}$
τ_{ST}	$\begin{pmatrix}2\omega\\-1\\2\omega^2\end{pmatrix}$	$\begin{pmatrix}2\omega^2\\2\omega\\-1\end{pmatrix}$	$\begin{pmatrix}2\omega^2\\2\omega\\-1\end{pmatrix}$	$\begin{pmatrix}-1\\2\omega^2\\2\omega\end{pmatrix}$	$\begin{pmatrix}-1\\2\omega^2\\2\omega\end{pmatrix}$	$\mathbf{0}$
τ_{STS}	$\begin{pmatrix}2\\2\\-1\end{pmatrix}$	$\begin{pmatrix}2\\-1\\2\end{pmatrix}$	$\begin{pmatrix}2\\-1\\2\end{pmatrix}$	$\begin{pmatrix}-1\\2\\2\end{pmatrix}$	$\begin{pmatrix}-1\\2\\2\end{pmatrix}$	$\mathbf{0}$

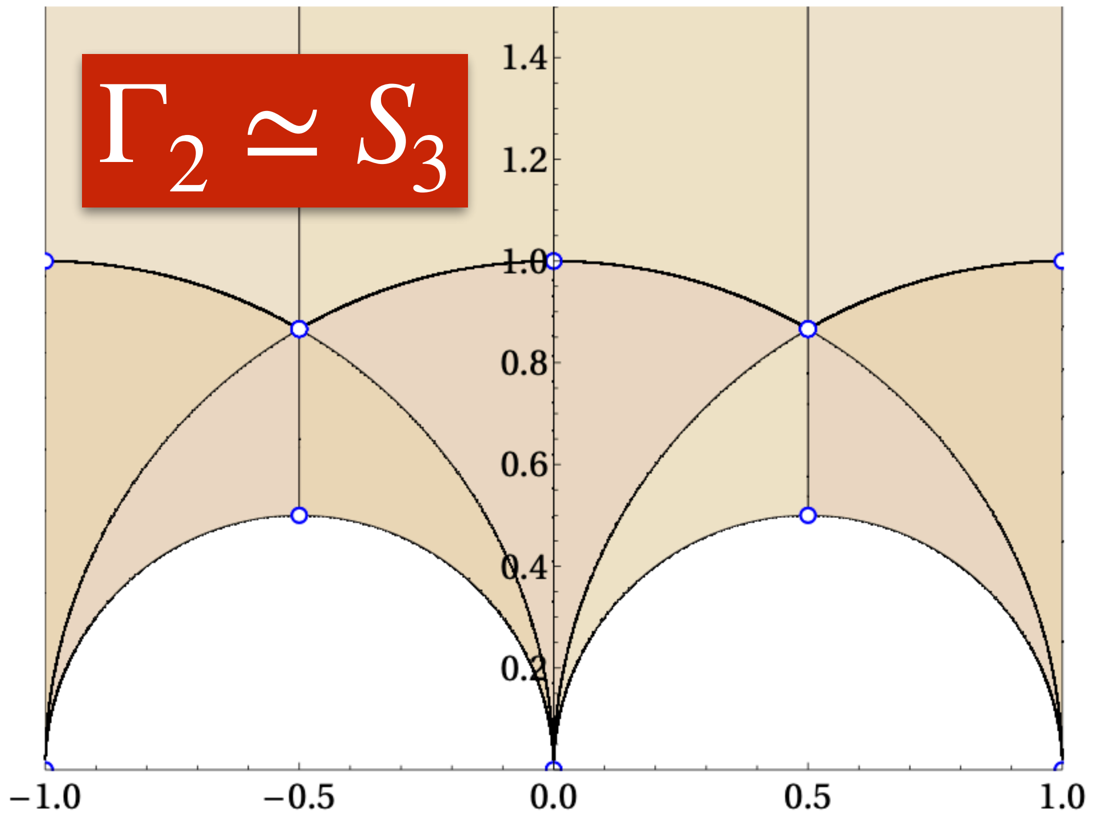
Stabilisers and residual symmetries

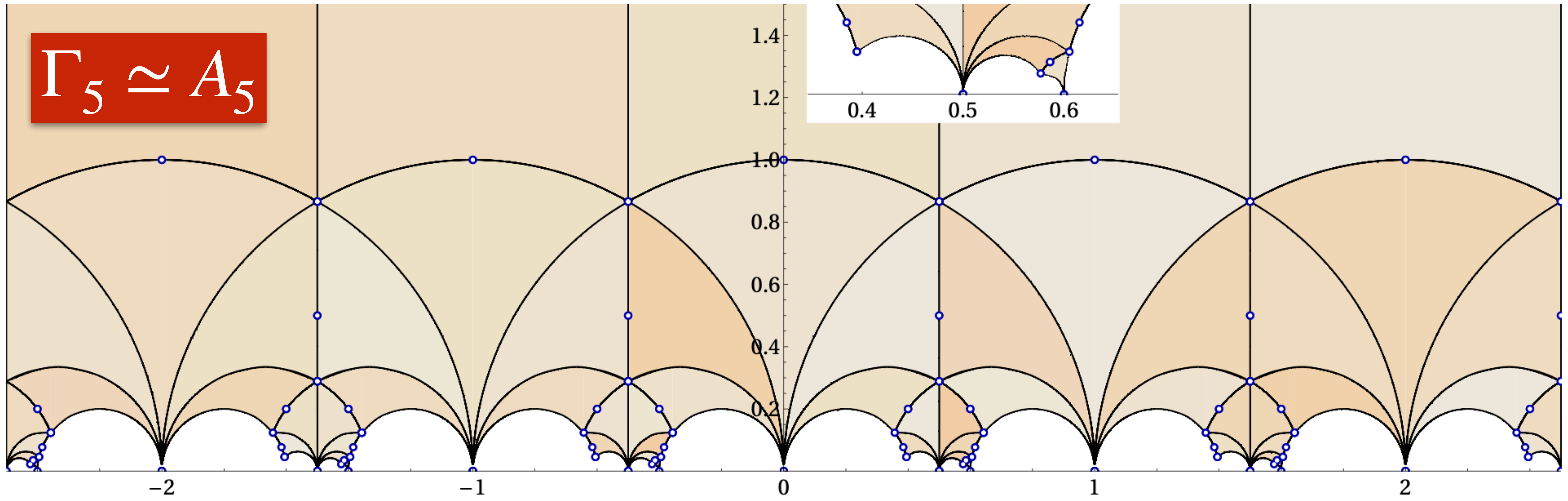
de Medeiros Varzielas, Levy, Zhou,2008.05329

Full list of stabilisers for each element of finite modular groups for Γ_N with $N = 2{,}3{,}4{,}5$ in the fundamental domain of $\bar{\Gamma}(N)$







