Beyond $\mathrm{SL}(2, \mathbb{Z})$ in the bottom-up approach

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May 2024

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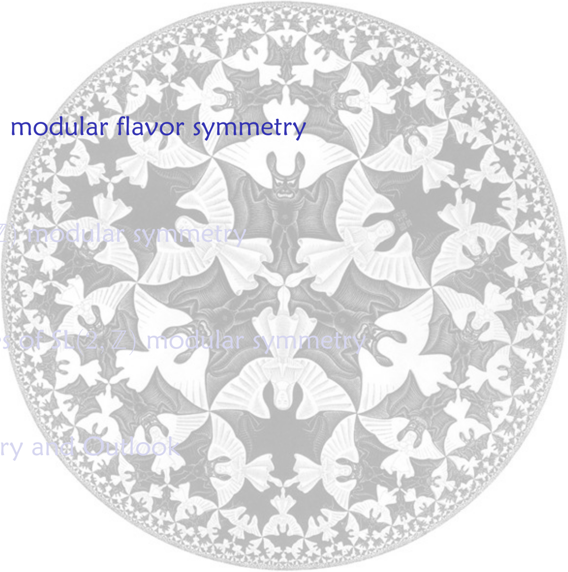
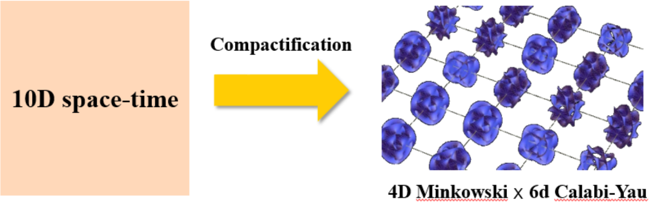
Outline

Outline



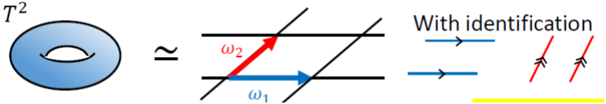
1. $\mathrm{SL}(2,\mathbb{Z})$ modular flavor symmetry
2. $\mathrm{Sp}(2g,\mathbb{Z})$ modular symmetry
3. Varieties of $\mathrm{SL}(2,\mathbb{Z})$ modular symmetry
4. Summary and Outlook

In string theory, extra dimension compactification leads to (target space) **Modular symmetery** (See the talk by Ramos-Sanchez, Otsuka, Leontaris and Nasu)



$$S=\int d^4x d^6y\, \mathcal{L}_{10D} \Rightarrow \int d^4x\, \mathcal{L}_{eff}(\varphi, \tau_i)$$

Example: Torus compactification ($6D \to 4D$):



Modular symmetry I

- The shape of a torus is characterized by a modulus

$$\tau = \omega_1/\omega_2, \quad \text{Im}(\tau) > 0$$

which is in the complex upper half plane $\mathcal{H} = \langle \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \rangle$.

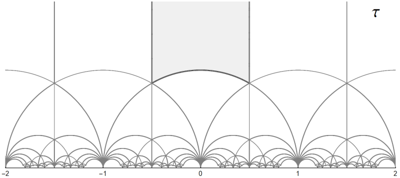
- The lattice (torus) is left invariant by modular transformations

$$\tau \mapsto \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$

$\Longrightarrow$ **(Target space) Modular symmetry!**

- The inequivalent moduli vacua: fundamental domain $\mathcal{F} = \mathcal{H}/\text{SL}(2, \mathbb{Z})$



Modular symmetry II

- The matter fields φ_I undergoes a non-linear transformation under modular group: (Ferrara et al. 1989; Lauer, Mas, and Nilles 1989; Feruglio 2017)

$$\varphi_I \mapsto (c\tau + d)^{-k_I} \rho_I(\gamma) \varphi_I$$

with

- ◇ k_I: modular weight of matter fields φ_I
- ◇ ρ_I: unitary irreps of SL$(2, \mathbb{Z})$

Remarks:

- ρ with finite image: SL$(2, \mathbb{Z})/$ ker$(\rho) \cong$ Im(ρ) is a finite (modular) group
- The finite modular group are SL$(2, \mathbb{Z})/ \pm \Gamma(N) \equiv \Gamma_N$ in Ref. (Feruglio 2017)
- Γ_N has extended to its double cover SL$(2, \mathbb{Z})/\Gamma(N) \equiv \Gamma'_N$ (Liu and Ding 2019)
- Γ_N has extended to the most general cases SL$(2, \mathbb{Z})/$Nor (Liu and Ding 2022)
- Modular weights are arbitrary
- (The modular weight or) The modular transformation lacks an intuitive physical or geometric picture (See the talk by Victor)

Modular symmetry III

The list of finite modular groups with order < 78:

Normal subgroups N			Finite modular groups Γ/N	
Index	Label	Additional relators	Group structure	GAP Id
6	$\Gamma(2) \equiv N_{[6,1]}$	T^2	S_3	$[6,1]$
12	$N_{[12,1]}$	S^2T^2	$Z_3 \rtimes Z_4 \cong 2D_3$	$[12,1]$
	$\pm\Gamma(3) \equiv N_{[12,3]}$	S^2, T^3	A_4	$[12,3]$
18	$N_{[18,3]}$	$ST^{-2}ST^2$	$S_3 \times Z_3$	$[18,3]$
24	$\Gamma(3) \equiv N_{[24,3]I}$	T^3	T'	$[24,3]$
	$N_{[24,3]II}$	S^2T^3		
	$\pm\Gamma(4) \equiv N_{[24,12]}$	S^2, T^4	S_4	$[24,12]$
	$N_{[24,13]}$	$S^2, (ST^{-1}ST)^2$	$A_4 \times Z_2$	$[24,13]$
36	$N_{[36,6]}$	$S^3T^{-2}ST^2$	$(Z_3 \rtimes Z_4) \times Z_3$	$[36,6]$
42	$N_{[42,1]I}$	$T^6, (ST^{-1}S)^2TST^{-1}ST^2$	$Z_7 \rtimes Z_6$	$[42,1]$
	$N_{[42,1]II}$	$T^6, ST^{-1}ST(ST^{-1}S)^2T^2$		
48	$N_{[48,28]}$	S^2T^4	$2O$	$[48,28]$
	$N_{[48,29]}$	T^8, ST^4ST^{-4}	$GL(2,3)$	$[48,29]$
	$\Gamma(4) \equiv N_{[48,30]}$	T^4	$A_4 \rtimes Z_4 \cong S'_4$	$[48,30]$
	$N_{[48,31]}$	$(ST^{-1}ST)^2$	$A_4 \times Z_4$	$[48,31]$
	$N_{[48,32]}$	$S^2(ST^{-1}ST)^2$	$T' \times Z_2$	$[48,32]$
	$N_{[48,33]}$	T^{12}, ST^3ST^{-3}	$((Z_4 \times Z_2) \rtimes Z_2) \rtimes Z_3$	$[48,33]$
54	$N_{[54,5]}$	$T^6, (ST^{-1}ST)^3$	$(Z_3 \times Z_3) \rtimes Z_6$	$[54,5]$
60	$\pm\Gamma(5) \equiv N_{[60,5]}$	S^2, T^5	A_5	$[60,5]$
72	$N_{[72,42]}$	T^{12}, ST^4ST^{-4}	$S_4 \times Z_3$	$[72,42]$
	$\pm\Gamma(6) \equiv N_{[72,44]}$	$S^2, T^6, (ST^{-1}STST^{-1}S)^2T^2$	$A_4 \times S_3$	$[72,44]$
...	...	...	...	..

Modular symmetry IV

We work in the $\mathcal{N}=1$ global (or local) SUSY theory

- (Minimal) Kähler potential

$$\mathcal{K}(\varphi_I,\tau) = -k_{\mathcal{W}}\Lambda^2 \log(-i\tau+i\bar{\tau}) + \sum_I (-i\tau+i\bar{\tau})^{-k_I}|\varphi_I|^2$$

- Superpotential

$$\mathcal{W}(\varphi_I,\tau) = Y_{IJK}(\tau)\,\varphi_I\varphi_J\varphi_K + \ldots$$

$$\begin{cases} \text{modular invariant in global SUSY:} \quad \mathcal{W} \xrightarrow{\gamma} \mathcal{W} \\ \text{modular covariant in local SUSY:} \quad \mathcal{W} \xrightarrow{\gamma} (c\tau+d)^{-k_{\mathcal{W}}}\rho_{\mathcal{W}}(\gamma)\mathcal{W} \end{cases}$$

$\Longrightarrow$ Yukawa couplings $Y_{IJK}(\tau)$ are **(vector-valued) modular forms**:

$$Y_{IJK}(\tau) \xrightarrow{\gamma} Y_{IJK}(\gamma\tau) = (c\tau+d)^{k_Y}\rho_Y(\gamma)Y_{IJK}(\tau)$$

with $\begin{cases} k_Y = k_I + k_J + k_K\,, \quad \rho_Y\otimes\rho_I\otimes\rho_J\otimes\rho_K \ni \mathbf{1} \quad \text{in global SUSY} \\ k_Y = -k_{\mathcal{W}} + k_I + k_J + k_K\,, \quad \rho_Y\otimes\rho_I\otimes\rho_J\otimes\rho_K \ni \rho_{\mathcal{W}} \quad \text{in local SUSY} \end{cases}$

