Top-down modular invariance approach to flavour II

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Outline

1. Modular weights

   S. Kikuchi, T. Kobayashi, H. O., M. Tanimoto, H. Uchida, 2201.04505
   T. Kobayashi, T. Nomura, H. Okada, H. O., 2310.10091

2. Metaplectic modular symmetries

   K. Ishiguro, T. Kai, H. Okada, T. Kobayashi, T. Nomura, H. Okada, H. O., 2310.10091

3. $\Gamma_0(N)$ modular symmetries

   T. Kobayashi, H. O., 2004.04518
   K. Ishiguro, T. Kai, T. Kobayashi, H. O., 2311.12425
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Modular invariant 4D supersymmetric EFT

Lauer, Mas, Nilles, 1989; Ferrara, Lust etal, 1989; Feruglio, 1706.08749

$$K = -\ln\big(i(\bar{\tau} - \tau)\big) + \sum_i \frac{|\phi_i|^2}{\big(i(\bar{\tau} - \tau)\big)^{k_i}}$$

$$W = \sum_n Y_{i_1 \ldots i_n}(\tau)\, \phi_{i_1} \cdots \phi_{i_n}$$

ϕ_i : chiral superfields with modular weight k_i

$Y_{i_1 \ldots i_n}(\tau)$: couplings

Modular invariant 4D supersymmetric EFT

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$$W = \sum_n Y_{i_1 \ldots i_n}(\tau)\, \phi_{i_1} \cdots \phi_{i_n}$$

- Moduli fields

$$\tau \to R(\tau) = \frac{a\tau + b}{c\tau + d}$$

$a, b, c, d \in \mathbb{Z}$
$ad - bc = 1$

- Chiral superfields

$$\phi_i \to (c\tau + d)^{-k_i} \rho_i(g) \phi_i$$

Automorphy factor

Weight $k_i \in \mathbb{Z}$

- Couplings

$$Y(\tau) \to (c\tau + d)^{k_Y} \rho_Y(g) Y(\tau)$$

Modular invariant 4D supersymmetric EFT

$$K = -\ln\big(i(\bar{\tau}-\tau)\big) + \sum_i \frac{|\phi_i|^2}{\big(i(\bar{\tau}-\tau)\big)^{k_i}}$$

$$W = \sum_n Y_{i_1 \ldots i_n}(\tau)\, \phi_{i_1} \cdots \phi_{i_n}$$

Question : Sign of modular weights for matter fields ?

$$\phi_i \to (c\tau + d)^{-k_i} \rho_i(\gamma) \phi_i$$

- Most of bottom-up researchers use negative weights $-k_i < 0$.
 (Indeed, negative modular weights is naturally realized in the top-down construction)

- Is it possible to consider $-k_i > 0$?

Modular weights of 4D matters

S. Kikuchi, T. Kobayashi, H. O., M. Tanimoto, H. Uchida, 2201.04505

Consider higher-dimensional scalars $\Phi(x, y)$ and spinors $\Psi(x, y)$

x : 4D coordinates
y : Coordinates of extra-dimensional space

KK reduction :

$$\Phi(x, y) = \sum_i \phi_i(x)\varphi_i(y) + \mathrm{KK\ modes}$$

$$\Psi(x, y) = \sum_i \psi_i(x)\chi_i(y) + \mathrm{KK\ modes}$$

- Internal background sources
  → degenerate massless modes ϕ_i, ψ_i

Modular weights of 4D matters

S. Kikuchi, T. Kobayashi, H. O., M. Tanimoto, H. Uchida, 2201.04505

Consider 6D theory on T^2 or its orbifolds with background sources

6D Kinetic term :

$$\int d^4x d^2y \sqrt{g_{6D}}\, \partial_m \Phi \partial^m \Phi^* + \cdots$$

$$m = 0,1,2,3,4,5$$

$$\Phi(x,y) = \sum_i \phi_i(x) \varphi_i(y) + \text{KK modes}$$

Normalization :

$$\int d^2y \sqrt{g_{2D}} \varphi_i(y) \varphi_j(y)^* = \frac{1}{\left(2\mathrm{Im}(\tau)\right)^{k_i}} \delta_{i,j}$$

4D Kinetic term :

$$\int d^4x \sqrt{g_{4D}} \left(\int d^2y \sqrt{g_{2D}} \varphi_i(y) \varphi_j(y)^* \right) \partial_\mu \phi_i \partial^\mu \phi_j^* = \int d^4x \sqrt{g_{4D}} \frac{1}{\left(2\mathrm{Im}(\tau)\right)^{k_i}} \partial_\mu \phi_i \partial^\mu \phi_i^*$$

Modular weights of 4D matters

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When we assume 4D N=1 SUSY,

$$\Phi(x,y) = \sum_i \phi_i(x)\varphi_i(y) + \text{KK modes}$$

$$\Psi(x,y) = \sum_i \psi_i(x)\chi_i(y) + \text{KK modes}$$

the same wavefunctions with the same degeneracy

Normalization :

$$\int d^2y \sqrt{g_{2D}}\varphi_i(y)\varphi_j(y)^* = \frac{1}{\left(2\mathrm{Im}(\tau)\right)^{k_i}}\delta_{i,j}$$

$$\int d^2y \sqrt{g_{2D}}\chi_i(y)\chi_j(y)^* = \frac{1}{\left(2\mathrm{Im}(\tau)\right)^{k_i}}\delta_{i,j}$$

