What can **modular flavour symmetries** do for you?

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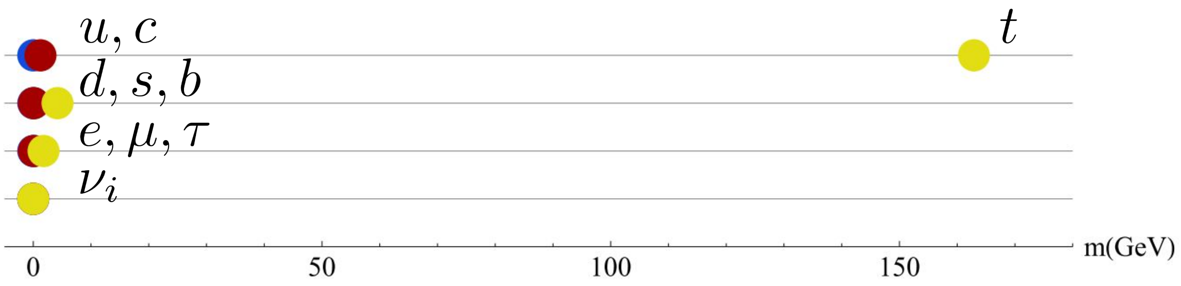
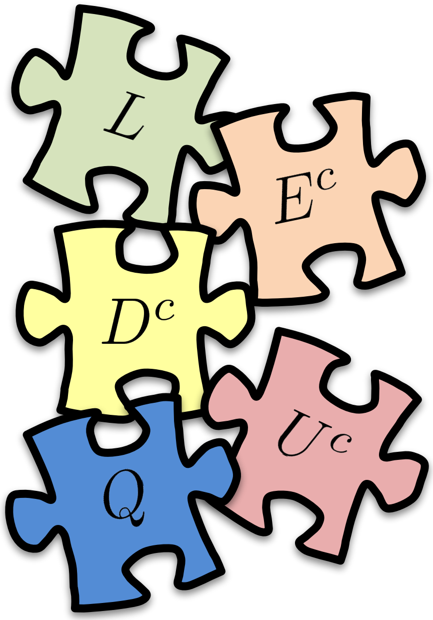
Mainz, ITP Seminar, Workshop on Modular Flavour Symmetries

14 May 2024

The flavour puzzle





adapted from R. Toorop's PhD thesis

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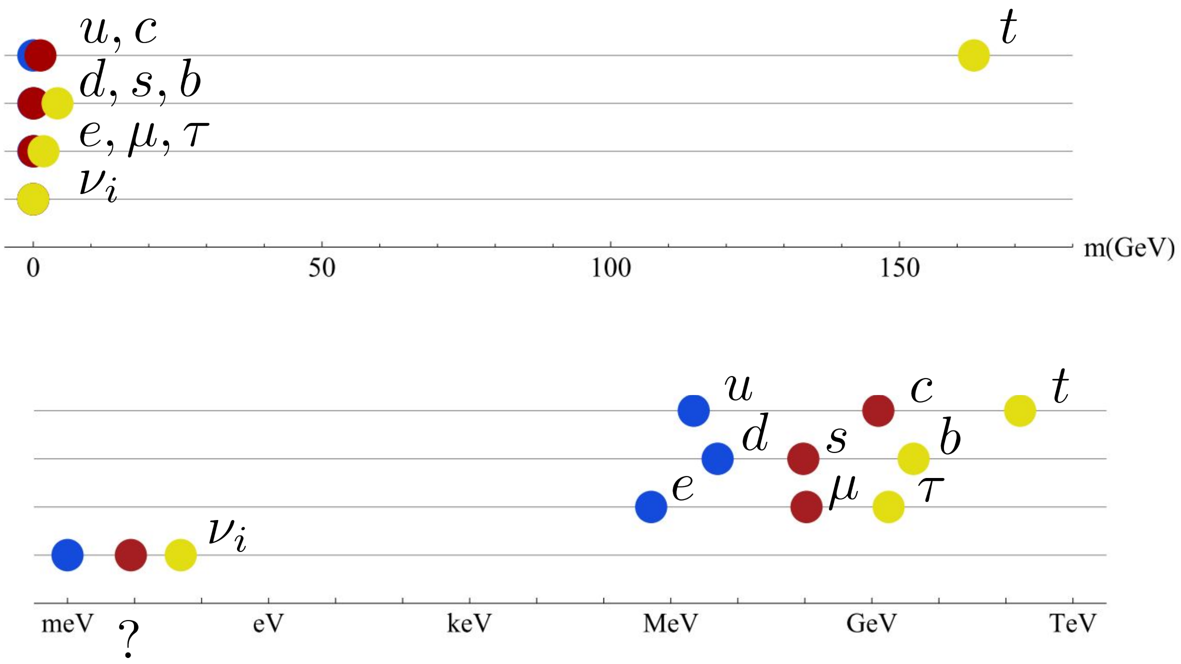

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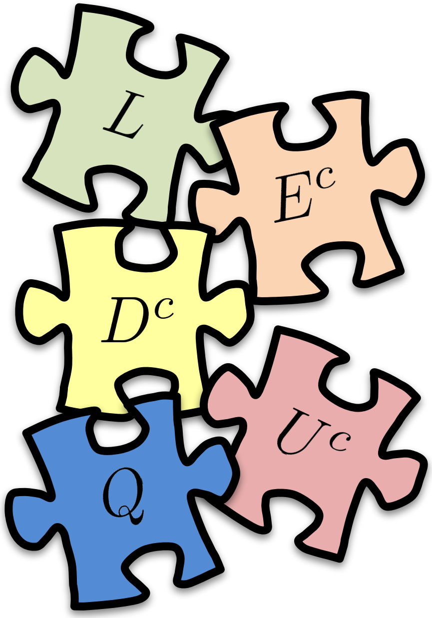
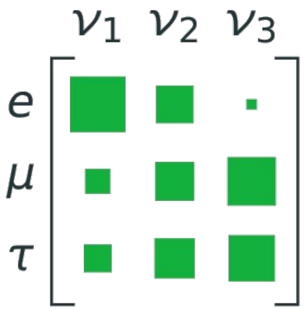
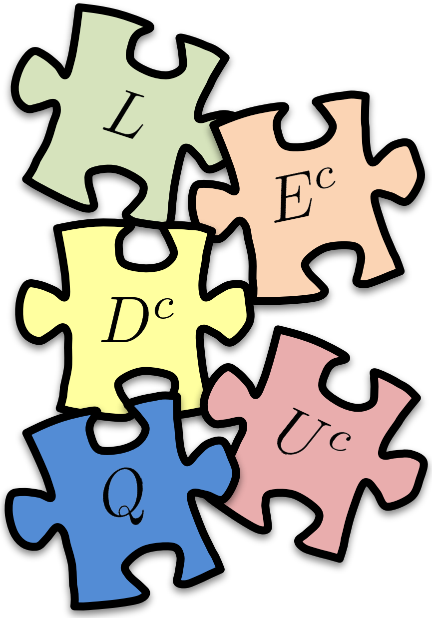


The flavour puzzle

$U_{\text{PMNS}} \sim$

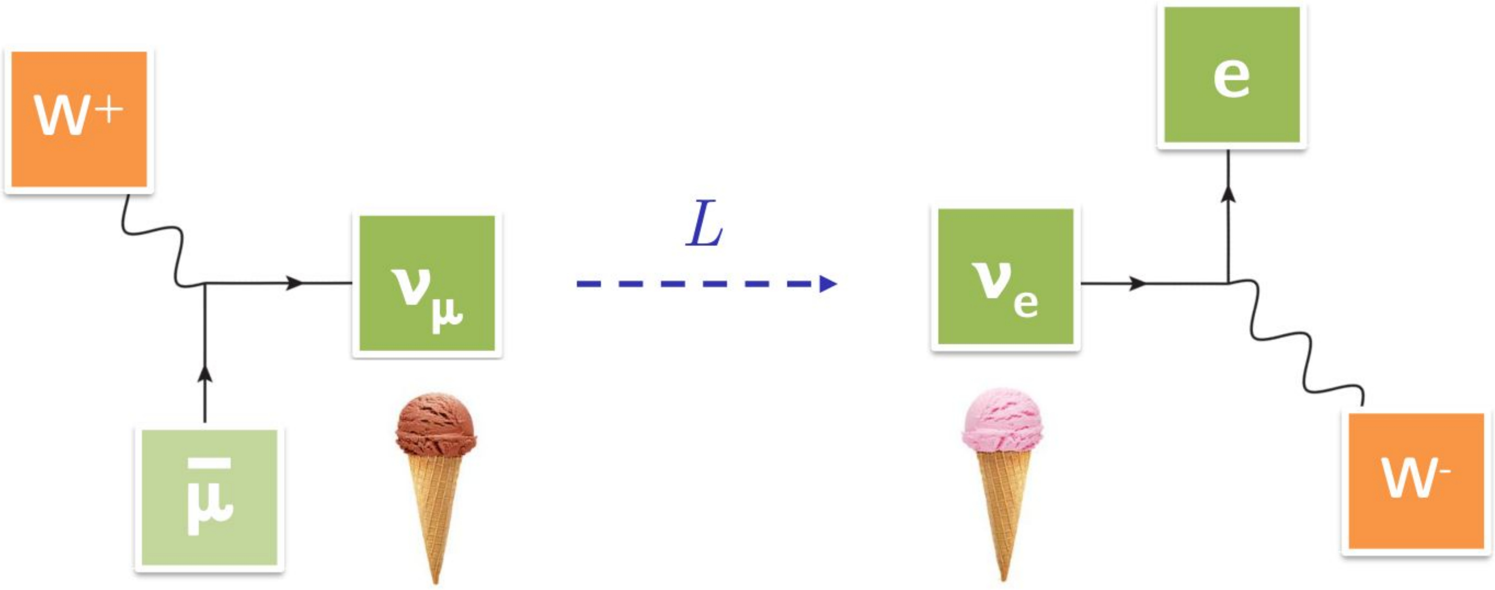




adapted from P. Novichkov's slides at PASCOS 2021

The flavour puzzle

Neutrinos are temperamental...



Neutrino oscillation: propagating neutrinos can change their flavour

The flavour puzzle

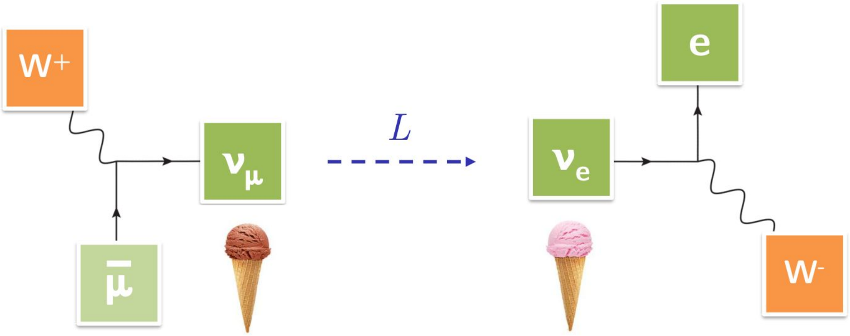
Neutrinos are temperamental…

Mixing matrix elements

$$|\nu_e\rangle = U^*_{e1}|\nu_1\rangle + U^*_{e2}|\nu_2\rangle + U^*_{e3}|\nu_3\rangle$$

$$|\nu_\mu\rangle = U^*_{\mu 1}|\nu_1\rangle + U^*_{\mu 2}|\nu_2\rangle + U^*_{\mu 3}|\nu_3\rangle$$

$$|\nu_\tau\rangle = U^*_{\tau 1}|\nu_1\rangle + U^*_{\tau 2}|\nu_2\rangle + U^*_{\tau 3}|\nu_3\rangle$$



In a *2-neutrino approximation*,

$$P(\nu_\mu \to \nu_\tau) = \sin^2 2\theta \ \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

depends on the elements *(U)αi* of the PMNS

difference of squares
of neutrino masses *mi*

$$\boxed{\Delta m^2 \neq 0}$$

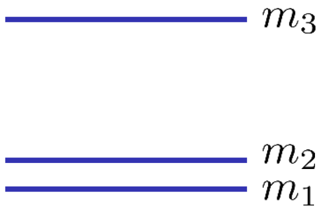
3ν flavour paradigm

Masses: ordering

$$\boxed{\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}}$$

Normal ordering (NO)

$m_1 < m_2 < m_3$

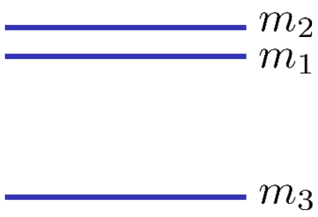


?

vs.

Inverted ordering (IO)

$m_3 < m_1 < m_2$



?

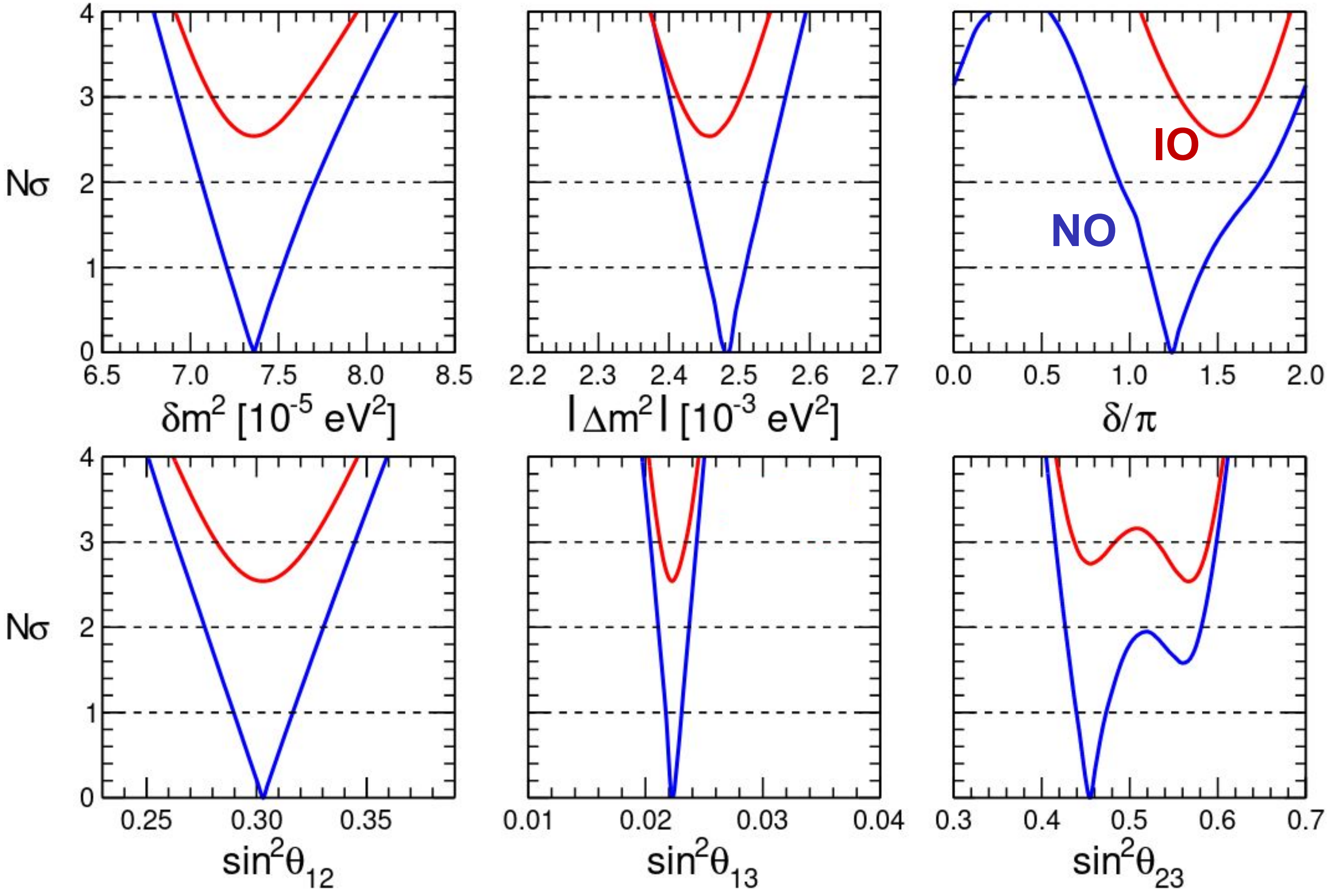
Mixing matrix parameterisation

$c_{ij} \equiv \cos\theta_{ij},\ s_{ij} \equiv \sin\theta_{ij}$

$$U_{\text{PMNS}} = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & e^{i\alpha_{21}/2} & \\ & & e^{i\alpha_{31}/2} \end{pmatrix}$$

3ν flavour paradigm

from Capozzi et al. [2107.00532],
see also València [2006.11237], NuFIT [2007.14792]



$\sim 2.5\sigma$

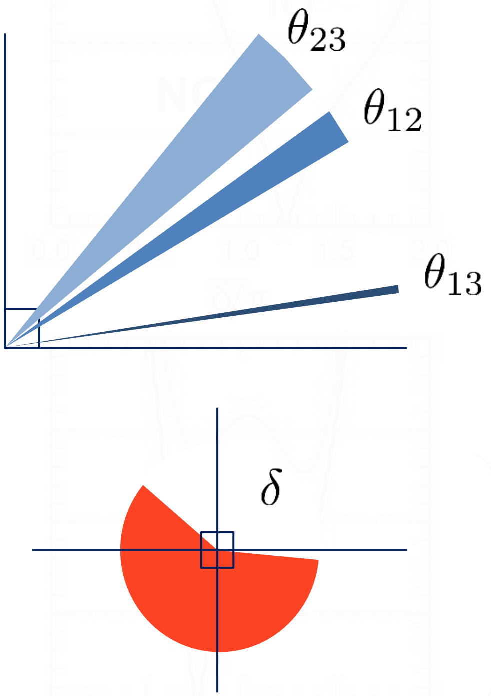
mostly driven
by SK atm. data

3ν flavour paradigm

from Capozzi et al. [2107.00532],
see also València [2006.11237], NuFIT [2007.14792]

For a spectrum with NO:

Parameter	Best-fit value
$\Delta m^2_\odot$	7.36×10^{-5} eV^2
$\lvert\Delta m^2_{\rm atm}\rvert$	2.49×10^{-3} eV^2
$\sin^2\theta_{12}$	0.303
$\sin^2\theta_{13}$	0.0223
$\sin^2\theta_{23}$	0.455
δ	$1.24\,\pi$



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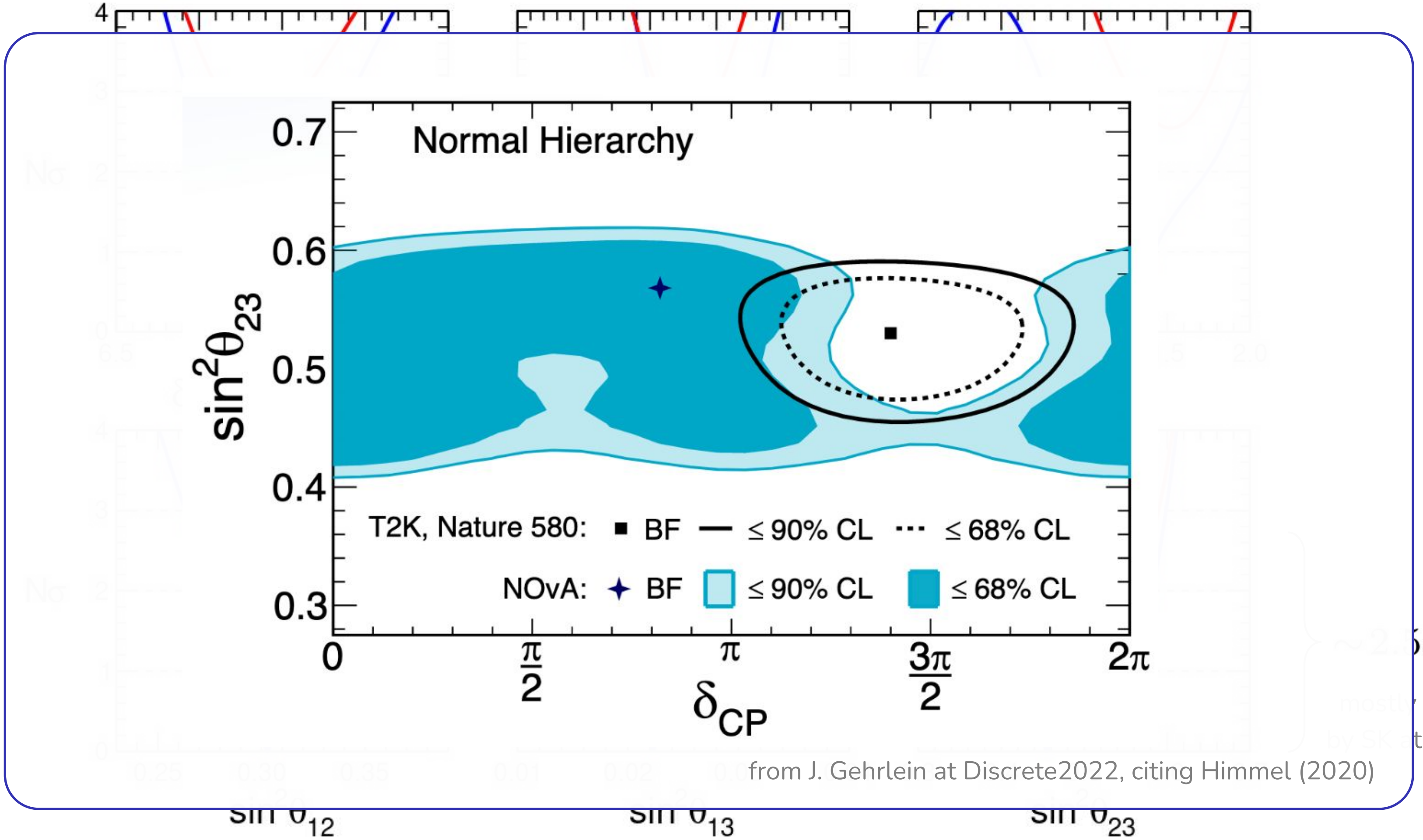

from J. Gehrlein at Discrete2022, citing Himmel (2020)

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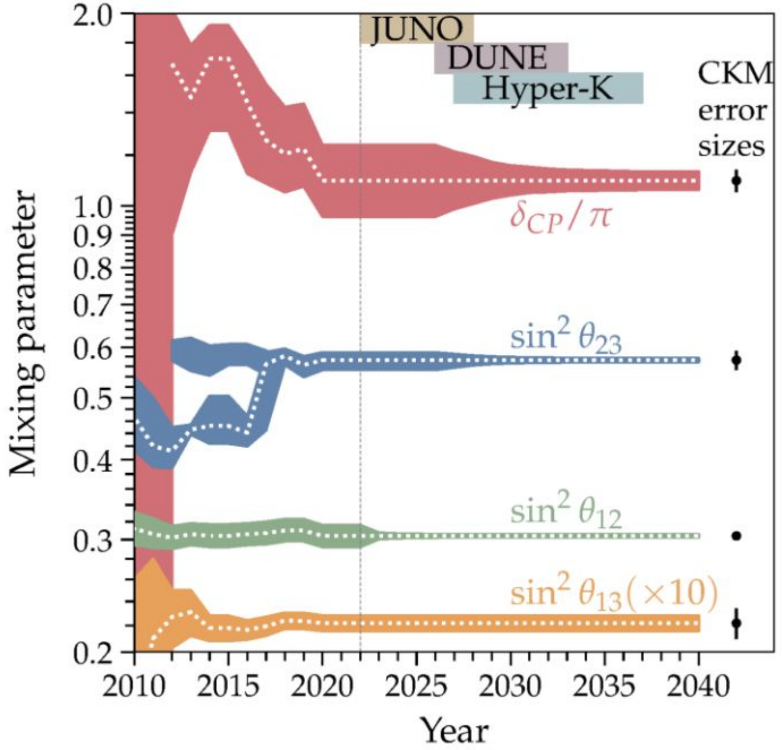


from M.-C. Chen's slides at the Bethe forum, citing [2012.12893]

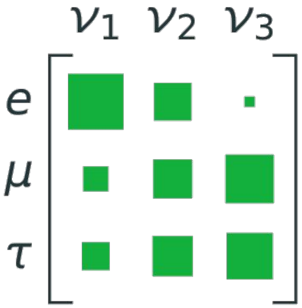
$\sim 2.5\sigma$

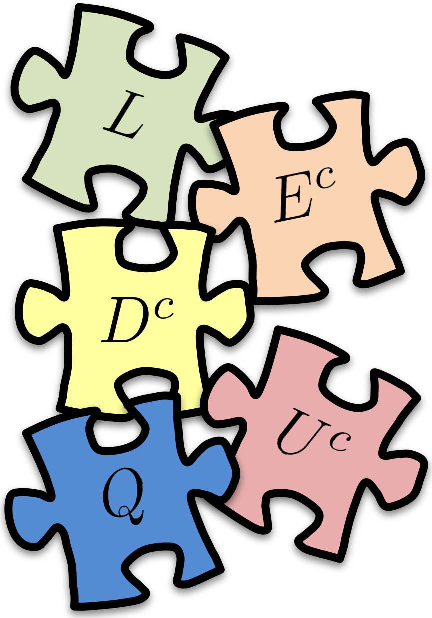
mostly driven by SK atm. data

$\sin^2\theta_{12}$ $\sin^2\theta_{13}$ $\sin^2\theta_{23}$

The flavour puzzle

$U_{\text{PMNS}} \sim$





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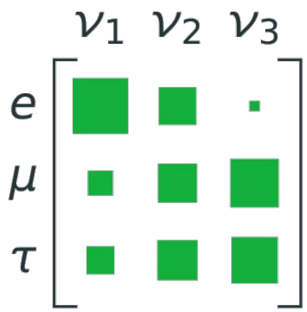
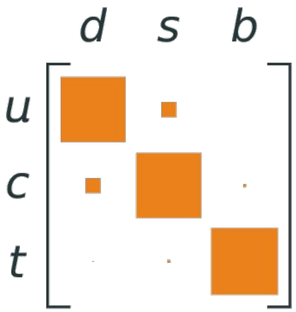
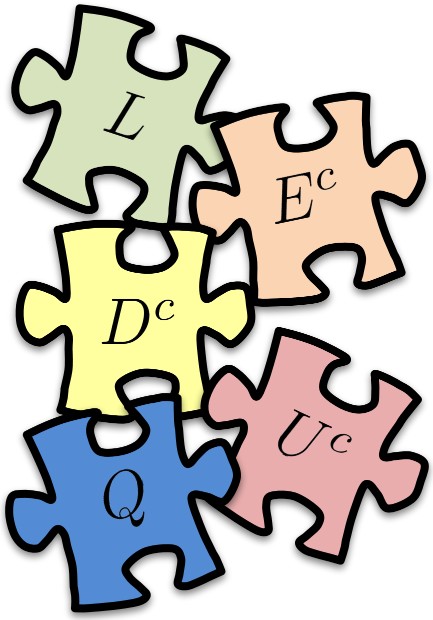


$V_{\text{CKM}} \sim$





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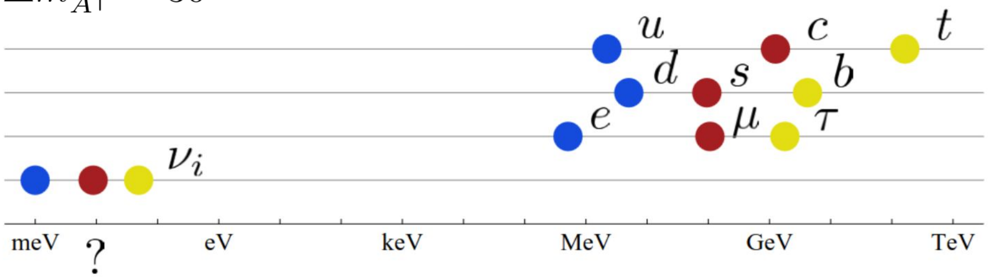
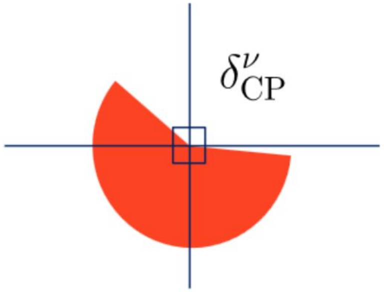
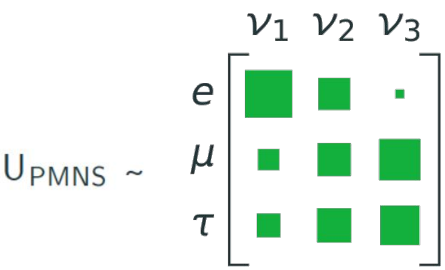
Motivation

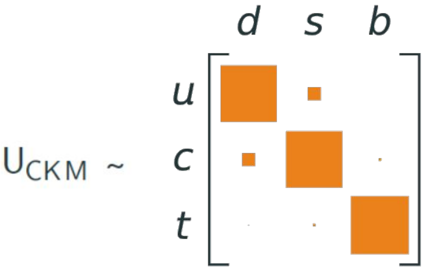
In search of an organising principle...

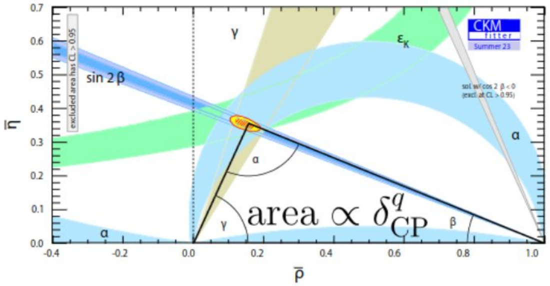
$$\frac{\Delta m^2_{\odot}}{|\Delta m^2_A|} \sim \frac{1}{30}$$





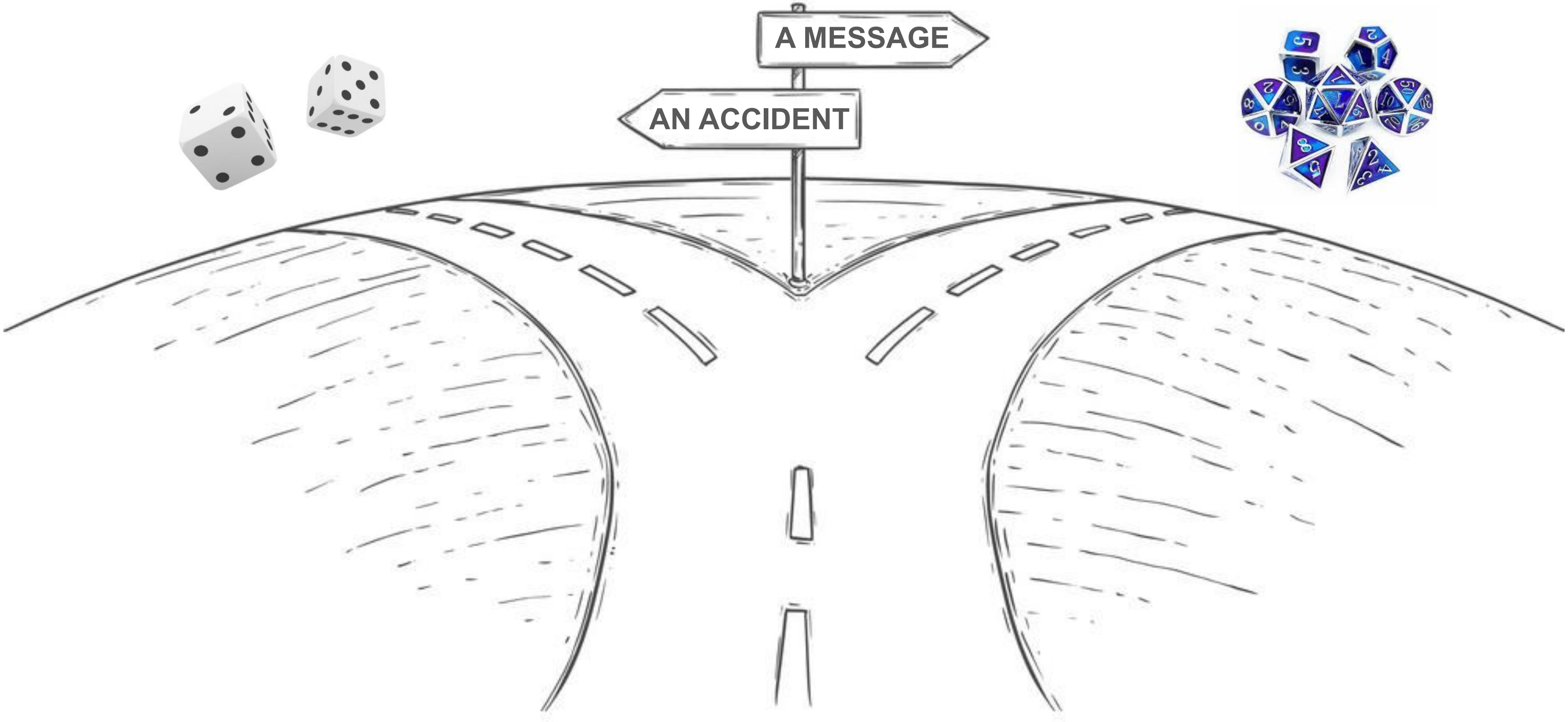




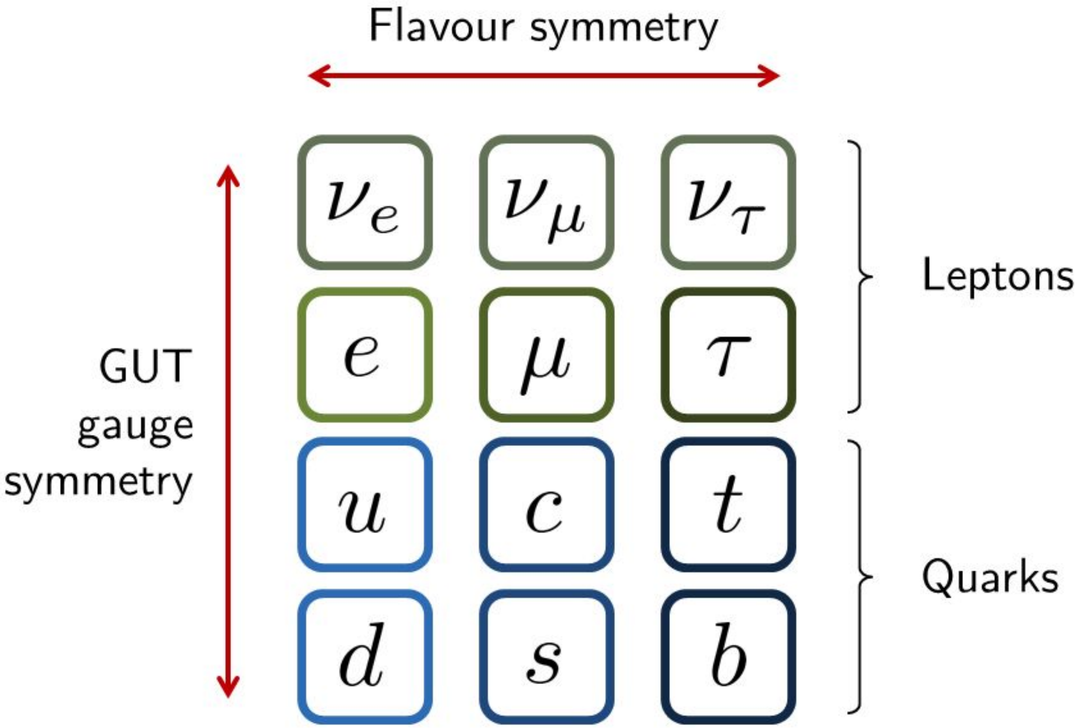


Motivation

Is there an organising principle?