- Freedom of model building: $\varphi_I,\ k_I,\ \rho_I$
- Three conditions on superpotential:
  - Modular invariance
  - Meromorphy
  - Finiteness
- For the given k_Y and ρ_Y, the modular forms space is finite-dimensional: Only a finite number of possible Yukawa couplings !

$$\dim \mathcal{M}_k(\rho) = \frac{(5+k)\dim\rho}{12} + \frac{i^k\,\mathrm{Tr}\left(\rho\left(S^3\right)\right)}{4} + \frac{(1+\omega)^k\,\mathrm{Tr}\left(\rho\left(S^3T\right)\right)}{3(1-\omega)} + \frac{\omega^k\,\mathrm{Tr}\left(\rho\left((ST)^2\right)\right)}{3\left(1-\omega^2\right)} - \frac{1}{2\pi i}\mathrm{Tr}(\log\rho(T))$$

- All higher-dimensional operators in τ are completely determined
- No additional flavons other than a modulus τ
- Modulus VEV $\langle\tau\rangle$ is treated as a free parameter

The soul is the (vector-valued) modular form !

Vector-valued modular forms

Lucky! The theory of vector-valued modular forms for SL$(2,\mathbb{Z})$ has been established by mathematicians in the last twenty years (Gannon 2014; Franc and Mason 2016; Franc and Mason 2018)

- Vector-valued modular forms (VVMFs) are defined by
  - Modularity: $Y(\gamma\tau) = (c\tau+d)^k \rho(\gamma) Y(\tau)$
  - Finiteness: Y_i is holomorphic at infinity.
- Vector-valued modular forms $\Longleftrightarrow$ modular form multiplets
  - The scalar modular forms on ker(ρ) can be organized into VVMFs in rep ρ.
  - Each component of VVMF $Y(\tau)$ in the rep ρ is scalar modular form on ker(ρ)
- VVMFs always have q-expansion:

$$Y(\tau) = \begin{pmatrix} Y_1(\tau) \\ \vdots \\ Y_d(\tau) \end{pmatrix} = \begin{pmatrix} q^{r_1}\sum_{n\geq 0} a_1(n) q^n \\ \vdots \\ q^{r_d}\sum_{n\geq 0} a_d(n) q^n \end{pmatrix}, \quad q = e^{2\pi i \tau}$$

for $\rho(T) = \mathtt{diag}(e^{2\pi i r_1}, \dots, e^{2\pi i r_d}),\ 0 \leq r_i < 1.$

How to construct VVMFs systematically? —— structure theorem

The structure of VVMFs I

- Denoting by $\mathcal{M}_k(\rho)$ the linear space of holomorphic VVMFs

Free-module theorem

The direct sums $\mathcal{M}(\rho) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\rho)$ is a **free module** over the ring $\mathcal{M}(\mathbf{1}) = \mathbb{C}[E_4, E_6]$ whose **rank** $= \dim \rho$ (Marks and Mason 2010)

Here E_4 and E_6 are the Eisenstein series

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \,, \quad E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \,,$$

with $\sigma_k(n) = \sum_{d|n} d^k$.

- Introducing modular differential operators $D_k \equiv \frac{1}{2\pi i} \frac{d}{d\tau} - \frac{kE_2}{12}$ where $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$.
- D_k preserves modularity of modular forms and increase its weight by 2, i.e. $D_k Y(\gamma\tau) = (c\tau + d)^{k+2} \rho(\gamma) D_k Y(\tau)$ for $Y \in \mathcal{M}_k(\rho)$
  The nth iteration of D_k is denoted by $D_k^n \equiv D_{k+2(n-1)} \circ \ldots \circ D_{k+2} \circ D_k$
- $\{F, D_k F, \ldots, D_k^{d-1} F\}$ are linearly independent over $\mathcal{M}(\mathbf{1})$

The structure of VVMFs II

- For dim $\rho \le 3$, $\{F, D_{k_0}F, \ldots, D_{k_0}^{d-1}F\}$ forms a basis for module $\mathcal{M}(\rho)$ where F is the VVMF of lowest weight k_0
- Higher weight VVMFs can be written as the linear combination of the generators $F, D_{k_0}F, \ldots, D_{k_0}^{d-1}F$ over ring $\mathbb{C}[E_4, E_6]$
  e.g.
  - dim $\rho = 1$: $D_{k_0}F = 0$
  - dim $\rho = 2$: $(D_{k_0}^2 + aE_4)F = 0$
  - dim $\rho = 3$: $(D_{k_0}^3 + aE_4 D_{k_0} + bE_6)F = 0$

  $\Rightarrow$ **Linear differential equation satisfied by VVMFs!**

  The solutions are
- dim $\rho = 1$: $F(\tau) = \eta^{24r_1}(\tau)$.
- dim $\rho = 2$: $F(\tau) = \begin{pmatrix} \eta^{12(r_1+r_2)-2} K^{\frac{6(r_1-r_2)+1}{12}} {}_2F_1(\frac{6(r_1-r_2)+1}{12}, \frac{6(r_1-r_2)+5}{12}; r_1-r_2+1; K) \\ \eta^{12(r_1+r_2)-2} K^{\frac{6(r_2-r_1)+1}{12}} {}_2F_1(\frac{6(r_2-r_1)+1}{12}, \frac{6(r_2-r_1)+5}{12}; r_2-r_1+1; K) \end{pmatrix}$

The structure of VVMFs III

- dim $\rho = 3$: $F(\tau) = \begin{pmatrix} \eta^{8(r_1+r_2+r_3)-4} K^{\frac{a_1+1}{6}} \ {}_3F_2(\frac{a_1+1}{6}, \frac{a_1+3}{6}, \frac{a_1+5}{6}; r_1 - r_2 + 1, r_1 - r_3 + 1; K) \\ \eta^{8(r_1+r_2+r_3)-4} K^{\frac{a_2+1}{6}} \ {}_3F_2(\frac{a_2+1}{6}, \frac{a_2+3}{6}, \frac{a_2+5}{6}; r_2 - r_3 + 1, r_2 - r_1 + 1; K) \\ \eta^{8(r_1+r_2+r_3)-4} K^{\frac{a_3+1}{6}} \ {}_3F_2(\frac{a_3+1}{6}, \frac{a_3+3}{6}, \frac{a_3+5}{6}; r_3 - r_1 + 1, r_3 - r_2 + 1; K) \end{pmatrix}$

  with $a_1 = 4r_1 - 2r_2 - 2r_3,\ a_2 = 4r_2 - 2r_1 - 2r_3,\ a_3 = 4r_3 - 2r_1 - 2r_2$

  Where

  - r_i come from $\rho(T) = \texttt{diag}(e^{2\pi i r_1}, \ldots, e^{2\pi i r_d})$
  - K is the inverse of the famous j-invariant: $K(\tau) = 1/j(\tau)$

    $$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \ldots \quad \text{with} \quad q = e^{2\pi i\tau}$$

  - nF_{n-1} is the generalized hypergeometric series

    $${}_nF_{n-1}(a_1, \ldots, a_n; b_1, \ldots, b_{n-1}; z) = \sum_{m\geq 0}^{\infty} \frac{\prod_{j=1}^{n} a_j(a_j+1)\ldots(a_j+m-1)}{\prod_{k=1}^{n-1} b_k(b_k+1)\ldots(b_k+m-1)} \frac{z^m}{m!}$$

  - The minimal weight k_0 can be read out from the power of $\eta(\tau)$

In summary, the module structure of $\mathcal{M}(\rho)$ for $\dim\rho \leq 3$:

$$\dim\rho = 1 : \mathcal{M}(\rho) = \mathbb{C}[E_4, E_6]F$$
$$\dim\rho = 2 : \mathcal{M}(\rho) = \mathbb{C}[E_4, E_6]F \oplus \mathbb{C}[E_4, E_6]D_{k_0}F$$
$$\dim\rho = 3 : \mathcal{M}(\rho) = \mathbb{C}[E_4, E_6]F \oplus \mathbb{C}[E_4, E_6]D_{k_0}F \oplus \mathbb{C}[E_4, E_6]D_{k_0}^3 F$$

We give an example based on a $A_4 \times Z_2$ instead of $\Gamma_N^{(\prime)}$:

- $A_4 \times Z_2 = \langle S, T \mid S^2 = (ST)^3 = (ST^{-1}ST)^2 = 1 \rangle$
- Irreps: $\mathbf{1}_0,\ \mathbf{1}'_0,\ \mathbf{1}''_0,\ \mathbf{1}_1,\ \mathbf{1}'_1,\ \mathbf{1}''_1,\ \mathbf{3}_0,\ \mathbf{3}_1$
- Irreps matrices ($\omega = e^{2\pi i/3}$):

$$
\begin{aligned}
\mathbf{1}_i &: \quad \rho(S) = (-1)^i\,, \qquad \rho(T) = (-1)^i \\
\mathbf{1}'_i &: \quad \rho(S) = (-1)^i\,, \qquad \rho(T) = (-1)^i\omega \\
\mathbf{1}''_i &: \quad \rho(S) = (-1)^i\,, \qquad \rho(T) = (-1)^i\omega^2 \\
\mathbf{3}_i &: \quad \rho(S) = \frac{(-1)^i}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad \rho(T) = (-1)^i \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}
\end{aligned}
$$

An example model —— VVMFs I

- There are two VVMFs in the irreps $\mathbf{1}_1''$ and $\mathbf{3}_0$ at weight 2

$$Y^{(2)}_{\mathbf{1}_1''}(\tau) = \eta^4(\tau)$$

$$Y^{(2)}_{\mathbf{3}_0}(\tau) = \begin{pmatrix} \eta^4(\tau)(\frac{K(\tau)}{1728})^{-\frac{1}{6}}\ {}_3F_2(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\frac{2}{3},\frac{1}{3};K(\tau)) \\ -6\eta^4(\tau)(\frac{K(\tau)}{1728})^{\frac{1}{6}}\ {}_3F_2(\frac{1}{6},\frac{1}{2},\frac{5}{6};\frac{4}{3},\frac{2}{3};K(\tau)) \\ -18\eta^4(\tau)(\frac{K(\tau)}{1728})^{\frac{1}{2}}\ {}_3F_2(\frac{1}{2},\frac{5}{6},\frac{7}{6};\frac{5}{3},\frac{4}{3};K(\tau)) \end{pmatrix}$$

Their q-expansions ($q = e^{2\pi i\tau}$)

$$Y^{(2)}_{\mathbf{1}_1''}(\tau) = q^{1/6}(1 - 4q + 2q^2 + 8q^3 - 5q^4 - 4q^5 - 10q^6 + \dots)$$

$$Y^{(2)}_{\mathbf{3}_0}(\tau) = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + 36q^6 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + 31q^5 + 20q^6 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + 6q^5 + 14q^6 + \dots) \end{pmatrix}$$

Higher weight VVMFs can be constructed by tensor product

$$Y^{(4)}_{\mathbf{1}_0} = (Y^{(2)}_{\mathbf{3}_0} Y^{(2)}_{\mathbf{3}_0})_{\mathbf{1}_0} = \left(Y^{(2)}_{\mathbf{3}_0,1}\right)^2 + 2Y^{(2)}_{\mathbf{3}_0,2} Y^{(2)}_{\mathbf{3}_0,3}$$

$$Y^{(4)}_{\mathbf{1}'_0} = (Y^{(2)}_{\mathbf{3}_0} Y^{(2)}_{\mathbf{3}_0})_{\mathbf{1}'_0} = \left(Y^{(2)}_{\mathbf{3}_0,3}\right)^2 + 2Y^{(2)}_{\mathbf{3}_0,1} Y^{(2)}_{\mathbf{3}_0,2}$$

$$Y^{(4)}_{\mathbf{3}_0} = \frac{1}{2}(Y^{(2)}_{\mathbf{3}_0} Y^{(2)}_{\mathbf{3}_0})_{\mathbf{3}_0} = \begin{pmatrix} \left(Y^{(2)}_{\mathbf{3}_0,1}\right)^2 - Y^{(2)}_{\mathbf{3}_0,2} Y^{(2)}_{\mathbf{3}_0,3} \\ \left(Y^{(2)}_{\mathbf{3}_0,3}\right)^2 - Y^{(2)}_{\mathbf{3}_0,1} Y^{(2)}_{\mathbf{3}_0,2} \\ \left(Y^{(2)}_{\mathbf{3}_0,2}\right)^2 - Y^{(2)}_{\mathbf{3}_0,1} Y^{(2)}_{\mathbf{3}_0,3} \end{pmatrix}$$

$$Y^{(4)}_{\mathbf{3}_1} = Y^{(2)}_{\mathbf{3}_0} Y^{(2)}_{\mathbf{1}''_1} = \begin{pmatrix} Y^{(2)}_{\mathbf{3}_0,2} Y^{(2)}_{\mathbf{1}''_1} \\ Y^{(2)}_{\mathbf{3}_0,3} Y^{(2)}_{\mathbf{1}''_1} \\ Y^{(2)}_{\mathbf{3}_0,1} Y^{(2)}_{\mathbf{1}''_1} \end{pmatrix}$$

...

- **New method of constructing all VVMFs:**

The minimal weight VVMFs in each $\rho \in \textbf{Rep}(A_4 \times Z_2)$:

$$\widetilde{Y}^{(6)}_{\mathbf{1}_1}(\tau) = \eta^{12}(\tau)\,,\ \widetilde{Y}^{(4)}_{\mathbf{1}'_0}(\tau) = \eta^{8}(\tau)\,,\ \widetilde{Y}^{(10)}_{\mathbf{1}'_1}(\tau) = \eta^{20}(\tau)\,,$$
$$\widetilde{Y}^{(8)}_{\mathbf{1}''_0}(\tau) = \eta^{16}(\tau)\,,\quad \widetilde{Y}^{(2)}_{\mathbf{1}''_1}(\tau) = \eta^{4}(\tau)\,,$$
$$\widetilde{Y}^{(2)}_{\mathbf{3}_0}(\tau) = \begin{pmatrix} \eta^4(\tau)(\frac{K(\tau)}{1728})^{-\frac{1}{6}}\ {}_3F_2(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\frac{2}{3},\frac{1}{3};K(\tau)) \\ -6\eta^4(\tau)(\frac{K(\tau)}{1728})^{\frac{1}{6}}\ {}_3F_2(\frac{1}{6},\frac{1}{2},\frac{5}{6};\frac{2}{3},\frac{4}{3};K(\tau)) \\ -18\eta^4(\tau)(\frac{K(\tau)}{1728})^{\frac{1}{2}}\ {}_3F_2(\frac{1}{2},\frac{5}{6},\frac{7}{6};\frac{5}{3},\frac{4}{3};K(\tau)) \end{pmatrix}\,,$$
$$\widetilde{Y}^{(4)}_{\mathbf{3}_1}(\tau) = \begin{pmatrix} -6\eta^8(\tau)(\frac{K(\tau)}{1728})^{\frac{1}{6}}\ {}_3F_2(\frac{1}{6},\frac{1}{2},\frac{5}{6};\frac{2}{3},\frac{4}{3};K(\tau)) \\ -18\eta^8(\tau)(\frac{K(\tau)}{1728})^{\frac{1}{2}}\ {}_3F_2(\frac{1}{2},\frac{5}{6},\frac{7}{6};\frac{5}{3},\frac{4}{3};K(\tau)) \\ \eta^8(\tau)(\frac{K(\tau)}{1728})^{-\frac{1}{6}}\ {}_3F_2(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\frac{2}{3},\frac{1}{3};K(\tau)) \end{pmatrix}\,.$$

An example model —— VVMFs IV

- Higher weight VVMFs can be obtained from module structure (Acting differential operator D_k^n and multiplying the polynomial of E_4, E_6)

$$
\begin{aligned}
&k=2: \widetilde{Y}^{(2)}_{\mathbf{3}_0}\,,\ \widetilde{Y}^{(2)}_{\mathbf{1}''_1}\,,\\
&k=4: E_4\,,\ \ \widetilde{Y}^{(4)}_{\mathbf{1}'_0}\,,\ \ D_2\widetilde{Y}^{(2)}_{\mathbf{3}_0}\,,\ \ \widetilde{Y}^{(4)}_{\mathbf{3}_1}\,,\\
&k=6: E_6\,,\ \ \widetilde{Y}^{(6)}_{\mathbf{1}_1}\,,\ \ E_4\widetilde{Y}^{(2)}_{\mathbf{1}''_1}\,,\ \ D_2^2\widetilde{Y}^{(2)}_{\mathbf{3}_0}\,,\ \ E_4\widetilde{Y}^{(2)}_{\mathbf{3}_0}\,,\ \ D_4\widetilde{Y}^{(4)}_{\mathbf{3}_1}\,,\\
&\ldots
\end{aligned}
$$

These two methods are equivalent !

$$
Y^{(4)}_{\mathbf{1}_0}=E_4\,,\quad Y^{(4)}_{\mathbf{1}'_0}=-12\widetilde{Y}^{(4)}_{\mathbf{1}'_0}\,,\quad Y^{(4)}_{\mathbf{3}_0}=-6D_2\widetilde{Y}^{(2)}_{\mathbf{3}_0}\,,\quad Y^{(4)}_{\mathbf{3}_1}=\widetilde{Y}^{(4)}_{\mathbf{3}_1}
$$