4D Kaehler potential :

$$K_{\text{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(\tau)\right)^{k}}$$

Modular weights of 4D matters

$$K_{\mathrm{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(\tau)\right)^k}$$

Corresponding to the normalization of internal wavefuncion

$$\int d^2y \sqrt{g}\, |\varphi(y)|^2 = \frac{1}{\left(2\mathrm{Im}(\tau)\right)^k}$$

Modular trf. :

$$\varphi(y) \to \rho(\gamma)(c\tau + d)^k \varphi(y)$$

$$\int d^2y \sqrt{g}\, |\varphi(y)|^2 \to |c\tau + d|^{2k} \int d^2y \sqrt{g}\, |\varphi(y)|^2$$

$$\left(2\mathrm{Im}(\tau)\right)^{-k} \to |c\tau + d|^{2k} \left(2\mathrm{Im}(\tau)\right)^{-k}$$

Modular weights of 4D matters

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Corresponding to the normalization of internal wavefuncion

$$\int d^2y \sqrt{g}\, |\varphi(y)|^2 = \frac{1}{\left(2\text{Im}(\tau)\right)^k}$$

Modular trf. :

$$\varphi(y) \rightarrow \rho(\gamma)(c\tau + d)^k \varphi(y)$$

Modular weight :

4D matters	$\phi(x), \psi(x)$	: $-k$
Internal fields	$\varphi(y), \chi(y)$	: k

Modular weights of 4D matters

$$K_{\mathrm{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(\tau)\right)^k}$$

Corresponding to the normalization of internal wavefuncion

$$\int d^2y \sqrt{g}\, |\varphi(y)|^2 = \frac{1}{\left(2\mathrm{Im}(\tau)\right)^k}$$

For instance,

・ In type IIB magnetized D-brane models on T^2 with magnetic fluxes, internal fields are described by the Jacobi-theta function (weight k= ½)

[Cremades-Ibanez-Marcesano'04; Abe-Kobayashi-Ohki'08,…]

・ In heterotic orbifold models, fields can have fractional weights

[Lauer-Mas-Nilles, (1989, 1991); Ferrara-Lust-Theisen (1989); Dixon-Kaplunovsky-Louis (1990);…]

Modular weights of 4D matters

$$K_{\text{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(\tau)\right)^k}$$

Applicable to D-dim.theory on T^{2n} or some manifolds with modualr sym.

$$\int d^d y \sqrt{g}\, |\varphi(y)|^2 = \frac{1}{\left(2\mathrm{Im}(\tau)\right)^k}$$

Modular weights of 4D matters

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$$\int d^d y \sqrt{g}\, |\varphi(y)|^2 = \frac{1}{\left(2\mathrm{Im}(\tau)\right)^k}$$

4D Yukawa couplings ：

$$\bar{\Psi}(x,y)\gamma^m(\partial_m - gA_m(x,y))\Psi(x,y)$$

$$y\,\bar{N}(x)H_u(x)L(x)$$

$$y = g\int dy\, \chi_{\bar{N}}(y)\varphi^*_{H_u}(y)\chi_L(y)$$

Normalized

Modular weights of 4D matters

$$K_{\mathrm{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(\tau)\right)^k} + \Delta K$$

Certain corrections : Chen, Ramos-Sanchez, Ratz, 1909.06910

- higher-dimensional higher-derivative and interaction terms

- loop corrections, correction from flavor symmetry breaking,...

Symmetry will be useful to control ΔK

ex., eclectic modular flaor symmetries

H.P. Nilles, S. Ramos-Sanchez, P.K.S. Vaudrevange 2001.01736, 2004.05200; ...

Sign of 4D modular weights ?

$$K_{\text{matter}} = \frac{|\phi(x)|^2}{\left(2\text{Im}(\tau)\right)^k}$$

Question : Sign of modular weights for 4D matter fields ?

$$\phi_i \to (c\tau + d)^{-k_i}\rho_i(\gamma)\phi_i$$

$k > 0$

- controls higher-order corrections in the Kaehler potential, e.x., $\frac{|\phi(x)|^4}{\left(2\text{Im}(\tau)\right)^{2k}}$

  (will correspond to the volume expansion of the torus)

$k < 0$

- may be out of control

Possibilities of 4D positive modular weights (1/2)

$$K_{\mathrm{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(T)\right)^p \left(2\mathrm{Im}(\tau)\right)^k}$$

Suppose that
- Modular flavor symmetry is originated from the modulus τ
- Overall volume is mainly determined by another T

Possibilities of 4D positive modular weights (1/2)

$$K_{\text{matter}} = \frac{|\phi(x)|^2}{\left(2\text{Im}(T)\right)^p \left(2\text{Im}(\tau)\right)^k}$$