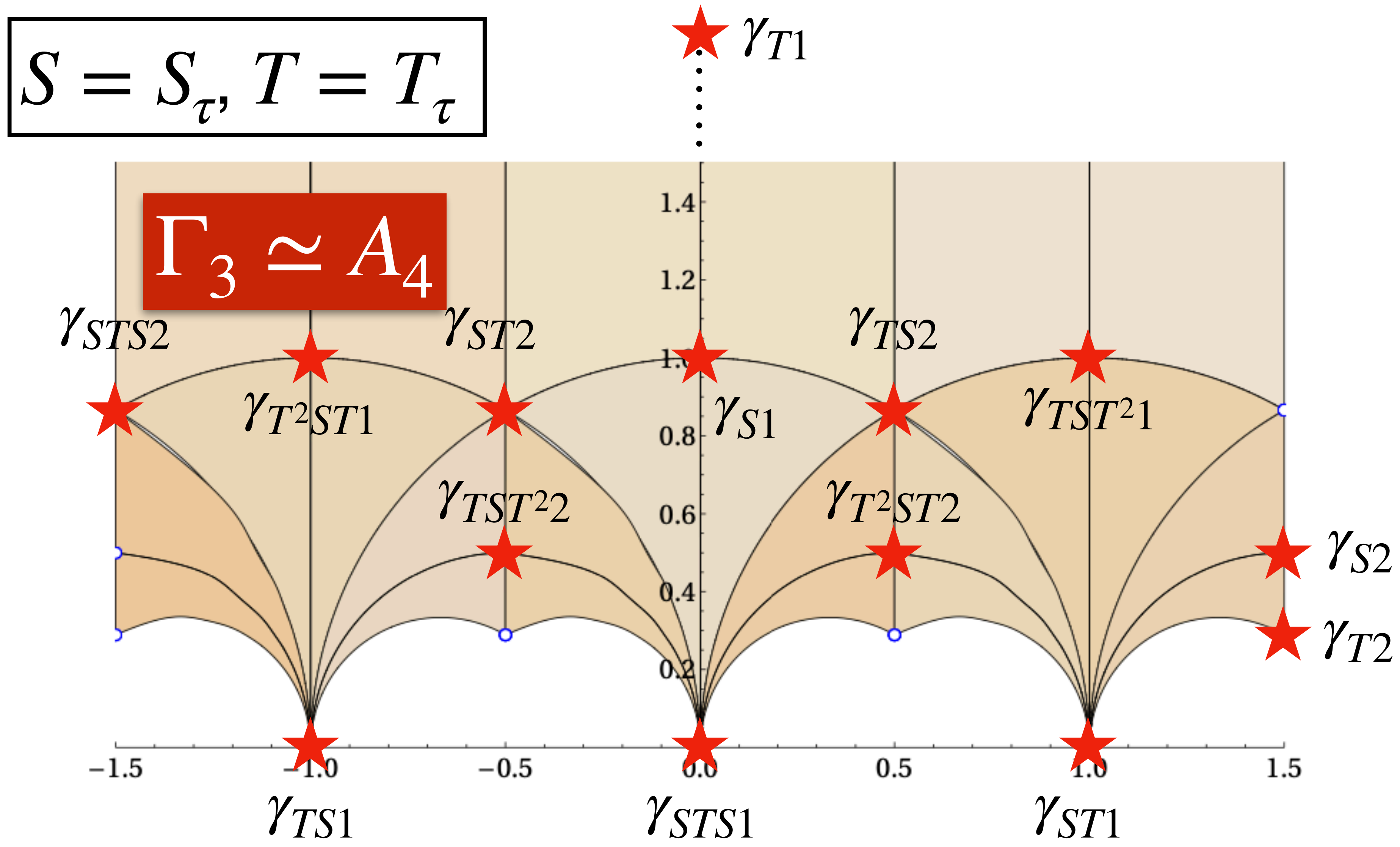


$\Gamma_N \simeq \bar{\Gamma}/\bar{\Gamma}(N)$

For A_4, S_4, see also Ding, King, Liu, Lu, 1910.03460

Scanning for all stabilisers in two modular A_4

$\boxed{S = S_\tau,\, T = T_\tau}$



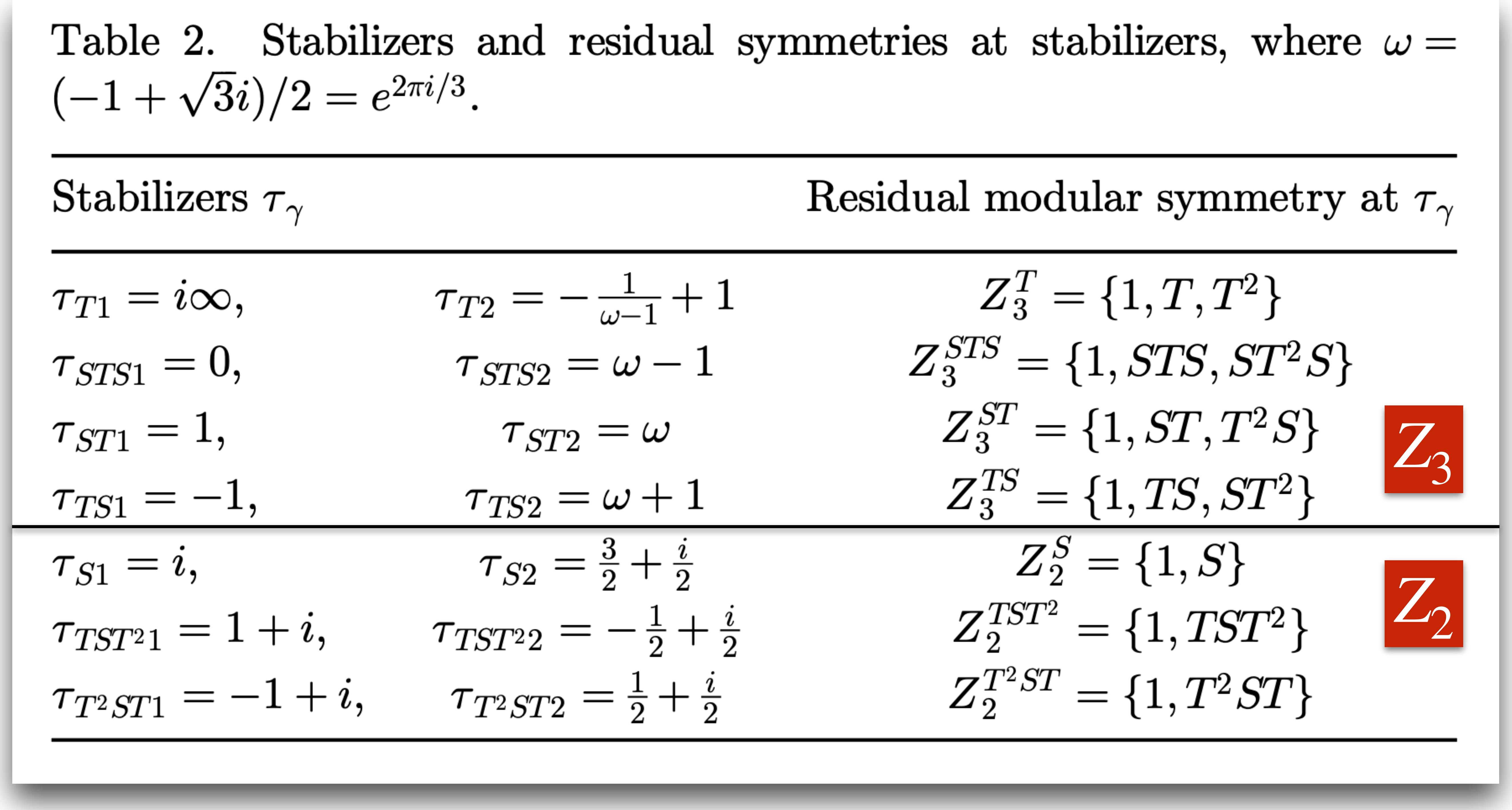


Table 2. Stabilizers and residual symmetries at stabilizers, where $\omega = (-1+\sqrt{3}i)/2 = e^{2\pi i/3}$.

Stabilizers τ_γ		Residual modular symmetry at τ_γ	
$\tau_{T1} = i\infty,$	$\tau_{T2} = -\frac{1}{\omega-1}+1$	$Z_3^T = \{1, T, T^2\}$	
$\tau_{STS1} = 0,$	$\tau_{STS2} = \omega - 1$	$Z_3^{STS} = \{1, STS, ST^2S\}$	