- Lepton mass model (neutrinos mass originate from the Type-I seesaw)
- Matter fields content: $\varphi_I = \{E^c_{1,2,3},\ L_{1,2,3},\ N_{1,2,3},\ H_u,\ H_d\}$.
- The assignments of weight k_I and irreps ρ_I of matter fields:

Fields	$E^c_{1,2,3}$	$L_{1,2,3}$	$N^c_{1,2,3}$	$H_{u,d}$	$Y^{(k_Y)}_{\mathbf{r}}(\tau)$
$\mathsf{SU}(2)_L \times \mathsf{U}(1)_{\mathsf{Y}}$	$(1,1)$	$(2,-1/2)$	$(1,0)$	$(2,\pm 1/2)$	$(1,0)$
$A_4 \times Z_2$	$\mathbf{1}''_0 \oplus \mathbf{1}'_0 \oplus \mathbf{1}''_1$	$\mathbf{3}_0$	$\mathbf{3}_1$	$\mathbf{1}_0$	$\mathbf{r}$
$-k_I$	$-1,1,1$	3	1	0	k_Y

- Modular invariant superpotential

$$\mathcal{W}_e = \alpha \left(Y^{(2)}_{\mathbf{3}_0} E^c_1 L\right)_{\mathbf{1}_0} H_d + \beta \left(Y^{(4)}_{\mathbf{3}_0} E^c_2 L\right)_{\mathbf{1}_0} H_d + \gamma \left(Y^{(4)}_{\mathbf{3}_1} E^c_3 L\right)_{\mathbf{1}_0} H_d$$

$$\mathcal{W}_\nu = g_1 \left(Y^{(4)}_{\mathbf{3}_1} (N^c L)_{\mathbf{3}_1,S}\right)_{\mathbf{1}_0} H_u + g_2 \left(Y^{(4)}_{\mathbf{3}_1} (N^c L)_{\mathbf{3}_1,A}\right)_{\mathbf{1}_0} H_u + \Lambda \left(Y^{(2)}_{\mathbf{3}_0} (N^c N^c)_{\mathbf{3}_0,S}\right)_{\mathbf{1}_0}$$

- Charged lepton and neutrino mass matrices:

$$M_e = \begin{pmatrix} \alpha Y^{(2)}_{\mathbf{3}_0,2} & \alpha Y^{(2)}_{\mathbf{3}_0,1} & \alpha Y^{(2)}_{\mathbf{3}_0,3} \\ \beta Y^{(4)}_{\mathbf{3}_0,3} & \beta Y^{(4)}_{\mathbf{3}_0,2} & \beta Y^{(4)}_{\mathbf{3}_0,1} \\ \gamma Y^{(4)}_{\mathbf{3}_1,2} & \gamma Y^{(4)}_{\mathbf{3}_1,1} & \gamma Y^{(4)}_{\mathbf{3}_1,3} \end{pmatrix} v_d$$

$$M_D = \begin{pmatrix} 2g_1 Y^{(4)}_{\mathbf{3}_1,1} & (-g_1+g_2) Y^{(4)}_{\mathbf{3}_1,3} & -(g_1+g_2) Y^{(4)}_{\mathbf{3}_1,2} \\ -(g_1+g_2) Y^{(4)}_{\mathbf{3}_1,3} & 2g_1 Y^{(4)}_{\mathbf{3}_1,2} & (-g_1+g_2) Y^{(4)}_{\mathbf{3}_1,1} \\ (-g_1+g_2) Y^{(4)}_{\mathbf{3}_1,2} & -(g_1+g_2) Y^{(4)}_{\mathbf{3}_1,1} & 2g_1 Y^{(4)}_{\mathbf{3}_1,3} \end{pmatrix} v_u$$

$$M_N = \Lambda \begin{pmatrix} 2Y^{(2)}_{\mathbf{3}_0,1} & -Y^{(2)}_{\mathbf{3}_0,3} & -Y^{(2)}_{\mathbf{3}_0,2} \\ -Y^{(2)}_{\mathbf{3}_0,3} & 2Y^{(2)}_{\mathbf{3}_0,2} & -Y^{(2)}_{\mathbf{3}_0,1} \\ -Y^{(2)}_{\mathbf{3}_0,2} & -Y^{(2)}_{\mathbf{3}_0,1} & 2Y^{(2)}_{\mathbf{3}_0,3} \end{pmatrix}$$

Light neutrino mass matrix: $M_\nu = -M_D^T M_N^{-1} M_D$

- The numerical best-fit values of input parameters:

$$\tau = 0.103786 + 1.34097i\,, \quad \beta/\alpha = 2321.27\,, \quad \gamma/\alpha = 798.326\,,$$
$$g_2/g_1 = 13.8646 - 4.23100i\,, \quad \alpha v_d = 0.536619 \text{ MeV}\,,$$
$$g_1^2 v_u^2/\Lambda = 5.84234 \text{ meV}\,.$$

- The lepton masses and flavor mixing parameters are predicted

$$\sin^2\theta_{12} = 0.315878\,, \quad \sin^2\theta_{13} = 0.021913\,, \quad \sin^2\theta_{23} = 0.531526\,,$$
$$\delta_{CP} = 1.13711\pi\,, \quad \alpha_{21} = 1.03704\pi\,, \quad \alpha_{31} = 1.15945\pi\,,$$
$$m_e/m_\mu = 0.00480007, \qquad m_\mu/m_\tau = 0.0566796\,,$$
$$m_1 = 6.47366 \text{ meV}\,, \quad m_2 = 10.7754 \text{ meV}\,, \quad m_3 = 49.9964 \text{ meV}\,.$$

Light neutrino masses are normal ordering. Neutrino mass sum $m_1 + m_2 + m_3 = 67.2454$ meV is well compatible with the latest upper bound $\sum_i m_i < 120$ meV.

What we have about $\mathrm{SL}(2,\mathbb{Z})$ (2017 ∼ 2024)

The original $\mathrm{SL}(2,\mathbb{Z})$ modular invariant theory has been extended to

- Include the odd weight modular forms, and $\Gamma_N \mapsto \Gamma'_N$. (Liu and Ding 2019)
- Include the rational weight modular forms, and $\Gamma'_N \mapsto \widetilde{\Gamma_N}$. (Liu et al. 2020)
- Combine with the CP symmetry: $\tau \xrightarrow{CP} -\tau^*$ (Baur et al. 2019; Novichkov et al. 2019)
- Reformulate in VVMFs, and $\Gamma_N \mapsto \Gamma/\mathrm{Nor}$ (Liu and Ding 2022)
- Eclectic flavor symmetry: tradition flavor ∪ modular flavor (Nilles, Ramos-Sánchez, and Vaudrevange 2020; Nilles, Ramos-Sanchez, and Vaudrevange 2020)

Minimal model building: (See the talk by Ding)

- The minimal lepton models (6 real input parameters). (Ding, Liu, and Yao 2023; Ding et al. 2023)
- The minimal quark models (8 real input parameters). (Ding, Liu, and Yao 2023; Ding et al. 2023)
- The minimal lepton + quark models (14 real input parameters) (Ding et al. 2023)

Other applications: (See the talk by Penedo)

- Applying to solve Strong CP problem. (Feruglio, Strumia, and Titov 2023; Petcov and Tanimoto 2024; Penedo and Petcov 2024)
- Modular inflation. (Ding, Jiang, and Zhao 2024) ...
- ...

Beyond $SL(2,\mathbb{Z})$

What have we learned from the $SL(2,\mathbb{Z})$ modular invariant framework?

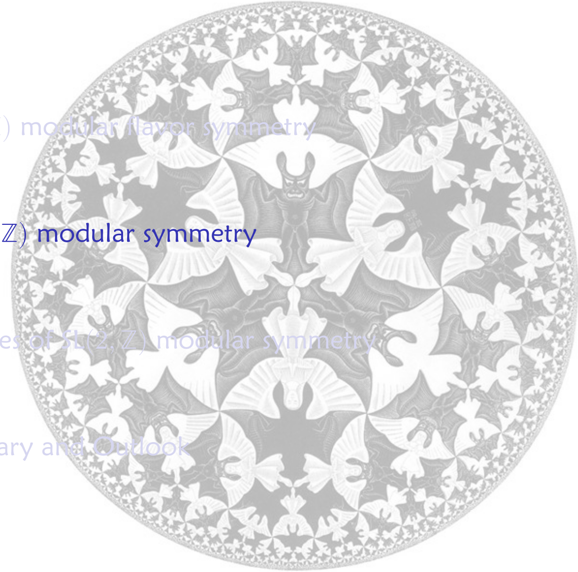
- From the perspective of field theory, $SL(2,\mathbb{Z})$ modular symmetry is a **nonlinear flavor symmetry** in nonlinear σ-model. (Feruglio and Romanino 2021) :
  - ◇ Scalar field(s) (or flavons) τ span the non-trivial moduli space $\mathcal{M}$
  - ◇ Flavons τ and matter fields φ_I have reparameterized under isometry group G
  - ◇ Interactions (e.g.Yukawa coupling) break G into its discrete subgroup $\Gamma \subset G$
- In original $SL(2,\mathbb{Z})$: $(\mathcal{M}, G, \Gamma) = (\mathcal{H},\ SL(2,\mathbb{R}),\ SL(2,\mathbb{Z}))$.
- Within SUSY (or SUGRA), $\mathcal{M}$ is in general (special) **Kähler manifold**. e.g.