Suppose that
- Modular flavor symmetry is originated from the modulus τ
- Overall volume is mainly determined by another T

Consider $E_8 \times E_8'$ heterotic string with standard embedding
On $T^6/\mathbb{Z}_3$ orbifold, kinetic terms of 27 matters ϕ^a [Ferrara-Kounnas-Porrati '86, Cvetic-Louis-Ovrut '88]

$$K = -\ln\, \det\left(2\text{Im}(\mathbb{T})\right) + \left(2\text{Im}\,(\mathbb{T})\right)_{a\bar{b}} \phi^a \bar{\phi}^{\bar{b}}$$

After diagonalizing $K_{a\bar{b}}$

$$K_{\text{matter}} \simeq \frac{2\text{Im}(\tau)}{2\text{Im}(T)} |\phi^a|^2$$

$$\mathbb{T} = \begin{pmatrix} T^1 & T^4 & T^5 \\ T^7 & T^2 & T^6 \\ T^8 & T^9 & T^3 \end{pmatrix}$$

$$T \equiv T^1 = T^2 = T^3$$
$$\tau \equiv T^4 = \cdots = T^9$$

Possibilities of 4D positive modular weights (1/2)

$$K_{\mathrm{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(T)\right)^p \left(2\mathrm{Im}(\tau)\right)^k}$$

Suppose that
- Modular flavor symmetry is originated from the modulus τ
- Overall volume is mainly determined by another T

On $T^6/\mathbb{Z}_3$ orbifold, kinetic terms of 27 matters ϕ^a

$$K_{\mathrm{matter}} \simeq \frac{2\mathrm{Im}(\tau)}{2\mathrm{Im}(T)} |\phi^a(x)|^2$$

- control matter Kaehler potential against higher-order corrections
- moderate hierarchy between overall volume and local cycle volume is important in realizing the 4D positive modular weight

Possibilities of 4D positive modular weights (2/2)

$$K_{\mathrm{matter}} = \frac{|\phi(x)|^2}{\left(2\mathrm{Im}(\tau)\right)^k}$$

In contrast to bulk modes, twisted modes (localized modes) may have a chance to have the 4D positive modular weights

[Dixon-Kaplunovsky-Louis (1990), Ibanez-Lust ’92, Kawabe-Kobayashi-Ohtsubo ‘94]

- The ground states of twisted modes on $T^2/\mathbb{Z}_N$ have the modular weights $-1/N$ for 4D matter fields

See talk by Saul

- The oscillators shift the modular weight by ± 1

- The matter Kaehler potential would be controlled in stringy calculations

A new lepton model building with positive modular weights

	Leptons				Higgs	
	(L_e, L_μ, L_τ)	$(\bar{e}, \bar{\mu}, \bar{\tau})$	$\bar{N}$	S	H_u	H_d
$SU(2)_L$	**2**	**1**	**1**	**1**	**2**	**2**
$U(1)_Y$	$-\frac{1}{2}$	$+1$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
A_4	$\{1\}$	$\{\bar{1}\}$	$\{\bar{1}\}$	3	1	1
$-k$	$+1$	-1	-1	-1	0	0

- 4D positive modular weights enable us to construct Inverse Seesaw and Linear Seesaw scenarios without any additional symmetries [Kobayashi-Nomura-Okada-Otsuka, ’23]

- Positive and negative modular weights → large kinetic mixing

- Positive modular weights will give new insights into the flavor structure of quarks/leptons and higher-dimensional operators in SMEFT

[Kobayashi-Otsuka-Tanimoto-Yamamoto ’21,22]

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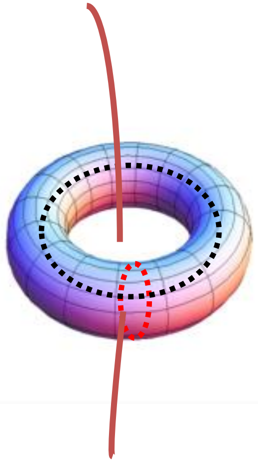
Magnetic fluxes and chiral fermions

- Let us consider 6D U(N) SYM on $R^{1,3} \times T^2$

$$L = -\frac{1}{4g^2}\ \mathrm{Tr}\,(F^{mn}F_{mn}) + \frac{i}{2g^2}\ \mathrm{Tr}\left(\bar{\lambda}\Gamma^m D_m \lambda\right)$$



$m, n = 0,1,2,3,4,5$

[E.Witten'84]

Constant U(1) magnetic flux F on T^2

$$\int_{T^2} F = M \ :\ \ \text{Integer}$$

Magnetic fluxes and chiral fermions

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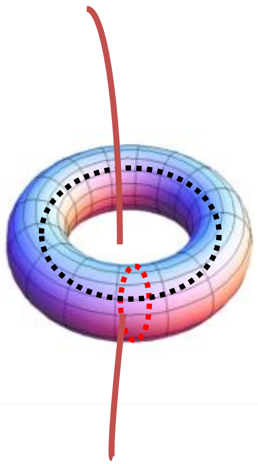
$$L = -\frac{1}{4g^2}\ \mathrm{Tr}\,(F^{mn}F_{mn}) + \frac{i}{2g^2}\ \mathrm{Tr}\,(\bar{\lambda}\Gamma^m D_m \lambda)$$