Flavour symmetries

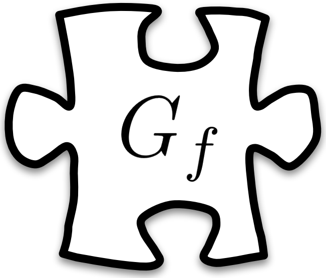


For reviews, see: Altarelli and Feruglio (2010), Ishimori et al. (2010), King and Luhn (2013), Petcov (2017), Feruglio and Romanino (2019), Ding and Valle (2024)

[see also talks by S. King, M. Mondragon]

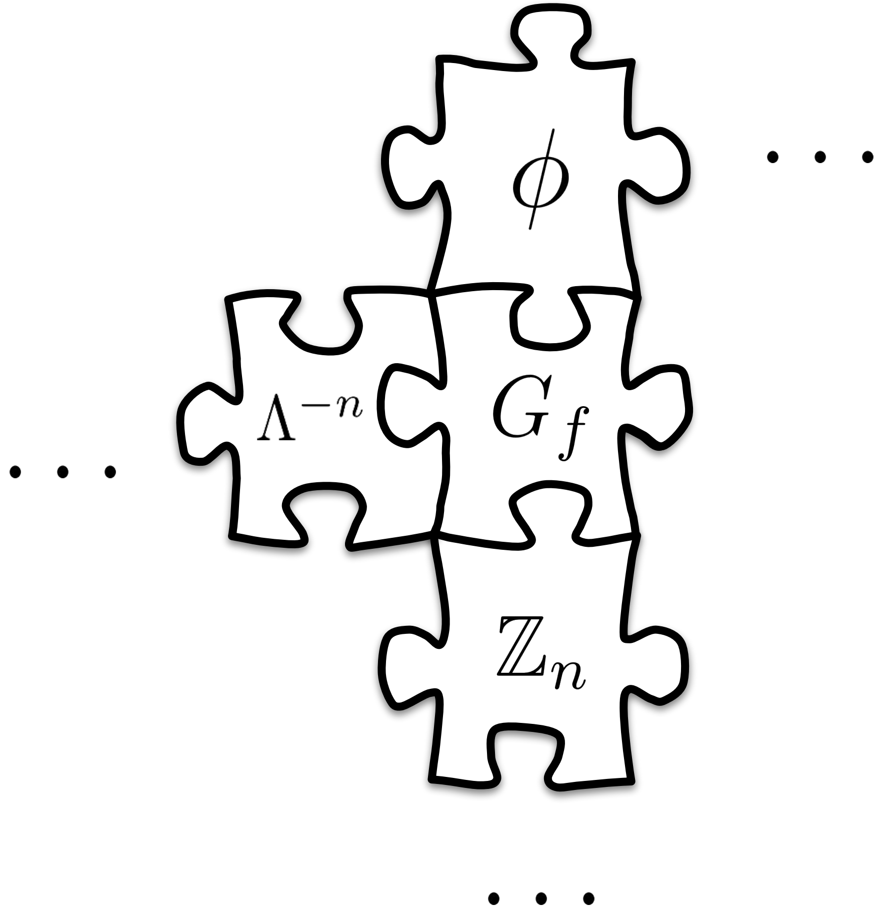
Flavour symmetries

Non-Abelian discrete flavour symmetries

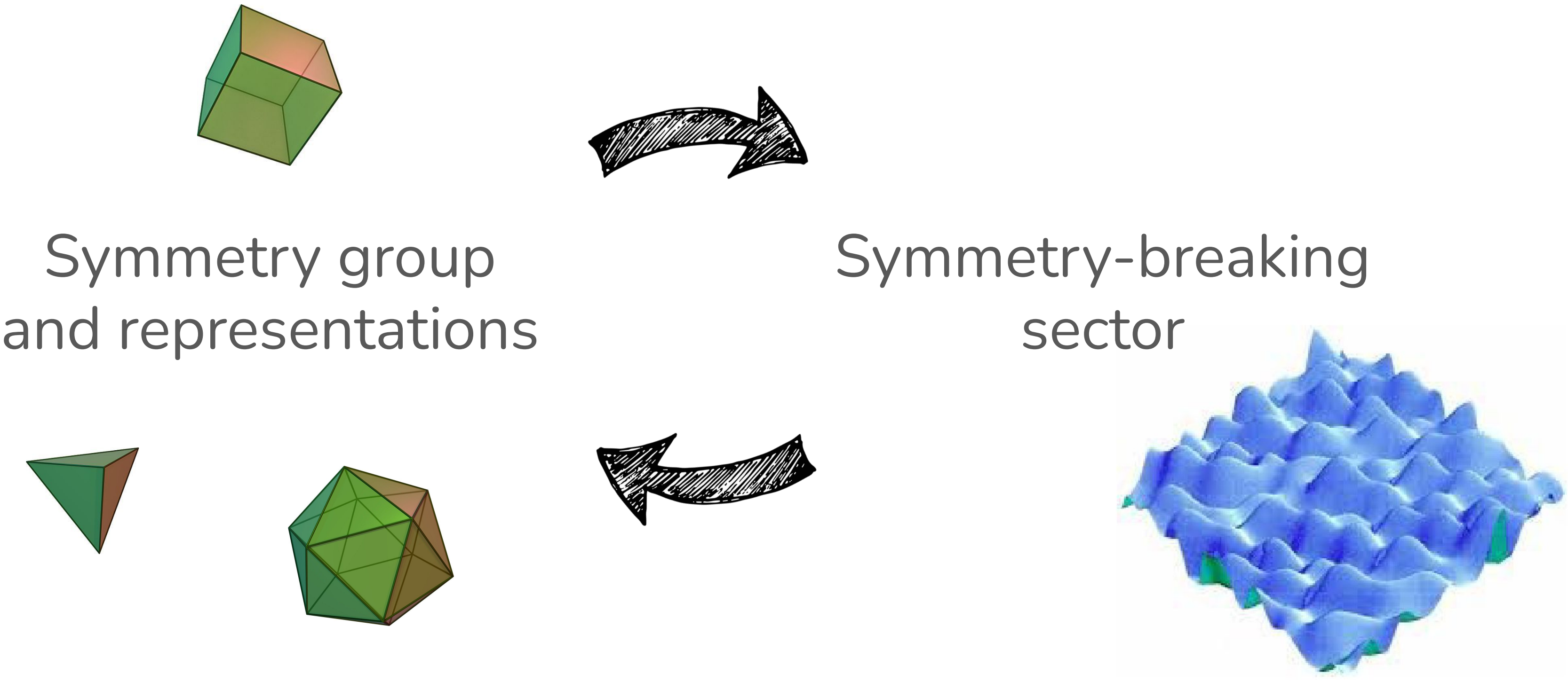


model-independent approaches relying on residual symmetries constrain mixing and the Dirac phase

Problems with the usual approach



A reversal of the usual logic

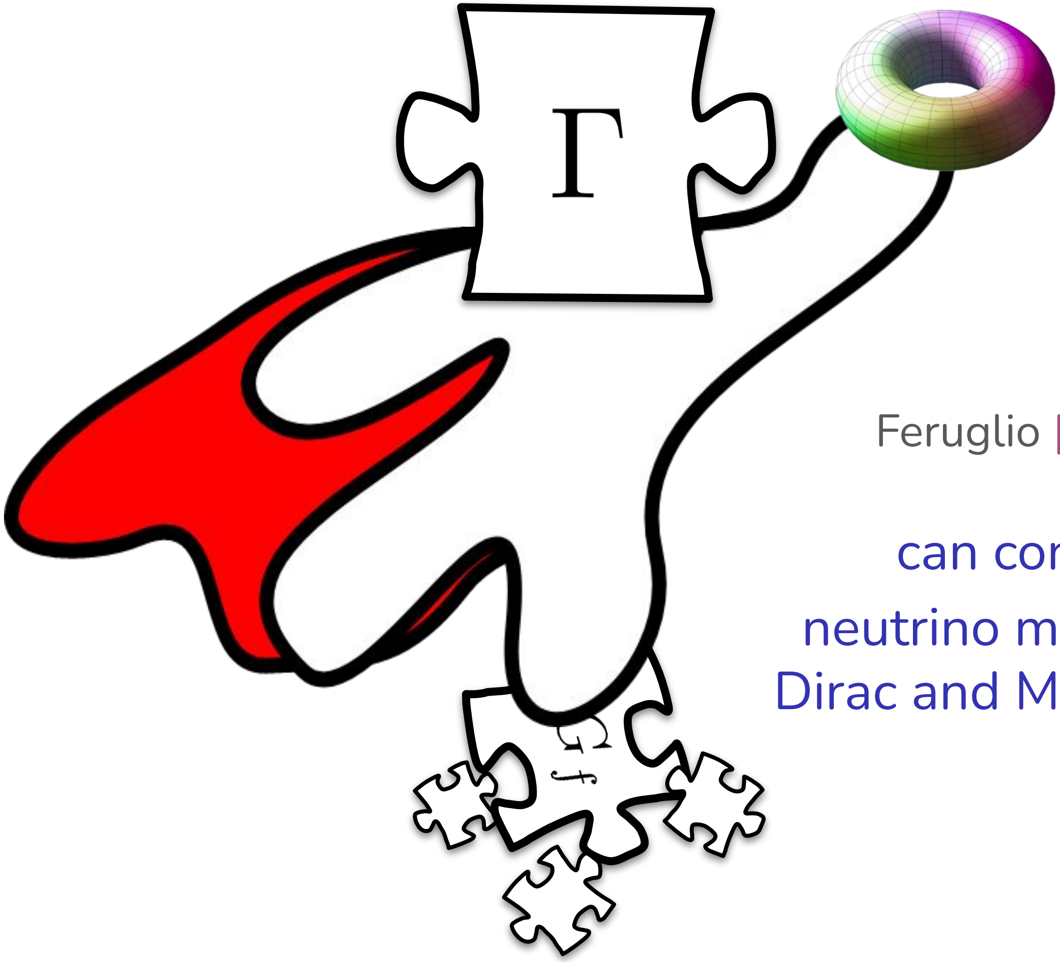


Symmetry group
and representations

Symmetry-breaking
sector

see Feruglio, PoS DISCRETE 2020-2021, 007

Modular symmetry to the rescue!

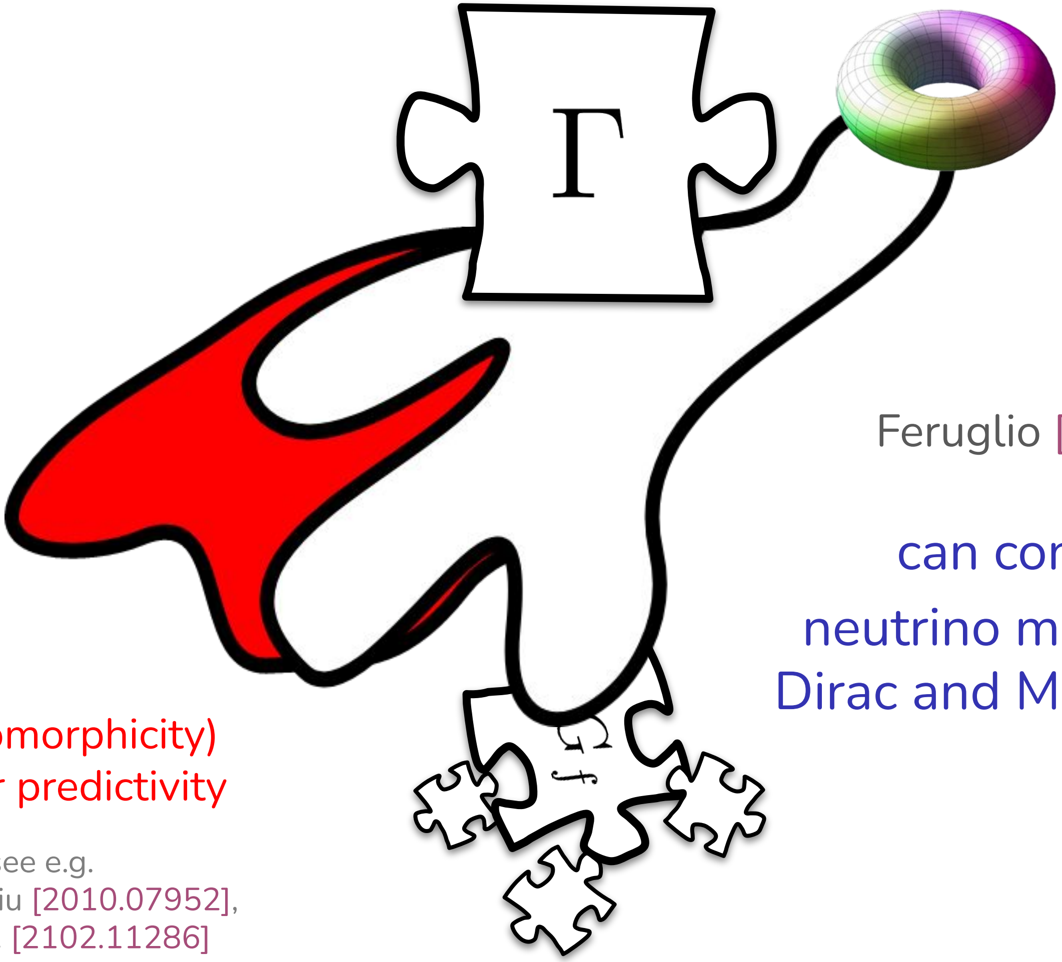


Feruglio [1706.08749]

can constrain all:

neutrino masses, mixing,
Dirac and Majorana phases

Modular symmetry to the rescue!



Feruglio [1706.08749]

can constrain all:

neutrino masses, mixing,
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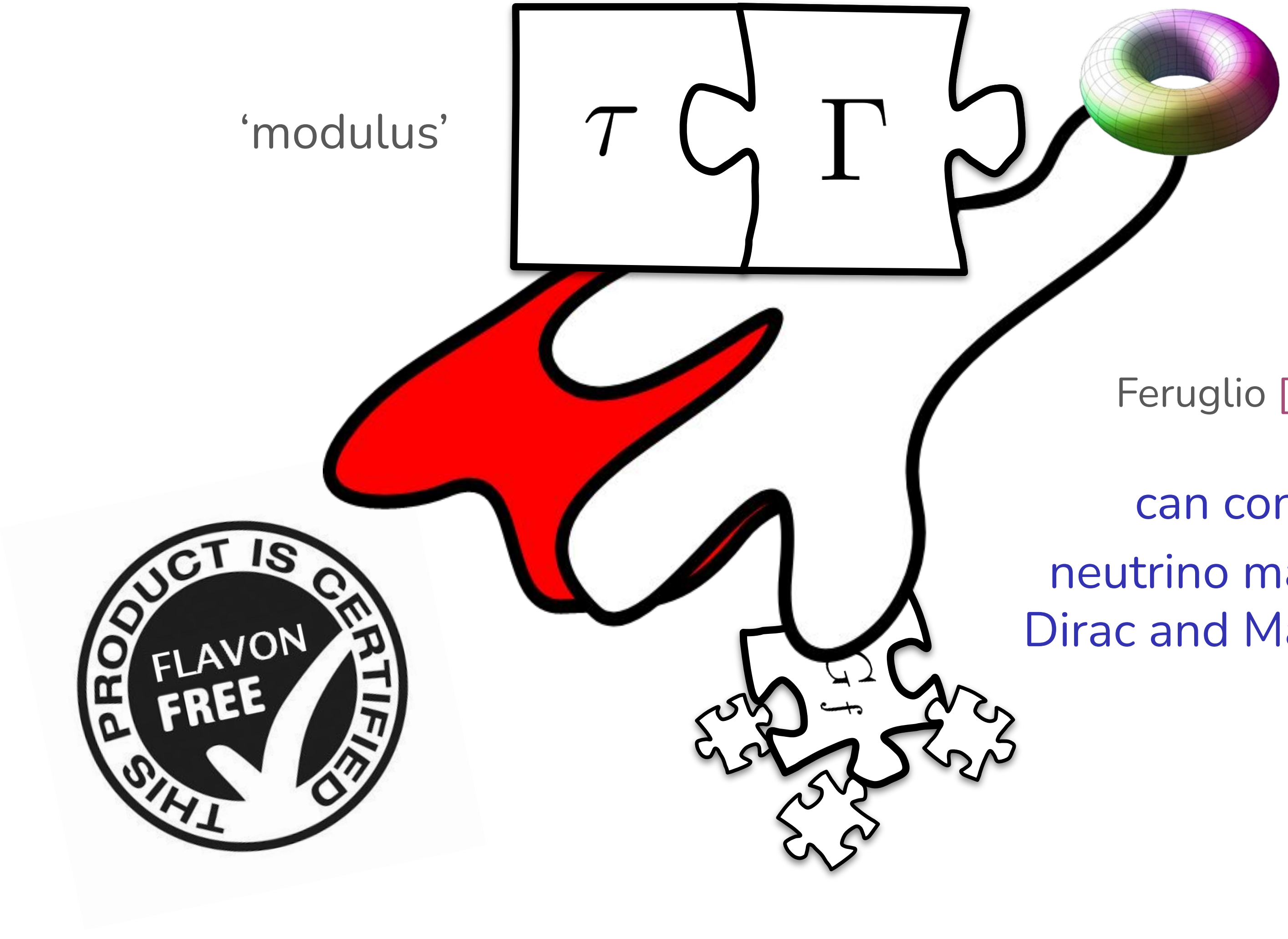
SUSY (holomorphicity)
required for predictivity

...but see e.g.
Ding, Feruglio, Liu [2010.07952],
Almumin et al. [2102.11286]

Modular symmetry to the rescue!



‘modulus’

Feruglio [1706.08749]

can constrain all:

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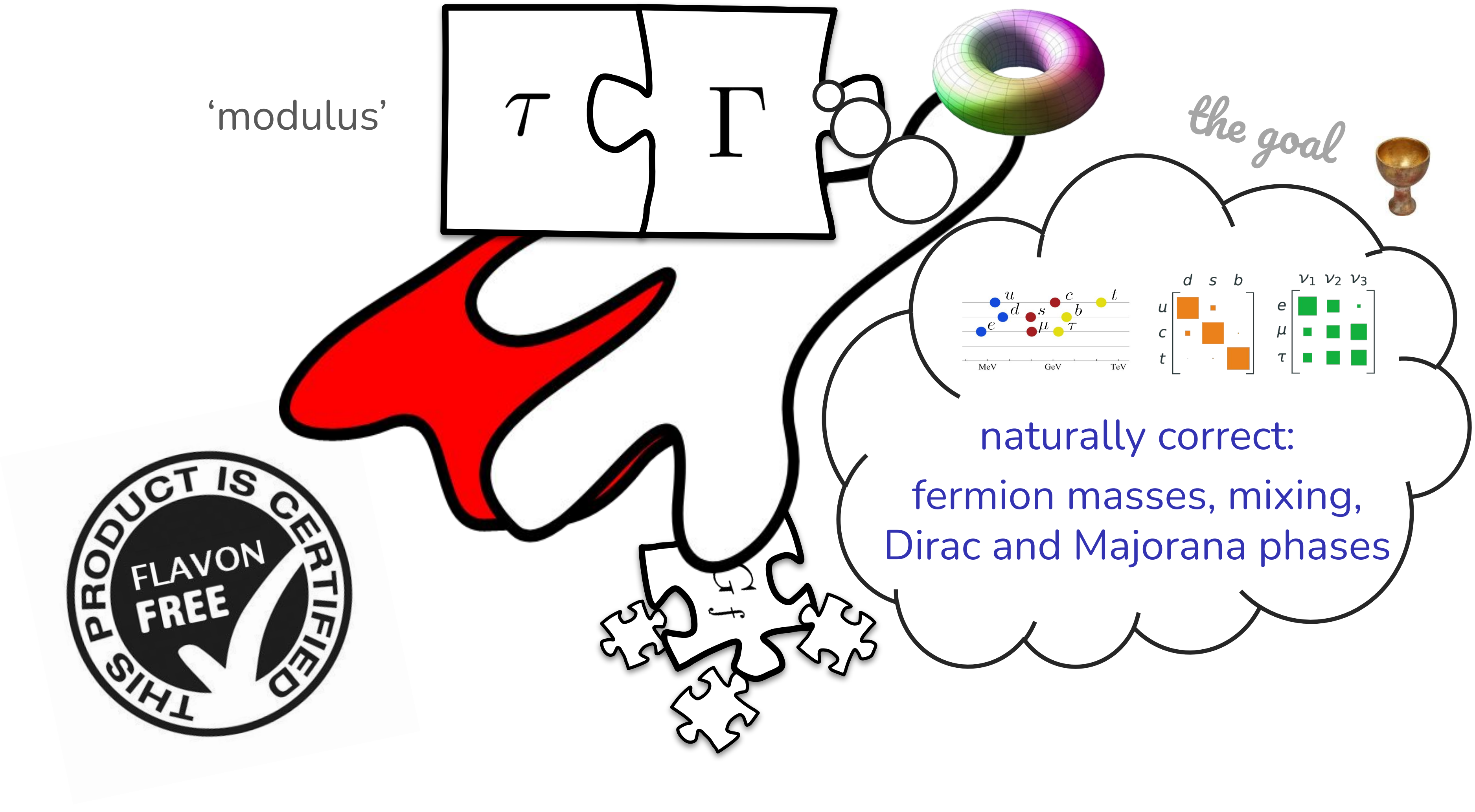
Modular symmetry to the rescue!



naturally correct:

fermion masses, mixing,
Dirac and Majorana phases

so… what *can*
modular (flavour) symmetries
do for you?

Modular symmetries can...

(the outline)

...offer a **predictive framework** for flavour

...provide an origin for **CP violation** (CPV)

...explain fermion **mass hierarchies**

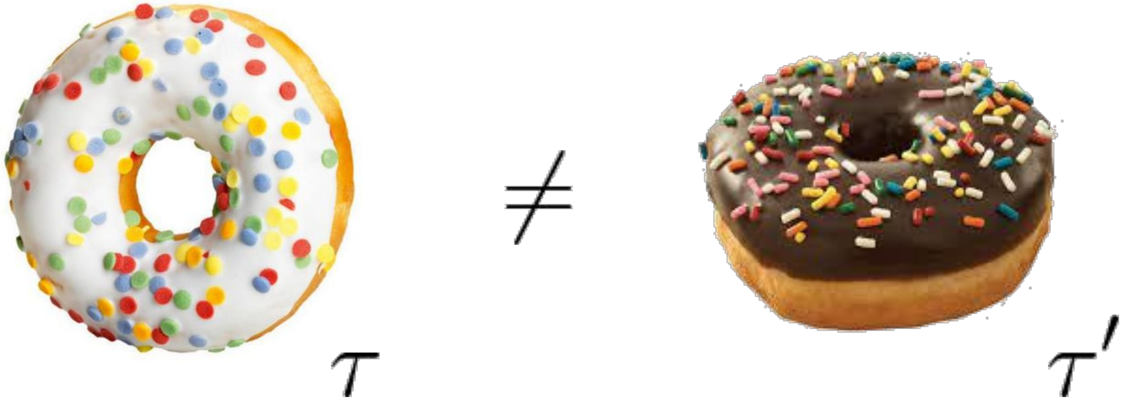
...help solve the **strong CP** problem

...bridge low-energy and **string** model building

The bottom-up framework

Image credit: sallysbakingaddiction.com

The modulus



τ may describe a torus compactification

(we assume only 1 unfrozen modulus)

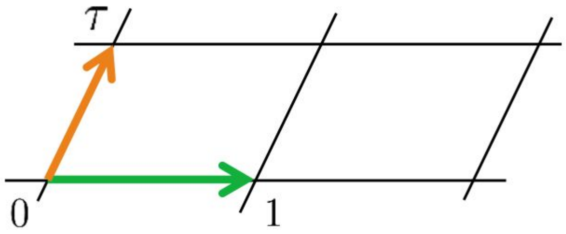
In the **bottom-up** modular approach τ is a dimensionless **spurion**

The modulus

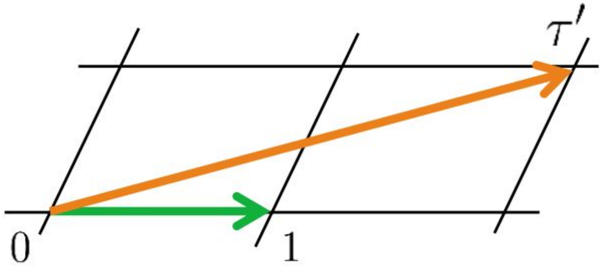
τ

$=$

$$\tau' = \frac{a\tau+b}{c\tau+d}$$

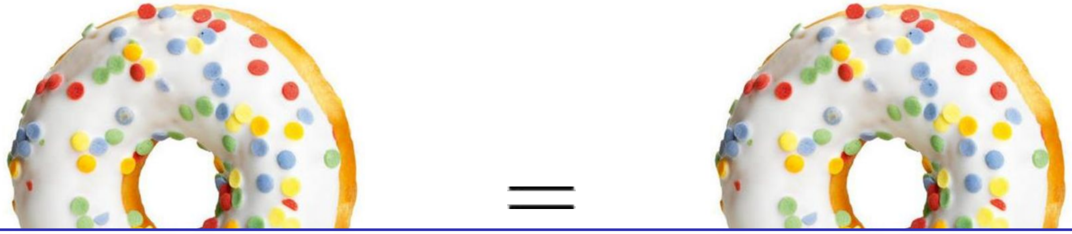


$$\operatorname{Im}\tau > 0$$



$$ad - bc = 1 \qquad a, b, c, d \in \mathbb{Z}$$

The modulus



$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\tau' = \frac{a\tau+b}{c\tau+d}$$

$$\det \gamma = 1$$

$$a, b, c, d \in \mathbb{Z}$$

*the **modular group***

$$\Gamma \equiv SL(2,\mathbb{Z}) = \{\gamma\}$$

The modular group

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

$$\Gamma \equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z},\ \det\gamma = 1 \right\}$$

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR$$

The modular group

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

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$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} :$$

$$\tau \rightarrow -1/\tau$$

inver**S**ion

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} :$$

$$\tau \rightarrow \tau + 1$$

Translation

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} :$$

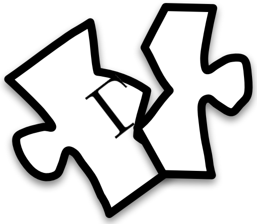
$$\tau \rightarrow \tau$$

Redundant

but can affect fields...

The modular group

$$\langle\tau\rangle \nrightarrow \frac{a\tau+b}{c\tau+d}$$



$$\equiv SL(2,\mathbb{Z}) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\ \det\gamma = 1\right\}$$

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Redundant

but can affect fields...

The field transformations

$$\psi \;\rightarrow\; (c\tau + d)^{-k}\,\rho(\gamma)\,\psi$$

The field transformations

$$\psi \;\to\; \overset{\textit{automorphy factor}}{\boxed{(c\tau + d)^{-k}}}\,\rho(\gamma)\,\psi$$

Weight $k \in \mathbb{Z}$

The field transformations

* not necessarily,
rare from top-down!

NEW!

$$\psi \to \overset{\text{automorphy factor}}{\boxed{(c\tau + d)^{-k}}}\,\rho(\gamma)\,\psi$$

Weight $k \in \mathbb{Z}^{*}$

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The field transformations

NEW!

$$\psi \to \overset{\textit{automorphy factor}}{\boxed{(c\tau + d)^{-k}}}\boxed{\rho(\gamma)}\,\psi$$

Weight $k \in \mathbb{Z}^*$

“Almost trivial” representation of the modular group

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$

$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$

$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$

$\cdots$

Feruglio [1706.08749]

The field transformations

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automorphy factor

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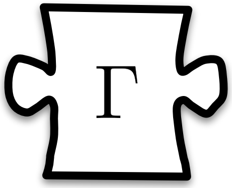
"Almost trivial" representation of the modular group



$$\Gamma(N) \subset \Gamma$$

Principal congruence subgroup of level *N*

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$

$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$

$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$

$$\cdots$$

Feruglio [1706.08749]

$\rho(\gamma)$ **is effectively a representation of** $\Gamma'_N \equiv \Gamma/\Gamma(N)$

other choices are possible: in general, *vector-valued modular forms*, see Liu, Ding [2112.14761]

The finite modular groups

$\Gamma'_N \equiv \Gamma/\Gamma(N)$ *behave like flavour groups*

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$

← drop the **R** generator

[work w/ PSL(2,Z)]

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR, \quad \boxed{T^N = \mathbb{1}}$$

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(in general there in TD)

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← drop the **R** generator

(in general there in TD)

[see talk by M. Parriciatu]

$\Gamma_2 \simeq S_3$
Kobayashi et al. [1803.10391]
Meloni, Parriciatu [2306.09028]

$\Gamma_3 \simeq A_4$
Feruglio [1706.08749]

$\Gamma_4 \simeq S_4$
JP, Petcov [1806.11040]

$\Gamma_5 \simeq A_5$
Novichkov et al. [1812.02158]

summary in Appendices of Novichkov, JP, Petcov, Titov [1905.11970]

$\Gamma'_3 \simeq A'_4$
Liu, Ding [1907.01488]

$\Gamma'_4 \simeq S'_4$
Novichkov, JP, Petcov [2006.03058]

$\Gamma'_5 \simeq A'_5$
Wang, Yu, Zhou [2010.10159]

For early top-down, see e.g.:

Kobayashi et al. [1804.06644]; Kobayashi, Tamba [1811.11384]; de Anda et al. [1812.05620]; Baur et al. [1901.03251, 1908.00805]; Kariyazono et al. [1904.07546]; Nilles et al. [2001.01736, 2004.05200, 2006.03059]; Kobayashi, Otsuka [2001.07972, 2004.04518]; Abe et al. [2003.03512]; Ohki et al. [2003.04174]; Kikuchi et al. [2005.12642]

by now, O(200) papers…

A lot of model building…

- models based on finite modular groups of higher *N*
- modular models of unification (also without GUTs)
- modular models of leptogenesis
- models with multiple moduli

$$\tau \to \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$$

  based on symplectic modular invariance (**Siegel modular group**)
  and automorphic forms

- models relating modular flavour symmetries and inflation
- models exploring the interplay of modular and gCP symmetries

...and a vast literature...

- models based on finite modular groups of higher *N*
  [2004.12662, 2108.02181, 2307.01419]
- modular models of unification (also without GUTs) [2312.09255] [see talk by O. Medina]
  [1906.10341, 2012.01397, 2101.02266, 2101.12724, 2103.02633, 2103.16311, 2108.09655, 2206.14675]
- modular models of leptogenesis
  [1909.06520, 2007.00545, 2103.07207, 2201.10429, 2204.08338, 2205.08269, 2206.14675, 2402.18547]
- models with multiple moduli [see talk by Y.-L. Zou]
  [1811.04933, 1812.11289, 1906.02208, 1908.02770, 2304.05958]

$$\tau \to \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$$

based on symplectic modular invariance (**Siegel modular group**)
and automorphic forms: Ding, Feruglio, Liu [2010.07952, 2402.14915]

From TD, see e.g.: Nilles et al. [2105.08078], Baur et al. [2012.09586], Kikuchi et al. [2305.16709]
[see talk by K. Nasu]

- models relating modular flavour symmetries and inflation
  [2208.10086], [2303.02947], Ding et al. [2405.06497] NEW!
  [see talk by X. Wang]
- models exploring the interplay of modular and gCP symmetries
  [1901.03251, 1905.11970, 1910.11553, 2006.03058, 2012.01688, 2012.13390, 2102.06716, 2106.11659]

But how does it work?

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(\psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \;\to (c\tau + d)^{-k} \, \rho_{\mathbf{r}}(\gamma) \, \psi$$

Assuming rigid SUSY (*W* not invariant in SUGRA)

Need modular forms

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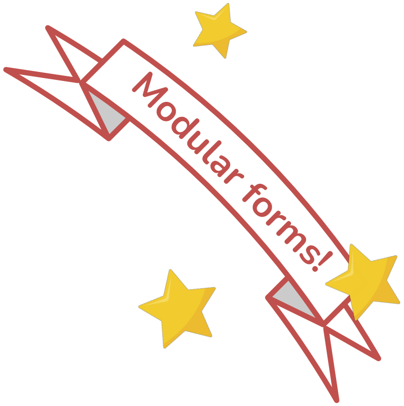
$$Y(\tau) \;\to (c\tau + d)^{k_Y}\, \rho_Y(\gamma)\, Y(\tau)$$



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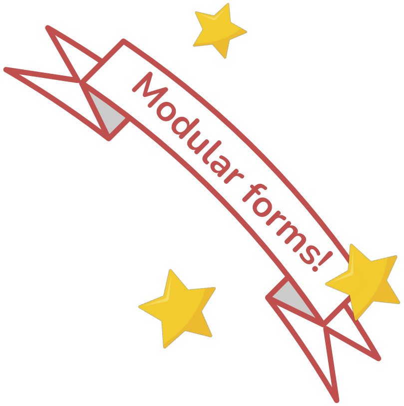
$$Y(\tau) \to \boxed{(c\tau + d)^{k_Y}\,\rho_Y(\gamma)}\,Y(\tau)$$

$$\begin{cases} k_Y = k_1 + \ldots + k_n \\ \rho_Y \otimes \rho_1 \otimes \ldots \otimes \rho_n \supset \mathbf{1} \end{cases}$$



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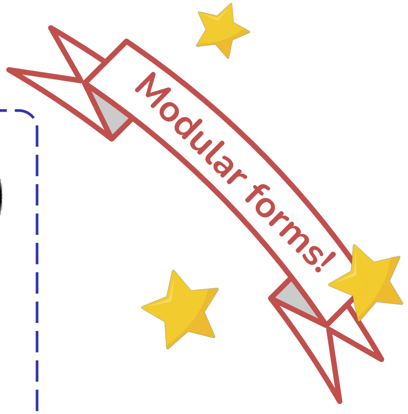

Assuming rigid SUSY (*W* not invariant in SUGRA)

The modular forms

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$
$\dim \mathcal{M}_k(\Gamma(N))$	$k/2+1$	$k+1$	$2k+1$	$5k+1$

Not so many available!