Hermitian symmetric space $\mathcal{M} = G/K$

$$\frac{U(m,n)}{U(m)\times U(n)},\quad \frac{SO^*(2m)}{U(m)},\quad \frac{Sp(2m)}{U(m)},\quad \frac{SO(m,2)}{SO(m)\times SO(2)}$$

$$\frac{E_{6(-14)}}{SO(10)\times SO(2)},\quad \frac{E_{7(-25)}}{E_6\times U(1)}$$

Outline



1. $\mathrm{SL}(2,\mathbb{Z})$ modular flavor symmetry
2. $\mathrm{Sp}(2g,\mathbb{Z})$ modular symmetry
3. Varieties of $\mathrm{SL}(2,\mathbb{Z})$ modular symmetry
4. Summary and Outlook

$\mathsf{Sp}(2g,\mathbb{Z})$ symplectic modular invariance I

(Ding, Feruglio, and Liu 2021a)

- The natural generalization of $\mathsf{SL}(2,\mathbb{Z})$ modular invariance is $\mathsf{Sp}(2g,\mathbb{Z})$ Symplectic modular invariance.
- $\mathsf{Sp}(2g,\mathbb{Z})$ often arises from string compactification (Baur et al. 2020) (See the talk by Ramos-Sanchez & Nasu)
- (non-compact) Moduli space: Siegel upper half plane

$$\mathcal{H}_g = \{\tau \in GL(g,\mathbb{C}) \mid \tau^T = \tau,\ \mathrm{Im}(\tau) > 0\} \cong \mathsf{Sp}(2g,\mathbb{R})/U(g)$$

- (Siegel) symplectic modular group $\mathsf{Sp}(2g,\mathbb{Z}) := \Gamma_g$:

$$\mathsf{Sp}(2g,\mathbb{Z}) = \left\{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \gamma^T J\gamma = J \text{ with } J = \begin{pmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{pmatrix}\right\}$$

- Generators:

$$S = \begin{pmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{pmatrix}, \quad T_i = \begin{pmatrix} \mathbb{I}_g & B_i \\ 0 & \mathbb{I}_g \end{pmatrix},$$

Remark: At genus $g=1$, $\mathsf{Sp}(2,\mathbb{Z}) = \mathsf{SL}(2,\mathbb{Z})$.

- Vector-valued Siegel modular forms

$$f(\tau)\xrightarrow{\gamma} f(\gamma\tau)=\det(C\tau+D)^{k}\rho(\gamma)f(\tau)$$

  - So far there is no systematic theory like the $g=1$ case.
  - There are only a few results on scalar modular forms for genus $g=2$ for $\Gamma_g(n)$ with level $n=1,2,3$.

- Principal congruence subgroup $\Gamma_g(n)$:

$$\Gamma_g(n)=\left\{\gamma\in\Gamma_g \,\middle|\, \gamma\equiv\mathbb{1}_{2g}\,\mathtt{mod}\,n\right\},$$

- Finite Siegel modular group $\Gamma_{g,n}=\Gamma_g/\Gamma_g(n)$
- Action of $\mathrm{Sp}(2g,\mathbb{Z})$ on τ and φ_I:

$$\begin{cases}\tau\to\gamma\tau=(A\tau+B)(C\tau+D)^{-1}\\ \varphi_I\to\det(C\tau+D)^{-k_I}\rho_I(\gamma)\varphi_I\end{cases}$$

Yukawa couplings are (vector-valued) Siegel modular forms.

Symplectic modular symmetry at $g = 2$

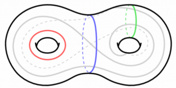
- At $g = 2$, $\mathrm{Sp}(4, \mathbb{Z})$ has four generators

$$T_1 = \begin{pmatrix} \mathbb{I}_2 & B_1 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \mathbb{I}_2 & B_2 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \quad T_3 = \begin{pmatrix} \mathbb{I}_2 & B_3 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix},$$

$$\text{with} \;\; B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- **Moduli** (three components τ_1, τ_2, τ_3):

$$\mathcal{H}_2 = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \middle| \; \det(\mathrm{Im}(\tau)) > 0, \;\; \mathrm{tr}(\mathrm{Im}(\tau)) > 0 \right\}$$



  - ◇ $\tau_3 = 0$: $\mathcal{H}_2 \simeq$ 2 factorized tori;
  - ◇ $\tau_3 \neq 0$: $\mathcal{H}_2 \simeq$ generic Riemann surface of genus 2;

- The simplest non-Abelian finite modular group is $\Gamma_{2,2} \cong S_6$, which has **no 3-d irreps to accommodate 3 families of fermions!**

Irreps: $[\mathbf{1}, 2]$, $[\mathbf{5}, 4]$, $[\mathbf{9}, 2]$, $[\mathbf{10}, 2]$, $[\mathbf{16}, 1]$

$g = 2, n = 2$ Siegel modular forms

- There are five Siegel modular forms at $k = 2, n = 2$: (Cacciatori and Dalla Piazza 2008)

$$
\begin{aligned}
p_0 &= \Theta[00]^4(\tau) + \Theta[01]^4(\tau) + \Theta[10]^4(\tau) + \Theta[11]^4(\tau)\,, \\
p_1 &= 2\left(\Theta[00]^2(\tau)\Theta[01]^2(\tau) + \Theta[10]^2(\tau)\Theta[11]^2(\tau)\right)\,, \\
p_2 &= 2\left(\Theta[00]^2(\tau)\Theta[10]^2(\tau) + \Theta[01]^2(\tau)\Theta[11]^2(\tau)\right)\,, \\
p_3 &= 2\left(\Theta[00]^2(\tau)\Theta[11]^2(\tau) + \Theta[01]^2(\tau)\Theta[10]^2(\tau)\right)\,, \\
p_4 &= 4\Theta[00](\tau)\Theta[01](\tau)\Theta[10](\tau)\Theta[11](\tau)\,.
\end{aligned}
$$

They form a quintet $Y_{\mathbf{5}} = (Y_1, Y_2, Y_3, Y_4, Y_5)^T$ of $S_6 \cong \Gamma_{2,2}$: (Ding, Feruglio, and Liu 2021a)

$$
\begin{aligned}
Y_1(\tau) &= p_0(\tau) + 3p_1(\tau)\,, \\
Y_2(\tau) &= \sqrt{3}\left[p_0(\tau) - p_1(\tau)\right]\,, \\
Y_3(\tau) &= \sqrt{3}\left[p_2(\tau) + p_3(\tau) - 2p_4(\tau)\right]\,, \\
Y_4(\tau) &= 3p_2(\tau) - p_3(\tau) + 2p_4(\tau)\,, \\
Y_5(\tau) &= 2\sqrt{2}\left[p_3(\tau) + p_4(\tau)\right]\,.
\end{aligned}
$$

New features at genus $g = 2$

- Restrict to (invariant) subspace Σ of $\mathcal{H}_2$:

$$H\tau = \tau\,, \qquad \forall\ \tau \in \Sigma$$

Symmetry $\mathrm{Sp}(4,\mathbb{Z}) \mapsto N(H)$ (finite modular group $\Gamma_{2,n} \mapsto N_n(H)$)

  - ◇ Two dimensional:

$$\begin{cases} \Sigma_1 : \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, & \Gamma_{2,2} \mapsto (S_3 \times S_3) \rtimes Z_2 \\ \Sigma_2 : \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix}, & \Gamma_{2,2} \mapsto S_4 \times Z_2 \end{cases}$$

  - ◇ One dimensional:

$$\begin{pmatrix} i & 0 \\ 0 & \tau_2 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \tau_2 \end{pmatrix}, \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_1 \end{pmatrix}, \begin{pmatrix} \tau_1 & \tau_1/2 \\ \tau_1/2 & \tau_1 \end{pmatrix}.$$

(See the talk by Ramos-Sanchez)

- We focus on the subspace Σ_2, i.e., we impose $\tau_1 = \tau_2$. The modular form quintet $Y_{\mathbf{5}}$ collapse to singlet and triplet of the $N_2(H) \cong S_4 \times Z_2$:

$$\mathbf{3}' : \quad Y_{\mathbf{3}'}(\tau) = \begin{pmatrix} p_0(\tau) + 4p_1(\tau) - p_3(\tau) \\ p_0(\tau) - 2p_1(\tau) - p_3(\tau) - 2i\sqrt{3}p_4(\tau) \\ p_0(\tau) - 2p_1(\tau) - p_3(\tau) + 2i\sqrt{3}p_4(\tau) \end{pmatrix} \equiv \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix},$$

$$\mathbf{1} : \quad Y_{\mathbf{1}}(\tau) = p_0(\tau) + 3p_3(\tau) \equiv Y_4(\tau)\,.$$