$m,n = 0,1,2,3,4,5$

Constant U(1) magnetic flux F on T^2

[E.Witten'84]

$$F_{45} = 2\pi \begin{pmatrix} M_a \mathbf{1}_{N_a \times N_a} & 0 \\ 0 & M_b \mathbf{1}_{N_b \times N_b} \end{pmatrix}.$$

$N = N_a + N_b$

$M_a, M_b \in \boldsymbol{Z}$

→ (i) Gauge symmetry breaking $U(N) \to U(N_a) \times U(N_b)$

(ii) Chiral fermions

Bi-fundamental field

$$\lambda(x,y) = \begin{pmatrix} \lambda^{aa}(x,y) & \lambda^{ab}(x,y) \\ \lambda^{ba}(x,y) & \lambda^{bb}(x,y) \end{pmatrix}.$$

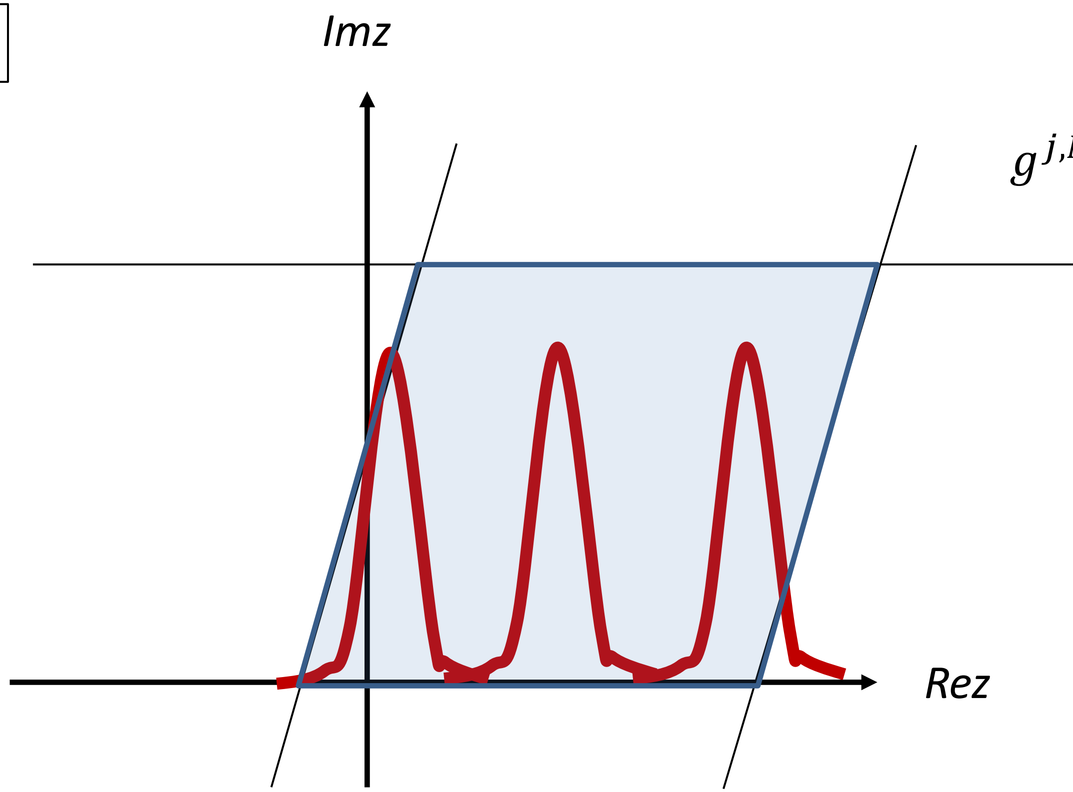
M degenerate zero-modes on T^2

[Cremades-Ibanez-Marcesano'04
Abe-Kobayashi-Ohki'08,…]

- After KK decomposition, zero-modes are quasi-localized on T^2
- Zero-mode solution is given by Jacobi-theta function
- Yukawa couplings (Jacobi-theta function)

modular forms with half-integral modular weights

$M = 3$



$$g^{j,M}(z) \propto \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (Mz, M\tau)$$

$$0 \leq j \leq M-1$$

Modular transformation on T^2

Kobayashi, Nagamoto '17, Kobayashi, Tamba '18;,⋯
Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla 2102.11286

Zero-mode wave functions :

$$\psi^{\tilde{\alpha},|M|}(z+\zeta,\tau)=\left(\frac{|M|}{A^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{Im(z+\zeta)}{Im\tau}}\vartheta\begin{bmatrix}\frac{\tilde{\alpha}}{M}\\0\end{bmatrix}(|M|(z+\zeta),|M|\tau),$$

(M: magnetic flux , $\tilde{\alpha}=0,\dots,|M|-1$)