*A **finite set** of functions for each k_Y*

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Lowest-weight *k* modular forms for each group:

$\Gamma_N^{(\prime)}$	$Y_{\mathbf{r}}^{(k)}$
$\Gamma_2 \simeq S_3$	$Y_{\mathbf{2}}^{(2)}$
$\Gamma'_3 \simeq A'_4$	$Y_{\hat{\mathbf{2}}}^{(1)}$
$\Gamma_3 \simeq A_4$	$Y_{\mathbf{3}}^{(2)}$
$\Gamma'_4 \simeq S'_4$	$Y_{\hat{\mathbf{3}}}^{(1)}$
$\Gamma_4 \simeq S_4$	$Y_{\mathbf{2}}^{(2)}, Y_{\mathbf{3'}}^{(2)}$
$\Gamma'_5 \simeq A'_5$	$Y_{\hat{\mathbf{6}}}^{(1)}$
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non-singular, unlike modular functions. Can still have an interpretation, see Feruglio, Strumia, Titov [2305.08908]

can generalize modular group to e.g. the larger metaplectic group and get half-integer weight forms, see Liu et al. [2007.13706]

[see X.-G. Liu, V. Knapp-Perez talks]

Example

$$W \supset N N$$

Let’s build a modular-invariant term!

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$$Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

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$$Y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right] = 1 + 12q + 36q^2 + 12q^3 + \dots$$

$$Y_2(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -6q^{1/3}(1 + 7q + 8q^2 + \dots)$$

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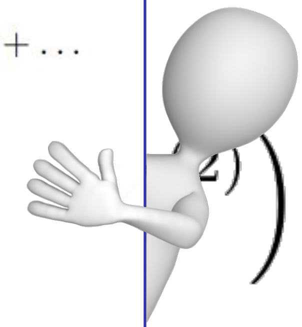
with

$$\omega = e^{2\pi i/3} \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n) \qquad q \equiv e^{i2\pi\tau}$$

obey $Y_2^2 + 2Y_1Y_3 = 0$

Feruglio [1706.08749]



$$\Big)_{\mathbf{1}}$$

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so now we can build models…

Example: an S4 lepton model

Novichkov, JP, Petcov, Titov [1811.04933]

Ingredients: **Choose group, field content** $\psi \sim (\mathbf{r}, k)$

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$$N^c \sim (\mathbf{3'}, 0)\,, \quad L \sim (\mathbf{3}, 2)$$

$$E^c \sim (\mathbf{1'}, 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1'}, 2)$$

Ingredients: **Choose group, field content**

$$\psi \sim (\mathbf{r}, k)$$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3'}}^{(4)}\right)_{\mathbf{1}} H_d$$

$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + \boxed{g'} \left(N^c L\, Y_{\mathbf{3'}}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}} ,$$

$\in \mathbb{C}$ only physical phase

Procedure: **Fit couplings and τ**

$$\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$$

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Procedure: **Fit couplings and *τ***

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$$\mathrm{gCP} \;\Rightarrow\; g' \in \mathbb{R}$$

Novichkov, JP, Petcov, Titov [1905.11970]

***τ* can be the only source of CPV**

Example: an S4 lepton model (results)

Novichkov, JP, Petcov, Titov
[1811.04933, 1905.11970]

$$\sin^2\theta_{23} \sim 0.49$$
$$\delta \sim 1.6\pi$$
$$\alpha_{21} \sim 0.3\pi$$
$$\alpha_{31} \sim 1.3\pi$$

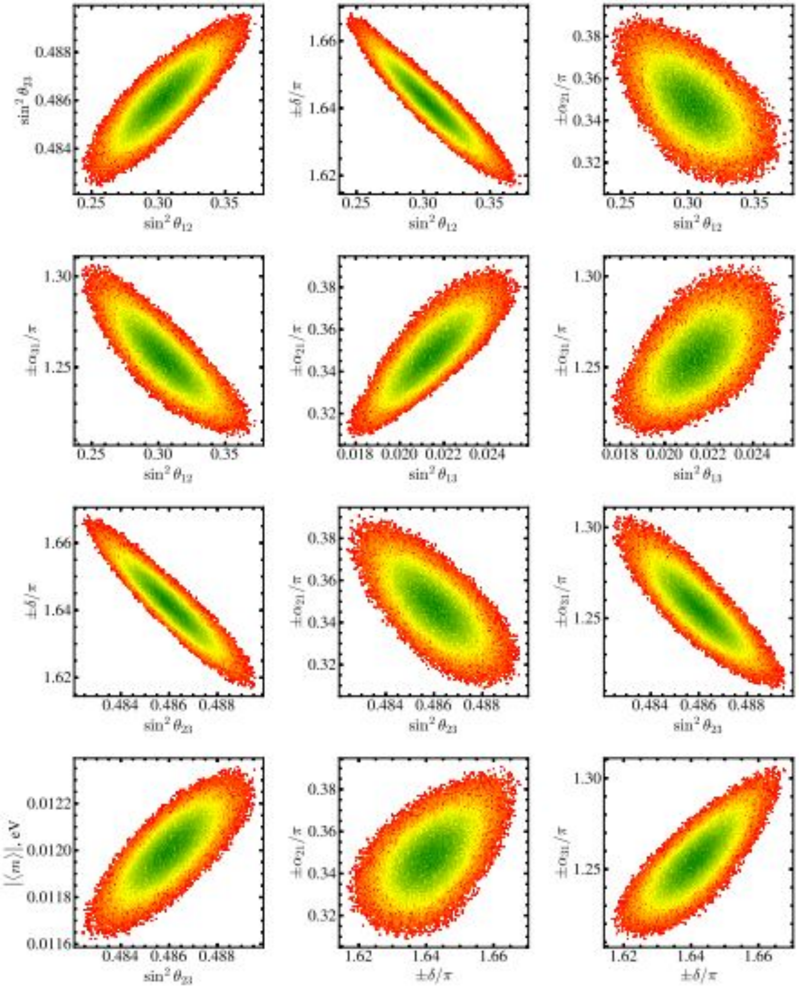
$$|\langle m \rangle|_{\beta\beta} \sim 12 \text{ meV}$$
$$\textstyle\sum_i m_i \sim 0.08 \text{ eV}$$

7 (4) parameters

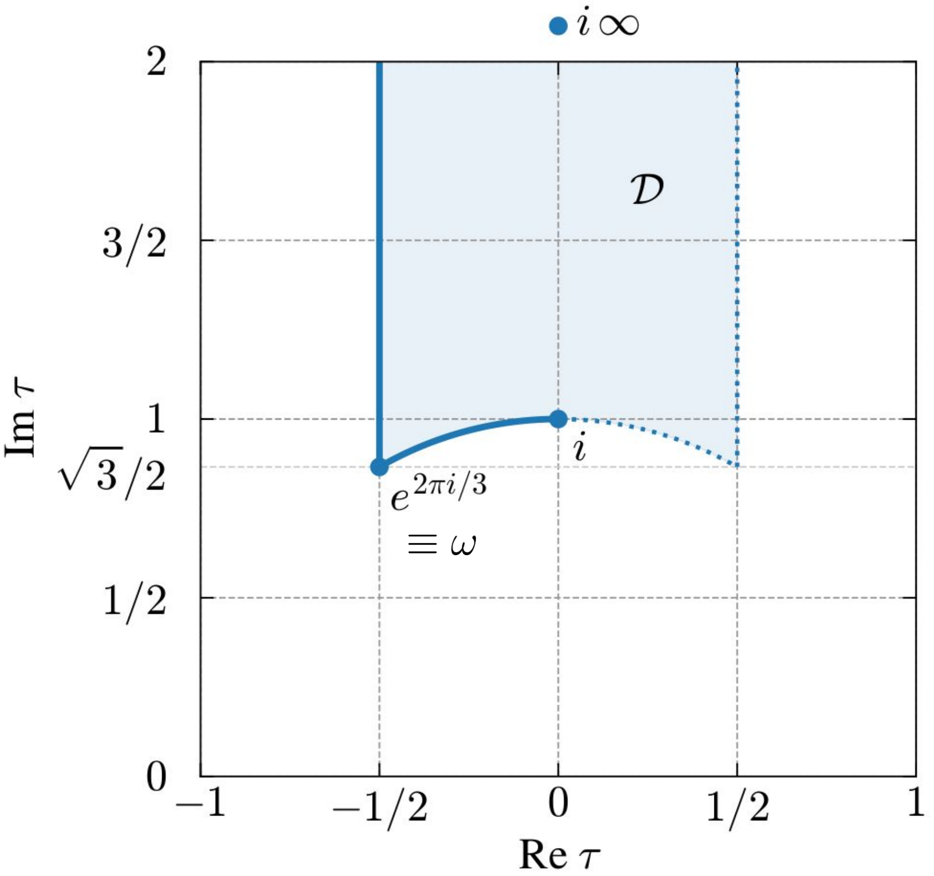
vs.

12 (9) observables

Minimal model found with one less parameter:
Ding, Liu, Yao [2211.04546]



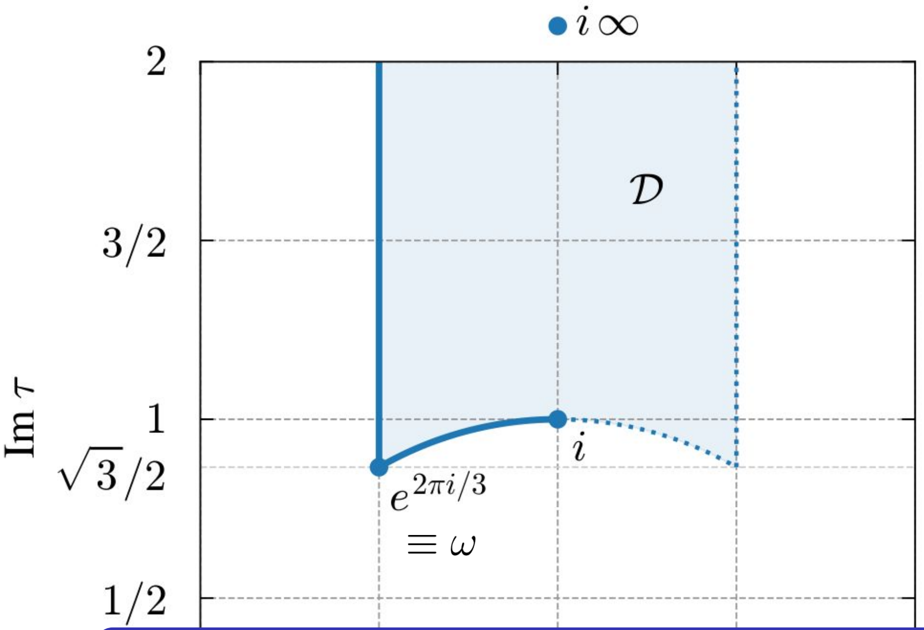
The fundamental domain



- **Any τ** breaks the full modular symmetry
- To **fit** a model which is invariant under the full modular group, it is enough to scan τ in the **fundamental domain**

In some cases, invariants can save a lot of time! Chen et al. [2211.04546] [see talk by X. Li]

The fundamental domain



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$$(\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots) \to$$

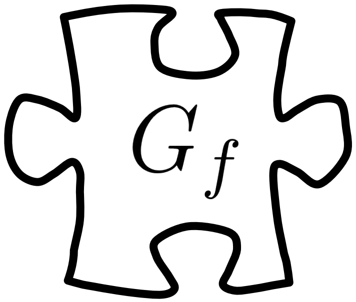
$$\left(\frac{a\tau+b}{c\tau+d},\, (c\tau+d)^{-2}\,\beta/\alpha,\, (c\tau+d)^{-2}\,\gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots\right)$$

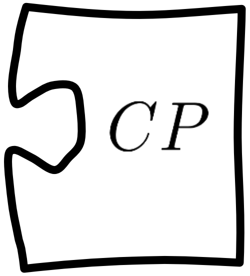
Novichkov, JP, Petcov, Titov [1811.04933]

CP violation

Image credit: www.davidharber.co.uk

Flavour symmetries + gCP (general~~ized~~ CP)





$$\psi(x) \to \rho_{\mathbf{r}}(g)\,\psi(x)$$

$$\psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}})$$

Branco, Lavoura, Rebelo (1986)
Harrison, Scott (2002)
Grimus, Lavoura (2003)
Farzan, Smirnov (2006)
Ferreira et al. (2012)
...

Flavour symmetries + gCP (general~~ized~~ CP)

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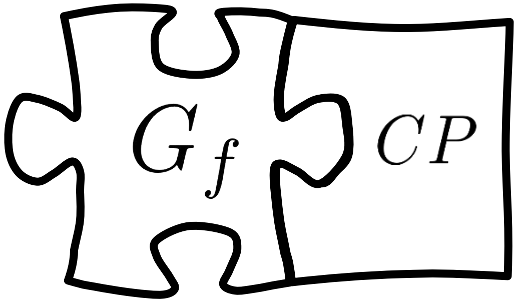


$\psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}})$

can constrain mixing, the Dirac **and the Majorana phases**

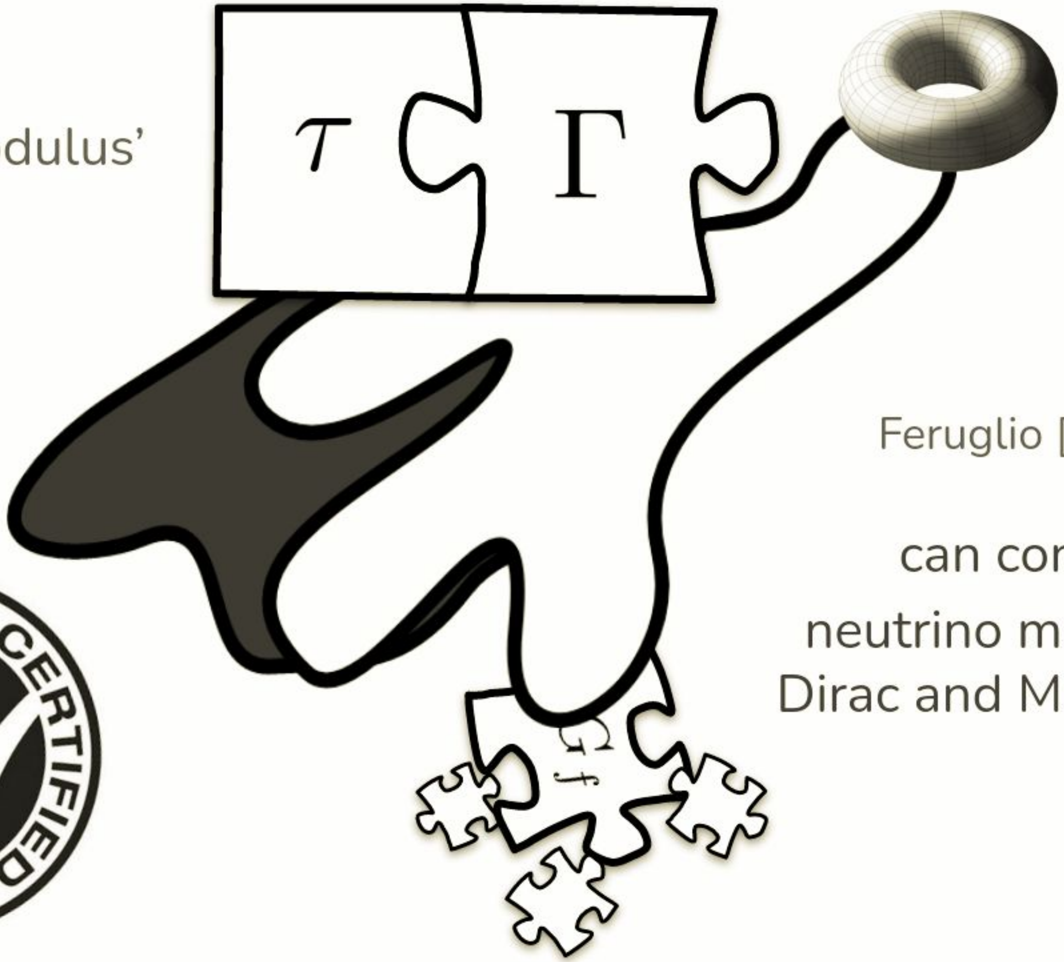
Consistency condition [Feruglio et al. (2012), Holthausen et al. (2012)]

$$X_{\mathbf{r}}^{\mathrm{CP}}\,\rho_{\mathbf{r}}^{*}(g)\,\left(X_{\mathbf{r}}^{\mathrm{CP}}\right)^{-1} = \rho_{\mathbf{r}}(u(g))$$

u is a class-inverting outer automorphism [Chen et al. (2014)]

Modular symmetry to the rescue!

‘modulus’

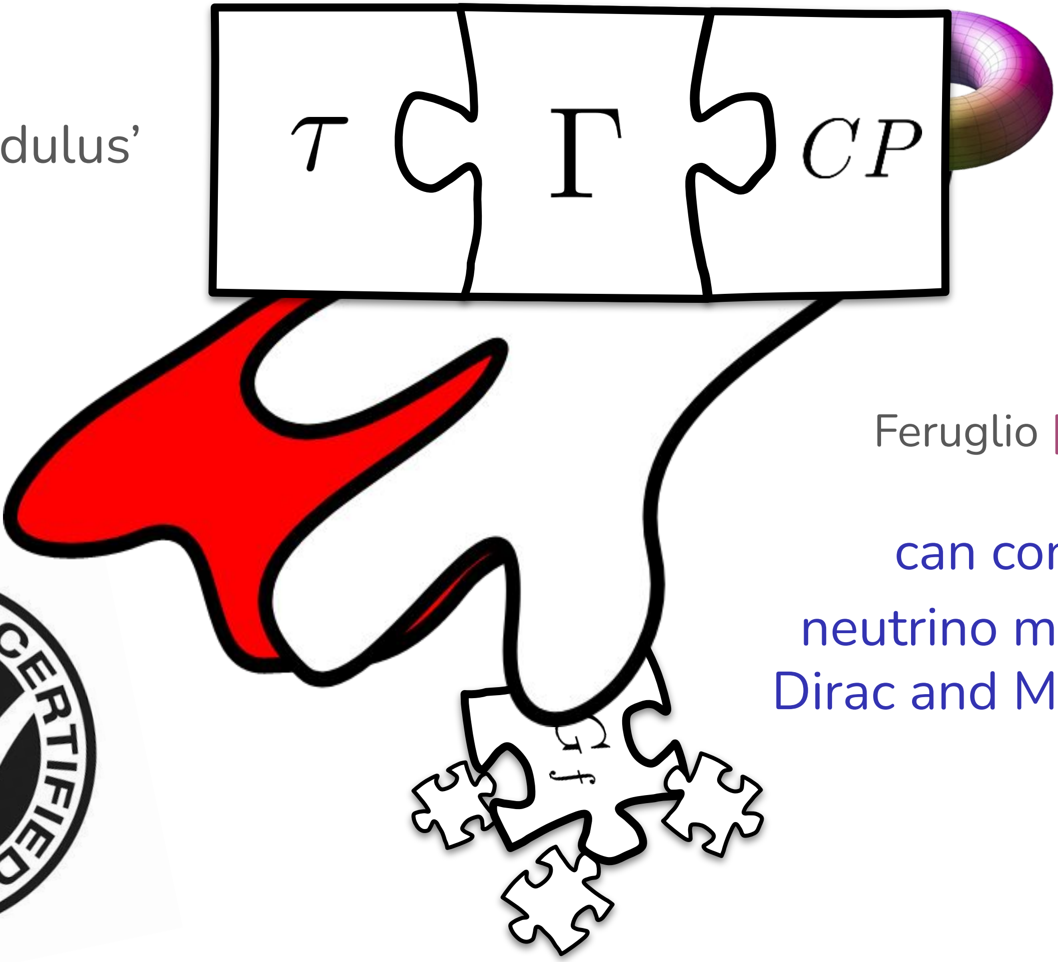


Feruglio [1706.08749]

can constrain all:

neutrino masses, mixing, Dirac and Majorana phases

Modular symmetry + gCP



Feruglio [1706.08749]

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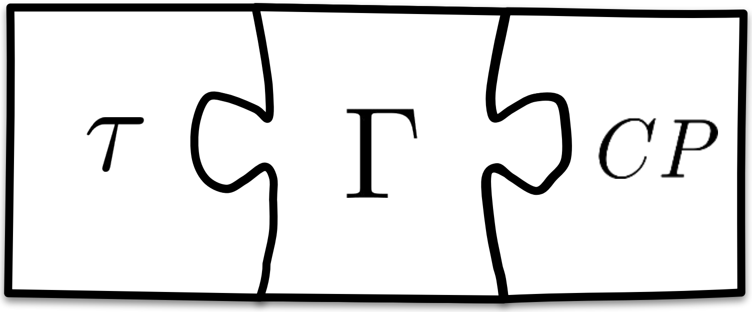
THIS PRODUCT IS CERTIFIED
FLAVON FREE

Modular symmetry + gCP

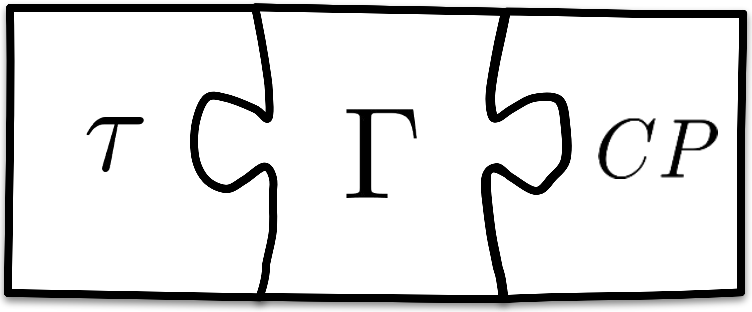


$\tau \xrightarrow{\text{CP}} ?$

$\psi \xrightarrow{\text{CP}} ?$

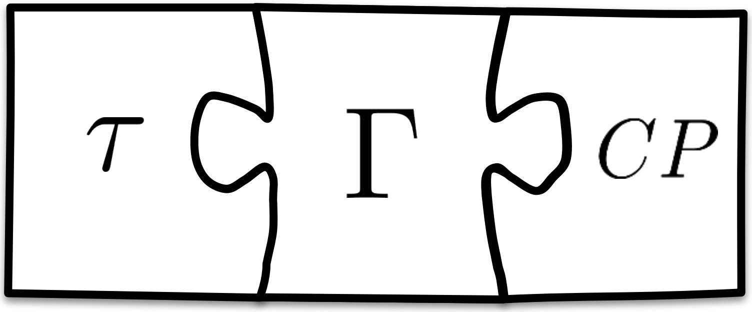
$Y(\tau) \xrightarrow{\text{CP}} ?$

Modular symmetry + gCP



$$\tau \xrightarrow{\mathrm{CP}} ? \qquad \psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}}) \qquad Y(\tau) \xrightarrow{\mathrm{CP}} ?$$

Modular symmetry + gCP



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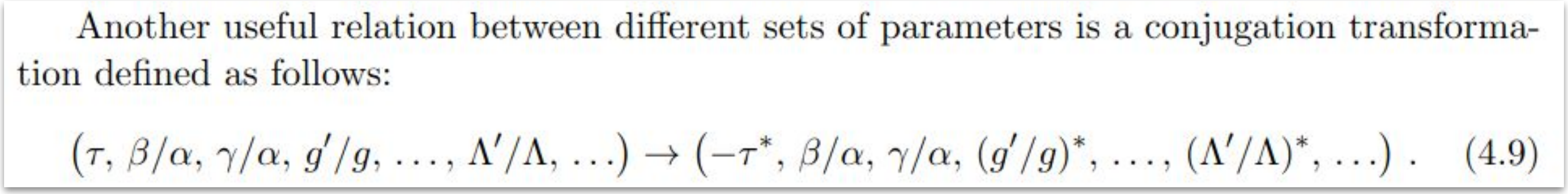
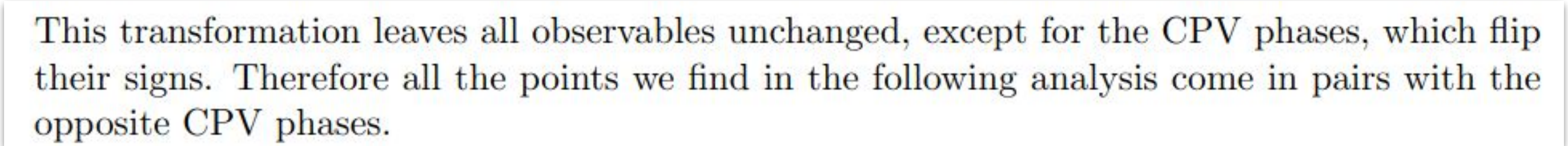


Another useful relation between different sets of parameters is a conjugation transformation defined as follows:

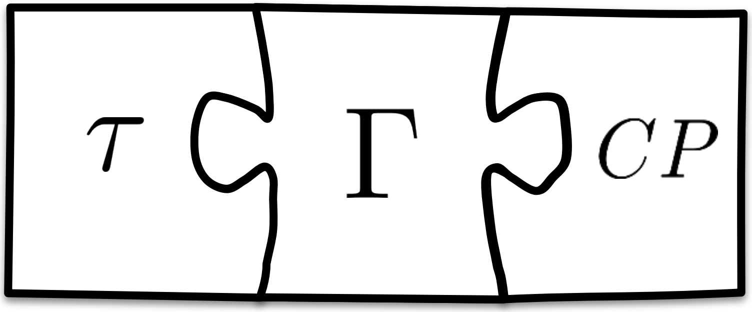
$$(\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots) \to (-\tau^*,\, \beta/\alpha,\, \gamma/\alpha,\, (g'/g)^*,\, \ldots,\, (\Lambda'/\Lambda)^*,\, \ldots)\,. \quad (4.9)$$



This transformation leaves all observables unchanged, except for the CPV phases, which flip their signs. Therefore all the points we find in the following analysis come in pairs with the opposite CPV phases.

Novichkov, JP, Petcov, Titov [1811.04933]

Modular symmetry + gCP



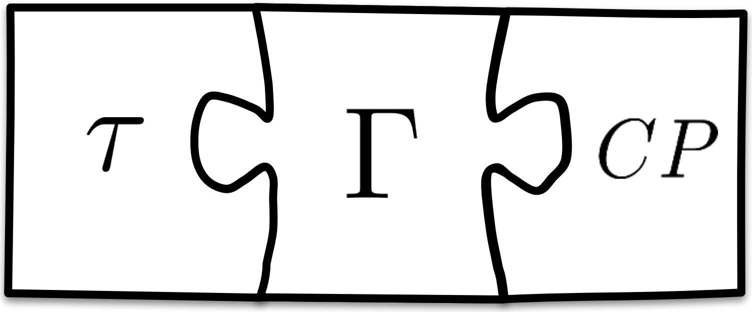
$$\tau \xrightarrow{\text{CP}} ? \qquad \psi(x) \to X_{\mathbf{r}}^{\text{CP}}\,\overline{\psi}(x_{\text{P}}) \qquad Y(\tau) \xrightarrow{\text{CP}} ?$$

$$\boxed{\tau \xrightarrow{\text{CP}} -\tau^*}$$

Dent [hep-ph/0105285]
Baur, Nilles, Trautner, Vaudrevange [1901.03251]
Novichkov, JP, Petcov, Titov [1905.11970]
Baur, Nilles, Trautner, Vaudrevange [1908.00805]

can be derived from the BU (group theory considerations)

Modular symmetry + gCP



$$\tau \xrightarrow{\mathrm{CP}} -\tau^* \qquad \psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}}) \qquad Y(\tau) \xrightarrow{\mathrm{CP}} \,?$$

- Consistency condition

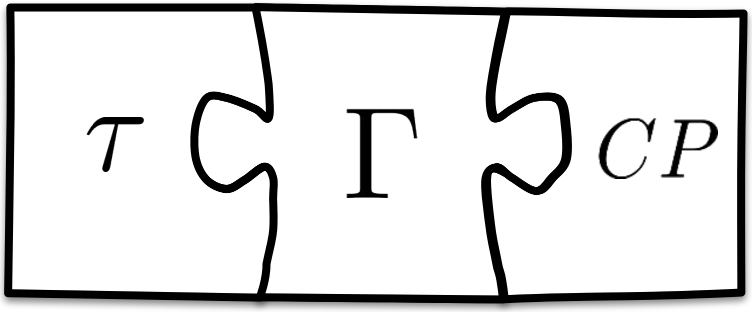
$$X_{\mathbf{r}}^{\mathrm{CP}}\,\rho_{\mathbf{r}}^*(\gamma)\left(X_{\mathbf{r}}^{\mathrm{CP}}\right)^{-1} = \rho_{\mathbf{r}}(u(\gamma)) \qquad u(\gamma) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

- SL(2,Z) → GL(2,Z)
- Can choose $X = 1$ in a **symmetric** basis

Novichkov, JP, Petcov, Titov [1905.11970]

in a symmetric basis
with real q-expansions

Modular symmetry + gCP



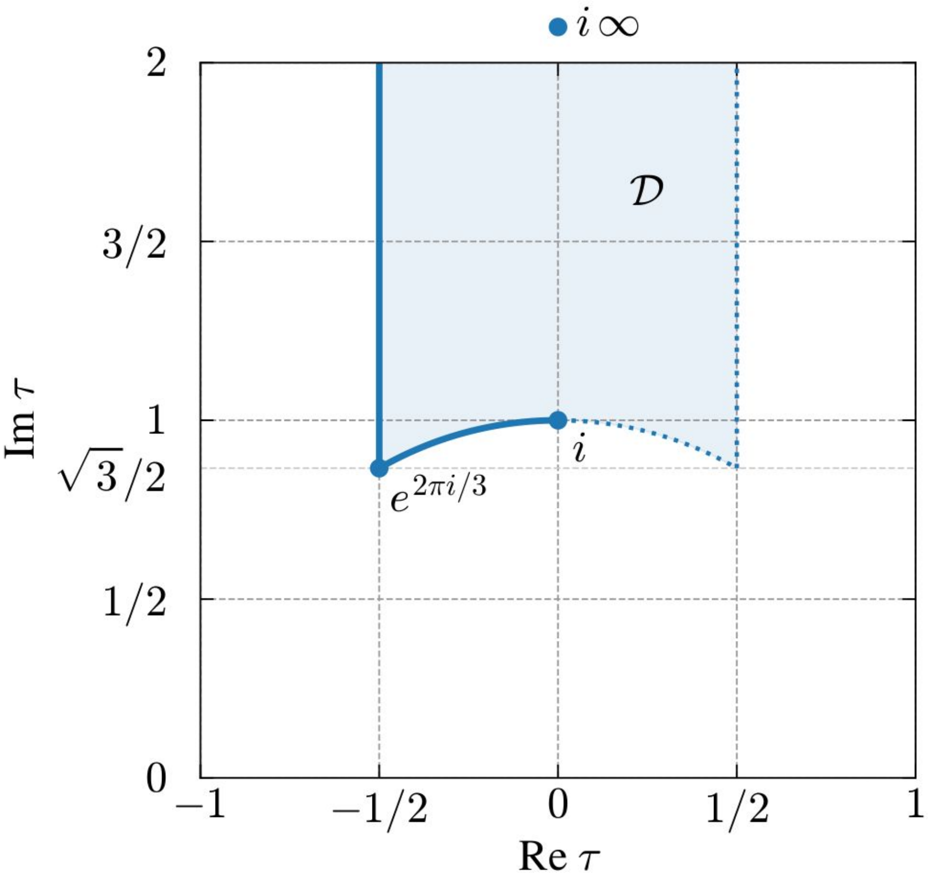
$$\tau \xrightarrow{\text{CP}} -\tau^* \qquad \psi(x) \to \overline{\psi}(x_{\text{P}}) \qquad Y(\tau) \xrightarrow{\text{CP}} Y^*(\tau)$$

- Given the reality of Clebsch-Gordan coefficients,

CP conservation $\Longrightarrow$ $\boxed{g \in \mathbb{R}}$ **real superpotential parameters**

- CP is violated by the modulus unless $\boxed{-\tau^* = \gamma\tau}$ **special regions of the fundamental domain**

Modular symmetry + gCP (back to the fundamental domain)



- **Any** ***τ*** breaks the full modular symmetry
- Special values of τ preserve the CP symmetry
- The modulus can be the **only source of CP violation**!

(recall the S4 model of slide 59...)