An example model I

- Lepton mass model (neutrinos mass originate from the Type-I seesaw)
- Matter fields content: $\varphi_I = \{E^c_{1,2,3},\ L_{1,2,3},\ N_{1,2,3},\ H_u,\ H_d\}$.
- The assignments of weight k_I and irreps ρ_I of matter fields:

Fields	$E^c_{1,2,3}$	$L_{1,2,3}$	$N^c_{1,2,3}$	$H_{u,d}$	$Y^{(k_Y)}_{\mathbf{r}}(\tau)$
$SU(2)_L \times U(1)_Y$	$(1,1)$	$(2,-1/2)$	$(1,0)$	$(2,\pm1/2)$	$(1,0)$
$S_4 \times Z_2$	$\mathbf{3}'$	$\mathbf{3}'$	$\mathbf{3}'$	$\mathbf{1}$	$\mathbf{r}$
$-k_I$	0	-2	0	0	k_Y

- Modular invariant superpotential

$$\mathcal{W}_e = \alpha \left(Y_{\mathbf{3}'} E^c L\right)_{\mathbf{1}} H_d + \beta \left(Y_{\mathbf{1}} E^c L\right)_{\mathbf{1}} H_d \,,$$
$$\mathcal{W}_\nu = g_1 \left(Y_{\mathbf{3}'} N^c L\right)_{\mathbf{1}} H_u + g_2 \left(Y_{\mathbf{1}} N^c L\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}} \,.$$

An example model II

- The charged lepton and neutrino mass matrices:

$$M_e = \begin{pmatrix} 2\alpha Y_1 + \beta Y_4 & -\alpha Y_3 & -\alpha Y_2 \\ -\alpha Y_3 & 2\alpha Y_2 & -\alpha Y_1 + \beta Y_4 \\ -\alpha Y_2 & -\alpha Y_1 + \beta Y_4 & 2\alpha Y_3 \end{pmatrix} v_d \,,$$

$$M_D = \begin{pmatrix} 2g_1 Y_1 + g_2 Y_4 & -g_1 Y_3 & -g_1 Y_2 \\ -g_1 Y_3 & 2g_1 Y_2 & -g_1 Y_1 + g_2 Y_4 \\ -g_1 Y_2 & -g_1 Y_1 + g_2 Y_4 & 2g_1 Y_3 \end{pmatrix} v_u \,,$$

$$M_N = \Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

The light neutrino mass matrix is $M_\nu = -M_D^T M_N^{-1} M_D$.

- The best fit values of the input parameters:

$$\tau_1 = 0.04017 + 0.89185i \,, \quad \tau_3 = 0.49053 + 0.00792i \,, \quad \beta/\alpha = 2.10415 - 0.14380i \,,$$
$$g_2/g_1 = -0.96942 - 3.32507i \,, \quad \alpha v_d = 136.26910 \text{ MeV} \,, \quad g_1^2 v_u^2/\Lambda = 2.71970 \text{ meV} \,,$$

An example model III

- The lepton mixing parameters and neutrino masses are predicted to be

$$\sin^2\theta_{12} = 0.3068\,, \quad \sin^2\theta_{13} = 0.02219\,, \quad \sin^2\theta_{23} = 0.5753\,, \quad \delta_{CP} = 1.09\pi\,,$$
$$\alpha_{21} = 0.05\pi\,, \quad \alpha_{31} = 0.03\pi\,, \quad m_e/m_\mu = 0.00476\,, \quad m_\mu/m_\tau = 0.06071\,,$$
$$m_1 = 120.75\text{ meV}\,, \quad m_2 = 121.06\text{ meV}\,, \quad m_3 = 130.69\text{ meV}\,,$$

Remarks:

- The best-fit value of τ close to the 1-d fixed locus:

$$\tau_{fit} = \begin{pmatrix} 0.04017 + 0.89185i & 0.49053 + 0.00792i \\ 0.49053 + 0.00792i & 0.04017 + 0.89185i \end{pmatrix} \approx \begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_1 \end{pmatrix}$$

where a $Z_2 \times Z_2$ subgroup is preserved, and M_e, M_ν become $\mu - \tau$ symmetric.

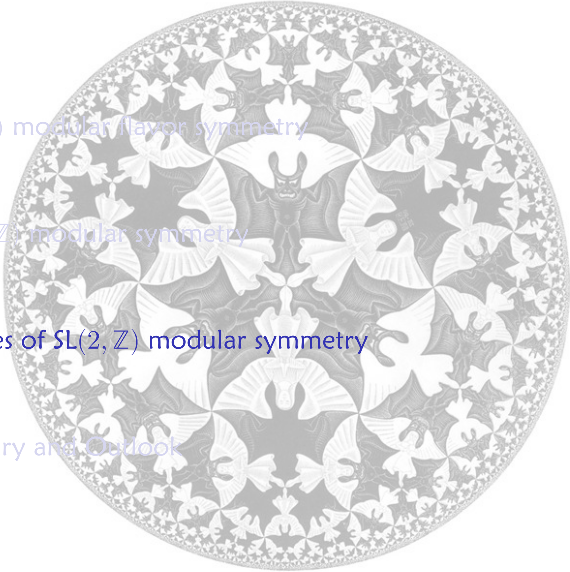
- All the dimensionless parameters of the model are of $\mathcal{O}(1)$.
- The mass hierarchy of charged lepton is achieved with Lagrangian parameters α, β of the same order.

Recent developments in symplectic framework

Relevant bottom-up work on symplectic modular flavor symmetry is still scarce, However:

- The gCP symmetry consistent with Sp$(2g, \mathbb{Z})$ is studied. (Ding, Feruglio, and Liu 2021b)
- New model construction based on $g = 2, n = 2$. (**RickyDevi:2024ijc**)
- Systematic study of near-critical behavior in Sp$(4, \mathbb{Z})$. (Ding, Feruglio, and Liu 2024)
- Moduli stabilization with symplectic modular symmetry Sp$(4, \mathbb{Z})$.(Working in progress)
- The new example based on $g = 2, n = 3$ is being worked (where $\Gamma_{2,3}$ double covering of Burkhardt group and $|\Gamma_{2,3}| = 51840$).(in collaboration with Michael, Hans Peter and Saul...)

Outline



"Siblings" of $SL(2,\mathbb{Z})$

- In the original single-modulus framework $(\mathcal{M}, G) = (\mathcal{H} \cong SL(2,\mathbb{R})/SO(2),\ \ SL(2,\mathbb{R}))$

- The choice of discrete flavor groups $\Gamma \subset SL(2,\mathbb{R})$ is still quite rich: e.g. Arithmetic subgroups or Fuchsian groups of the first kind.

Some choices of discrete flavor groups Γ

- $SL(2,\mathbb{Z})$ and all its possible subgroups, e.g. $\Gamma(N), \Gamma_0(N), \Gamma_1(N)$ ...
- Triangle modular groups $\Gamma(l, m, n)$
- ...

We refer to these discrete groups as "siblings" of $SL(2,\mathbb{Z})$.

Interestingly, there are also exist non-trivial vector-valued modular forms for those infinite discrete groups (Gannon 2014)

The classification and structures of the subgroups of SL$(2,\mathbb{Z})$ are still unclear.

- But we know the normal subgroups N of SL$(2,\mathbb{Z})$ in some sense.
- Starting from N, we can construct subgroups K that include N as its normal subgroups: $N \triangleleft K \subset \mathrm{SL}(2,\mathbb{Z})$

The correspondence theorem

If N is a normal subgroup of a group Γ , then there exists a bijection from the set of all subgroups of Γ containing N, onto the set of all subgroups of the quotient group Γ/N.

$$\{\text{subgroups of } \Gamma \text{ containing } N\} \overset{1:1}{\longleftrightarrow} \{\text{subgroups of } \Gamma/N\}\,.$$

$$\begin{array}{ccccccccc}
\Gamma & \text{———} & K & \text{———} & \cdots & \text{———} & H & \text{———} & N \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
\Gamma/N & \text{——} & K/N & \text{———} & \cdots & \text{———} & H/N & \text{——} & \{N\}
\end{array}$$

where $K \subset \Gamma,\ K/N \subset \Gamma/N, \dots$.