S-transformation :

$$\psi^{\tilde{\alpha},|M|}(z+\zeta,\tau)\to\psi^{\tilde{\alpha},|M|}\left(-\frac{z+\zeta}{\tau},-\frac{1}{\tau}\right)=(-\tau)^{1/2}e^{i\pi\frac{3|M|+1}{4}}\sum_{\tilde{\beta}=0}^{|M|-1}\frac{1}{\sqrt{|M|}}e^{2\pi i\frac{\tilde{\alpha}\tilde{\beta}}{|M|}}\psi^{\tilde{\beta},|M|}(z+\zeta,\tau),$$

automorphy factor

$= \rho(S)$

T-transformation :

$$\psi^{\tilde{\alpha},|M|}(z+\zeta,\tau)\to\psi^{\tilde{\alpha},|M|}\left(z+\zeta+\frac{1}{2},\tau+1\right)=e^{i\pi|M|\frac{Im\,(z+\zeta)}{2Im\,\tau}}\sum_{\tilde{\beta}=0}^{|M|-1}e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)}\delta_{\tilde{\alpha},\tilde{\beta}}\,\psi^{\tilde{\beta},|M|}(z+\zeta,\tau),$$

$= \rho(T)$

Modular transformation on $T^2/\mathbb{Z}_2$

$\mathbb{Z}_2$-even and -odd modes :

$$\psi_{\mathrm{even}}^{\tilde{\alpha},|M|} \propto \psi_{\mathrm{T}^2}^{\tilde{\alpha},|M|}(z) + \psi_{\mathrm{T}^2}^{\tilde{\alpha},|M|}(-z) \qquad \psi_{\mathrm{odd}}^{\tilde{\alpha},|M|} \propto \psi_{\mathrm{T}^2}^{\tilde{\alpha},|M|}(z) - \psi_{\mathrm{T}^2}^{\tilde{\alpha},|M|}(-z)$$

Modular trf. for $\mathbb{Z}_2$-even modes

$$\rho(\tilde{S})_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{\sqrt{|M|}} \mathrm{e}^{i\pi\frac{3|M|+1}{4}} \cos\left(\frac{2\pi\tilde{\alpha}\tilde{\beta}}{|M|}\right), \qquad \rho(\tilde{T})_{\tilde{\alpha}\tilde{\beta}} = e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)} \delta_{\tilde{\alpha},\tilde{\beta}}$$

Modular trf. for $\mathbb{Z}_2$-odd modes

$$\rho(\tilde{S})_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{\sqrt{|M|}} \mathrm{e}^{i\pi\frac{3|M|+1}{4}} \sin\left(\frac{2\pi\tilde{\alpha}\tilde{\beta}}{|M|}\right), \qquad \rho(\tilde{T})_{\tilde{\alpha}\tilde{\beta}} = e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)} \delta_{\tilde{\alpha},\tilde{\beta}}$$

Metaplectic modular flavor symmetry

Finite metaplectic modular symmetry $\tilde{\Gamma}_{4N} = \tilde{\Gamma}/\tilde{\Gamma}(4N)$

$\tilde{\Gamma} \equiv Mp(2,\mathbb{Z})$: Metaplectic group

The generators of $\tilde{\Gamma}_{4N}$

$$\tilde{S}^2 = \tilde{R}, \qquad (\tilde{S}\tilde{T})^3 = \tilde{R}^4 = \mathbb{I}, \qquad \tilde{T}\tilde{R} = \tilde{R}\tilde{T}, \qquad \tilde{T}^{4N} = \mathbb{I},$$

For $N > 1$, additional relations are required to ensure the finiteness, e.g.,

$$\tilde{S}^5\tilde{T}^6\tilde{S}\tilde{T}^4\tilde{S}\tilde{T}^2\tilde{S}\tilde{T}^4 = \mathbb{I}, \qquad (\text{for } N = 2)$$

ex.,,Liu, Yao, Qu, Ding, 2007.13706

$M = 4$:

Modular trf. for three $\mathbb{Z}_2$-even modes

$$\rho(\widetilde{S}_{\rm even}) = -\frac{1}{2}\begin{pmatrix} (-1)^{1/4} & 1+i & (-1)^{1/4} \\ 1+i & 0 & -1-i \\ (-1)^{1/4} & -1-i & (-1)^{1/4} \end{pmatrix}, \qquad \rho(\widetilde{T}_{\rm even}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(-1)^{1/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Modular trf. for a $\mathbb{Z}_2$-odd mode

$$\rho(\widetilde{S}_{\rm odd}) = (-1)^{3/4}, \qquad \rho(\widetilde{T}_{\rm odd}) = -(-1)^{1/4}$$

Representation matrix of $\tilde{\Gamma}_8$

Metaplectic modular flavor symmetry

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For $N > 1$, additional relations are required to ensure the finiteness, e.g.,

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ex.,,Liu, Yao, Qu, Ding, 2007.13706

M	2	3	4	5	6
$\tilde{\Gamma}_N$	$\tilde{\Gamma}_4$	$\tilde{\Gamma}_{12}$	$\tilde{\Gamma}_8$	$\tilde{\Gamma}_{20}$	$\tilde{\Gamma}_{12}$