- CP is violated by the modulus unless

$$\boxed{-\tau^* = \gamma\tau}$$

special regions of the fundamental domain

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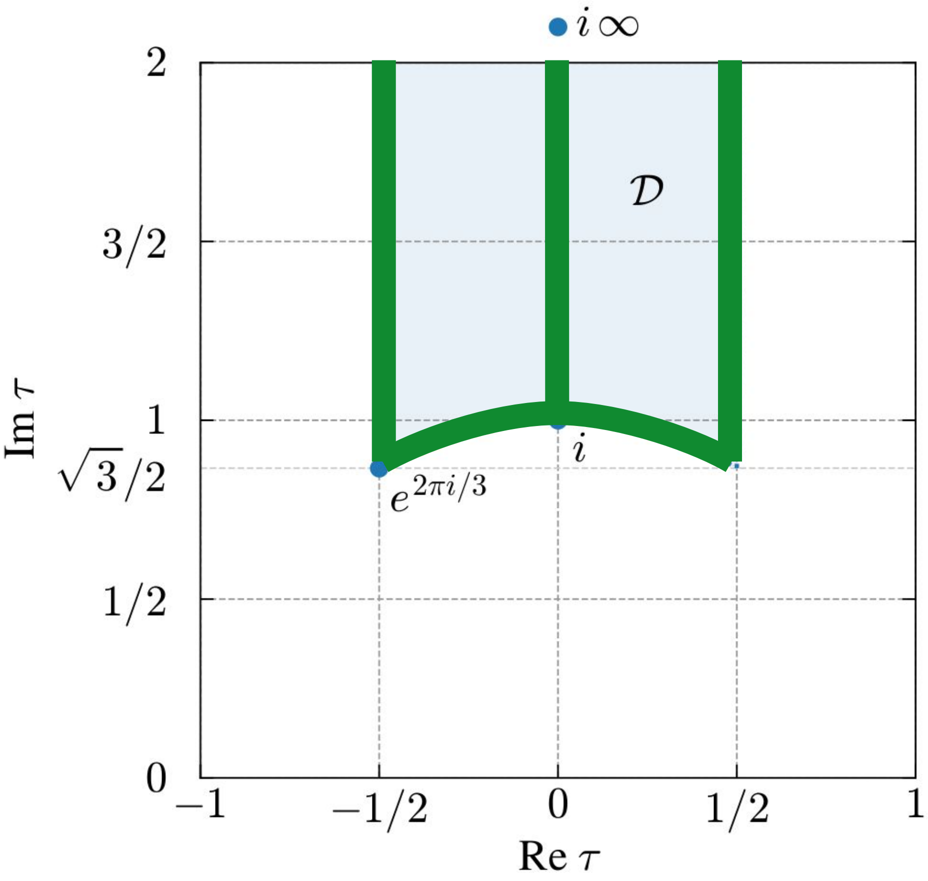

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(recall the S4 model of slide 59…)

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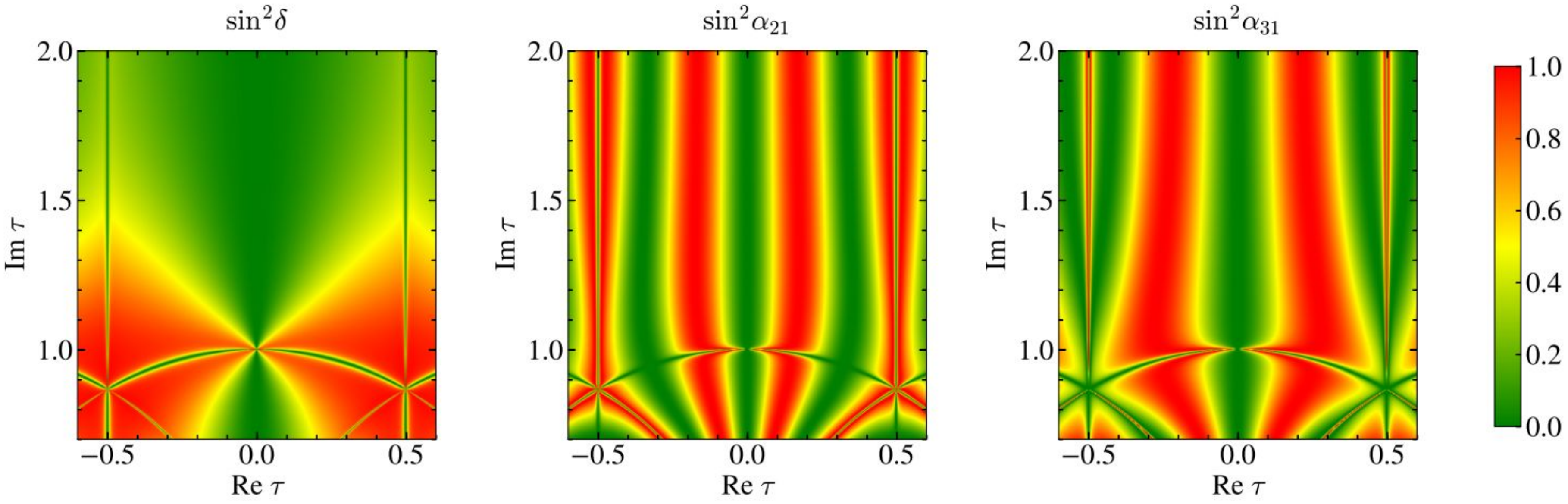
special regions of the fundamental domain

Large CPV for small departures?

[recall M. Parriciatu's talk]



Novichkov, JP, Petcov, Titov [1905.11970]

Large CPV for small departures?

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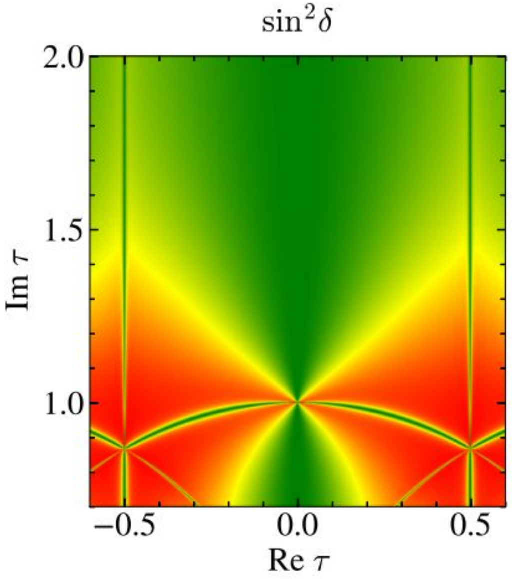
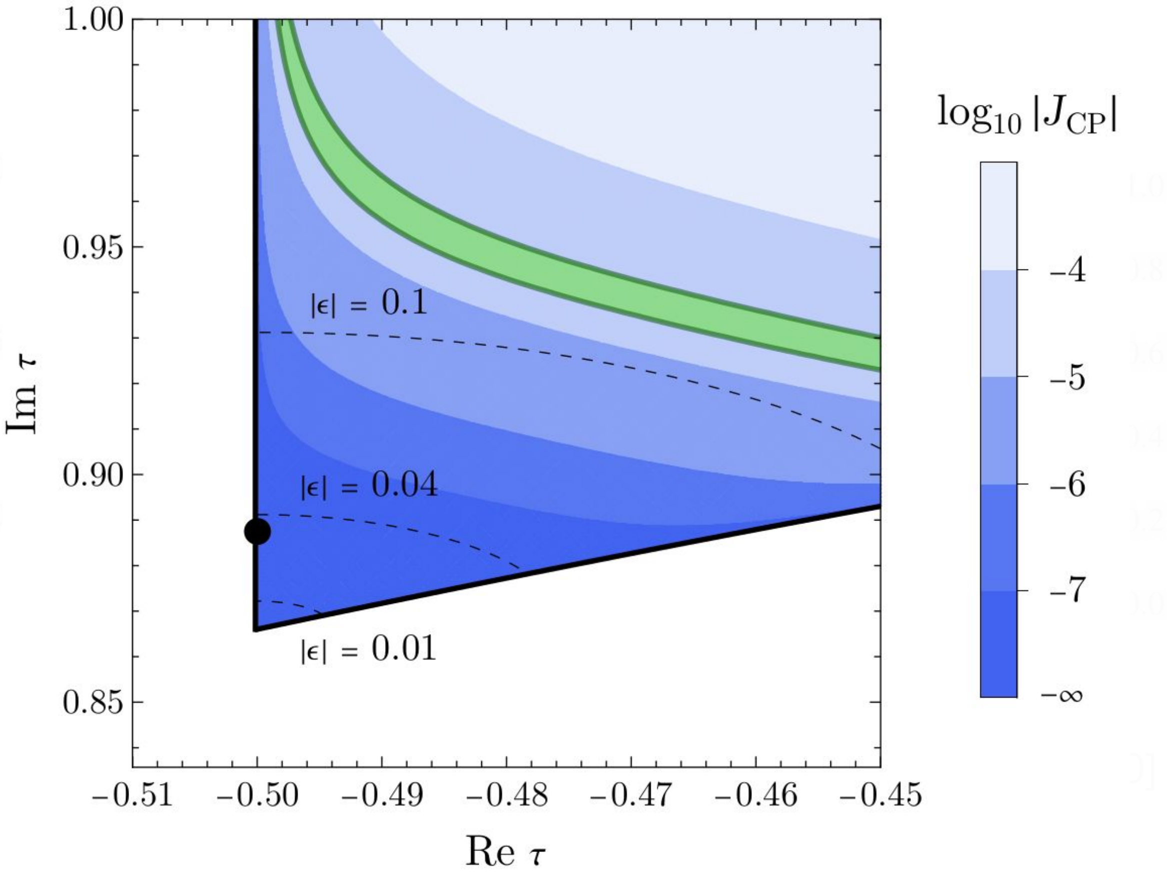


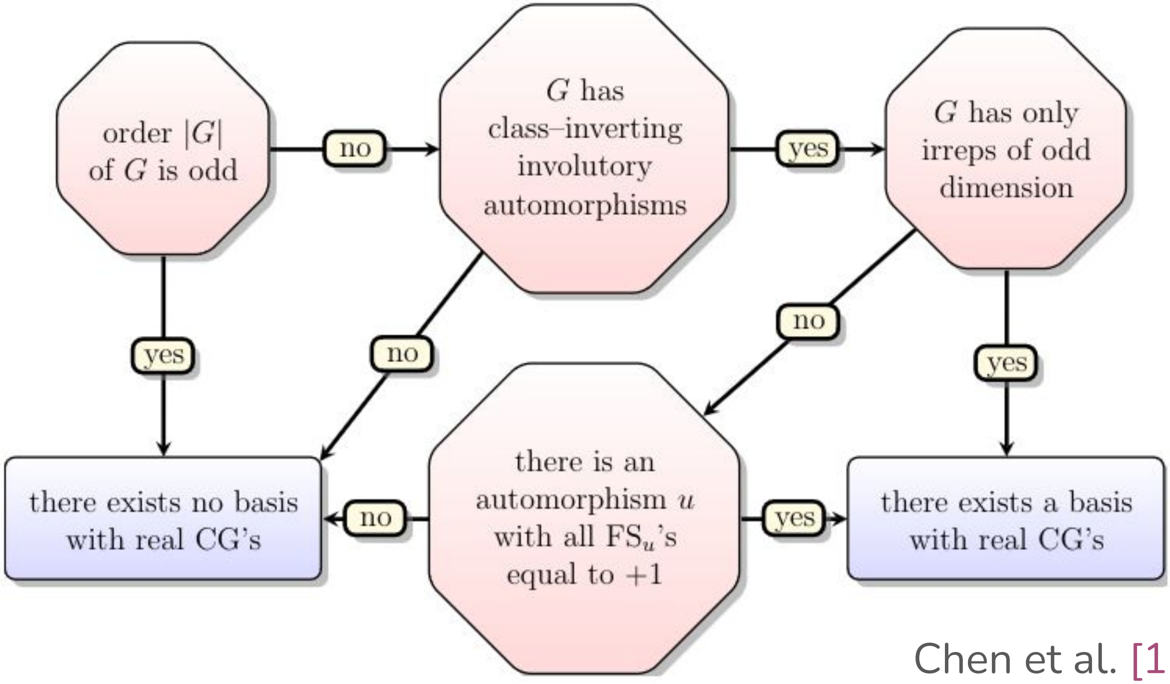
an example with quarks



Varzielas, Levy, JP, Petcov [2307.14410]

Other groups, other CPs

- **For N > 5**, conclusions still apply directly if:
    - At the lowest weight, there is ≤1 form for each irrep
    - There is a symmetric basis with real CGCs



Chen et al. [1402.0507]

Other groups, other CPs

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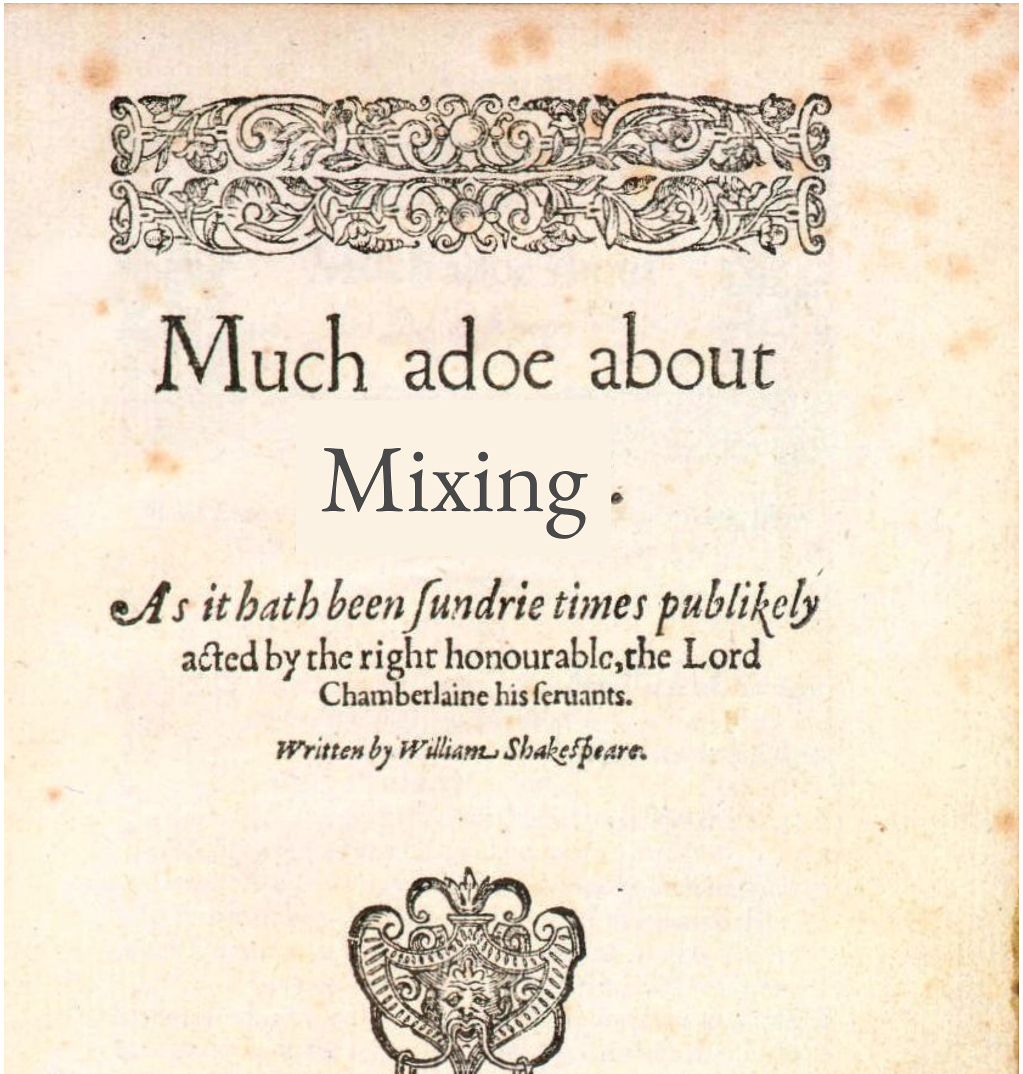
    - At the lowest weight, there is ≤1 form for each irrep
    - There is a symmetric basis with real CGCs

- **Non-standard** class-inverting automorphisms (CPs) can be found in double and metaplectic covers of the modular group:
    - CP_2 for SL(2,Z) in the S4' context [2006.03058]
      only valid for even *N* and specific irreps
      (may be incompat. with pheno)
    - $\mathrm{CP}_{2,3,4}$ for Mp(2,Z) [see Ding slides at Bethe Workshop]

Fermion mass hierarchies

Image credit: prompted StabilityAI's Stable Diffusion 2.1

Mass hierarchies from modular symmetry?



Much adoe about

Mixing.

As it hath been ſundrie times publikely
acted by the right honourable, the Lord
Chamberlaine his ſeruants.

Written by William Shakeſpeare.

Mass hierarchies from modular symmetry?

- Usually fermion mass hierarchies are put in **by hand**: hierarchies (or cancellations) between superpotential parameters

e.g. in the previously shown S4 model, $\gamma \ll \alpha \ll \beta$

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- **Other approaches** - New (weighted) scalars enter mass matrices a la Froggatt-Nielsen. Weights are analogous to FN charges Criado, Feruglio, King [1908.11867]; King^2 [2002.00969]

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- **Other approaches** - New (weighted) scalars enter mass matrices a la Froggatt-Nielsen. Weights are analogous to FN charges Criado, Feruglio, King [1908.11867]; King^2 [2002.00969]

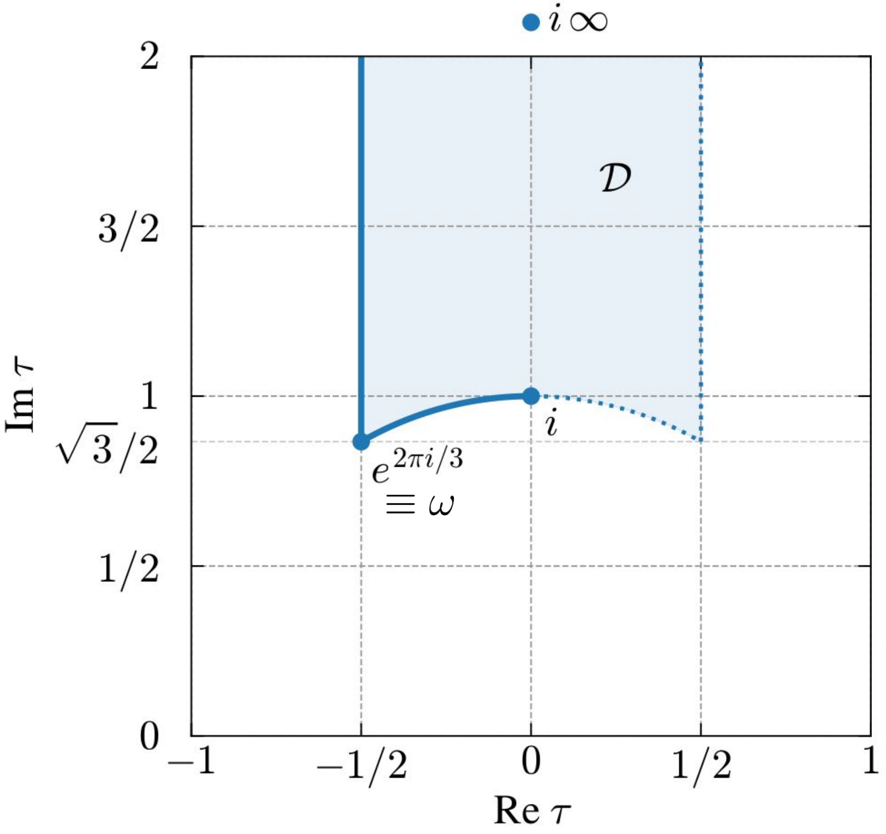
- **Our approach** - No new scalars, mechanism uses **only τ**, with common weights across generations (unlike FN charges)

Novichkov, JP, Petcov [2102.07488]

Idea by now applied in several works, see:
Petcov, Tanimoto [2212.13336]; Abe, Higaki, Kawamura, Kobayashi [2301.07439, 2302.11183]; Kikuchi, Kobayashi, Nasu, Takada, Uchida [2301.03737, 2302.03326]

[see also talk by M. Tanimoto]

Residual modular symmetries (the fund. domain yet again)



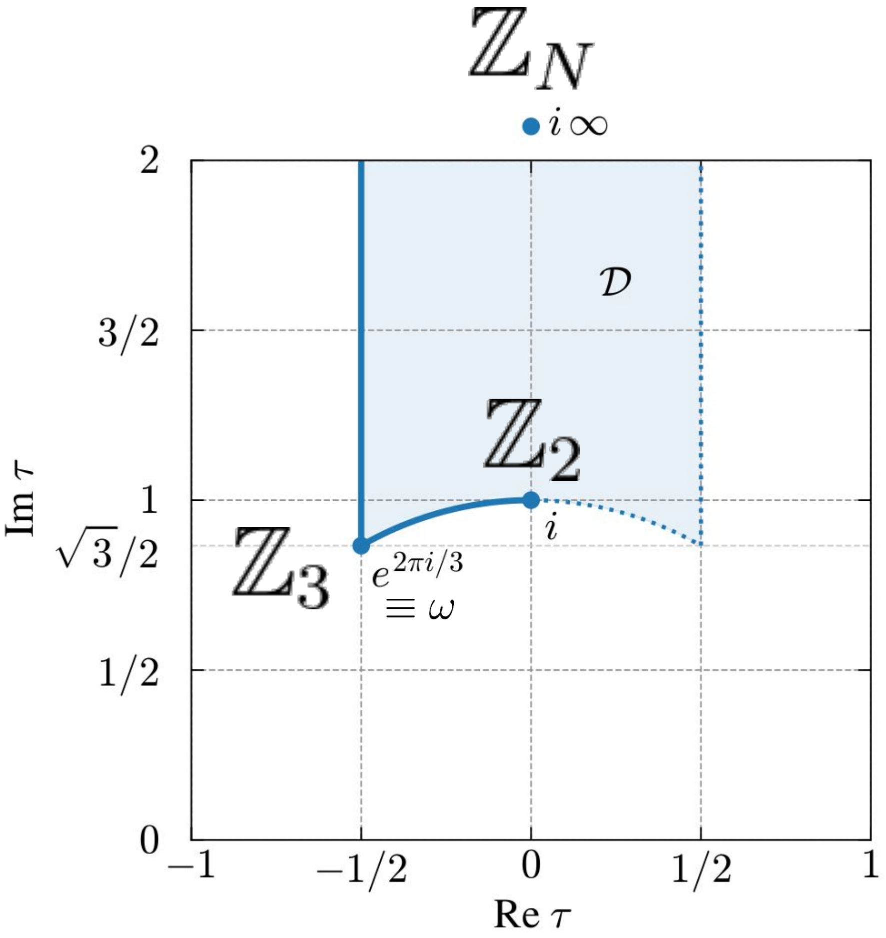
- At special values of τ, some residual symmetry remains

see e.g. Novichkov et al. [1811.04933]; Novichkov et al.' [1812.11289]

- Near them, these symmetries are linearly realized

see e.g. Feruglio [2302.11580]

Residual modular symmetries (the fund. domain yet again)



if the base symmetry is smaller, more stabilizers arise, see e.g. Varzielas, Levy, Zhou [2008.05329]
[recall talk by Y.-L. Zhou]

- At special values of τ, some **residual symmetry** remains

see e.g. Novichkov et al. [1811.04933]; Novichkov et al.' [1812.11289]

- Near them, these symmetries are linearly realized

see e.g. Feruglio [2302.11580]

Key idea:

some couplings vanish as we approach a symmetric point

Can be used for texture zeros, see [2207.04609]

Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}}$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\psi^c \, M \, \psi$$

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Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}}$$

$$\epsilon \sim |\tau - \tau_{\text{sym}}| > 0$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \to \quad M \sim \begin{pmatrix} 1 & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \end{pmatrix}$$

$$\psi^c \, M \, \psi$$

In the vicinity of the sym. point, the couplings are $\mathcal{O}(\epsilon^l)$

Key idea:

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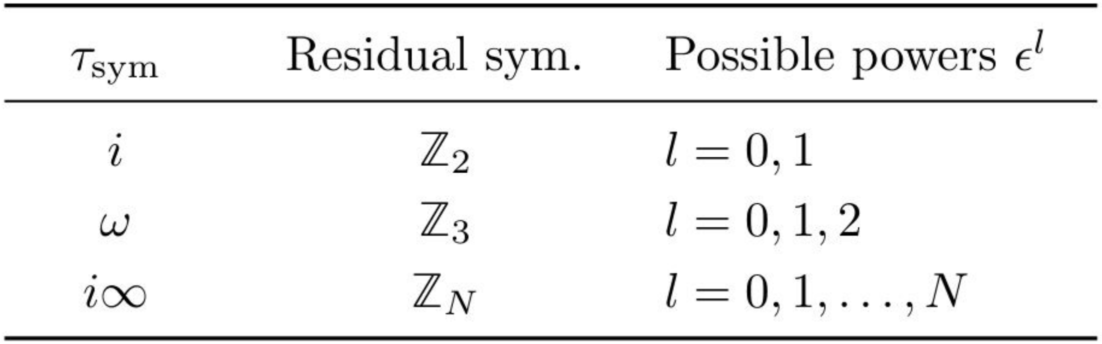
Can be used for texture zeros, see [2207.04609]

Decompositions under residual groups

(determine $\mathcal{O}(\epsilon^l)$)



τ_{sym}	Residual sym.	Possible powers ϵ^l
i	$\mathbb{Z}_2$	$l = 0, 1$
ω	$\mathbb{Z}_3$	$l = 0, 1, 2$
$i\infty$	$\mathbb{Z}_N$	$l = 0, 1, \ldots, N$

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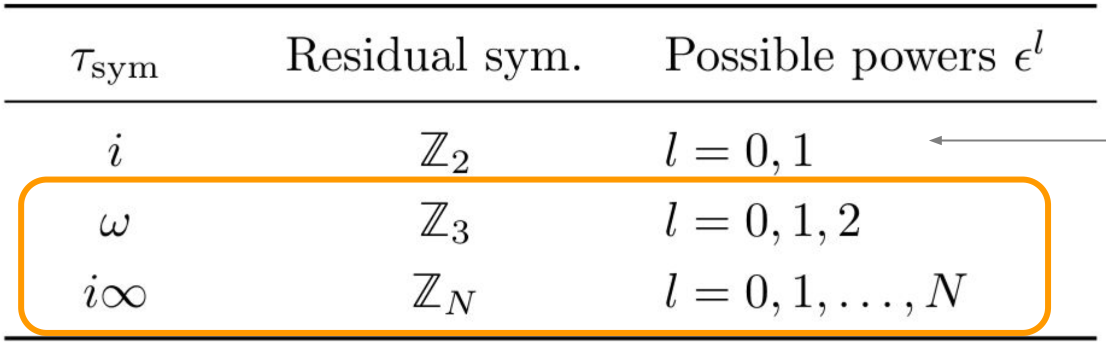

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Feruglio, Gherardi, Romanino, Titov [2101.08718] (for A_4, m_e=0)

$$\psi^c\, M\, \psi$$

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma)\, \psi$$

$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma)\, \psi^c$$

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger$$

$$\psi \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

$$\psi^c \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

In general, depend on weights

Determined for all $N \leqslant 5$

Example: hierarchical mass matrix (A5)

$$\begin{aligned} \psi &\sim (\mathbf{3}, k) \\ \psi^c &\sim (\mathbf{3}', k^c) \end{aligned}$$

$$\Rightarrow$$

Under the residual group of

$$\tau_{\text{sym}} = i\infty$$

$$\begin{aligned} \psi &\rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4 \\ \psi^c &\rightsquigarrow 1_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3 \end{aligned}$$

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$$\psi \rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4$$
$$\psi^c \rightsquigarrow 1_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3$$

For $\psi^c \, M \, \psi$, we expect:

$$M \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \quad \Rightarrow$$

with $\epsilon = e^{-2\pi \operatorname{Im} \tau / 5}$

fermion spectrum

$$\sim (1, \epsilon, \epsilon^4)$$ ✓

Indeed the case, provided enough modular forms contribute to *M*

(otherwise, $m_e = 0$)

Example: hierarchical mass matrix (A5)

$$\psi \sim (\mathbf{3}, k)$$

$$\Rightarrow$$

Under the residual group of

$$\tau_{\text{sym}} = i\infty$$

Not like Froggatt-Nielsen. Instead, it is an **improvement**!

Explicit example at weight 2

$$W \supset \sum_s \alpha_s \left(Y_{\mathbf{5}}^{(5,2)}(\tau)\psi^c\psi\right)_{\mathbf{1},s} \quad \Rightarrow \quad M(\tau) = \alpha \begin{pmatrix} \sqrt{3}Y_1 & Y_5 & Y_2 \\ Y_4 & -\sqrt{2}Y_3 & -\sqrt{2}Y_5 \\ Y_3 & -\sqrt{2}Y_2 & -\sqrt{2}Y_4 \end{pmatrix}_{Y_{\mathbf{5}}^{(5,2)}}$$

$$(Y_1, Y_2, Y_3, Y_4, Y_5) \simeq \mathcal{N}\left(-1/\sqrt{6},\ q,\ 3q^2,\ 4q^3,\ 7q^4\right)$$

$$\epsilon \quad \epsilon \quad \epsilon$$

$$\text{with } \epsilon = e^{-2\pi\, \mathrm{Im}\,\tau/5}$$

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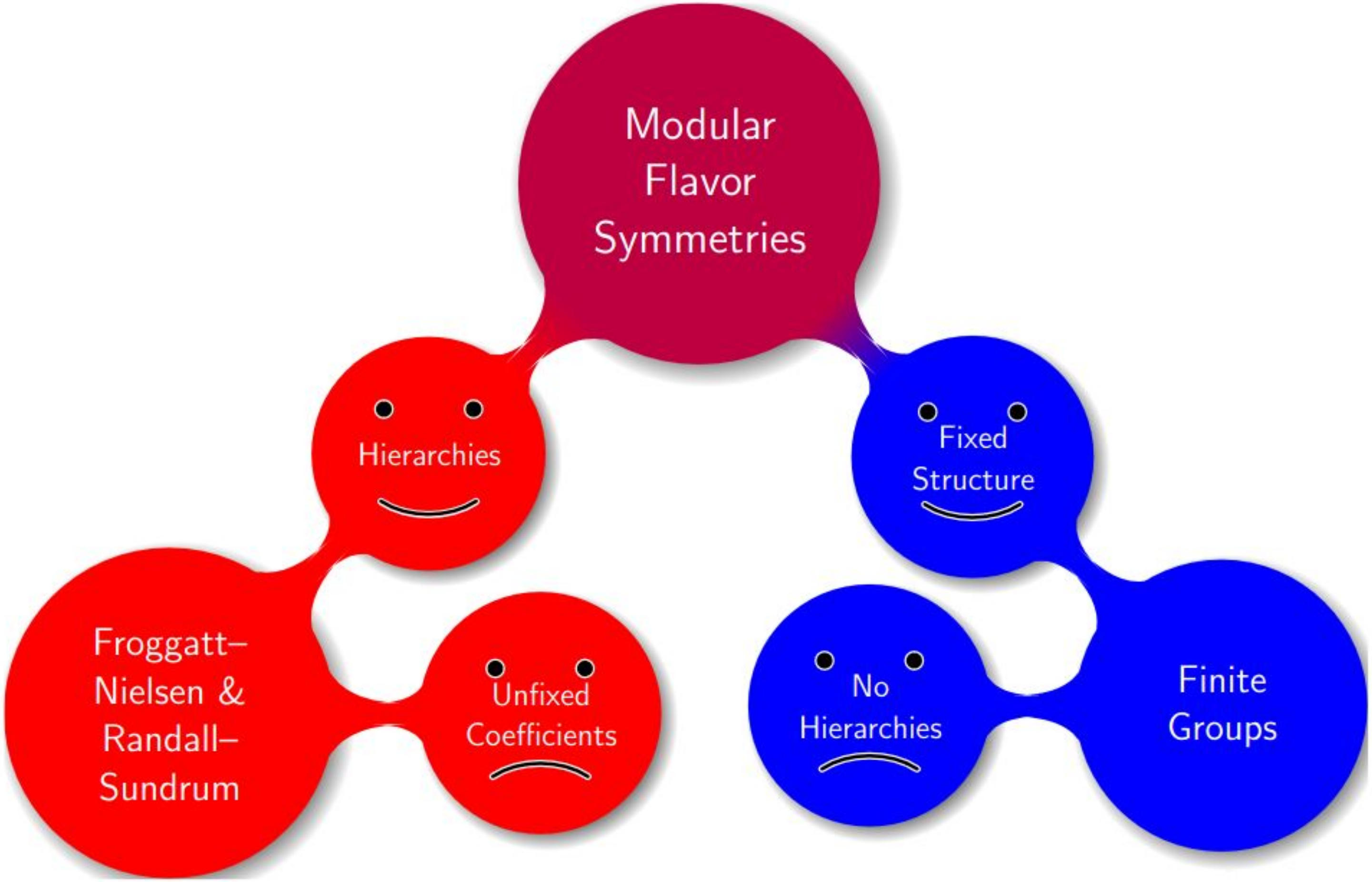


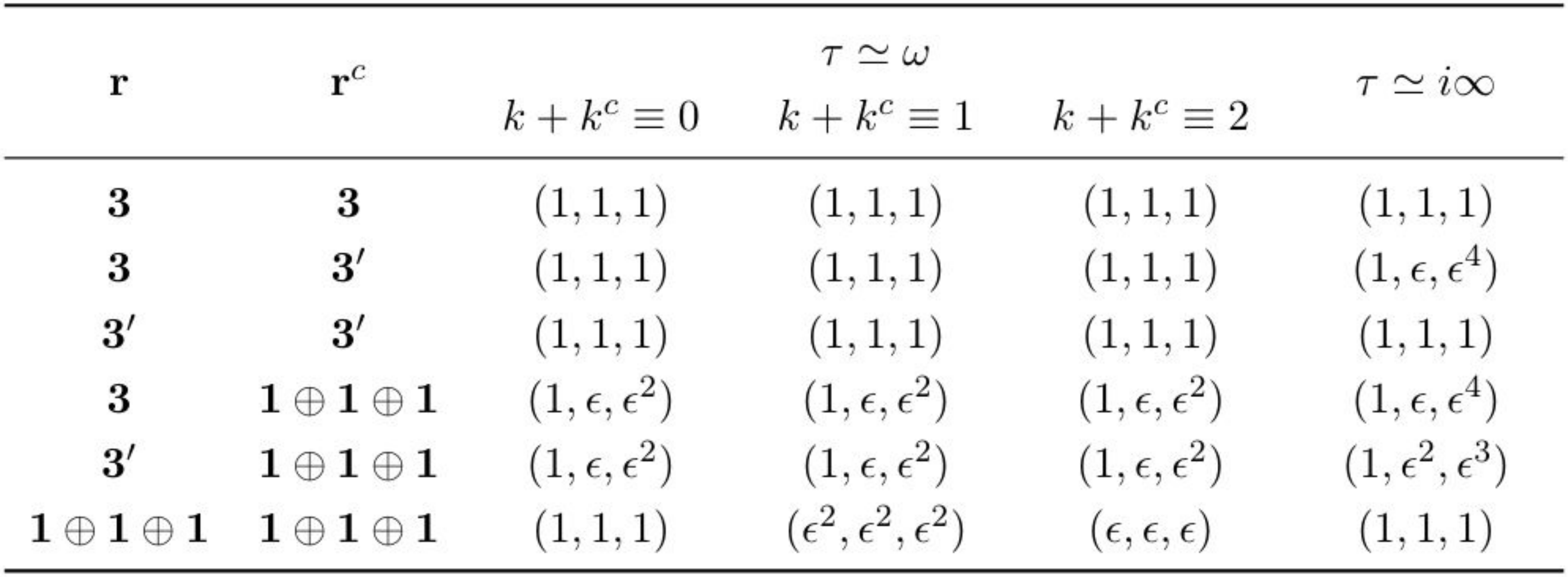
from M. Ratz [2405.00870]

Scan of possible mass patterns

Performed for 3 generations, for all $N \leqslant 5$

(simplifying assumption: family-blind weights)

e.g. fermion spectra for multiplets of modular A5



$\mathbf{r}$	$\mathbf{r}^c$		$\tau \simeq \omega$		$\tau \simeq i\infty$
		$k+k^c \equiv 0$	$k+k^c \equiv 1$	$k+k^c \equiv 2$	
$\mathbf{3}$	$\mathbf{3}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,\epsilon,\epsilon^4)$
$\mathbf{3}'$	$\mathbf{3}'$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
$\mathbf{3}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^4)$
$\mathbf{3}'$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon,\epsilon^2)$	$(1,\epsilon^2,\epsilon^3)$
$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$	$(1,1,1)$	$(\epsilon^2,\epsilon^2,\epsilon^2)$	$(\epsilon,\epsilon,\epsilon)$	$(1,1,1)$