Subgroups of $\mathrm{SL}(2, \mathbb{Z})$ II

- Coset decomposition of Γ in terms of its normal N:

$$\Gamma = eN \cup \gamma_1 N \cup \gamma_2 N \cup \cdots \cup \gamma_n N = \left\{ \bigcup_i \gamma_i N \mid \gamma_i \in \Gamma/N \right\} \equiv (\Gamma/N) * N \,,$$

  The subgroup K in Γ can be constructed:

$$K = eN \cup \gamma_1 N \cup \gamma_2 N \cup \cdots \cup \gamma_m N = \left\{ \bigcup_i \gamma_i N \mid \gamma_i \in K/N \right\} \equiv (K/N) * N \,,$$

- The modular transformation can be generalized to

$$\begin{cases} \tau \xrightarrow{\gamma} \dfrac{a\tau + b}{c\tau + d} \,, \\ \varphi \xrightarrow{\gamma} (c\tau + d)^{-k_\varphi} \rho(\gamma) \varphi \,, \end{cases} \quad \text{with} \quad \begin{array}{l} \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \,, \\ \rho \in \textbf{Rep}(K/N) \end{array}$$

- The original scalar modular forms on N form multiplets of K/N under K; In other words, the multiplets of Γ/N decompose into multiplets of the subgroup K/N, following the branching rule.

Some examples

1. $(K, N, \rho) = (A_4 * (\pm\Gamma(4)),\ \pm\Gamma(4),\ \rho \in \textbf{Rep}(A_4))$:

$$S_4 \cong \Gamma/\pm\Gamma(4) = \langle S, T \mid S^2 = (ST)^3 = T^4 = 1 \rangle$$

$$S_4 \supset A_4 = \langle s = T^2, t = ST \mid s^2 = (st)^3 = t^3 = 1 \rangle$$

The modular subgroup $K = \text{Mult}\left(\langle T^2, ST \rangle, \pm\Gamma(4)\right) \equiv A_4 * (\pm\Gamma(4))$.
Modular form multiplets decomposition:

$$Y^{(2)}_{S_4\mathbf{2}} \to Y^{(2)}_{A_4\mathbf{1}''} \oplus Y^{(2)}_{A_4\mathbf{1}'}\,, \qquad Y^{(2)}_{S_4\mathbf{3}} \to Y^{(2)}_{A_4\mathbf{3}}\,.$$

2. $(K, N, \rho) = (\Gamma(2),\ \Gamma(6),\ \rho \in \textbf{Rep}(T'))$: (Li, Liu, and Ding 2021)

$$S_3 \times T' \cong \Gamma/\Gamma(6) = \langle S, T \mid S^4 = (ST)^3 = T^6 = ST^2ST^3ST^4ST^3 = 1, S^2T = TS^2 \rangle$$

$$S_3 \times T' \supset T' = \langle \tilde{a} = TST^4S^3T,\ \tilde{b} = T^4 \mid \tilde{a}^4 = \tilde{b}^3 = (\tilde{a}\tilde{b})^3 = 1, \tilde{a}^2\tilde{b} = \tilde{b}\tilde{a}^2 \rangle$$

The modular subgroup $K = \text{Mult}\left(\langle TST^4S^3T, T^4 \rangle, \Gamma(6)\right) = \Gamma(2)$.
Modular form multiplets decomposition:

$$Y^{(1)}_{\Gamma'_6\mathbf{2}^0_2} \to Y^{(1)}_{T'\mathbf{2}''}\,, \qquad Y^{(2)}_{\Gamma'_6\mathbf{4}_1} \to Y^{(2)}_{T'\mathbf{2}'I} \oplus Y^{(2)}_{T'\mathbf{2}'II}\,.$$

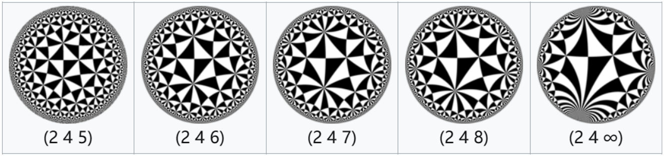
Finally we introduce another set of siblings of modular group SL$(2, \mathbb{Z})$:

Triangle group $\Delta(l, m, n)$ (Working in progress...)

which are the symmetry groups of tilings of the Euclidean plane, the sphere, or the hyperbolic plane by congruent triangles.

- The Euclidean case $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} = 1$: Only three cases and $\Delta(l, m, n)$ are infinite groups
- The spherical case $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1$: Infinitely many cases and $\Delta(l, m, n)$ are finite groups (of regular polyhedra)
- The hyperbolic case $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$: Infinitely many cases and $\Delta(l, m, n)$ are infinite groups.

  Examples of the tilings(in Poincaré disk):



Triangle modular group II

- We call the triangle group in hyperbolic case **triangle modular groups** denoted as $\Gamma(m_1, m_2, m_3)$.
- $\Gamma(m_1, m_2, m_3)$ has only two generators g_1 and g_2, and

$$\Gamma(m_1, m_2, m_3) = \langle g_1, g_2 \mid g_1^{m_1} = g_2^{m_2} = (g_1 g_2)^{m_3} \rangle \, .$$

Examples:

$$\Gamma(2,3,\infty) = \mathrm{PSL}(2,\mathbb{Z}), \qquad \Gamma(2,\infty,\infty) = \Gamma_0(2),$$
$$\Gamma(\infty,\infty,\infty) = \Gamma(2) \cong \Gamma_0(4), \ldots$$

- **Hecke group $\Gamma(2, m, \infty) \equiv H(m)$**:

$$\Gamma(2, m, \infty) = \left\langle S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 2\cos(\frac{\pi}{m}) \\ 0 & 1 \end{pmatrix} \,\middle|\, S^2 = (ST)^m = 1 \right\rangle \, .$$

...

Triangle modular group III

- Triangle modular group also act on $\mathscr{H}$ through Möbius transformation:

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad \gamma \in \Gamma(l, m, n)$$

- Triangle modular groups are in fact genus-0 Fuchsian groups of the first kind.
- Triangle modular groups have unitary finite dimensional irreps (finite groups).
- Triangle modular groups possesses vector-valued modular form!
- Unfortunately, there is no systematic theory of VVMFs on $\Gamma(l, m, n)$.
- A lot of work is being done ...

Outline

1. $\mathsf{SL}(2,\mathbb{Z})$ modular flavor symmetry
2. $\mathsf{Sp}(2g,\mathbb{Z})$ modular symmetry
3. Varieties of $\mathsf{SL}(2,\mathbb{Z})$ modular symmetry
4. Summary and Outlook

Summary

- The original SL$(2,\mathbb{Z})$ framework has been extended to the most general case, encompassing not only $\Gamma_N^{(\prime)}$. (Liu and Ding 2022)
- The SL$(2,\mathbb{Z})$ framework is already clear in the bottom-up approach, because the theory of vector-valued modular forms of SL$(2,\mathbb{Z})$ is well established.
- In the bottom-up approach, lack of more constraints, the freedom of modular theory construction is quite large: $(\mathcal{M}, G, \Gamma)$.
- We review modular symmetries beyond SL$(2,\mathbb{Z})$, including Sp$(2g,\mathbb{Z})$ symplectic modular symmetry and SL$(2,\mathbb{Z})$'s siblings.
- In the Sp$(2g,\mathbb{Z})$ symplectic modular framework, there are few concrete examples and phenomenological applications until now.
- The work on the varieties of SL$(2,\mathbb{Z})$ is still ongoing (in collaboration with Mu-Chun, Michael, Xueqi and Saul...)
- There are still considerable potential theories beyond SL$(2,\mathbb{Z})$ that have not been concretely constructed.

Outlook

- For the model building, perhaps these frameworks beyond SL$(2, \mathbb{Z})$ can provide more useful non-trivial applications.
- The mathematical tools needed to construct these theories are scarce, especially the corresponding modular form theory.
- The interesting thing is how to derive these constructs from a top-down approach. (See the talk by Kaito Nasu)
- Find more principles to limit the possible choices for $(\mathcal{M}, G, \Gamma)$, the swampland conjecture ?
- Find a specific relationship between the q-expansion coefficients of Yukawa couplings and the flavor observables: ”Flavor moonshine” ? (Some hints were hidden in Omar and Xue-Qi’s talk)

Thank you for your attention!

There is still sooooooooooooooooooo muchhhhhhhhhhhhhhhhhhhhhhh work to be done in the future. Let's push together! (2024∼?)

Baur, Alexander et al. (2019). "Unification of Flavor, CP, and Modular Symmetries". In: Phys. Lett. B 795, pp. 7–14. DOI: 10.1016/j.physletb.2019.03.066. arXiv: 1901.03251 [hep-th].

Baur, Alexander et al. (Dec. 2020). "Siegel modular flavor group and CP from string theory". In: arXiv: 2012.09586 [hep-th].

Cacciatori, Sergio L. and Francesco Dalla Piazza (2008). "Two loop superstring amplitudes and S(6) representations". In: Lett. Math. Phys. 83, pp. 127–138. DOI: 10.1007/s11005-007-0213-8. arXiv: 0707.0646 [hep-th].

Ding, Gui-Jun, Ferruccio Feruglio, and Xiang-Gan Liu (2021a). "Automorphic Forms and Fermion Masses". In: JHEP 01, p. 037. DOI: 10.1007/JHEP01(2021)037. arXiv: 2010.07952 [hep-th].

— (2021b). "CP symmetry and symplectic modular invariance". In: SciPost Phys. 10.6, p. 133. DOI: 10.21468/SciPostPhys.10.6.133. arXiv: 2102.06716 [hep-ph].