K. Ishiguro, T. Kai, H. Okada, H. O., 2310.10091

Pati-Salam-like modes on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$

K. Ishiguro, T. Kai, H. Okada, H. O., 2310.10091

D−brane configuration (visible sector) :

N_α	Gauge group	(m_α^1, n_α^1)	(m_α^2, n_α^2)	$(\tilde{m}_\alpha^3, n_\alpha^3)$
$N_a = 8$	$U(4)_C$	$(0, -1)$	$(1, 1)$	$(1/2, 1)$
$N_b = 4$	$U(2)_L$	$(M, 1)$	$(1, 0)$	$(1/2, -1)$
$N_c = 4$	$U(2)_R$	$(M, -1)$	$(0, 1)$	$(1/2, -1)$

n_a^i: wrapping number on $(T^2)_i$
m_a^i: units of magnetic flux on $(T^2)_i$

$$\tilde{m}_\alpha^3 = m_\alpha^3 + \frac{1}{2} n_\alpha^3$$

The magnetic flux M on the first torus
determines the flavor structure of quarks/leptons

e.g., $M = 4$ → three $\mathbb{Z}_2$−even modes controlled by $\tilde{\Gamma}_8$

Eclectic Flavor symmetry on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$

Heterotic string on toroidal orbifolds:
H.P. Nilles, S. Ramos-Sanchez, P.K.S. Vaudrevange 2001.01736, 2004.05200

See talk by Saul

1. Metaplectic modular symmetry ($\mathrm{G}_{\mathrm{modular}} = \tilde{\Gamma}_8$ for $M = 4$)

Eclectic Flavor symmetry on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$

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H.P. Nilles, S. Ramos-Sanchez, P.K.S. Vaudrevange 2001.01736, 2004.05200

1. Metaplectic modular symmetry ($\mathrm{G}_{\mathrm{modular}} = \tilde{\Gamma}_8$ for $M = 4$)

2. Traditional flavor symmetry ($\mathrm{G}_{\mathrm{flavor}} \equiv \mathbb{Z}_4 \times \mathbb{Z}_2^{\mathrm{P}} \times \mathbb{Z}_2^{\mathrm{C}} \times \mathbb{Z}_2^{\mathrm{Z}}$)

$\mathbb{Z}_2$−even mode

$$\rho(Z'_{\mathrm{even}}) = i\mathbb{I}_3, \qquad \rho(P_{\mathrm{even}}) = \mathbb{I}_3, \qquad \rho(C_{\mathrm{even}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho(Z_{\mathrm{even}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathbb{Z}_2$−odd mode

$$\rho(Z'_{\mathrm{odd}}) = i, \qquad \rho(P_{\mathrm{odd}}) = -1, \qquad \rho(C_{\mathrm{odd}}) = -1, \qquad \rho(Z_{\mathrm{odd}}) = -1,$$

Eclectic Flavor symmetry on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$

Heterotic string on toroidal orbifolds: See talk by Saul
H.P. Nilles, S. Ramos-Sanchez, P.K.S. Vaudrevange 2001.01736, 2004.05200

1. Metaplectic modular symmetry ($\mathrm{G}_{\mathrm{modular}} = \tilde{\Gamma}_8$ for $M = 4$)

2. Traditional flavor symmetry ($\mathrm{G}_{\mathrm{flavor}} \equiv \mathbb{Z}_4 \times \mathbb{Z}_2^{\mathrm{P}} \times \mathbb{Z}_2^{\mathrm{C}} \times \mathbb{Z}_2^{\mathrm{Z}}$)

$\mathbb{Z}_2$−even mode

$$\rho(Z'_{\mathrm{even}}) = i\mathbb{I}_3, \qquad \rho(P_{\mathrm{even}}) = \mathbb{I}_3, \qquad \rho(C_{\mathrm{even}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho(Z_{\mathrm{even}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathbb{Z}_2$−odd mode

$$\rho(Z'_{\mathrm{odd}}) = i, \qquad \rho(P_{\mathrm{odd}}) = -1, \qquad \rho(C_{\mathrm{odd}}) = -1, \qquad \rho(Z_{\mathrm{odd}}) = -1,$$

3. CP symmetry ($G_{\mathrm{CP}} \equiv \mathbb{Z}_2^{\mathrm{CP}}$)

$$\psi^{\tilde{\alpha},|M|} \to \varphi(\widetilde{CP},\tau)\rho(\widetilde{CP})_{\tilde{\alpha}\tilde{\beta}}\overline{\psi^{\tilde{\beta},|M|}(z,\tau)} \qquad \varphi(\widetilde{CP},\tau) = i, \qquad \rho(\widetilde{CP})_{\tilde{\alpha}\tilde{\beta}} = -i\delta_{\tilde{\alpha},\tilde{\beta}}.$$