$\tau_{ST1} = 1,$	$\tau_{ST2} = \omega$	$Z_3^{ST} = \{1, ST, T^2S\}$	Z_3
$\tau_{TS1} = -1,$	$\tau_{TS2} = \omega + 1$	$Z_3^{TS} = \{1, TS, ST^2\}$	
$\tau_{S1} = i,$	$\tau_{S2} = \frac{3}{2}+\frac{i}{2}$	$Z_2^S = \{1, S\}$	
$\tau_{TST^21} = 1+i,$	$\tau_{TST^22} = -\frac{1}{2}+\frac{i}{2}$	$Z_2^{TST^2} = \{1, TST^2\}$	Z_2
$\tau_{T^2ST1} = -1+i,$	$\tau_{T^2ST2} = \frac{1}{2}+\frac{i}{2}$	$Z_2^{T^2ST} = \{1, T^2ST\}$	

$$\begin{aligned}\tau_{T2} &= (ST^2)^3\tau_{T2} = (ST^2)^3\left(\frac{-1}{\omega-1}+1\right) = (ST^2)^2(\omega-1)\\ &= (ST^2)\left(\frac{-1}{\omega+1}\right) = \frac{-1}{\frac{-1}{\omega+1}+2} = \frac{-1}{\omega-1} - 1\end{aligned}$$