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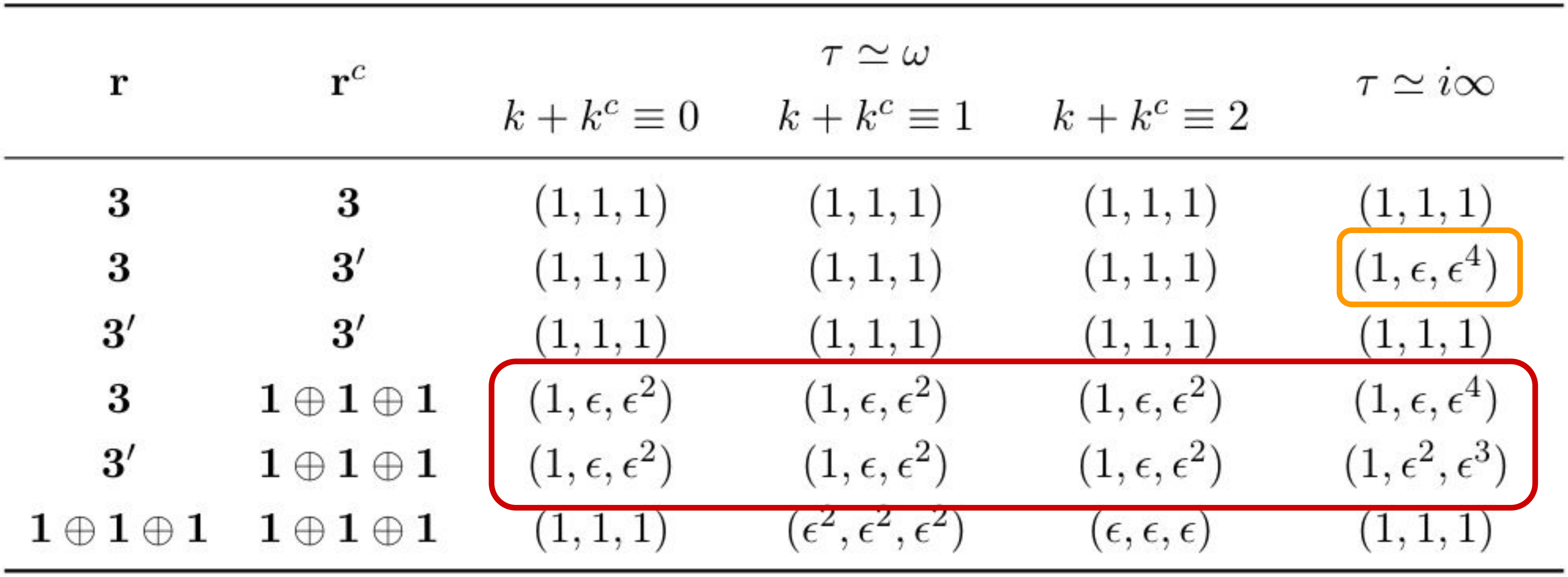

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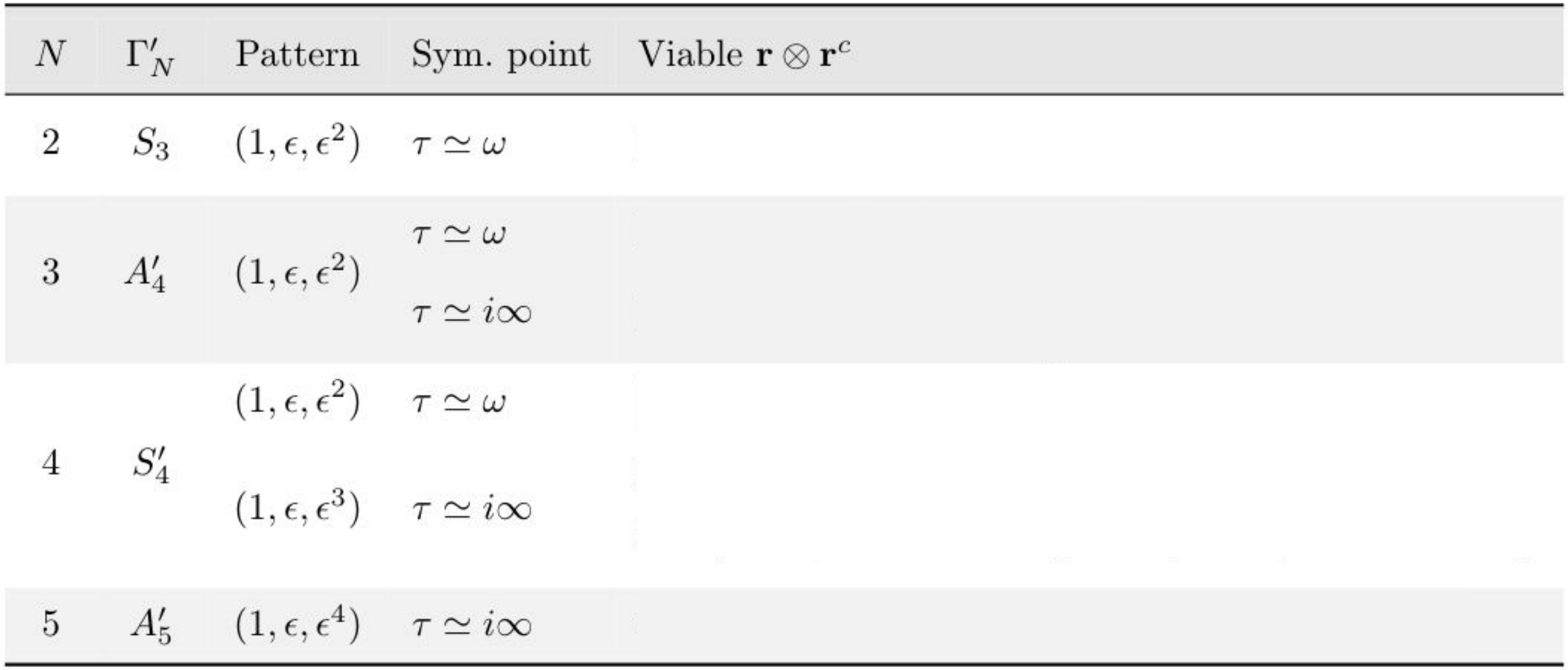
$\mathbf{3}$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3}'$	$\mathbf{3}'$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$	$(1, 1, 1)$
$\mathbf{3}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^4)$
$\mathbf{3}'$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon, \epsilon^2)$	$(1, \epsilon^2, \epsilon^3)$
$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$	$(1, 1, 1)$	$(\epsilon^2, \epsilon^2, \epsilon^2)$	$(\epsilon, \epsilon, \epsilon)$	$(1, 1, 1)$

Promising hierarchical patterns

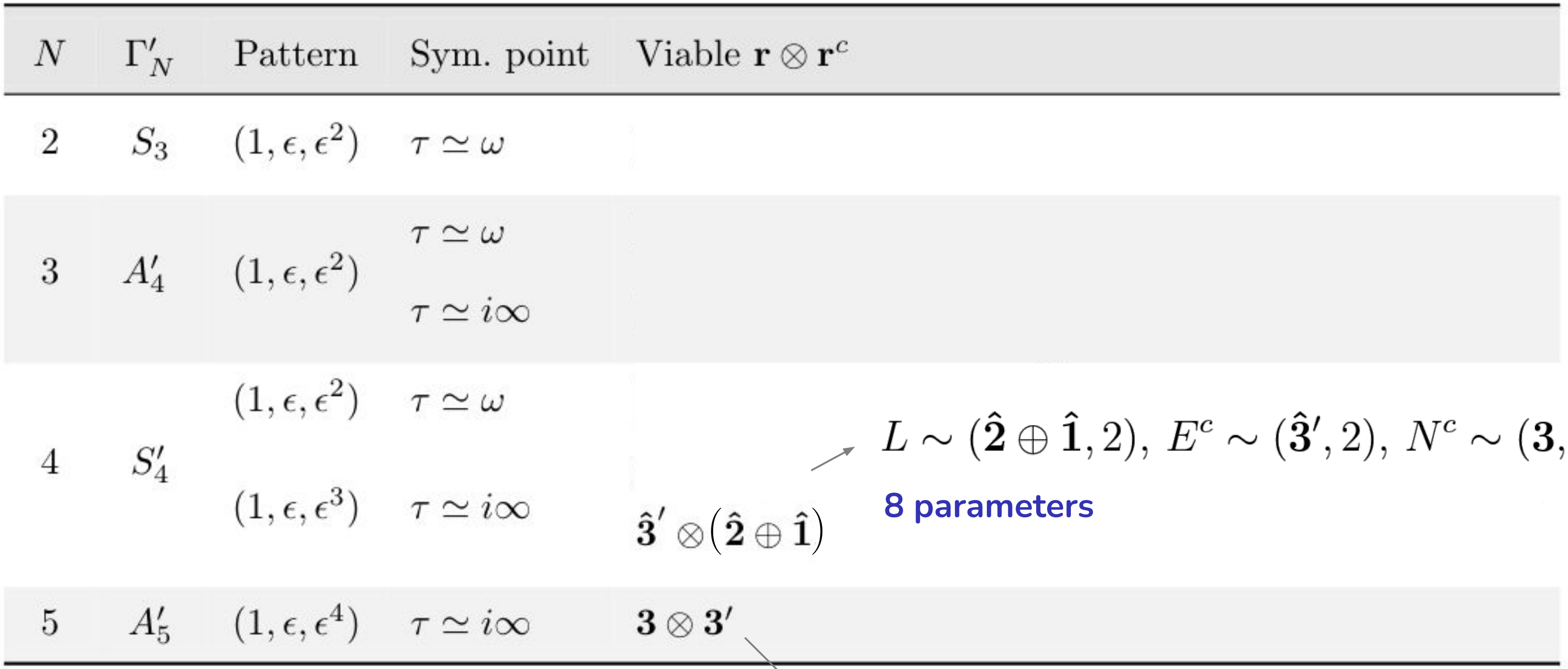


N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$ $\tau \simeq i\infty$	
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	

Promising hierarchical patterns

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r} \otimes \mathbf{r}^c$
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$
			$\tau \simeq i\infty$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$ with $\mathbf{1}_a \neq (\mathbf{1}_b)^*$
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$
		$(1, \epsilon, \epsilon^3)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes [\mathbf{2} \oplus \mathbf{1}$, or $\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}']$, $\mathbf{3}' \otimes [\mathbf{2} \oplus \mathbf{1}'$, or $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}']$, $\hat{\mathbf{3}}' \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}']$, $\hat{\mathbf{3}} \otimes [\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$, or $\hat{\mathbf{1}} \oplus \hat{\mathbf{1}}' \oplus \hat{\mathbf{1}}']$
5	A'_5	$(1, \epsilon, \epsilon^4)$	$\tau \simeq i\infty$	$\mathbf{3} \otimes \mathbf{3}'$

Promising hierarchical patterns (try leptons)



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		$(1,\epsilon,\epsilon^3)$	$\tau\simeq i\infty$	$\hat{\mathbf{3}}'\otimes(\hat{\mathbf{2}}\oplus\hat{\mathbf{1}})$
5	A'_5	$(1,\epsilon,\epsilon^4)$	$\tau\simeq i\infty$	$\mathbf{3}\otimes\mathbf{3}'$

$L\sim(\hat{\mathbf{2}}\oplus\hat{\mathbf{1}},2),\ E^c\sim(\hat{\mathbf{3}}',2),\ N^c\sim(\mathbf{3},1)$

8 parameters

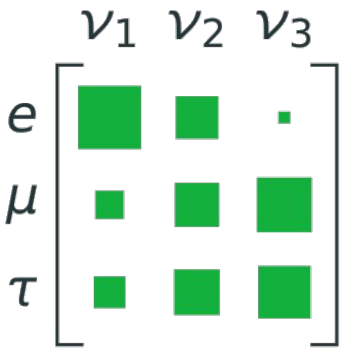
$L\sim(\mathbf{3},3),\ E^c\sim(\mathbf{3}',1),\ N^c\sim(\hat{\mathbf{2}},2)$

8 parameters

Masses are OK, but mixing is tuned :(

Wrong PMNS in the symmetric limit:
parameters are driven into cancellations

How to avoid fine-tuning (in the lepton sector)



$$\xrightarrow{\tau \to \tau_{\text{sym}}} \begin{bmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \text{ or } \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}$$

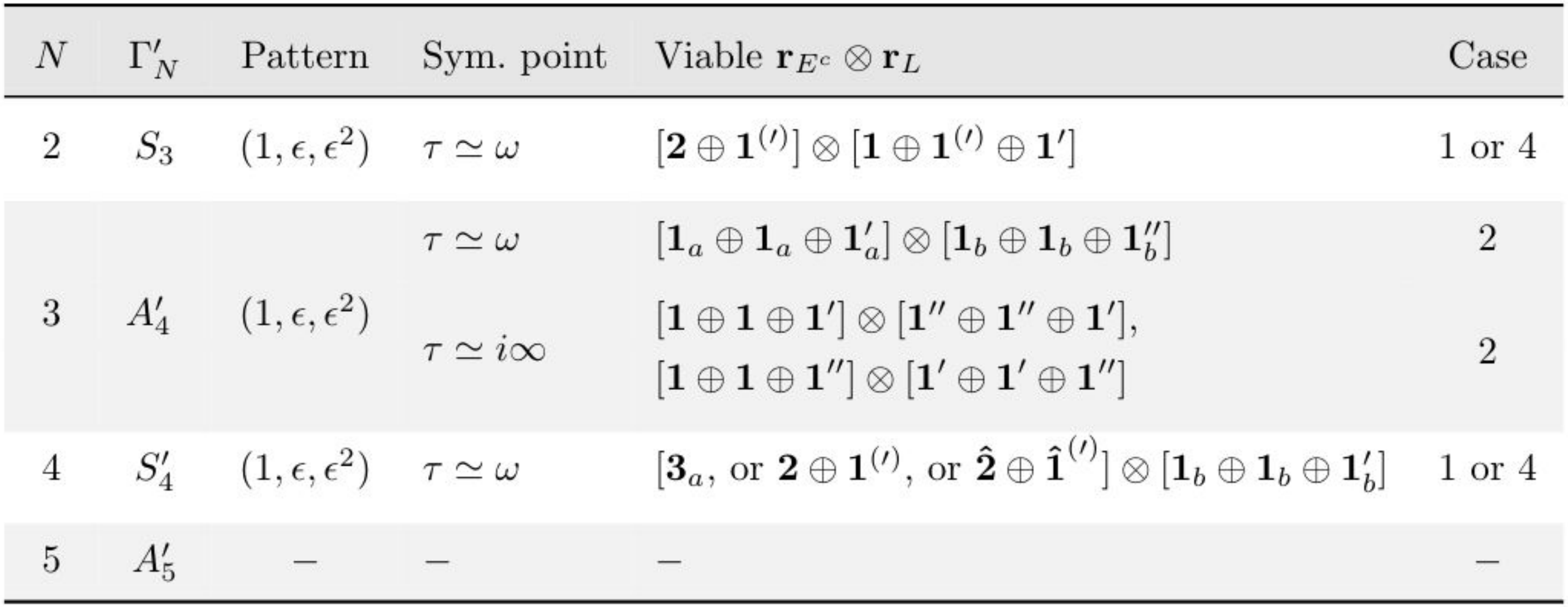
$$1. \begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases} \qquad 2. \begin{cases} L \sim \mathbf{1} \oplus \mathbf{1} \oplus \bar{\mathbf{1}} \\ E^c \sim \bar{\mathbf{1}} \oplus \mathbf{r} \not\supset \mathbf{1}, \bar{\mathbf{1}} \end{cases}$$

$$3.\ m_e = m_\mu = m_\tau = 0$$

$$4.\ m_{\nu_1} = m_{\nu_2} = m_{\nu_3} = 0$$

Reyimuaji, Romanino [1801.10530]

Promising hierarchical patterns (leptons)



N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}_{E^c} \otimes \mathbf{r}_L$	Case
2	S_3	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{2} \oplus \mathbf{1}^{(\prime)}] \otimes [\mathbf{1} \oplus \mathbf{1}^{(\prime)} \oplus \mathbf{1}']$	1 or 4
3	A'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{1}_a \oplus \mathbf{1}_a \oplus \mathbf{1}'_a] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}''_b]$	2
			$\tau \simeq i\infty$	$[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}'] \otimes [\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}']$, $[\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}''] \otimes [\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}'']$	2
4	S'_4	$(1, \epsilon, \epsilon^2)$	$\tau \simeq \omega$	$[\mathbf{3}_a$, or $\mathbf{2} \oplus \mathbf{1}^{(\prime)}$, or $\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}^{(\prime)}] \otimes [\mathbf{1}_b \oplus \mathbf{1}_b \oplus \mathbf{1}'_b]$	1 or 4
5	A'_5	–	–	–	–

$$1. \begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases} \qquad 2. \begin{cases} L \sim \mathbf{1} \oplus \mathbf{1} \oplus \bar{\mathbf{1}} \\ E^c \sim \bar{\mathbf{1}} \oplus \mathbf{r} \not\supset \mathbf{1}, \bar{\mathbf{1}} \end{cases}$$

$$3.\ m_e = m_\mu = m_\tau = 0$$

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Reyimuaji, Romanino [1801.10530]

Example: lepton model close to ω

Only S_4' model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

$$L \sim (\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}', 2),\ E^c \sim (\hat{\mathbf{3}}, 4),\ N^c \sim (\mathbf{3}', 1)$$

Example: lepton model close to ω

Only S_4' model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

$$L \sim (\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}', 2),\ E^c \sim (\hat{\mathbf{3}}, 4),\ N^c \sim (\mathbf{3}', 1)$$

Superpotential:

$$\begin{aligned} W = &\left[\alpha_1 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_1\right)_{\mathbf{1}} + \alpha_3 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_2\right)_{\mathbf{1}} + \alpha_4 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_2\right)_{\mathbf{1}} + \alpha_5 \left(Y^{(4,6)}_{\mathbf{3}} E^c L_3\right)_{\mathbf{1}}\right] H_d \\ &+ \left[g_1 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_1\right)_{\mathbf{1}} + g_2 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_2\right)_{\mathbf{1}} + g_3 \left(Y^{(4,3)}_{\hat{\mathbf{3}}'} N^c L_3\right)_{\mathbf{1}}\right] H_u \\ &+ \Lambda \left(Y^{(4,2)}_{\mathbf{2}} (N^c)^2\right)_{\mathbf{1}} . \end{aligned}$$

with gCP imposed

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$$M_e \propto \begin{pmatrix} 1 & \alpha - 2\beta & 2\sqrt{3} i\gamma \\ \sqrt{3}\epsilon & \sqrt{3}(\alpha + 2\beta)\epsilon & 2i\gamma\epsilon \\ \frac{5}{2}\epsilon^2 & \left(\frac{5}{2}\alpha - \beta\right)\epsilon^2 & -\frac{5}{\sqrt{3}} i\gamma\epsilon^2 \end{pmatrix}$$

$$|\epsilon| \simeq 2.8 \left|\frac{\tau - \omega}{\tau - \omega^2}\right| \sim \left|\tau - e^{2\pi i/3}\right|$$

$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}} b \end{pmatrix}$$

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$$u \equiv \frac{\tau - \omega}{\tau - \omega^2}$$

$|u|$ quantifies the deviation of τ from the left cusp (*do* u-expansions…)

Example: lepton model close to ω

Only S₄' model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

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$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}}b \end{pmatrix}$$

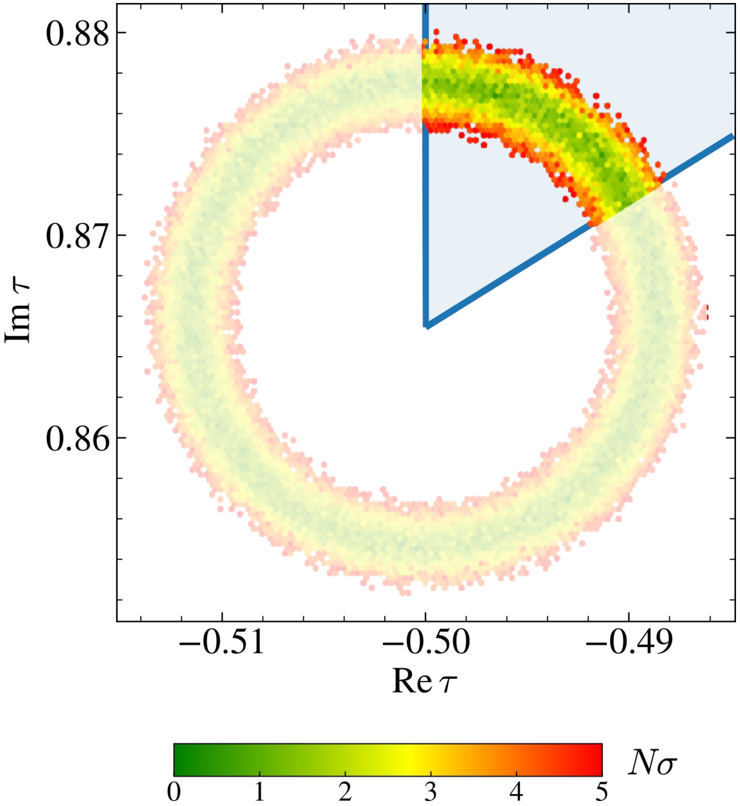
$|\epsilon| \simeq 0.02$ $\quad$ $\alpha = 2.45 \pm 0.44$

$a = 1.5 \pm 0.15$ $\quad$ $\beta = 2.14 \pm 0.32$

$b = 2.22 \pm 0.17$ $\quad$ $\gamma = 0.91 \pm 0.05$

Example: lepton model close to ω

$$|\epsilon| \simeq 0.02 \Leftrightarrow |u| \simeq 0.007$$



$$m_e = \mathcal{O}(\epsilon^2)$$
$$m_\mu = \mathcal{O}(\epsilon) \quad ✓$$
$$m_\tau = \mathcal{O}(1)$$

$$\text{NO}, \quad m_{\nu_1} = 0 \qquad \delta \simeq \pi$$
$$m_{\beta\beta} = (1.44 \pm 0.33)\ \text{meV}$$

Naturally allows for **hierarchies**, **large mixing**, and some **predictivity**

Example: lepton model close to ω

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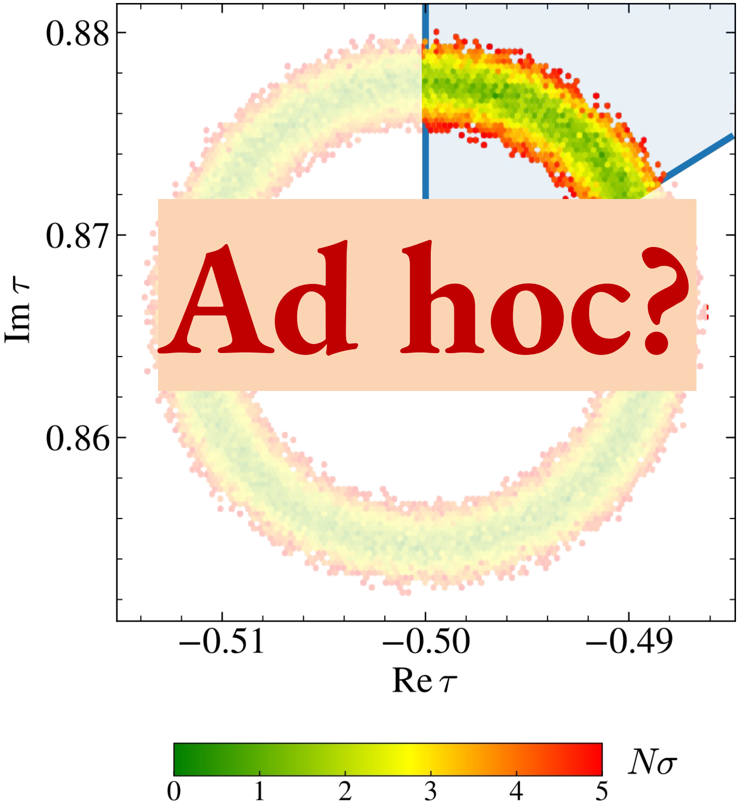

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Naturally allows for **hierarchies**, **large mixing**, and some **predictivity**

[recall talk by M. Parriciatu]

But what if I normalize *this* way?

A comment on normalizations

mod. form of weight *K*

$$W \supset \sum_i \left(\psi \boxed{\alpha_i Y_i^{(K)}} \psi^c\right)_{\mathbf{1}} H_q$$

[recall talk by M. Parriciatu]

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A comment on normalizations

$$W \supset \sum_i \left(\psi \overset{\text{mod. form of weight } K}{\boxed{\alpha_i Y_i^{(K)}}} \psi^c\right)_{\mathbf{1}} H_q$$

1	**100**
10	**10**
100	**1**
$\vert\alpha\vert$	$\Vert Y\Vert_{\text{E}}$

Same model predictions!

how can we discuss natural α's?

how do we interpret a hierarchy between α's?

how can we claim modular symmetries are responsible for hierarchies, not α's?

(norms of Y's are not fixed by group theory)

But what if I normalize *this* way?

A comment on normalizations

can trust these:

$$Y_{\mathbf{3}} \sim \begin{pmatrix} 1 \\ \epsilon \\ \epsilon^2 \end{pmatrix}$$

cannot trust these:

$$M \sim \left(\begin{array}{cc|c} 1 & \epsilon & 1 \\ \epsilon & \epsilon^2 & \epsilon \\ \hline \epsilon & \epsilon^2 & \epsilon \end{array}\right)$$

how can we claim modular symmetries are responsible for hierarchies, not α's?

discussion in Varzielas, Levy, JP, Petcov [2307.14410]

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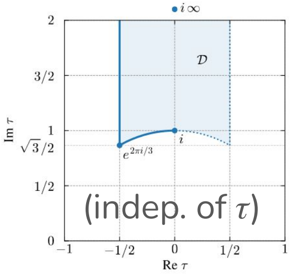
discussion in Varzielas, Levy, JP, Petcov [2307.14410]

A normalization proposal / choice

Petcov [2311.04185], D. Zagier (1981)

A “global” normalisation based on the Petersson inner product

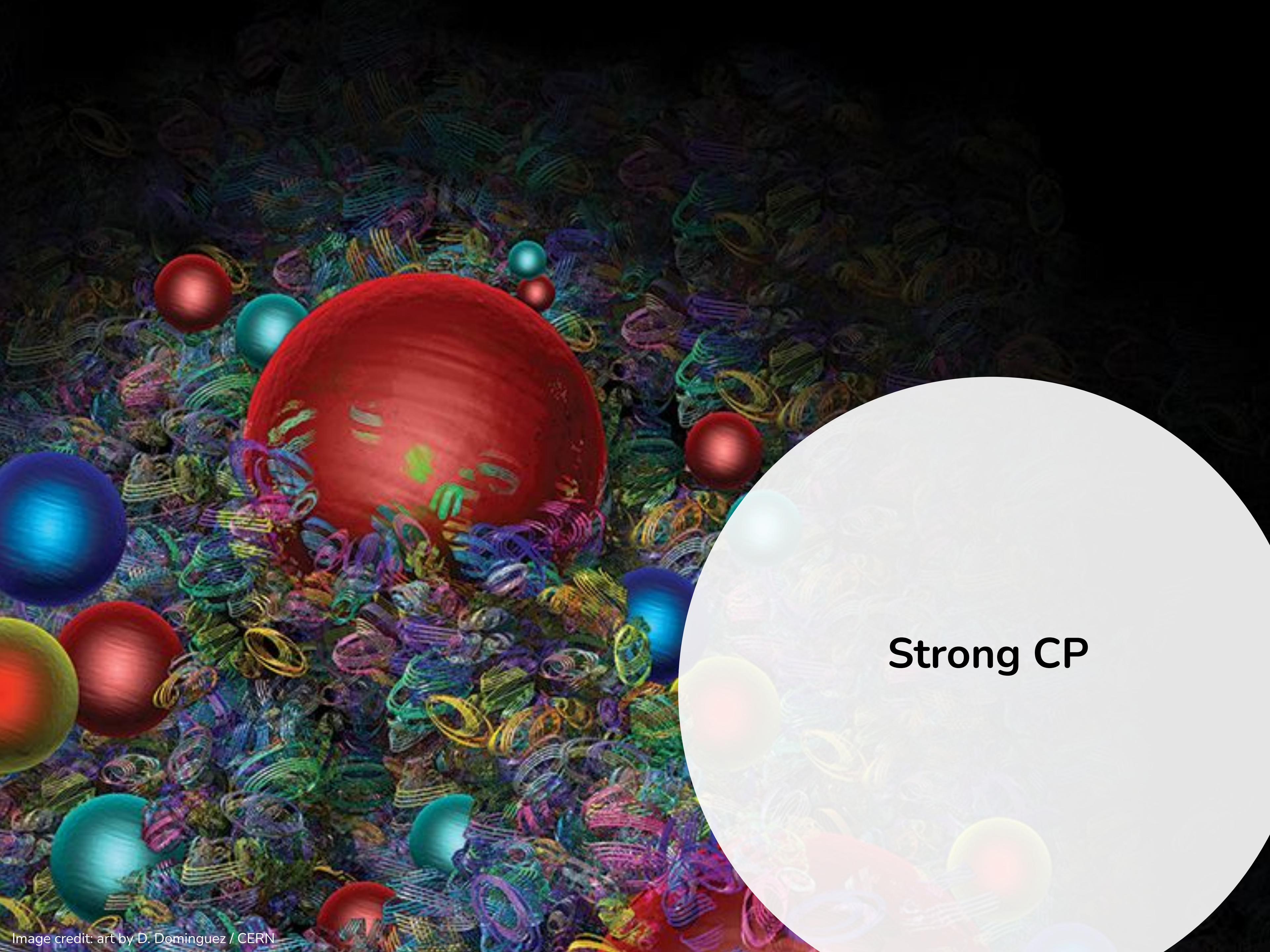
$$\boxed{\mathrm{N}\left[Y^{(K)}\right]^2 \equiv \iint_{\mathcal{D}} \sum_i \left|Y_i^{(K)}(x+iy)\right|^2 (2y)^K \frac{dx\,dy}{y^2} \overset{!}{=} 1}$$



(indep. of τ)

different prescription for non-cusp forms (yet another if K=1)

- Is there a general top-down recipe?
- Basis ambiguity if there are several forms of the same weight and irrep



Strong CP

Image credit: art by D. Dominguez / CERN

The idea by Feruglio, Strumia, Titov [2305.08908]

- No axions! see instead Higaki, Kawamura, Kobayashi [2402.02071] for a modular origin of the axion
- Need to **produce quark CPV phase** in the CKM mixing matrix
- Need to **suppress**:

$$\bar{\theta} = \theta_{\mathrm{QCD}} + \arg\det M_u M_d$$

vanishes due to
imposed gCP

vanishes due to
special structure

- It turns out to be holomorphic → insensitive to Kähler!
- The determinants of the mass matrices are modular forms

The idea by Feruglio, Strumia, Titov [2305.08908]

- Taking weightless Higgses for simplicity, the determinants have weights

  $$k^q_{\text{det}} \equiv \sum_i k_i + k_i^c$$

$$Q_i \xrightarrow{\gamma} (c\tau + d)^{-k_i} \rho_{ij}(\gamma) \, Q_j$$
$$q_i^c \xrightarrow{\gamma} (c\tau + d)^{-k_i^c} \rho_{ij}^c(\gamma) \, q_j^c$$

- To avoid massless quarks, must have $k^q_{\text{det}} \geq 0$

- To cancel both the modular QCD anomaly and $\arg\det M_u M_d$ we require

  $$k^u_{\text{det}} = k^d_{\text{det}} = 0$$

  making both determinants τ-independent **constants** (weight 0) and **real**, due to the imposed gCP!

incompatible w/ hierarchy mechanism
of a few slides ago :(

JP, Petcov [2404.08032]

Which matrices work?

(weight structures)

$$M_u \;:\; \begin{pmatrix} k+k' & k' & 0 \\ k & 0 & -k' \\ 0 & -k & -k-k' \end{pmatrix}$$

$$M_d \;:\; \begin{pmatrix} k+k' & k' & 0 \\ k & 0 & -k' \\ 0 & -k & -k-k' \end{pmatrix}$$

at least one of *k*, *k*’ non-zero

up to *simultaneous* permutations from the left
and independent permutations from the right (weak basis transformations)

JP, Petcov [2404.08032]

Which matrices work?

- Irreps beyond 1D imply extra relations between weights
- Cannot have triplet irreps
- Potentially viable non-singlet case:

$$\begin{aligned} Q &\sim (\mathbf{2}_Q, k_2) \oplus (\mathbf{1}_Q, k_1)\,, \\ u^c &\sim (\overline{\mathbf{2}_Q}, -k_2) \oplus (\overline{\mathbf{1}_Q}, -k_1)\,, \\ d^c &\sim (\overline{\mathbf{2}_Q}, -k_2) \oplus (\overline{\mathbf{1}_Q}, -k_1)\,, \end{aligned}$$

$$M_q \propto \begin{pmatrix} |\alpha_1^q/\beta_q| & 0 & \cos\theta_q\, e^{i\phi_1^q} \\ 0 & |\alpha_1^q/\beta_q| & \sin\theta_q\, e^{i\phi_2^q} \\ 0 & 0 & |\alpha_2^q/\beta_q| \end{pmatrix}^{(T)}$$

too much strain on the model, does not work → **quarks must furnish 1D irreps**

[like the first textures in M. Mondragon’s talk]

Which matrices work?