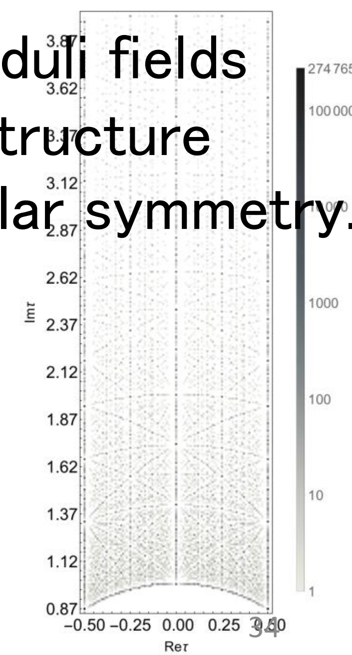
$$\left(G_{\mathrm{flavor}} \rtimes G_{\mathrm{modular}}\right) \rtimes G_{\mathrm{CP}}$$

Eclectic Flavor symmetry on $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$

The traditional flavor, modular flavor and CP symmetries are uniformly described in the context of eclectic flavor symmetry

$$(G_{\text{flavor}} \rtimes G_{\text{modular}}) \rtimes G_{\text{CP}}$$

Modular symmetry is broken by the stabilization of moduli fields
In flux compactifications, the distribution of complex structure moduli VEVs clusters at fixed points of $SL(2,\mathbb{Z})$ modular symmetry.



Ishiguro, Kobayashi, Otsuka 2011.09154

Outline

1. Modular weights

S. Kikuchi, T. Kobayashi, H. O., M. Tanimoto, H. Uchida, 2201.04505
T. Kobayashi, T. Nomura, H. Okada, H. O., 2310.10091

2. Metaplectic modular symmetries

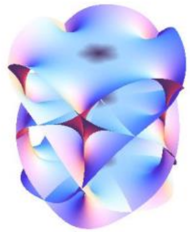
K. Ishiguro, T. Kai, H. Okada, T. Kobayashi, T. Nomura, H. Okada, H. O., 2310.10091

3. $\Gamma_0(N)$ modular symmetries

T. Kobayashi, H. O., 2004.04518
K. Ishiguro, T. Kai, T. Kobayashi, H. O., 2311.12425
K. Ishiguro, T. Kobayashi, S. Nishimura, H. O., 2402.13563

4D SUSY E_6 GUT from Heterotic string on 6D Calabi-Yau

Candelas-Horowitz-Strominger-Witten ('85)



- 4D gauge symmetry :

$$E_8 \times E_8^{(\mathrm{hidden})} \to E_6 \times SU(3) \times E_8^{(\mathrm{hidden})}$$

- Matters (E_6 : 27 or $\overline{27}$) $\approx$ Moduli

$$27^i \approx \text{Kahler Moduli } t^i$$

(2-cycle volume)

$(i = 1,2, \ldots, h^{1,1})$

- Yukawa couplings ($27^{\mathbf{3}}$)

$$W = F_{ijk} 27^i 27^j 27^k$$

$F_{ijk} = \partial_{t^i} \partial_{t^j} \partial_{t^k} F$ （$F(t)$: prepotential）

Flavor structure is controlled by $\mathrm{Sp}(2h^{1,1} + 2, \mathbb{Z})$ modular symmetry

Ishiguro-Kobayashi-Otsuka, 2107.00487

- Yukawa couplings

Classical level

Prepotential : $F = \frac{\kappa_{ijk}}{6} t^i t^j t^k$

→ Constant Yukawa coupling : $\partial_i \partial_j \partial_k F = \kappa_{ijk}$

$$W = \kappa_{ijk} 27^i 27^j\, 27^k$$

e.g., S^4 symmetry in *2107.00487*

Quantum level

Prepotential : $F = \frac{\kappa_{ijk}}{6} t^i t^j t^k + O(e^{2\pi i t})$ Instanton effects

→ Yukawa coupling :$\partial_i \partial_j \partial_k F = \kappa_{jik} + O(e^{2\pi i t})$

- Instanton effects will lead to modular forms

K. Ishiguro, T. Kobayashi, S. Nishimura, H.O., 2402.13563

Instanton-corrected Yukawa couplings on 6D CY

Hosono, Klemm, Theisen, Yau,
9308122, 9406055;...

$$y_{ijk} = \kappa_{ijk} + \sum_{d_1, d_2, \ldots, d_n = 0}^{\infty} \frac{(d_i d_j d_k) n_{d_1, d_2, \ldots, d_m}}{1 - \prod_{l=1}^{m} q_l^{d_l}} \prod_{l=1}^{m} q_l^{d_l}$$

$$q_l \equiv e^{2\pi i t_l}$$

Gromov-Witten invariants

We discuss two examples :
- (a part of) $SL(2,\mathbb{Z})$ modular symmetry
  emerges in asymptotic regions of the CY moduli space

Ex.1 Instanton-corrected Yukawa couplings

$P^{1,1,1,6,9}[18]$ with two Kahler moduli（ $h^{1,1}=2$ ）

- Moduli Kaehler potential :

$$K=-\ln\left[i\left(\frac{3}{8}(t-\bar{t})^3+\frac{1}{2}(t-\bar{t})(s-\bar{s})^2\right)\right]$$

$\mathrm{Im}(s)\gg\mathrm{Im}(t)$

$$K\simeq-\ln(t-\bar{t})-2\ln(s-\bar{s})+\cdots$$

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$$K\simeq-\ln(t-\bar{t})-2\ln(s-\bar{s})+\cdots$$