— (Feb. 2024). "Universal predictions of Siegel modular invariant theories near the fixed points". In: arXiv: 2402.14915 [hep-ph].

Ding, Gui-Jun, Si-Yi Jiang, and Wenbin Zhao (May 2024). "Modular Invariant Slow Roll Inflation". In: arXiv: 2405.06497 [hep-ph].

Ding, Gui-Jun, Xiang-Gan Liu, and Chang-Yuan Yao (2023). "A minimal modular invariant neutrino model". In: JHEP 01, p. 125. DOI: 10.1007/JHEP01(2023)125. arXiv: 2211.04546 [hep-ph].

Ding, Gui-Jun et al. (July 2023). "Modular binary octahedral symmetry for flavor structure of Standard Model". In: arXiv: 2307.14926 [hep-ph].

- Ferrara, S. et al. (1989). "Modular Invariance in Supersymmetric Field Theories". In: Phys. Lett. B 225, p. 363. DOI: 10.1016/0370-2693(89)90583-2.
- Feruglio, Ferruccio (2017). "Are neutrino masses modular forms?" In: From My Vast Repertoire ...: Guido Altarelli's Legacy. Ed. by Aharon Levy, Stefano Forte, and Giovanni Ridolfi, pp. 227–266. DOI: 10.1142/9789813238053_0012. arXiv: 1706.08749 [hep-ph].
- Feruglio, Ferruccio and Andrea Romanino (2021). "Lepton flavor symmetries". In: Rev. Mod. Phys. 93.1, p. 015007. DOI: 10.1103/RevModPhys.93.015007. arXiv: 1912.06028 [hep-ph].
- Feruglio, Ferruccio, Alessandro Strumia, and Arsenii Titov (2023). "Modular invariance and the QCD angle". In: JHEP 07, p. 027. DOI: 10.1007/JHEP07(2023)027. arXiv: 2305.08908 [hep-ph].
- Franc, Cameron and Geoffrey Mason (2016). "Hypergeometric series, modular linear differential equations and vector-valued modular forms". In: The Ramanujan Journal 41.1, pp. 233–267. arXiv: 1503.05519 [math.NT].
- — (2018). "On the structure of modules of vector-valued modular forms". In: The Ramanujan Journal 47.1, pp. 117–139. arXiv: 1509.07494 [math.NT].
- Gannon, Terry (2014). "The theory of vector-valued modular forms for the modular group". In: Conformal field theory, automorphic forms and related topics. Springer, pp. 247–286. arXiv: 1310.4458 [math.NT].
- Lauer, J., J. Mas, and Hans Peter Nilles (1989). "Duality and the Role of Nonperturbative Effects on the World Sheet". In: Phys. Lett. B 226, pp. 251–256. DOI: 10.1016/0370-2693(89)91190-8.

References III

Li, Cai-Chang, Xiang-Gan Liu, and Gui-Jun Ding (2021). "Modular symmetry at level 6 and a new route towards finite modular groups". In: JHEP 10, p. 238. DOI: 10.1007/JHEP10(2021)238. arXiv: 2108.02181 [hep-ph].

Liu, Xiang-Gan and Gui-Jun Ding (2019). "Neutrino Masses and Mixing from Double Covering of Finite Modular Groups". In: JHEP 08, p. 134. DOI: 10.1007/JHEP08(2019)134. arXiv: 1907.01488 [hep-ph].

— (2022). "Modular flavor symmetry and vector-valued modular forms". In: JHEP 03, p. 123. DOI: 10.1007/JHEP03(2022)123. arXiv: 2112.14761 [hep-ph].

Liu, Xiang-Gan et al. (2020). "Half-integral weight modular forms and application to neutrino mass models". In: Phys. Rev. D 102.11, p. 115035. DOI: 10.1103/PhysRevD.102.115035. arXiv: 2007.13706 [hep-ph].

Marks, Christopher and Geoffrey Mason (2010). "Structure of the module of vector-valued modular forms". In: Journal of the London Mathematical Society 82.1, pp. 32–48. arXiv: 0901.4367 [math.NT].

Mason, Geoffrey (2008). "2-dimensional vector-valued modular forms". In: The Ramanujan Journal 17.3, pp. 405–427.

Nilles, Hans Peter, Saul Ramos-Sanchez, and Patrick K. S. Vaudrevange (2020). "Lessons from eclectic flavor symmetries". In: Nucl. Phys. B 957, p. 115098. DOI: 10.1016/j.nuclphysb.2020.115098. arXiv: 2004.05200 [hep-ph].

Nilles, Hans Peter, Saúl Ramos-Sánchez, and Patrick K.S. Vaudrevange (2020). "Eclectic Flavor Groups". In: JHEP 02, p. 045. DOI: 10.1007/JHEP02(2020)045. arXiv: 2001.01736 [hep-ph].

References IV

Novichkov, P.P. et al. (2019). "Generalised CP Symmetry in Modular-Invariant Models of Flavour". In: JHEP 07, p. 165. DOI: 10.1007/JHEP07(2019)165. arXiv: 1905.11970 [hep-ph].

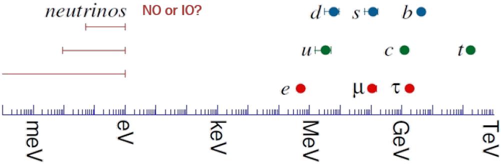
Penedo, J. T. and S. T. Petcov (Apr. 2024). "Finite modular symmetries and the strong CP problem". In: arXiv: 2404.08032 [hep-ph].

Petcov, S. T. and M. Tanimoto (Mar. 2024). "A_4 modular invariance and the strong CP problem". In: arXiv: 2404.00858 [hep-ph].

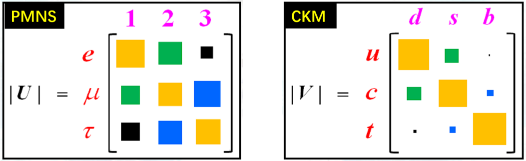
Backup

Flavor puzzle

- What is the origin of the mass hierarchies of leptons & quarks ?



- How to understand the flavor mixing patterns of leptons & quarks ?

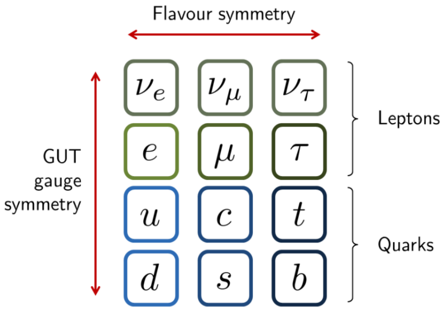


Remarks:

- ◇ In SM, the fermion masses and flavor mixing are determined by Yukawa coupling constants which are completely unconstrained
- ◇ No guiding principles
- ◇ Symmetries?

Theory of Flavor—Traditional flavor symmetry

Flavor symmetry: Horizontal symmetry linking different families of fermions.



◇ **Flavor transformation**:

$$\varphi \xrightarrow{g} \rho(g)\varphi$$

with $g \in G_f$, $\rho(g) \in \mathbf{Rep}(G_f)$.

◇ **Flavor group G_f** can be Abelian or non-Abelian, continuous Lie groups or discrete finite groups...

e.g.:

- Froggatt-Nielsen models: $G_f = \mathrm{U}(1)_{\mathrm{FN}}$.
- Non-Abelian discrete flavor symmetry: $G_f = A_4,\ S_4,\ A_5, \dots$.

New game in town: **Modular flavor symmetries**

Irreps of SL$(2,\mathbb{Z})$

The finite image irreps of SL$(2,\mathbb{Z})$ can be obtained from these (infinitely many) general finite modular groups : $\Gamma \xrightarrow{\text{natural}} \Gamma/N \xrightarrow{\tilde{\rho}} GL(V)$

- SL$(2,\mathbb{Z})$ has **12 one-dimensional** irreps

$$\mathbf{1}_p : \qquad \rho_{\mathbf{1}_p}(S) = i^p \,, \qquad \rho_{\mathbf{1}_p}(T) = e^{\frac{i\pi}{6}p} \,,$$

  with $p = 0, \ldots 11$

- SL$(2,\mathbb{Z})$ has **54 two-dimensional** irreps with finite image, determined by $\rho(T) = \texttt{diag}(e^{2\pi i r_1}, e^{2\pi i r_2})$ (Mason 2008)
  e.g. $(r_1, r_2) = (3/4, 1/4), (5/6, 1/3), (0, 1/2) \ldots$

- SL$(2,\mathbb{Z})$ has **> 156 three-dimensional** irreps with finite image, also determined by $\rho(T) = \texttt{diag}(e^{2\pi i r_1}, e^{2\pi i r_2}, e^{2\pi i r_3})$
  e.g. $(r_1, r_2, r_3) = (0, 1/3, 2/3), (0, 1/4, 3/4), (1/7, 2/7, 4/7) \ldots$