- Irreps beyond 1D imply extra relations betwee
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$$M_q \propto \left(\begin{matrix} |\alpha_1^q \\ (\\ (\end{matrix} \right.$$

too much strain on the model, does not work → **qu**

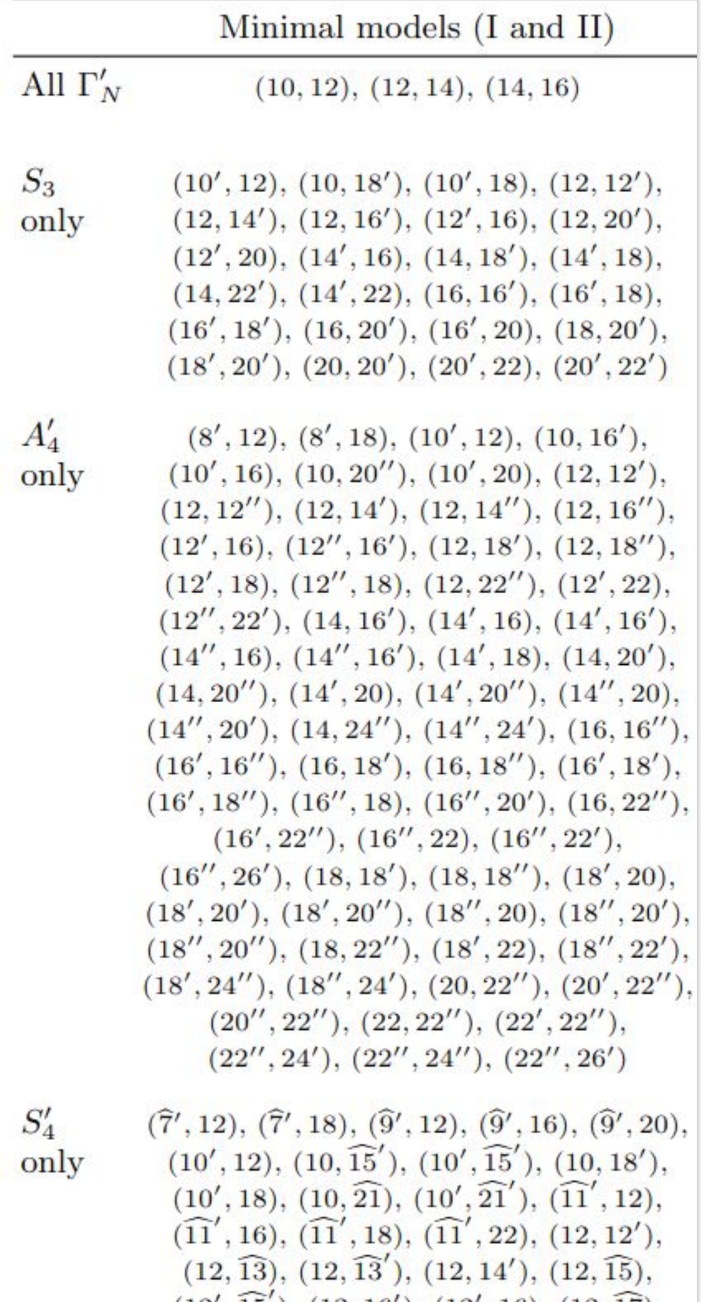
[like the first textures in M. Mondragon's talk]

- Minimal models (6+6+2=14 parameters):

$$\begin{pmatrix} \alpha_1^q & 0 & \tilde{\alpha}_{13}^q\, \mathcal{Y}_{q,13}^{(k+k')} \\ 0 & \alpha_2^q & \tilde{\alpha}_{23}^q\, \mathcal{Y}_{q,23}^{(k)} \\ 0 & 0 & \alpha_3^q \end{pmatrix}, \quad k > 0,\ k' \geq 0$$

and its transpose

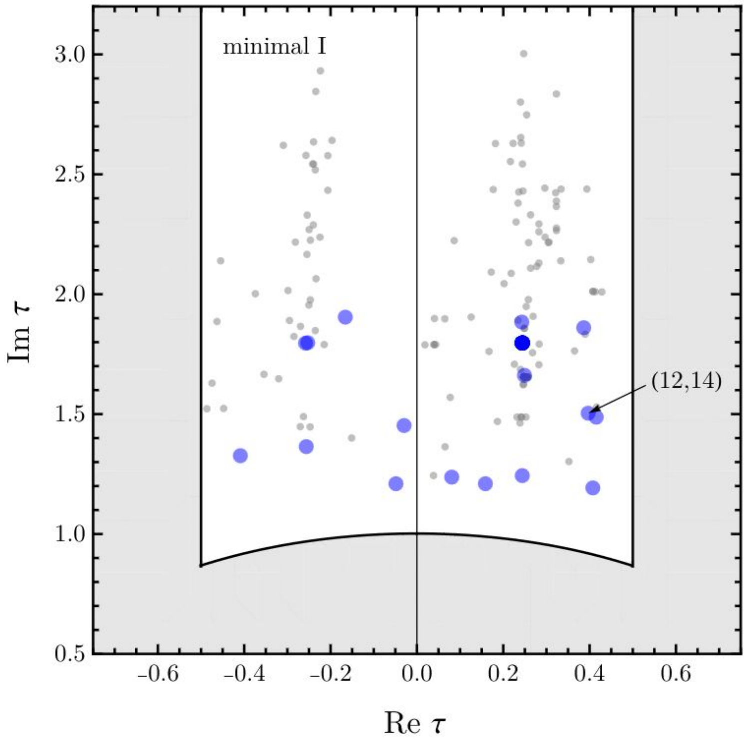
listed all 462 one can get with Γ'N



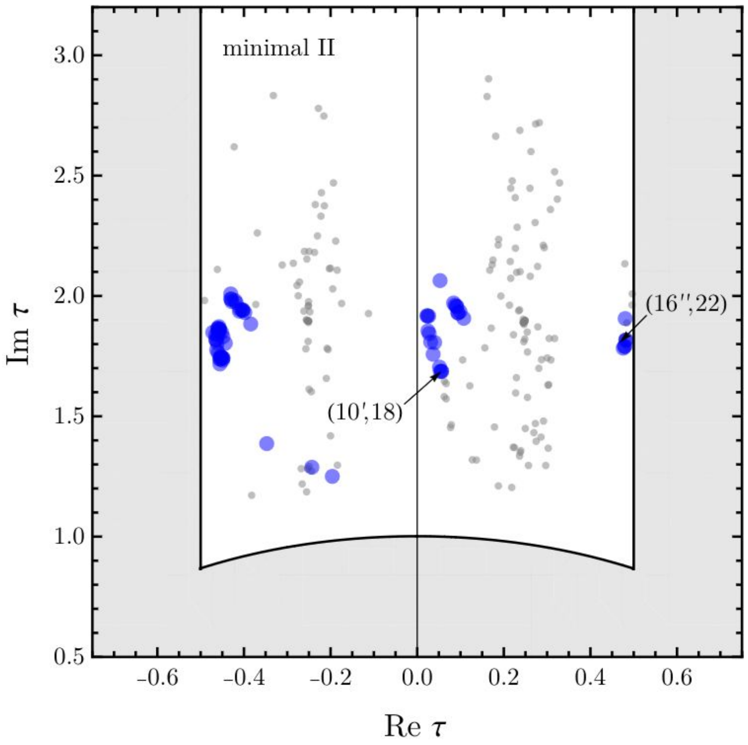
	Minimal models (I and II)
All Γ'_N	$(10, 12)$, $(12, 14)$, $(14, 16)$
S_3 only	$(10', 12)$, $(10, 18')$, $(10', 18)$, $(12, 12')$, $(12, 14')$, $(12, 16')$, $(12', 16)$, $(12, 20')$, $(12', 20)$, $(14', 16)$, $(14, 18')$, $(14', 18)$, $(14, 22')$, $(14', 22)$, $(16, 16')$, $(16', 18)$, $(16', 18')$, $(16, 20')$, $(16', 20)$, $(18, 20')$, $(18', 20')$, $(20, 20')$, $(20', 22)$, $(20', 22')$
A'_4 only	$(8', 12)$, $(8', 18)$, $(10', 12)$, $(10, 16')$, $(10', 16)$, $(10, 20'')$, $(10', 20)$, $(12, 12')$, $(12, 12'')$, $(12, 14')$, $(12, 14'')$, $(12, 16'')$, $(12', 16)$, $(12'', 16')$, $(12, 18')$, $(12, 18'')$, $(12', 18)$, $(12'', 18)$, $(12, 22'')$, $(12', 22)$, $(12'', 22')$, $(14, 16')$, $(14', 16)$, $(14', 16')$, $(14'', 16)$, $(14'', 16')$, $(14', 18)$, $(14, 20')$, $(14, 20'')$, $(14', 20)$, $(14', 20'')$, $(14'', 20)$, $(14'', 20')$, $(14, 24'')$, $(14'', 24')$, $(16, 16'')$, $(16', 16'')$, $(16, 18')$, $(16, 18'')$, $(16', 18')$, $(16', 18'')$, $(16'', 18)$, $(16'', 20')$, $(16, 22'')$, $(16', 22'')$, $(16'', 22)$, $(16'', 22')$, $(16'', 26')$, $(18, 18')$, $(18, 18'')$, $(18', 20)$, $(18', 20')$, $(18', 20'')$, $(18'', 20)$, $(18'', 20')$, $(18'', 20'')$, $(18, 22'')$, $(18', 22)$, $(18'', 22')$, $(18', 24'')$, $(18'', 24')$, $(20, 22'')$, $(20', 22'')$, $(20'', 22'')$, $(22, 22'')$, $(22', 22'')$, $(22'', 24')$, $(22'', 24'')$, $(22'', 26')$
S'_4 only	$(\hat{7}', 12)$, $(\hat{7}', 18)$, $(\hat{9}', 12)$, $(\hat{9}', 16)$, $(\hat{9}', 20)$, $(10', 12)$, $(10, \widehat{15}')$, $(10', \widehat{15}')$, $(10, 18')$, $(10', 18)$, $(10, \widehat{21})$, $(10', \widehat{21}')$, $(\widehat{11}', 12)$, $(\widehat{11}', 16)$, $(\widehat{11}', 18)$, $(\widehat{11}', 22)$, $(12, 12')$, $(12, \widehat{13})$, $(12, \widehat{13}')$, $(12, 14')$, $(12, \widehat{15})$,

A peek into the minimal model landscape

models without transposition



models with transposition

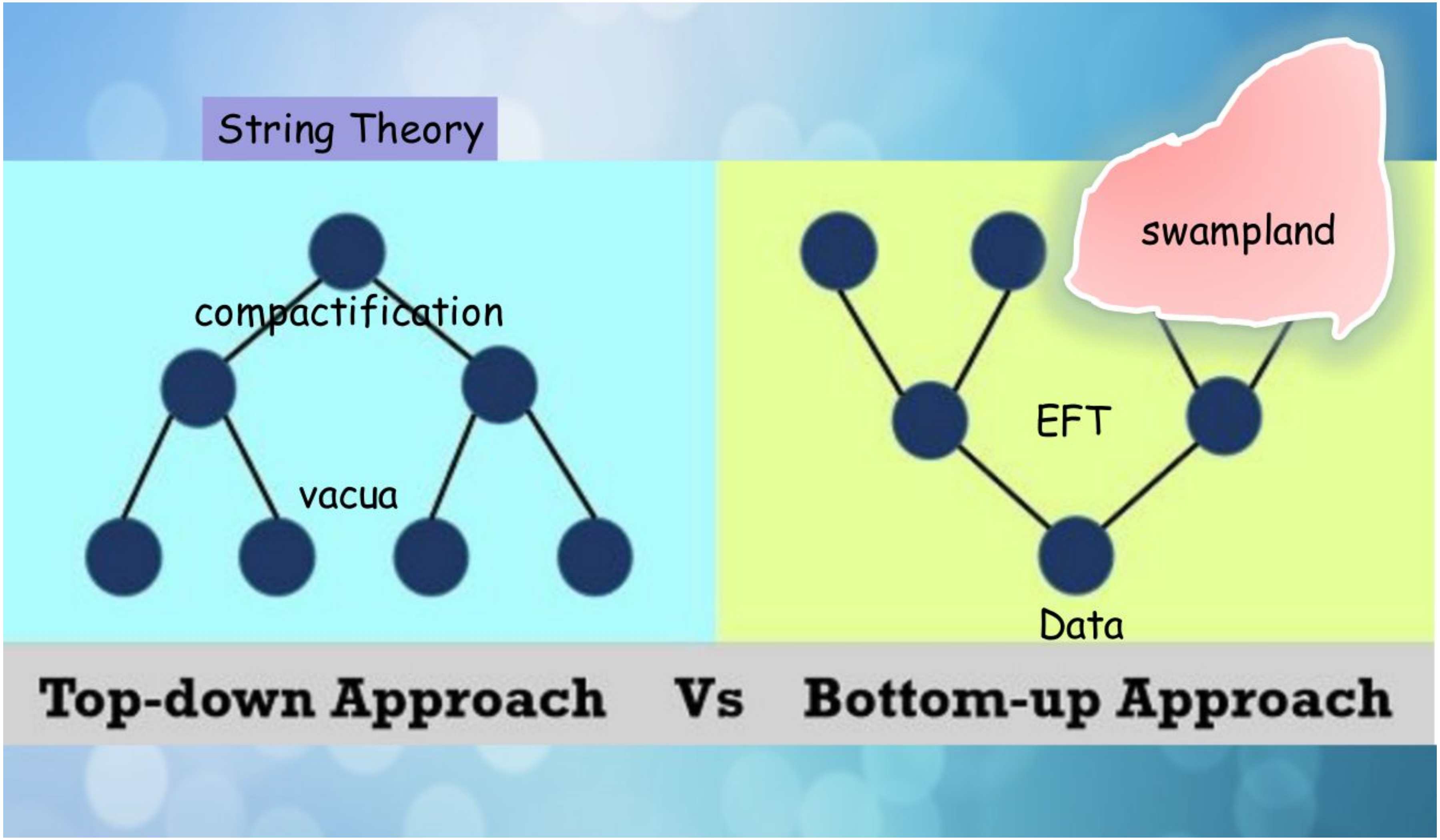


Moduli selection from the minimization of parameter hierarchies (log spread), using the normalization proposal in Petcov [2311.04185]

The top-down

Image credit: adapted from www.rayzorsedgetreeservice.com

[see talks by S. Ramos-Sanchez, H. Otsuka, G. Leontaris]



from F. Feruglio's slides at Mod. Symmetry Bethe Workshop (2022)

State of the art

What about... the Kähler?

- **Not holomorphic**: unconstrained by the symmetry!
- Under a modular transformation, invariant up to:

$$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) \;\to\; K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) + f(\chi_i; \tau) + f(\overline{\chi}_i; \overline{\tau})$$

- Minimal choice:

$$K(\chi_i, \overline{\chi}_i; \tau, \overline{\tau}) = -h\,\Lambda_0^2\,\log(-i(\tau - \overline{\tau})) + \sum_i \frac{|\chi_i|^2}{(-i(\tau - \overline{\tau}))^{k_i}}$$

impacts pheno → should be justified from the top-down

Chen, Ramos-Sánchez and Ratz [1909.06910]

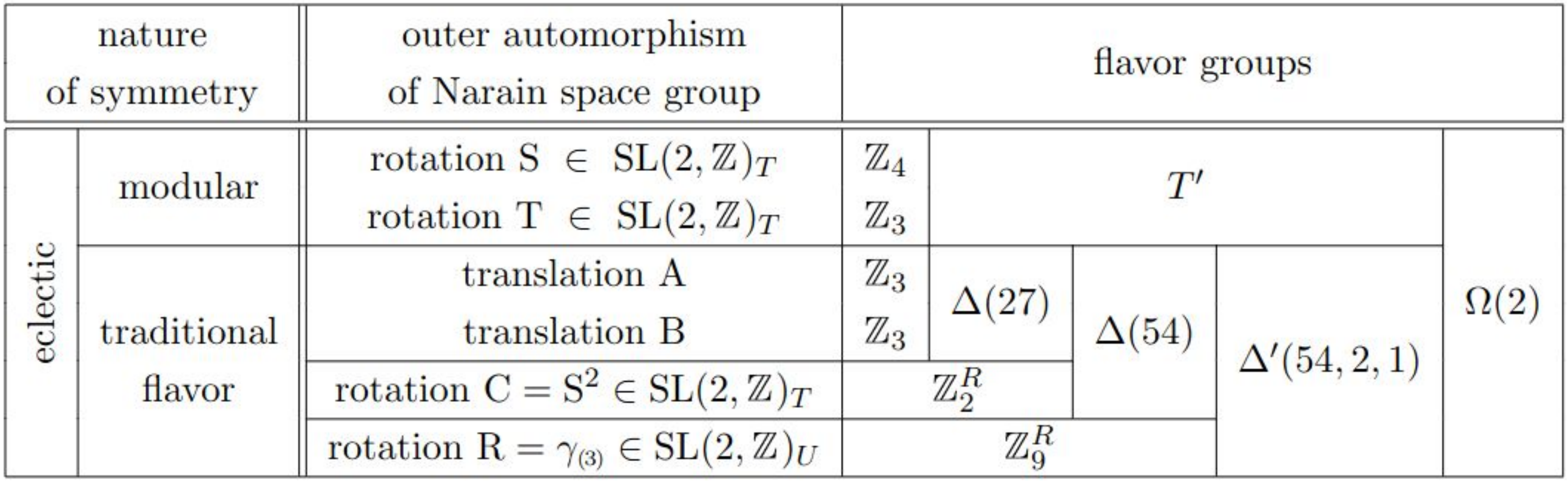
- Further constraints may arise from the (unavoidable...) combination of modular + traditional flavour symmetries

Nilles, Ramos-Sanchez, Vaudrevange [2004.05200]

Lessons from eclectic flavor symmetries

see [1901.03251, 1908.00805, 2001.01736, **2004.05200**, 2006.03059, 2112.06940]
and Nilles, Ramos-Sánchez [2404.16933] for a recent summary

- There is no possible scheme with just modular flavor symmetries
- Also discrete R-symmetries seem unavoidable
- A limited type of groups appear (e.g. *T*')



<table>
<tr><th colspan="2">nature
of symmetry</th><th>outer automorphism
of Narain space group</th><th colspan="5">flavor groups</th></tr>
<tr><td rowspan="6">eclectic</td><td rowspan="2">modular</td><td>rotation S $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_4$</td><td colspan="3" rowspan="2">T'</td><td rowspan="6">$\Omega(2)$</td></tr>
<tr><td>rotation T $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td rowspan="4">traditional
flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td><td rowspan="4">$\Delta'(54,2,1)$</td></tr>
<tr><td>translation B</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td>rotation $\mathrm{C} = \mathrm{S}^2 \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td colspan="2">$\mathbb{Z}_2^R$</td></tr>
<tr><td>rotation $\mathrm{R} = \gamma_{(3)} \in \mathrm{SL}(2,\mathbb{Z})_U$</td><td colspan="3">$\mathbb{Z}_9^R$</td></tr>
</table>

Nilles, Ramos-Sanchez, Vaudrevange [2006.03059]

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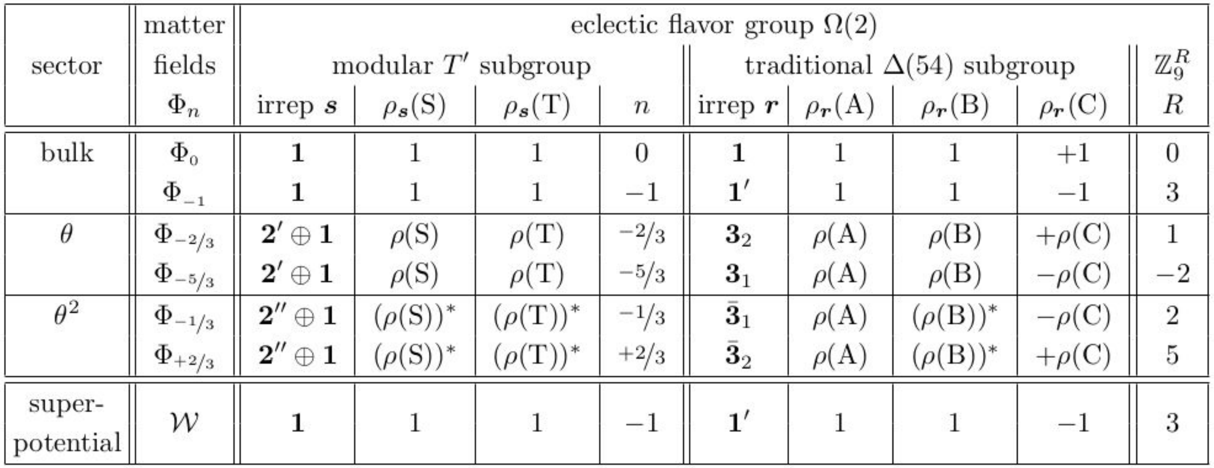
- Weights (typically fractional) correlated with irreps



	matter	eclectic flavor group $\Omega(2)$								
sector	fields	modular T' subgroup				traditional $\Delta(54)$ subgroup				$\mathbb{Z}_9^R$
	Φ_n	irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}' \oplus \mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}' \oplus \mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}'' \oplus \mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}'' \oplus \mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

Nilles, Ramos-Sanchez, Vaudrevange [2004.05200]

Lessons from eclectic flavor symmetries

see [1901.03251, 1908.00805, 2001.01736, **2004.05200**, 2006.03059, 2112.06940]
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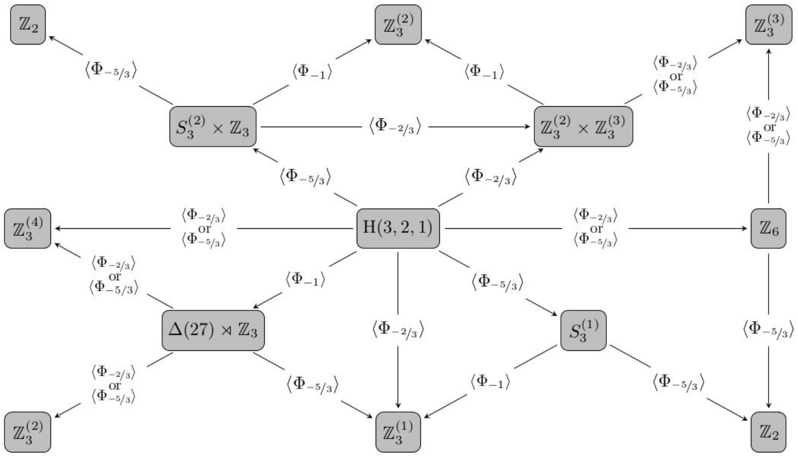

Baur et al. [2112.06940]

Lessons from eclectic flavor symmetries

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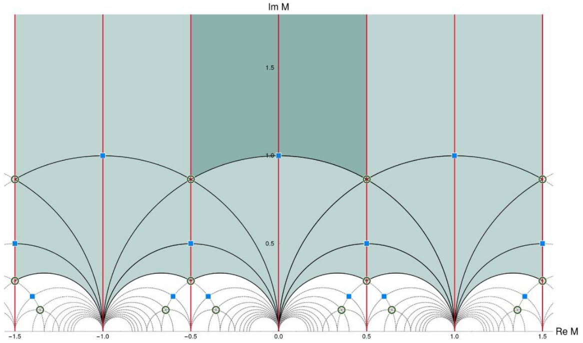
- Both Kähler and superpotential play a crucial role
- Larger fundamental domains ($\Gamma(N)$ instead of Γ?)



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- Breaking reintroduces flavons :(
- Both Kähler and superpotential play a crucial role
- Larger fundamental domains (Γ(N) instead of Γ?)
- Top-down and bottom-up do **not yet** meet

but there are a few BU attempts:
Chen et al. [2108.02240]; Ding et al. [2303.02071]; Li, Ding [2308.16901]

Moduli stabilization

early attempts: [1909.05139, 1910.11553]

[very incomplete discussion, please see talks by N. Righi, M. Ratz, J. Kawamura, X. Wang]

Simplest modular-invariant potentials?

- Studied by Cvetič, Font, Ibáñez, Lüst and Quevedo (1991) $\mathcal{N} = 1$ SUGRA

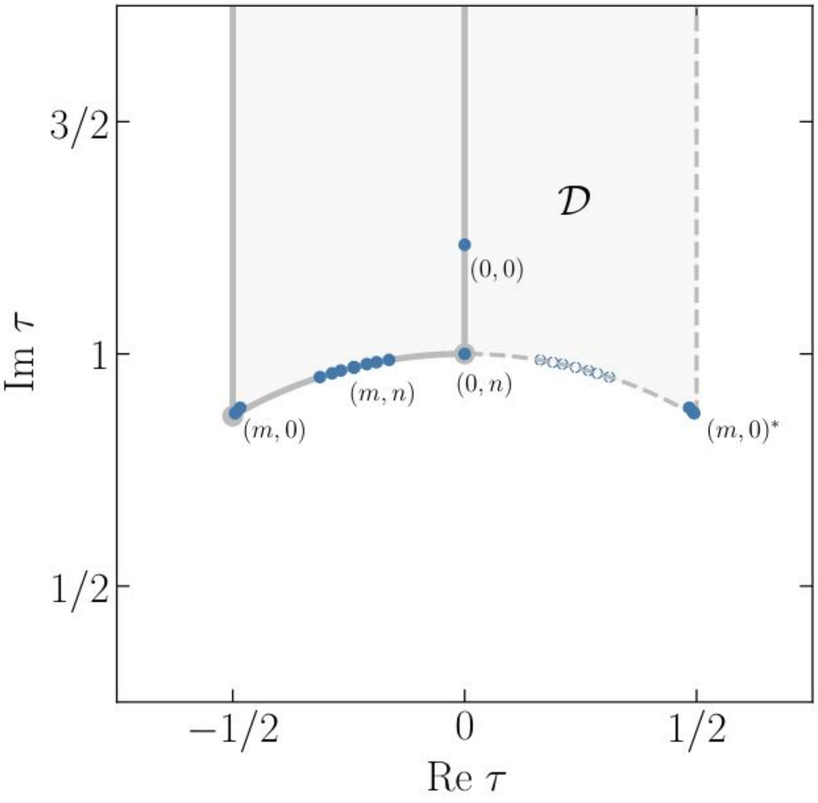
$$V(\tau,\overline{\tau}) = \frac{\Lambda_V^4}{8(\operatorname{Im}\tau)^3|\eta|^{12}}\left[\frac{4}{3}\left|iH' + \frac{3}{2\pi}H\hat{G}_2\right|^2(\operatorname{Im}\tau)^2 - 3|H|^2\right]$$

$$H(\tau) = (j(\tau) - 1728)^{\boxed{m}/2}\, j(\tau)^{\boxed{n}/3} \qquad W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^6}$$

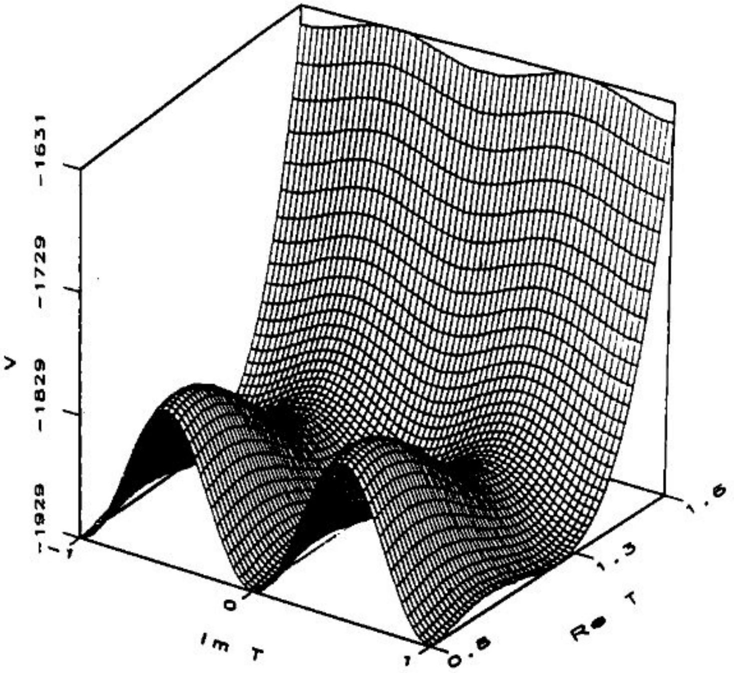
$m, n = 0, 1, 2, \ldots$

- This potential is **modular-** and **CP-invariant**
- Simplified model, independent of the level *N*

Global minima for (*m*,*n*)-potentials

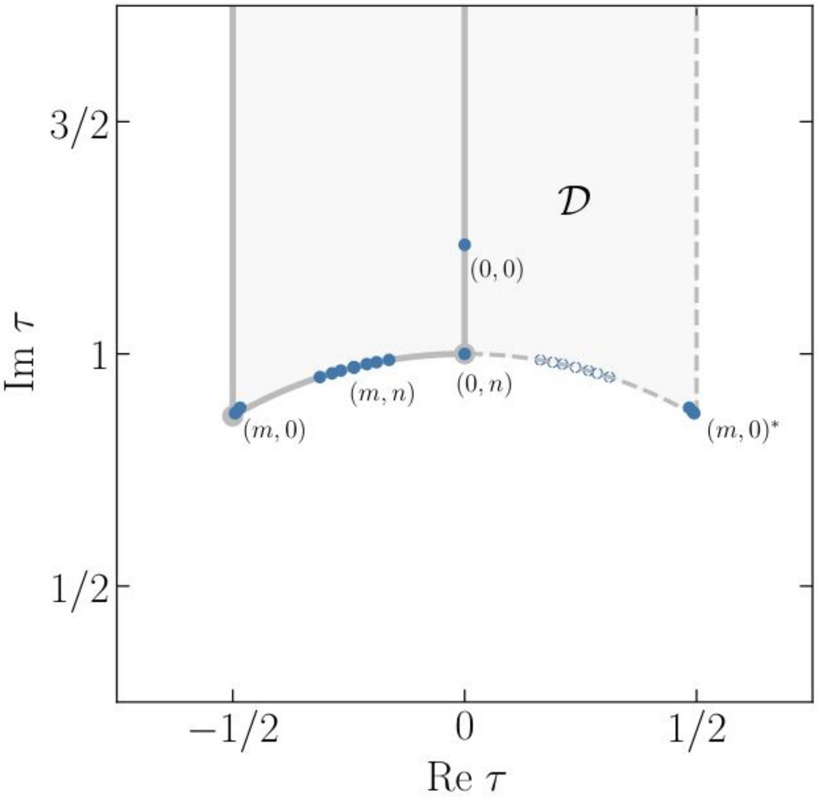


(0,0) ✓

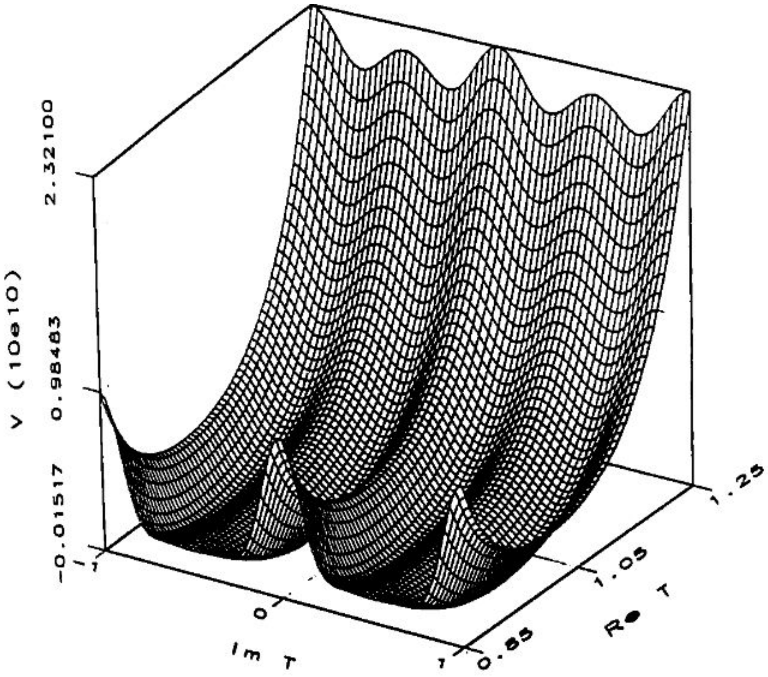


“(...) we conjecture that all extrema of V entirely lie on [the boundary].” — Cvetič et al.