- Matter Kaehler metric :

Dixon-Kaplunovsky-Louis ('90)

$$K^{(27)}_{t\bar{t}}\sim e^{-\frac{K}{3}}K_{t\bar{t}}\propto\frac{1}{(t-\bar{t})^{5/3}}$$

$$K^{(27)}_{s\bar{s}}\sim e^{-\frac{K}{3}}K_{s\bar{s}}\propto(t-\bar{t})^{1/3}$$

- Matter modular weight :

$$A^{(27)}_t:-\mathrm{k}=-5/3$$

$$A^{(27)}_s:-\mathrm{k}=1/3$$

Ex.1 Instanton-corrected Yukawa couplings

$P^{1,1,1,6,9}[18]$ with two Kahler moduli ($h^{1,1}=2$)

- Yukawa couplings :

$$y_{ijk}=\kappa_{ijk}+\sum_{d_1,d_2=0}^{\infty} c_{ijk}(d_1,d_2)n_{d_1,d_2}\frac{q_1^{d_1}q_2^{d_2}}{1-q_1^{d_1}q_2^{d_2}}$$

$q_1=e^{2\pi it}$

$q_2\simeq e^{2\pi is}\to 0$

Candelas-Font-Katz-Morrison, 9403187

n_{d_1,d_2} :

$d_1 \setminus d_2$	0	1	2	3
0		3	-6	27
1	540	-1080	2700	-17280
2	540	143370	-574560	5051970
3	540	204071184	74810520	-913383000

Table 1: Instanton numbers up to $d_1,d_2\le 3$.

$y_{ttt}=\frac{9}{4}E_4(t)$ (weight 4)

$y_{tss}=1$, otherwise 0

$E_4(t)=1+240\sum_{k=0}^{\infty}\frac{k^3q^k}{1-q^k}$

The action is invariant under $SL(2,\mathbb{Z})_t$: $t\to\frac{at+b}{ct+d}$

Ex.2 Instanton-corrected Yukawa couplings

Two Kahler moduli（ $h^{1,1} = 2$ ）: $\begin{matrix} \mathbb{CP}^2 \\ \mathbb{CP}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

- Moduli Kahler potential :

$$K = -\ln\left[i\left(-\frac{9}{4}(t-\bar{t})^3 + \frac{1}{2}(t-\bar{t})(s-\bar{s})^2\right)\right]$$

$\mathrm{Im}(s) \gg \mathrm{Im}(t)$

$$K \simeq -\ln(t-\bar{t}) - 2\ln(s-\bar{s}) + \cdots$$

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Two Kahler moduli ($h^{1,1} = 2$) : $\begin{matrix} \mathbb{CP}^2 \\ \mathbb{CP}^2 \end{matrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

- Yukawa couplings :

$$y_{ijk} = \kappa_{ijk} + \sum_{d_1,d_2=0}^{\infty} c_{ijk}(d_1,d_2) n_{d_1,d_2} \frac{q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}}$$

$q_1 = e^{2\pi i t}$

$q_2 \simeq e^{2\pi i s} \to 0$

n_{d_1,d_2} :

$d_1 \setminus d_2$	0	1	2	3	4	5	6
0		189	189	162	189	189	162
1	189	8262	142884	1492290	11375073	69962130	
2	189	142884	13108392	516953097	12289326723		
3	162	1492290	516953097	55962304650			
4	189	11375073	12289326723				
5	189	69962130					
6	162						

Table 3: Instanton numbers up to bidegree $d_1 + d_2 \leq 6$. Note that there exists the exchange symmetry $n_{d_1,d_2} = n_{d_2,d_1}$.

$$y_{ttt} = \frac{63}{80} E_4(t) - \frac{243}{80} E_4(3t)$$

$$y_{tss} = 9$$

($\Gamma_0(3)$ modular form of weight 4)

$$E_4(t) = 1 + 240 \sum_{k=0}^{\infty} \frac{k^3 q^k}{1-q^k}$$

The action is invariant under $\Gamma_0(3)_t$: $t \to \frac{at+b}{ct+d}$ $\quad c \equiv 0 \pmod 3$

Conclusion

- Modular weights

  - important to control the matter Kaehler potential
  - Positive and negative modular weights → large kinetic mixing
  - new insights into the flavor structure of quarks/leptons

- Metaplectic modular symmetries, eclectic flavor symmetries

  - naturally realized in type IIB magnetized D-brane models
  - important to control the Kaehler potential

- $\Gamma_0(N)$ modular symmetries

  - Heterotic Calabi-Yau compactifications K. Ishiguro, T. Kobayashi, S. Nishimura, H. O., 2402.13563
  - Type IIB flux compactifications on $\mathrm{T}^6/\mathbb{Z}_2 \times \mathbb{Z}_2'$, $\mathbb{Z}_{6-II}$ orbifolds

T. Kobayashi, H. O., 2004.04518
K. Ishiguro, T. Kai, T. Kobayashi, H. O., 2311.12425