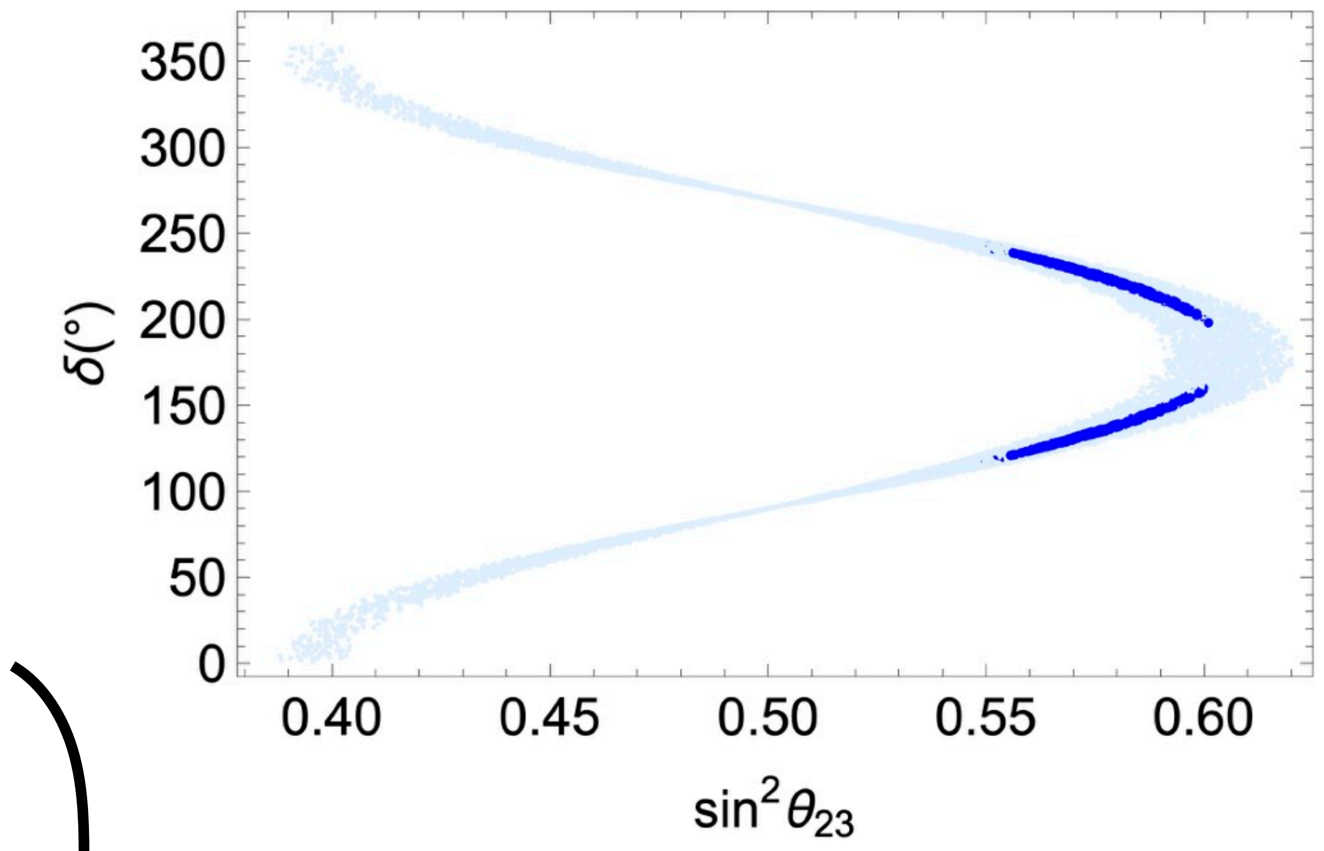
$$\text{Case A :}\quad (2k_l = 2{,}4)\quad \begin{cases} \tau_l = \tau_{T1}, & Y = (1,0,0)^{\mathrm{T}}, \quad M_l = M_l^T, \\ \tau_l = \tau_{STS1}, & Y \propto (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})^{\mathrm{T}}, \quad M_l = SM_l^T, \\ \tau_l = \tau_{ST1}, & Y \propto (-\frac{1}{3}, \frac{2}{3}\omega, \frac{2}{3}\omega^2)^{\mathrm{T}}, \quad M_l = TST^2M_l^T, \\ \tau_l = \tau_{TS1}, & Y \propto (-\frac{1}{3}, \frac{2}{3}\omega^2, \frac{2}{3}\omega)^{\mathrm{T}}, \quad M_l = T^2STM_l^T; \end{cases}$$