Global minima for (*m*,*n*)-potentials



(*1*,*1*) ✓



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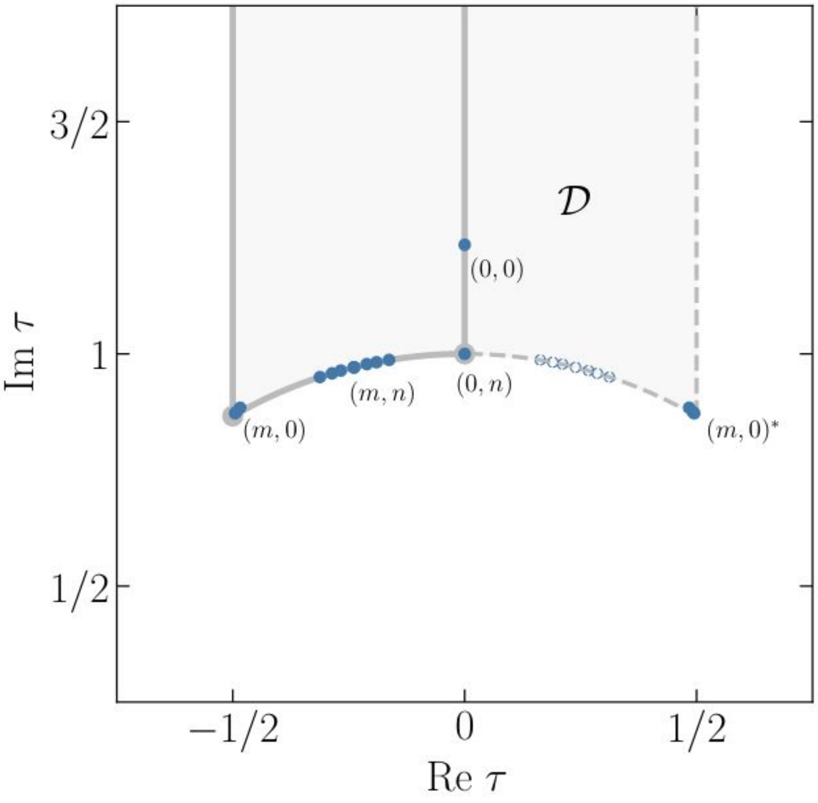
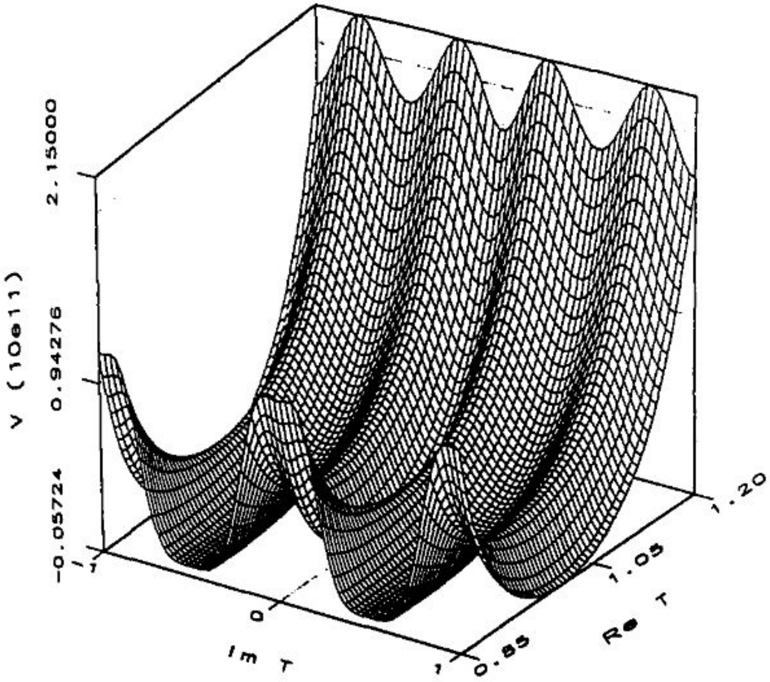
Global minima for (*m*,*n*)-potentials



(0,3) ✓



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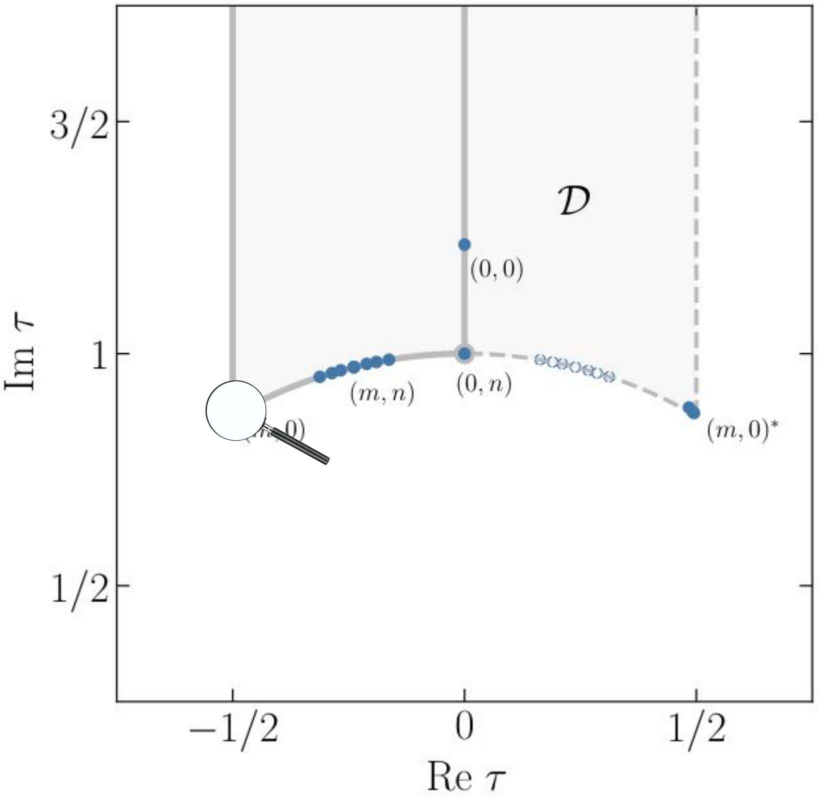
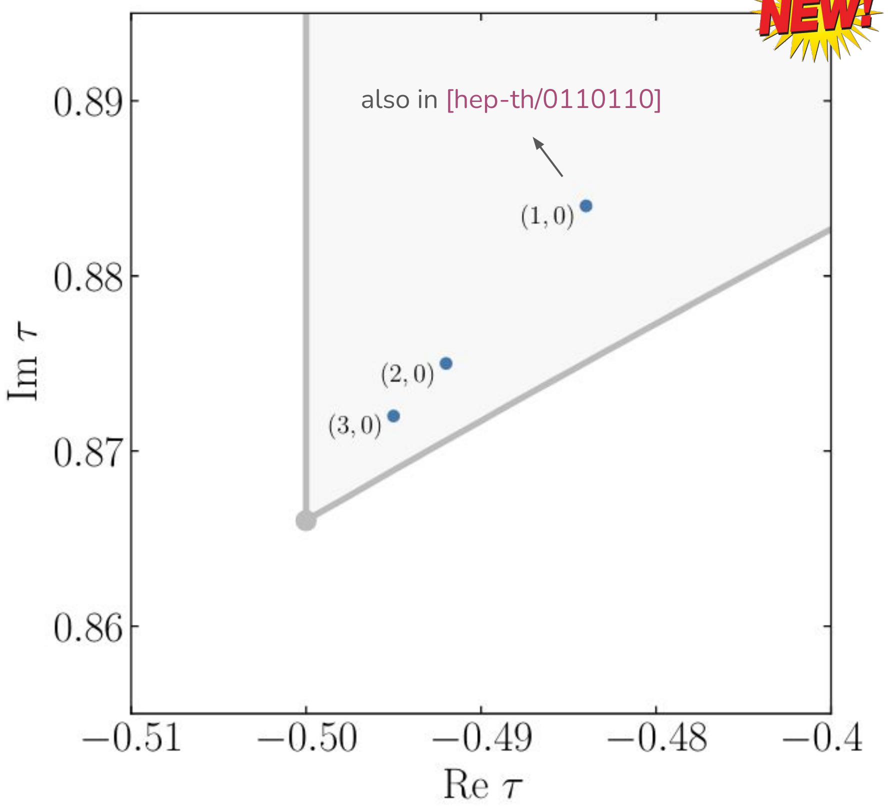
Global minima for (*m*,*n*)-potentials





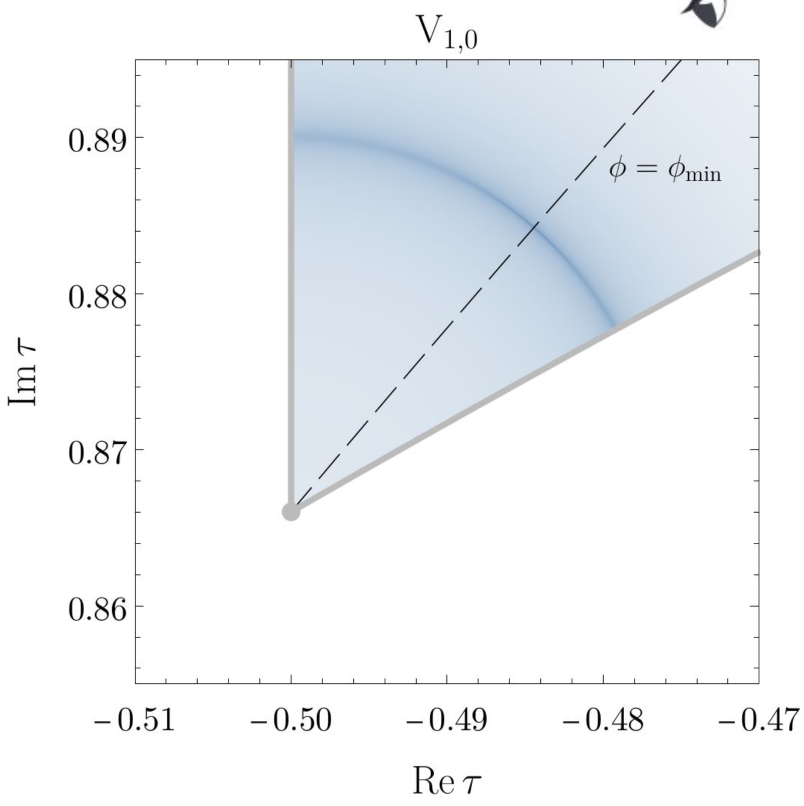
also in [hep-th/0110110]

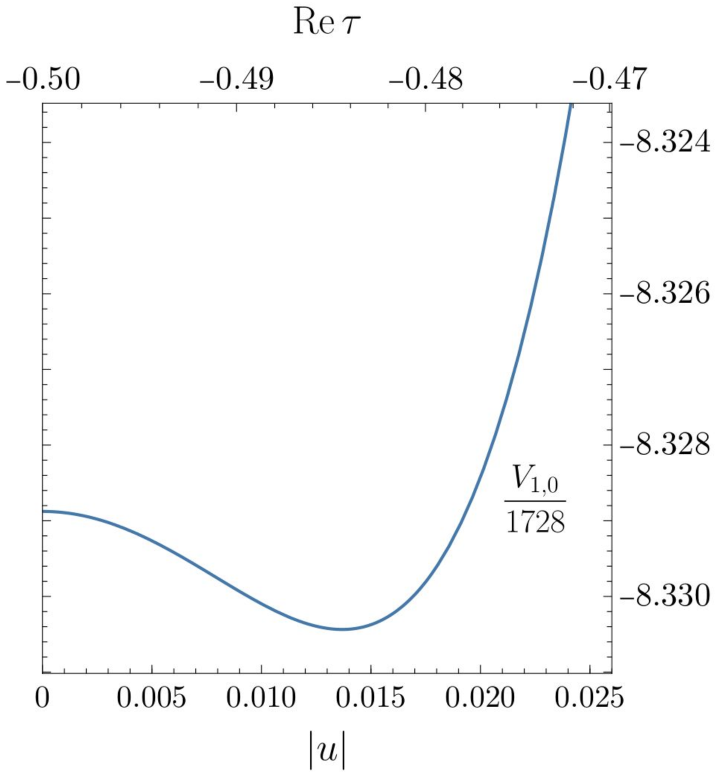
⚠

“(...) we conjecture that all extrema of V entirely lie on [the boundary].” — Cvetič et al.

these results later confirmed by Leedom, Righi, Westphal [2212.03876]

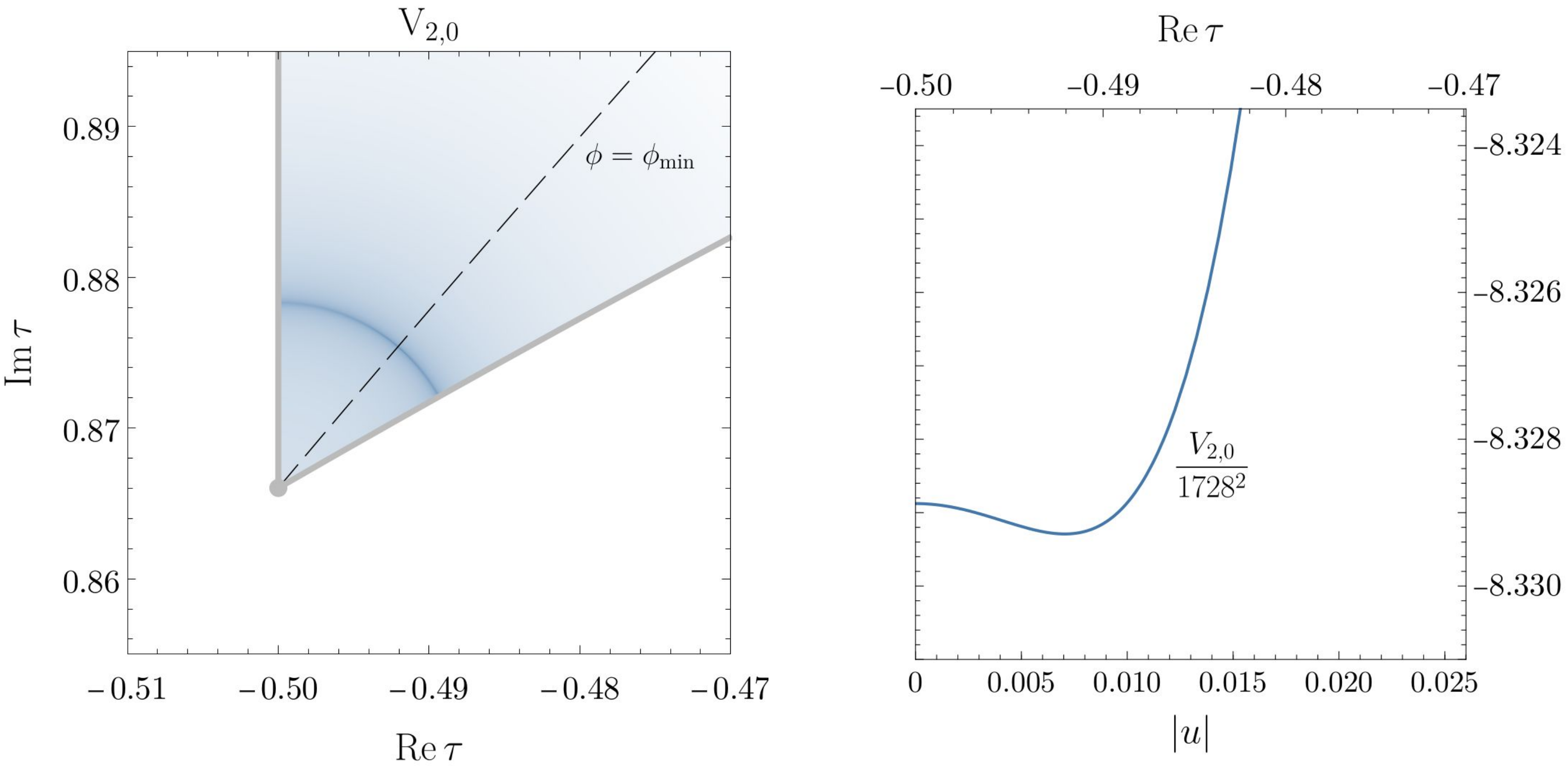
The (m,0) family of potentials ($m = 1$)



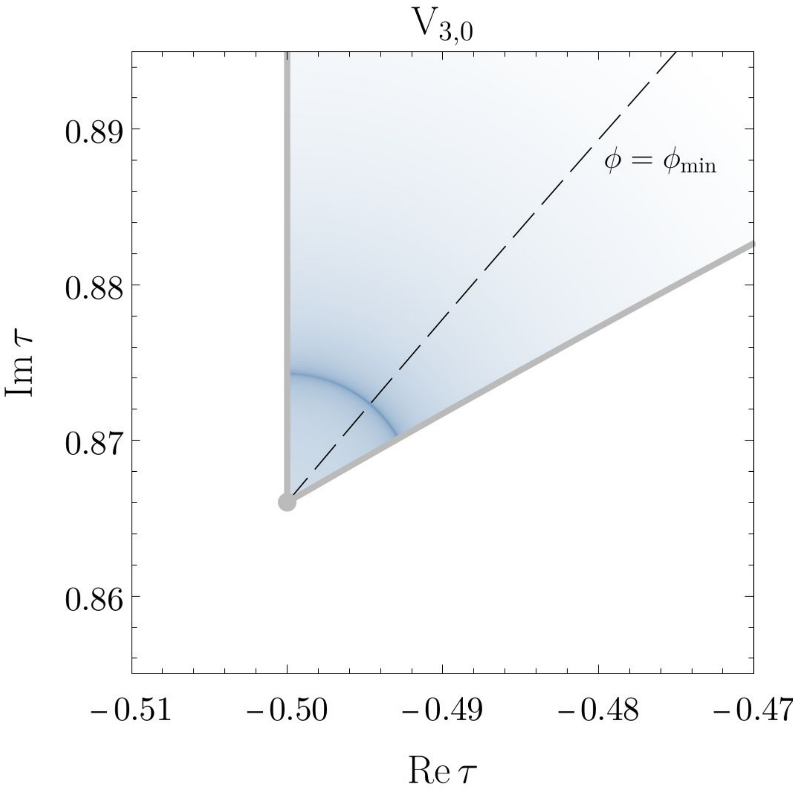


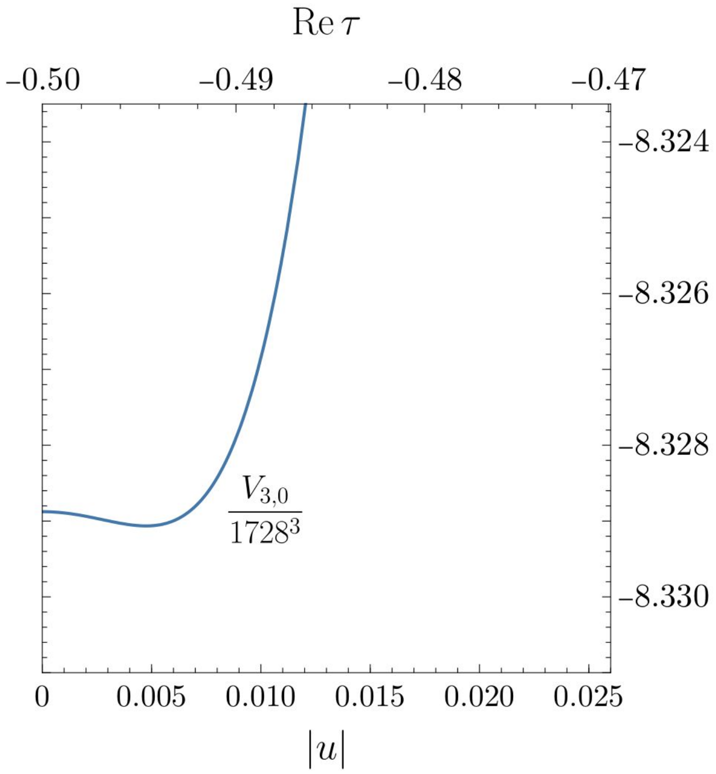
“Mexican”-hat potential

The (*m*,0) family of potentials (*m* = 2)

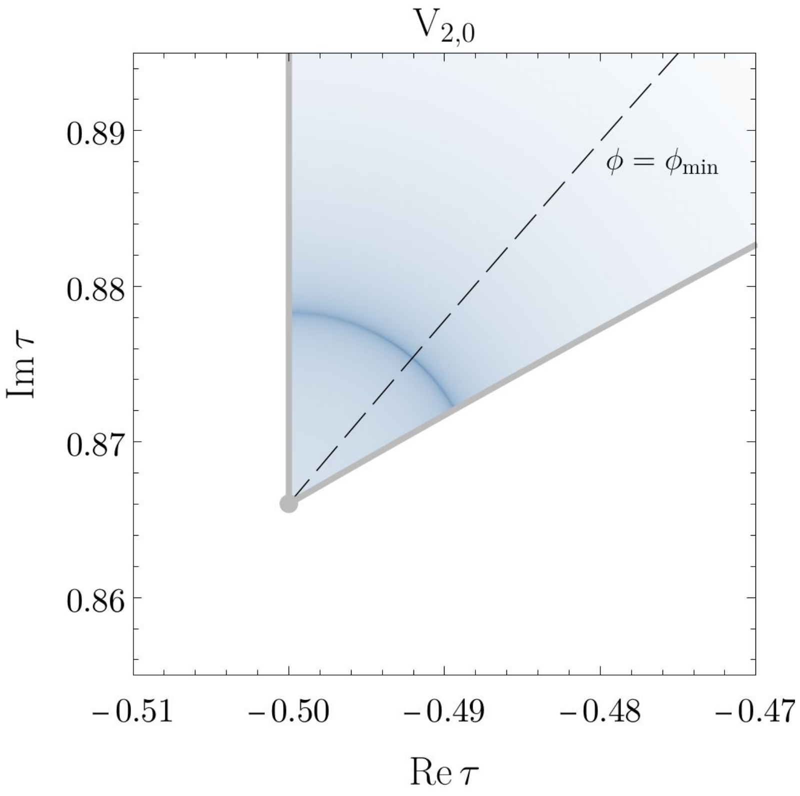


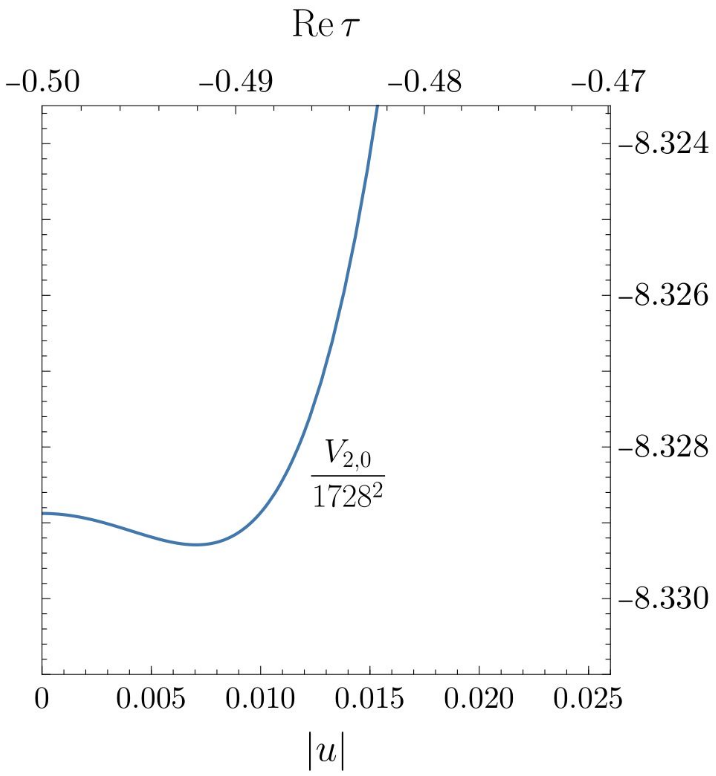
The (*m*,0) family of potentials (*m* = 3)





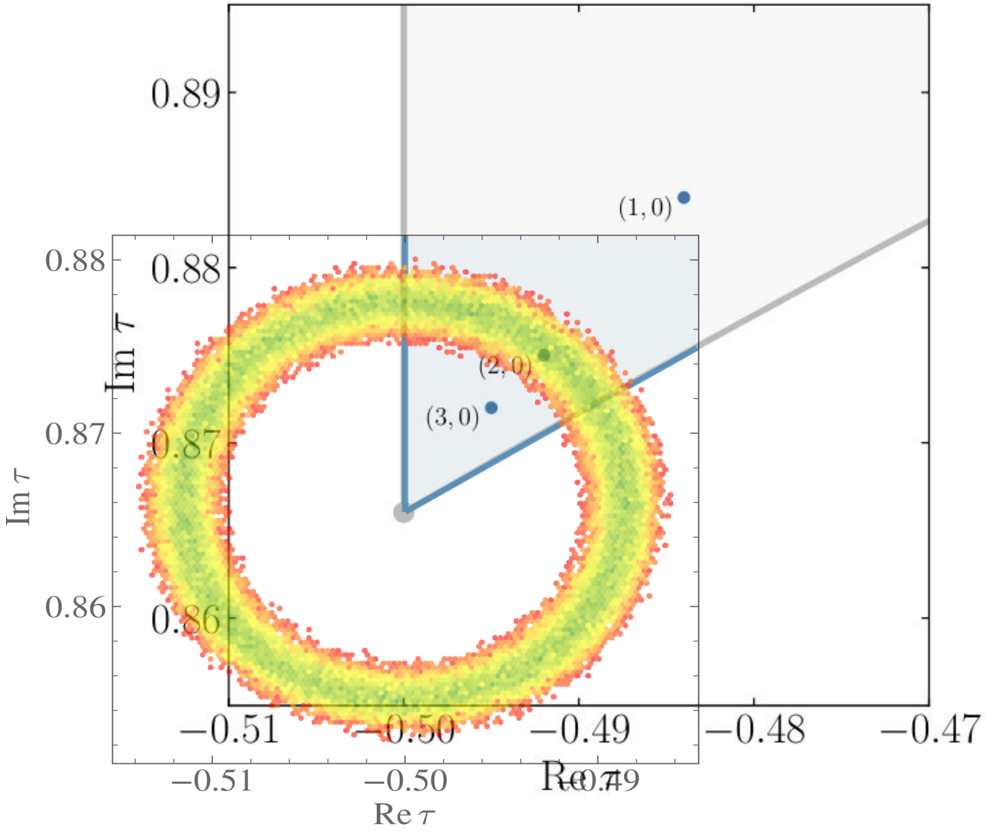
The (m,0) family of potentials (m = 2)





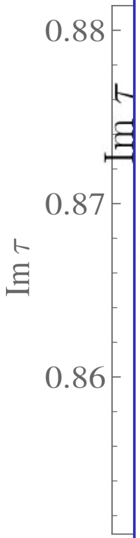
$$\boxed{|\epsilon| \simeq 0.02 \;\Leftrightarrow\; |u| \simeq 0.007}$$

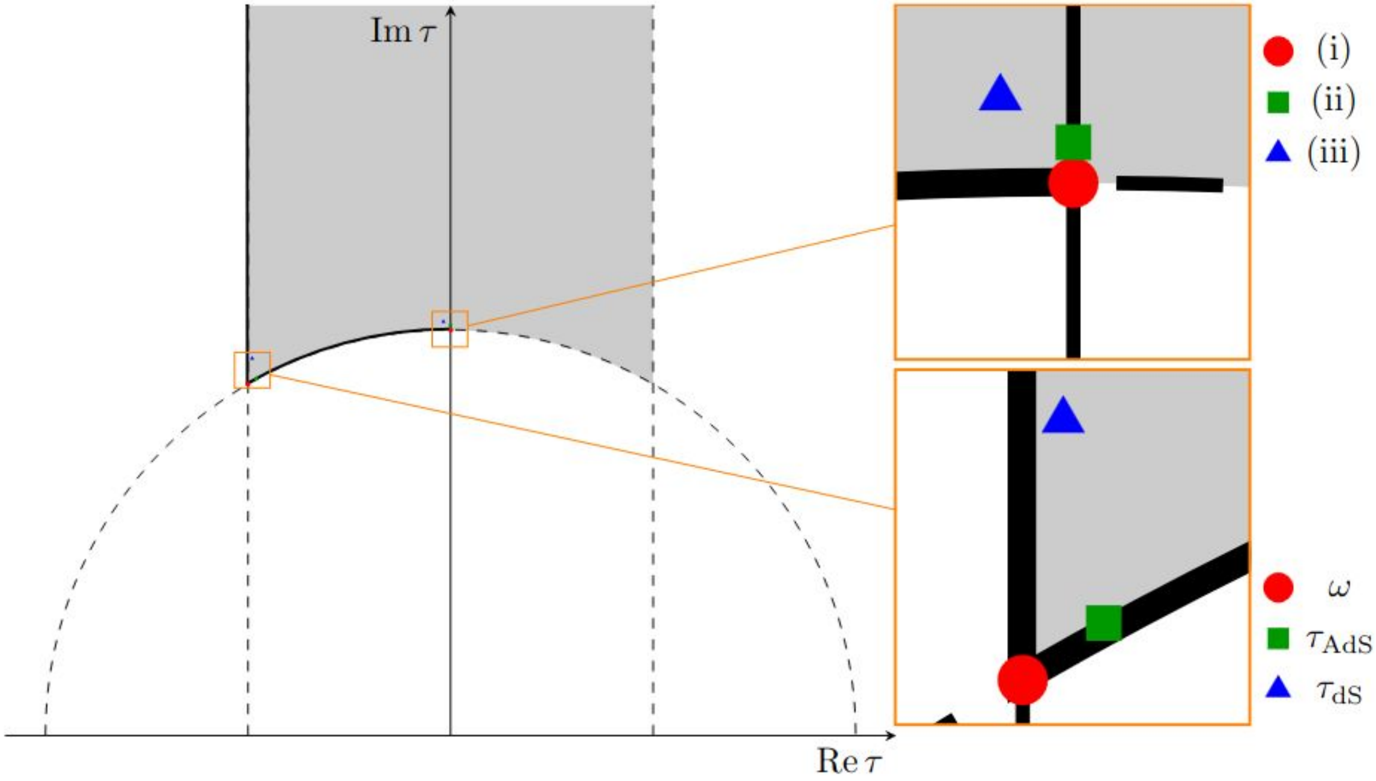
Matching puzzle pieces?



$$|\epsilon| \simeq 0.02 \Leftrightarrow |u| \simeq 0.007$$

Matching puzzle pieces?





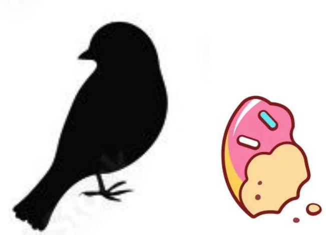
making the model more realistic by **uplifting AdS vacuum**
Knapp-Perez et al. [2304.14437]

$|\epsilon| \simeq 0.02 \Leftrightarrow |u| \simeq 0.007$

to conclude…

Modular symmetries can…

(in lieu of conclusions)



…offer a **predictive framework** for flavour

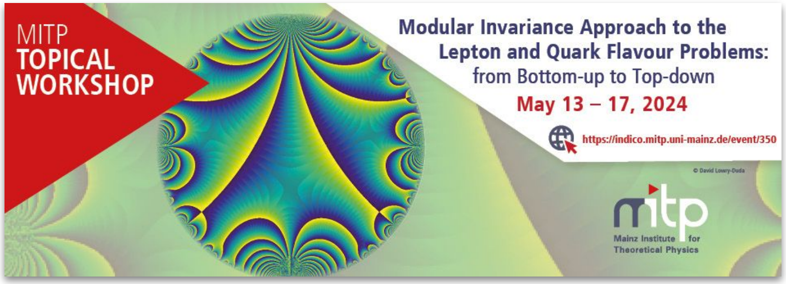
…provide an origin for **CP violation** (CPV)

…explain fermion **mass hierarchies**

…help solve the **strong CP** problem

…bridge low-energy and **string** model building

Parting words (i.e. what next?)



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Modular Invariance Approach to the Lepton and Quark Flavour Problems:
from Bottom-up to Top-down
May 13 – 17, 2024
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- modular symmetry breaking as the only source of CPV?
- natural origin of mass hierarchies?
- hints of universality? [see talk by G.-J. Ding]
- use TD to fix Kähler, irreps, weights!
  [see talks by S. Ramos-Sanchez, H. Otsuka]
- pheno beyond masses and mixing?
  [see talk by G.-J. Ding]
- do away with SUSY?

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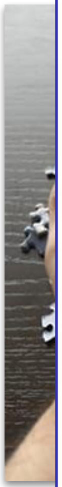
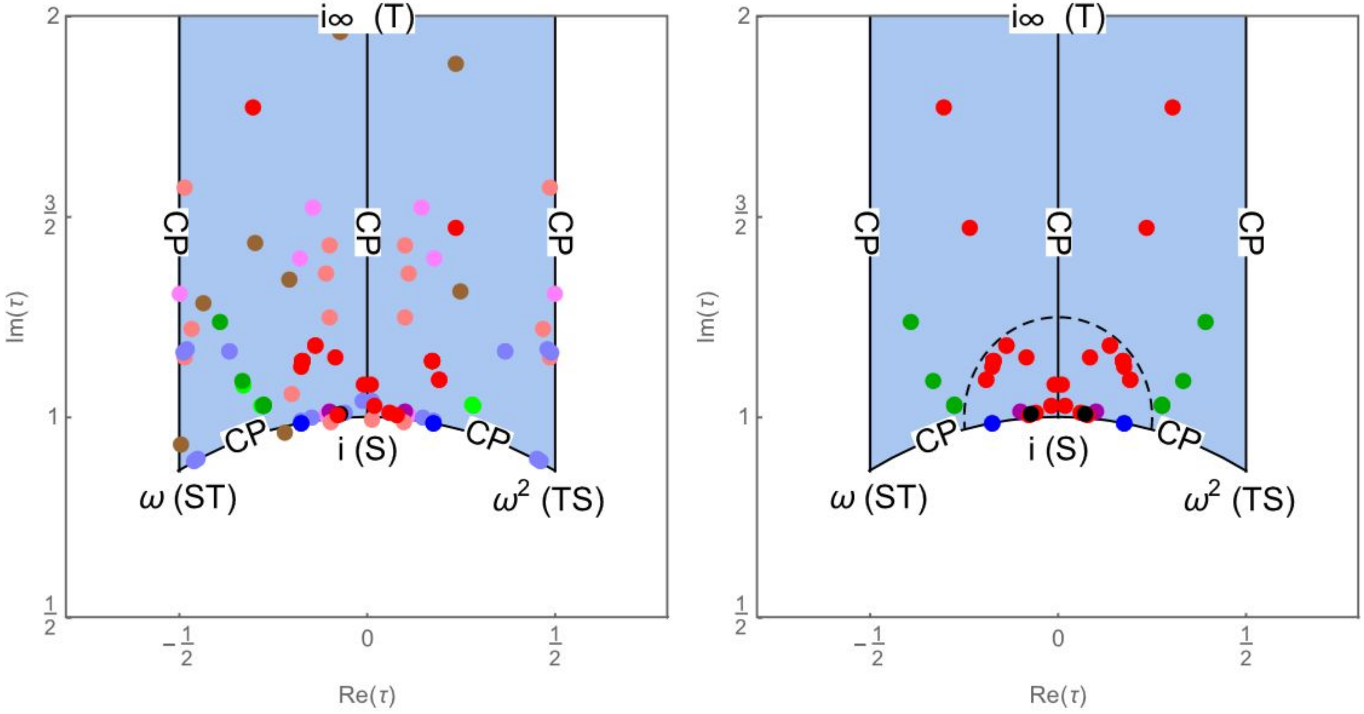


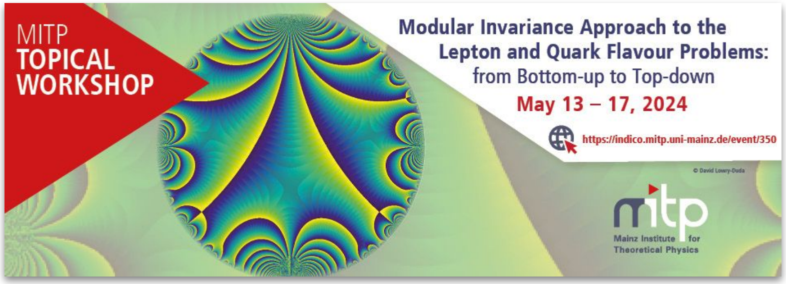
Hints of universality?



only gCP-invariant models

from Feruglio [2211.00659]
see also Feruglio [2302.11580] and Ding, Feruglio, Liu [2402.14915]

Parting words (i.e. what next?)