$$\text{Case B :}\quad (2k_l = 2)\quad \begin{cases} \tau_l = \tau_{T2}, & Y \propto (0,0,1)^{\mathrm{T}}, \quad M_l = P^2M_l^T, \\ \tau_l = \tau_{STS2}, & Y \propto (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})^{\mathrm{T}}, \quad M_l = SP^2M_l^T, \\ \tau_l = \tau_{ST2}, & Y \propto (\frac{2}{3}, \frac{2}{3}\omega, -\frac{1}{3}\omega^2)^{\mathrm{T}}, \quad M_l = TST^2P^2M_l^T, \\ \tau_l = \tau_{TS2}, & Y \propto (\frac{2}{3}, \frac{2}{3}\omega^2, -\frac{1}{3}\omega)^{\mathrm{T}}, \quad M_l = T^2STP^2M_l^T. \end{cases}$$

$$\text{Case C :}\quad (2k_l = 4)\quad \begin{cases} \tau_l = \tau_{T2}, & Z \propto (0,1,0)^{\mathrm{T}}, \quad M_l = PM_l^T, \\ \tau_l = \tau_{STS2}, & Z \propto (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})^{\mathrm{T}}, \quad M_l = SPM_l^T, \\ \tau_l = \tau_{ST2}, & Z \propto (\frac{2}{3}, -\frac{1}{3}\omega, \frac{2}{3}\omega^2)^{\mathrm{T}}, \quad M_l = TST^2PM_l^T, \\ \tau_l = \tau_{TS2}, & Z \propto (\frac{2}{3}, -\frac{1}{3}\omega^2, \frac{2}{3}\omega)^{\mathrm{T}}, \quad M_l = T^2STPM_l^T. \end{cases}$$

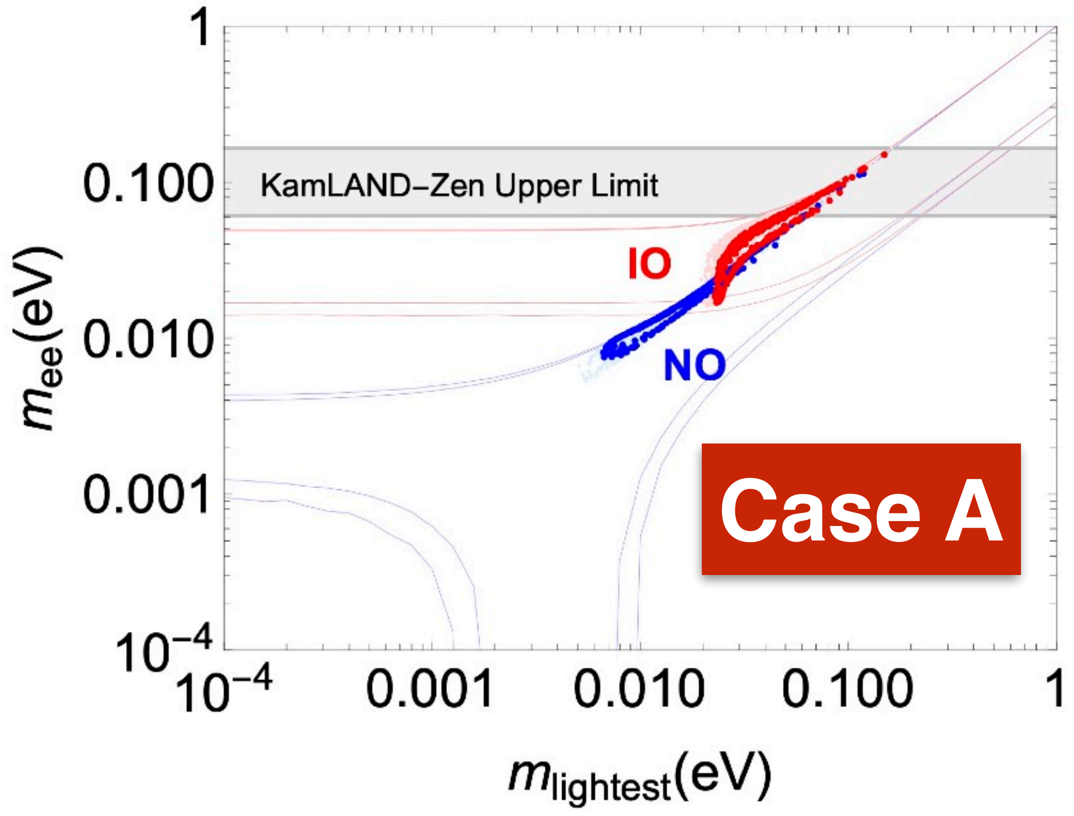
Scanning for all stabilisers in two modular A_4

$$(2k_\nu = 4) \quad \begin{cases} \tau_\nu = \tau_{S1}, \tau_{S2}, & M_R = M_R^S, \quad M_\nu = M_\nu^S, \\ \tau_\nu = \tau_{TST^21}, \tau_{TST^22}, & M_R = TM_R^ST, \quad M_\nu = TM_\nu^ST, \\ \tau_\nu = \tau_{T^2ST1}, \tau_{T^2ST2}, & M_R = T^2M_R^ST^2, \quad M_\nu = T^2M_\nu^ST^2. \end{cases}$$



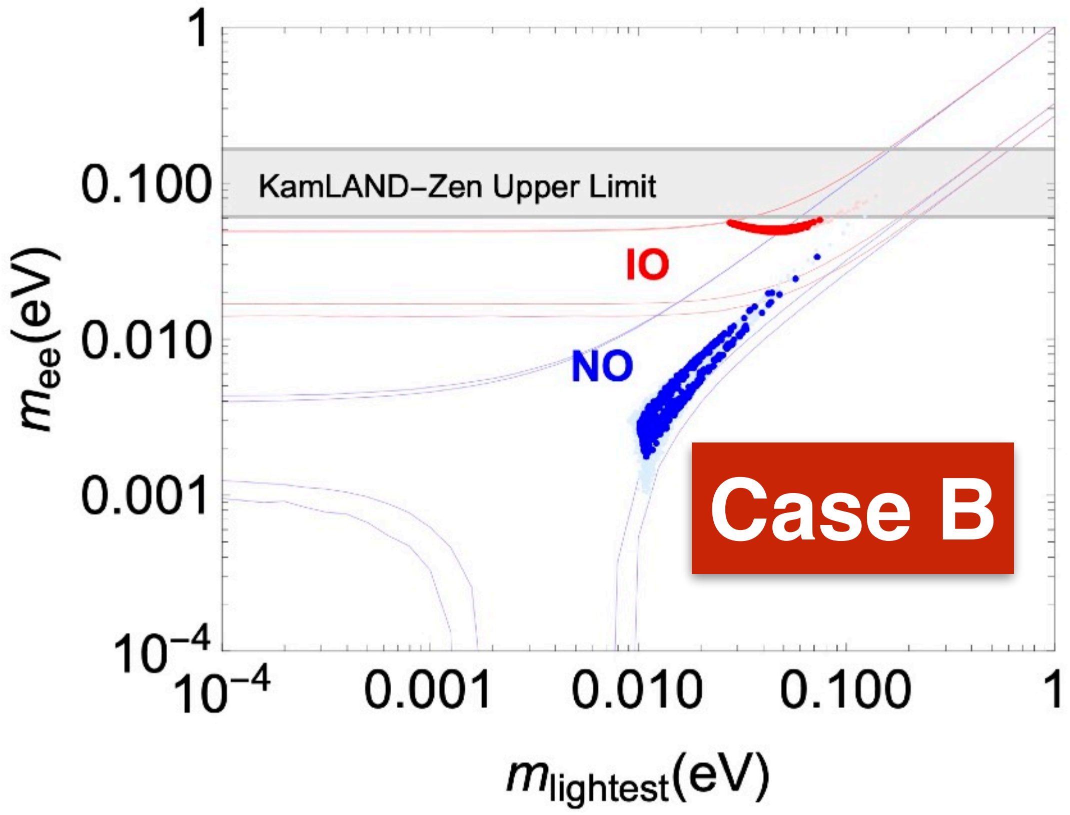
In flavour basis,

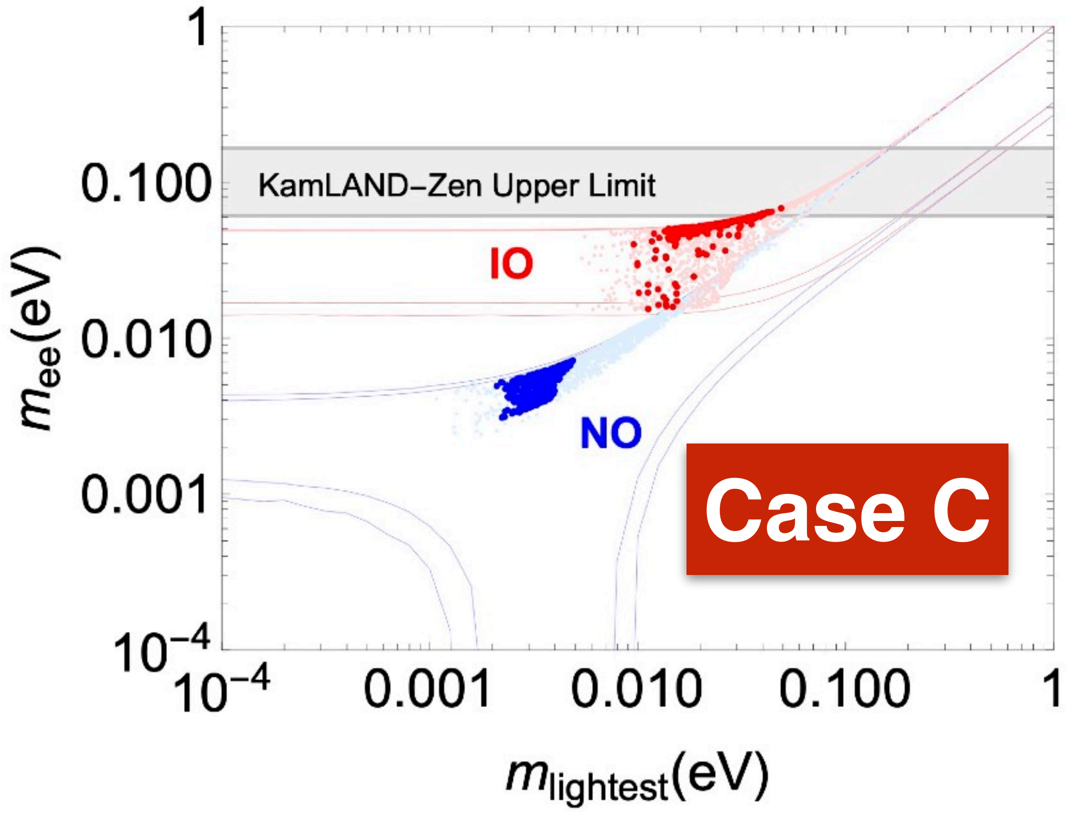
$$\left.\begin{aligned}
\widetilde{M}_\nu^A &= m_0 T^m \left[\pm \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \frac{3-g^2}{g+g'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{3-g'^2}{g+g'} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{gg'+3}{g+g'} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right] T^m, \\
\widetilde{M}_\nu^B &= m_0 T^m \left[\pm \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \frac{3-g^2}{g+g'} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{3-g'^2}{g+g'} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{gg'+3}{g+g'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right] T^m, \\
\widetilde{M}_\nu^C &= m_0 T^m \left[\pm \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \frac{3-g^2}{g+g'} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{3-g'^2}{g+g'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{gg'+3}{g+g'} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] T^m.
\end{aligned}\right\} \mathrm{TM}_2$$



de Medeiros Varzielas, Lourenco, 2107.04042

Zhang, YLZ, 2401.17810





Summary

- ☑ Framework of multiple modular symmetries as origin of lepton mixing
- ☑ Trimaximal TM_1 mixing realised in two S_4 in Approach I
  … and applied to SU(5) GUT
- ☑ Trimaximal TM_1 mixing realised in three S_4 in Approach II
- ☑ Trimaximal TM_2 mixing realised in two A_4 in Approach I
- ☑ More mixing patterns are expected in the framework of multiple modular symmetries

Thank you for your listening!