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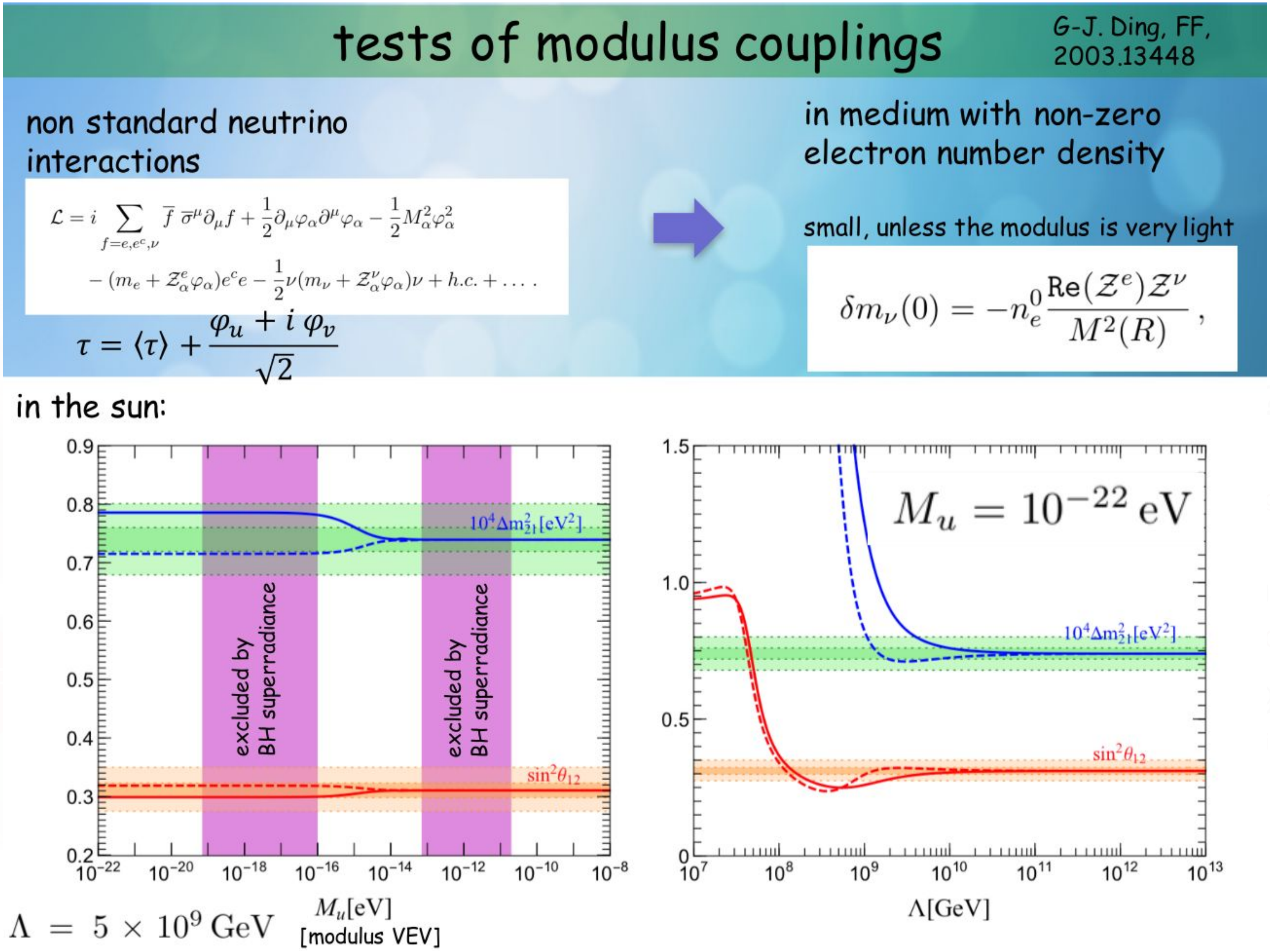


tests of modulus couplings

G-J. Ding, FF, 2003.13448

non standard neutrino interactions

$$\mathcal{L} = i \sum_{f=e,e^c,\nu} \overline{f}\,\overline{\sigma}^\mu \partial_\mu f + \frac{1}{2}\partial_\mu \varphi_\alpha \partial^\mu \varphi_\alpha - \frac{1}{2} M_\alpha^2 \varphi_\alpha^2 - (m_e + \mathcal{Z}^e_\alpha \varphi_\alpha) e^c e - \frac{1}{2}\nu (m_\nu + \mathcal{Z}^\nu_\alpha \varphi_\alpha)\nu + h.c. + \dots .$$

$$\tau = \langle\tau\rangle + \frac{\varphi_u + i\,\varphi_v}{\sqrt{2}}$$

in medium with non-zero electron number density

small, unless the modulus is very light

$$\delta m_\nu(0) = -n_e^0 \frac{\mathrm{Re}(\mathcal{Z}^e)\mathcal{Z}^\nu}{M^2(R)} ,$$

in the sun:

$$\Lambda = 5 \times 10^9\ \mathrm{GeV}$$

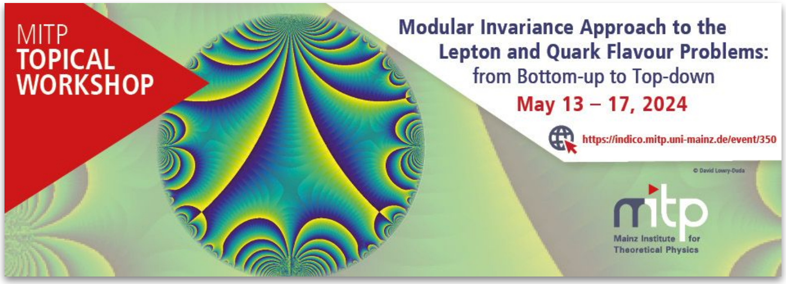
M_u[eV] [modulus VEV]

$$M_u = 10^{-22}\ \mathrm{eV}$$

Λ[GeV]

from Feruglio's slides at Bethe Workshop, citing [2003.13448]

Parting words (i.e. what next?)



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Vielen Dank!

Backup slides

Modular-invariant SUSY actions

Ferrara et al, '89

$$W(\psi;\tau) = \sum_{n} \sum_{\{i_1,\ldots,i_n\}} \sum_{s} g_{i_1\,\ldots\,i_n,s} \left(Y_{i_1\,\ldots\,i_n,s}(\tau)\, \psi_{i_1} \cdots \psi_{i_n}\right)_{\mathbf{1},s}$$

$$\boxed{\mathcal{S} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\; K(\psi,\overline{\psi};\tau,\overline{\tau}) + \int d^4x\, d^2\theta\; W(\psi;\tau) + \text{h.c.}}$$

τ is a dimensionless spurion: once its value is fixed,
it **parameterizes all** modular sym. breaking

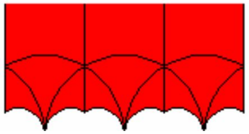
One may argue that *Y*'s play the role of flavons, but
structures are **completely fixed** given the modulus VEV

SUSY breaking effects?

- **RGEs & threshold corrections** need to be considered, depend on tan β and unknown SUSY spectrum
- **SUSY-breaking** corrections can be made negligible via separation of scales (power counting argument)
- Under reasonable conditions, predictions may be unaffected

Feruglio and Criado [1807.01125]

Larger fundamental domains?



- Despite working with representations of the quotients, theories in the BU are typically **fully modular invariant**
- To have canonical kinetic terms,

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \Rightarrow \quad g_i \to (c\tau + d)^{-k_{Y_i}} g_i$$

- e.g. in a particular model, see sec. 4 of Novichkov, JP, Petcov, Titov [1811.04933]

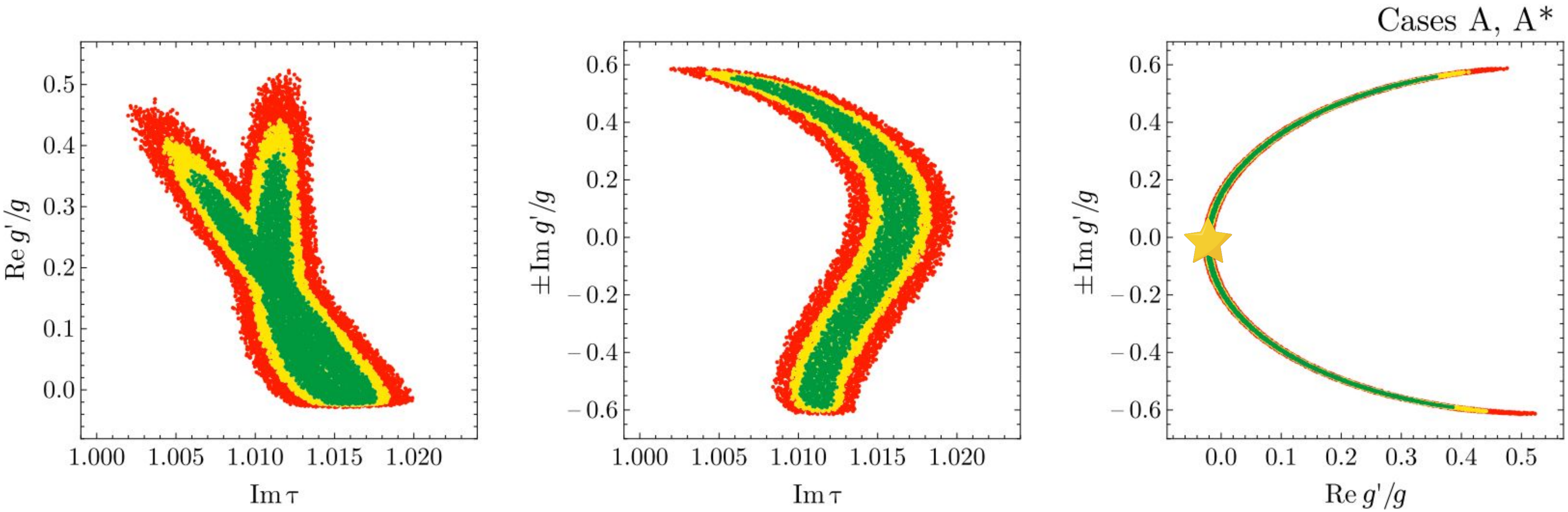
$$(\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots) \to$$

$$\left(\frac{a\tau + b}{c\tau + d},\, (c\tau + d)^{-2}\, \beta/\alpha,\, (c\tau + d)^{-2}\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots\right)$$

these different parameter sets lead to the same observables

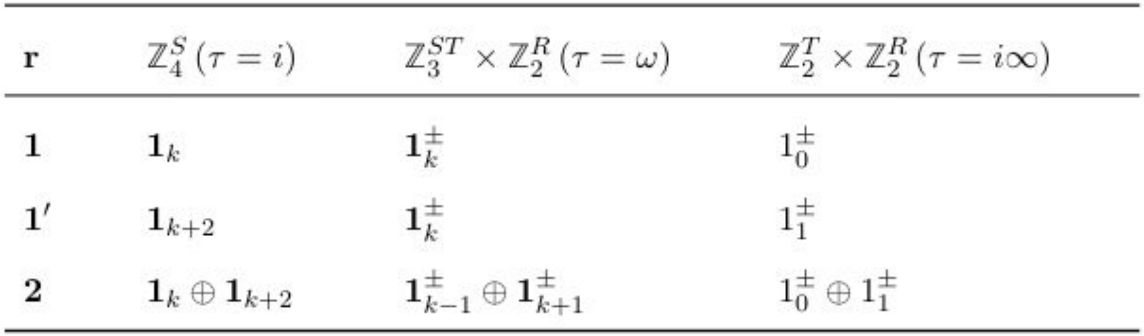
- Things may be different if **flavons** are present!

Correlations between parameters in the first S4 example model

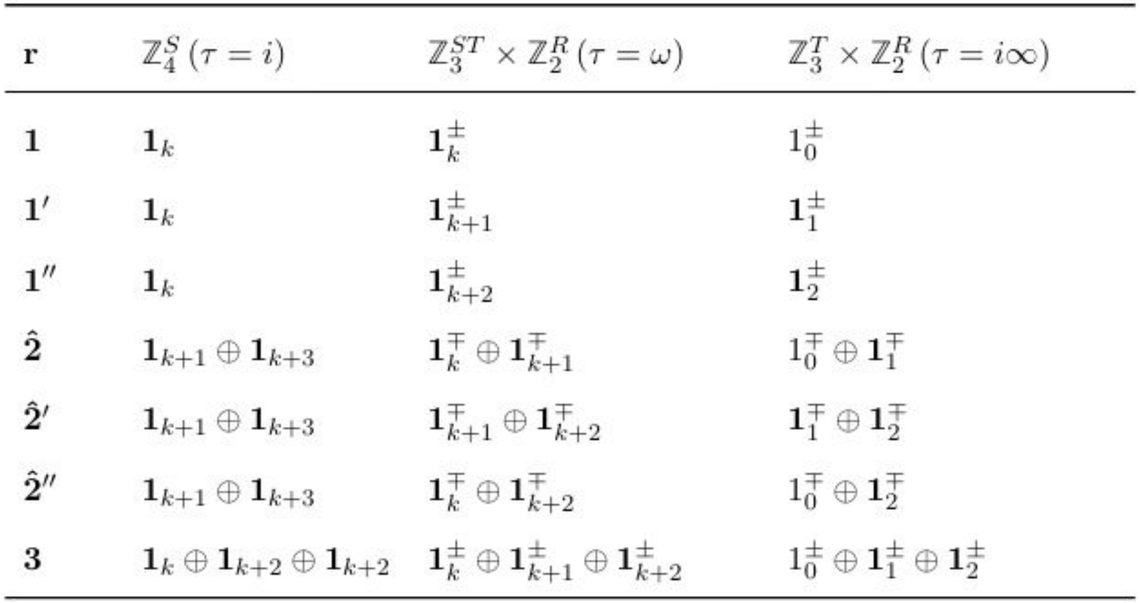


Novichkov, JP, Petcov, Titov [1811.04933]

Decompositions under residual groups: S3, A4’

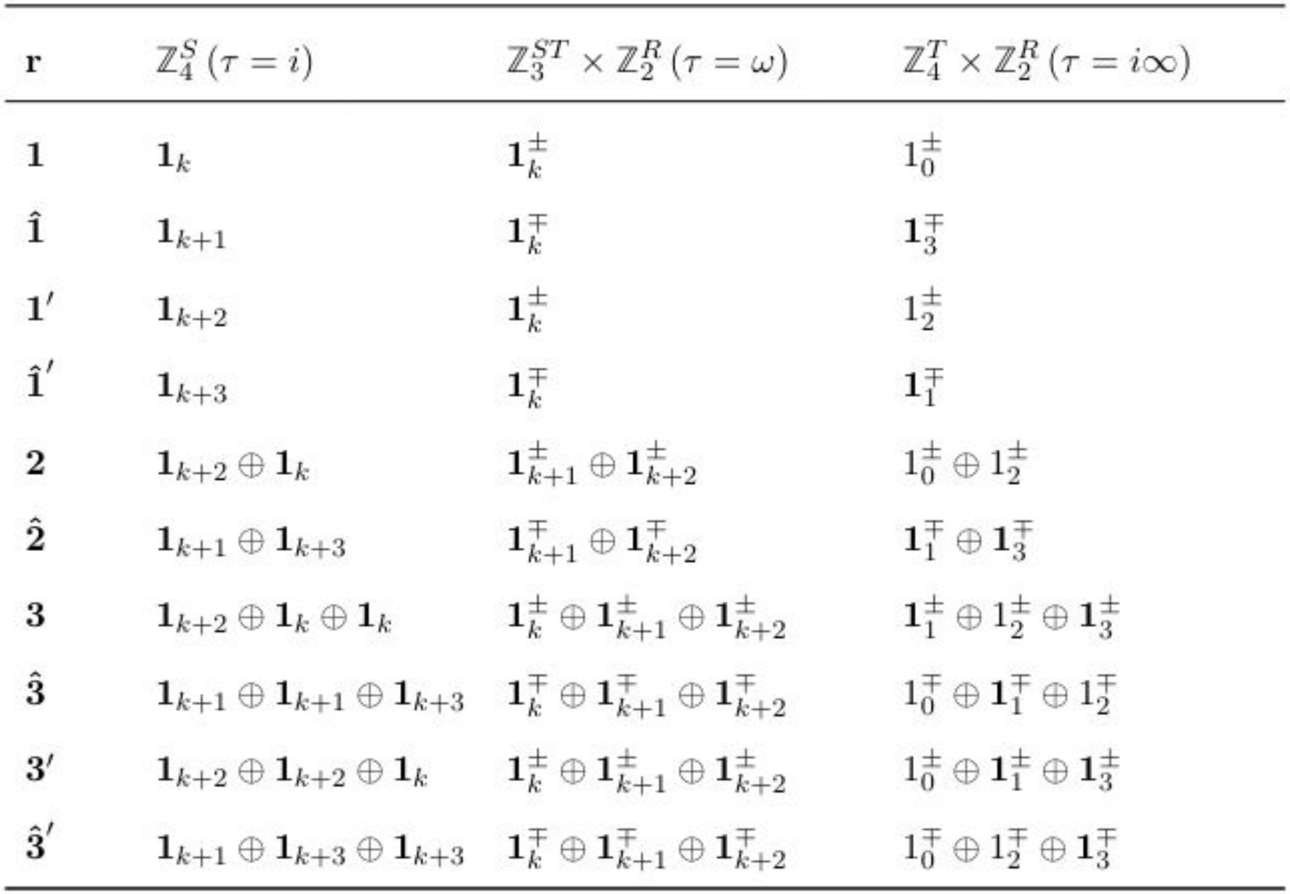


$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_2^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\mathbf{1}'$	$\mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm}$	$1_1^{\pm}$
$\mathbf{2}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_{k-1}^{\pm} \oplus \mathbf{1}_{k+1}^{\pm}$	$1_0^{\pm} \oplus 1_1^{\pm}$



$\mathbf{r}$	$\mathbb{Z}_4^S\,(\tau = i)$	$\mathbb{Z}_3^{ST} \times \mathbb{Z}_2^R\,(\tau = \omega)$	$\mathbb{Z}_3^T \times \mathbb{Z}_2^R\,(\tau = i\infty)$
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\mathbf{1}'$	$\mathbf{1}_k$	$\mathbf{1}_{k+1}^{\pm}$	$\mathbf{1}_1^{\pm}$
$\mathbf{1}''$	$\mathbf{1}_k$	$\mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_2^{\pm}$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp}$	$1_0^{\mp} \oplus \mathbf{1}_1^{\mp}$
$\hat{\mathbf{2}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_2^{\mp}$
$\hat{\mathbf{2}}''$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus \mathbf{1}_2^{\mp}$
$\mathbf{3}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm}$

Decompositions under residual groups: S4’



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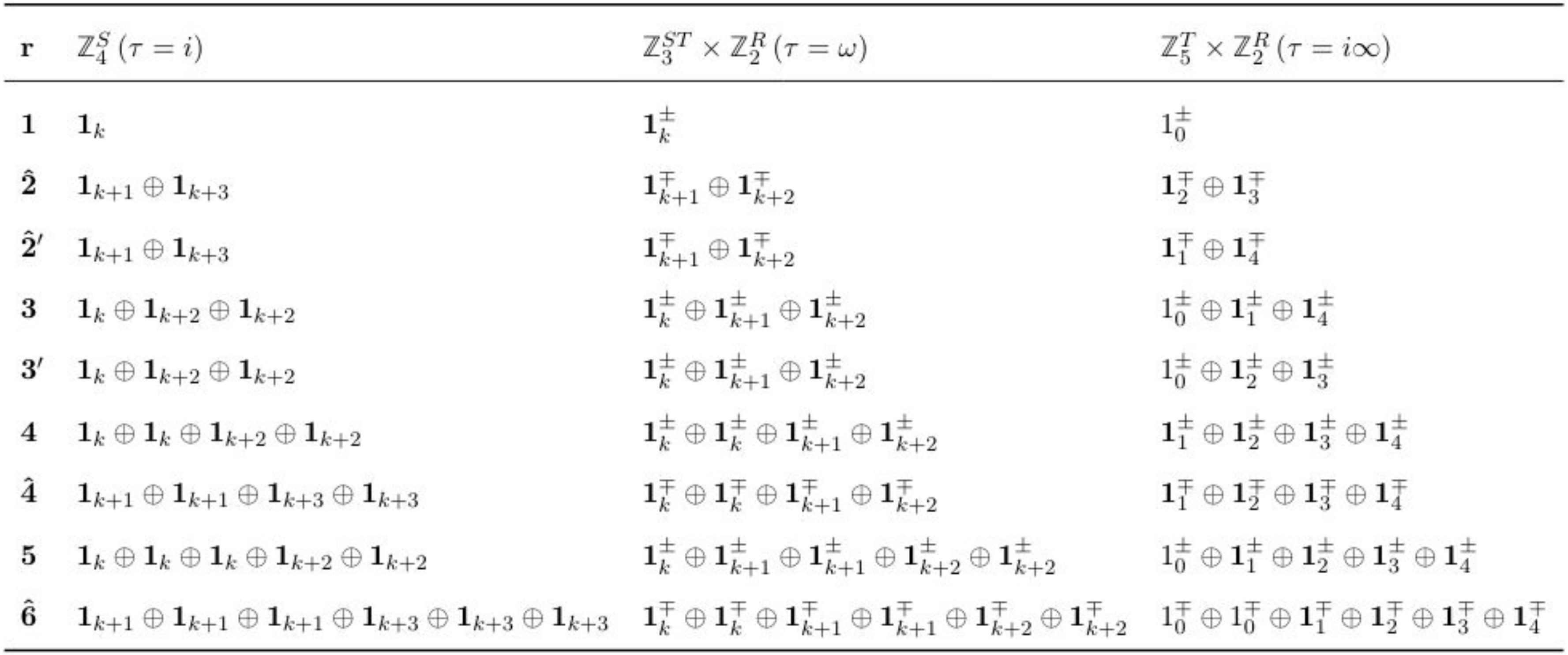
$\mathbf{1}$	$\mathbf{1}_k$	$\mathbf{1}_k^{\pm}$	$1_0^{\pm}$
$\hat{\mathbf{1}}$	$\mathbf{1}_{k+1}$	$\mathbf{1}_k^{\mp}$	$\mathbf{1}_3^{\mp}$
$\mathbf{1}'$	$\mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm}$	$1_2^{\pm}$
$\hat{\mathbf{1}}'$	$\mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp}$	$\mathbf{1}_1^{\mp}$
$\mathbf{2}$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_k$	$\mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus 1_2^{\pm}$
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_3^{\mp}$
$\mathbf{3}$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_k \oplus \mathbf{1}_k$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_1^{\pm} \oplus 1_2^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\hat{\mathbf{3}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus \mathbf{1}_1^{\mp} \oplus 1_2^{\mp}$
$\mathbf{3}'$	$\mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_k$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\hat{\mathbf{3}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus 1_2^{\mp} \oplus \mathbf{1}_3^{\mp}$

Decompositions under residual groups: A5’



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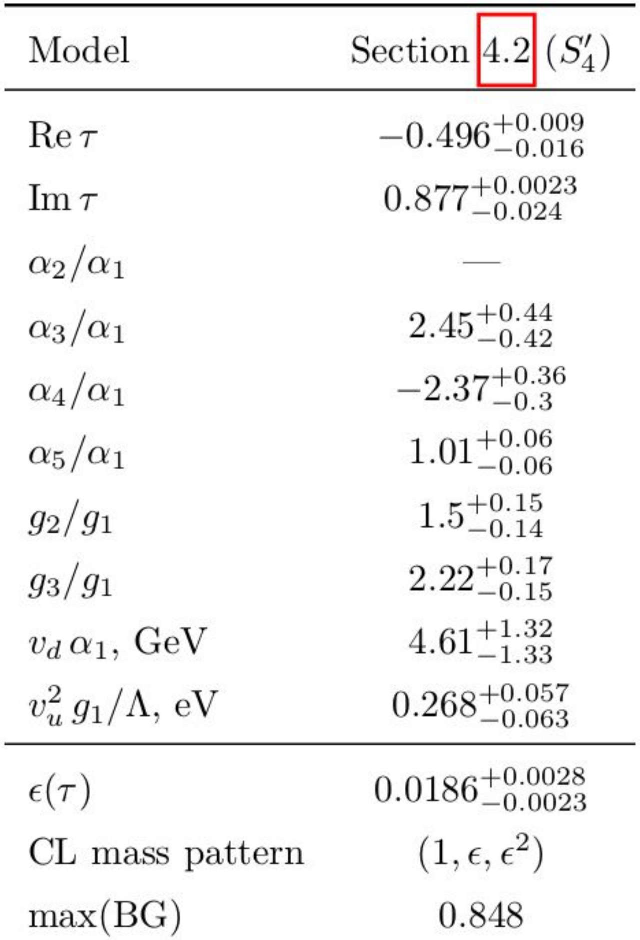
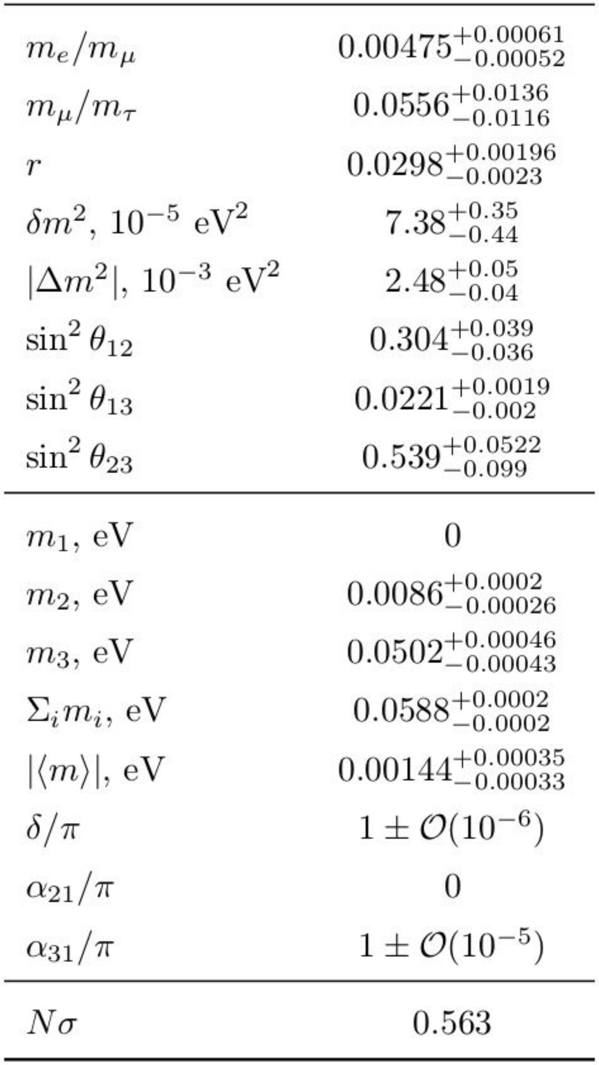
$\hat{\mathbf{2}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp}$
$\hat{\mathbf{2}}'$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_4^{\mp}$
$\mathbf{3}$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\mathbf{3}'$	$\mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm}$
$\mathbf{4}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$\mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\hat{\mathbf{4}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$\mathbf{1}_1^{\mp} \oplus \mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp} \oplus \mathbf{1}_4^{\mp}$
$\mathbf{5}$	$\mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_k \oplus \mathbf{1}_{k+2} \oplus \mathbf{1}_{k+2}$	$\mathbf{1}_k^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+1}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm} \oplus \mathbf{1}_{k+2}^{\pm}$	$1_0^{\pm} \oplus \mathbf{1}_1^{\pm} \oplus \mathbf{1}_2^{\pm} \oplus \mathbf{1}_3^{\pm} \oplus \mathbf{1}_4^{\pm}$
$\hat{\mathbf{6}}$	$\mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+1} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3} \oplus \mathbf{1}_{k+3}$	$\mathbf{1}_k^{\mp} \oplus \mathbf{1}_k^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+1}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp} \oplus \mathbf{1}_{k+2}^{\mp}$	$1_0^{\mp} \oplus 1_0^{\mp} \oplus \mathbf{1}_1^{\mp} \oplus \mathbf{1}_2^{\mp} \oplus \mathbf{1}_3^{\mp} \oplus \mathbf{1}_4^{\mp}$

Details of the model fit

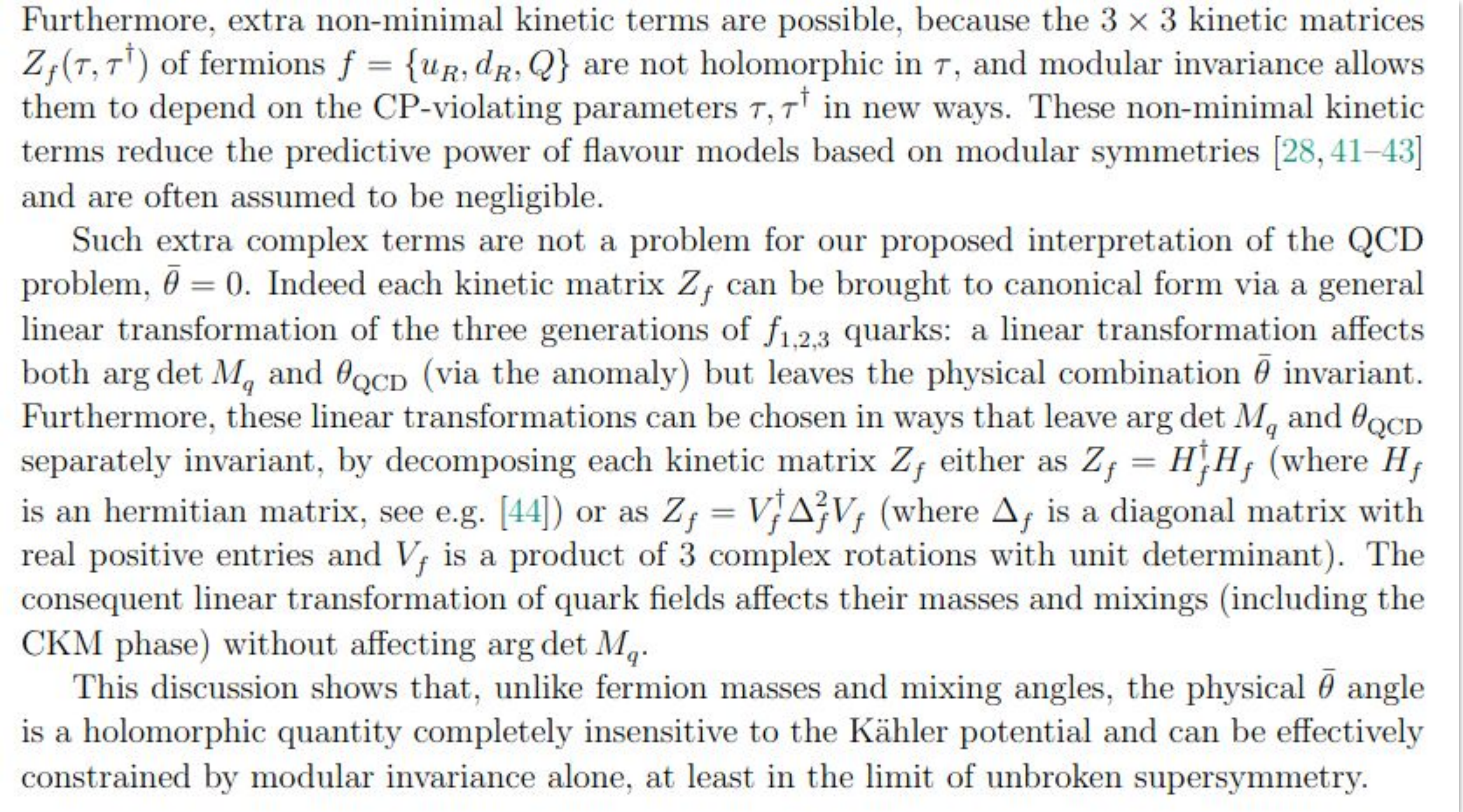


Model	Section $\boxed{4.2}$ (S_4')
$\mathrm{Re}\,\tau$	$-0.496^{+0.009}_{-0.016}$
$\mathrm{Im}\,\tau$	$0.877^{+0.0023}_{-0.024}$
α_2/α_1	—
α_3/α_1	$2.45^{+0.44}_{-0.42}$
α_4/α_1	$-2.37^{+0.36}_{-0.3}$
α_5/α_1	$1.01^{+0.06}_{-0.06}$
g_2/g_1	$1.5^{+0.15}_{-0.14}$
g_3/g_1	$2.22^{+0.17}_{-0.15}$
$v_d\,\alpha_1$, GeV	$4.61^{+1.32}_{-1.33}$
$v_u^2\,g_1/\Lambda$, eV	$0.268^{+0.057}_{-0.063}$
$\epsilon(\tau)$	$0.0186^{+0.0028}_{-0.0023}$
CL mass pattern	$(1, \epsilon, \epsilon^2)$
max(BG)	0.848



m_e/m_μ	$0.00475^{+0.00061}_{-0.00052}$
m_μ/m_τ	$0.0556^{+0.0136}_{-0.0116}$
r	$0.0298^{+0.00196}_{-0.0023}$
δm^2, 10^{-5} eV^2	$7.38^{+0.35}_{-0.44}$
$\lvert\Delta m^2\rvert$, 10^{-3} eV^2	$2.48^{+0.05}_{-0.04}$
$\sin^2\theta_{12}$	$0.304^{+0.039}_{-0.036}$
$\sin^2\theta_{13}$	$0.0221^{+0.0019}_{-0.002}$
$\sin^2\theta_{23}$	$0.539^{+0.0522}_{-0.099}$
m_1, eV	0
m_2, eV	$0.0086^{+0.0002}_{-0.00026}$
m_3, eV	$0.0502^{+0.00046}_{-0.00043}$
$\Sigma_i m_i$, eV	$0.0588^{+0.0002}_{-0.0002}$
$\lvert\langle m\rangle\rvert$, eV	$0.00144^{+0.00035}_{-0.00033}$
δ/π	$1 \pm \mathcal{O}(10^{-6})$
α_{21}/π	0
α_{31}/π	$1 \pm \mathcal{O}(10^{-5})$
$N\sigma$	0.563

The QCD angle is holomorphic



Furthermore, extra non-minimal kinetic terms are possible, because the 3×3 kinetic matrices $Z_f(\tau, \tau^\dagger)$ of fermions $f = \{u_R, d_R, Q\}$ are not holomorphic in τ, and modular invariance allows them to depend on the CP-violating parameters $\tau, \tau^\dagger$ in new ways. These non-minimal kinetic terms reduce the predictive power of flavour models based on modular symmetries [28, 41–43] and are often assumed to be negligible.

Such extra complex terms are not a problem for our proposed interpretation of the QCD problem, $\bar{\theta} = 0$. Indeed each kinetic matrix Z_f can be brought to canonical form via a general linear transformation of the three generations of $f_{1,2,3}$ quarks: a linear transformation affects both $\arg\det M_q$ and $\theta_{\rm QCD}$ (via the anomaly) but leaves the physical combination $\bar{\theta}$ invariant. Furthermore, these linear transformations can be chosen in ways that leave $\arg\det M_q$ and $\theta_{\rm QCD}$ separately invariant, by decomposing each kinetic matrix Z_f either as $Z_f = H_f^\dagger H_f$ (where H_f is an hermitian matrix, see e.g. [44]) or as $Z_f = V_f^\dagger \Delta_f^2 V_f$ (where Δ_f is a diagonal matrix with real positive entries and V_f is a product of 3 complex rotations with unit determinant). The consequent linear transformation of quark fields affects their masses and mixings (including the CKM phase) without affecting $\arg\det M_q$.

This discussion shows that, unlike fermion masses and mixing angles, the physical $\bar{\theta}$ angle is a holomorphic quantity completely insensitive to the Kähler potential and can be effectively constrained by modular invariance alone, at least in the limit of unbroken supersymmetry.

Feruglio, Strumia, Titov [2305.08908]

Simplest modular-invariant potentials?

- Studied by Cvetič, Font, Ibáñez, Lüst and Quevedo (1991)
  $\mathcal{N} = 1$ SUGRA

$$
\begin{aligned}
K(\tau,\bar{\tau}) &= -\Lambda_K^2 \log(2\ \mathrm{Im}\,\tau) \\
G(\tau,\bar{\tau}) &= \kappa^2 K(\tau,\bar{\tau}) + \log\left|\kappa^3 W(\tau)\right|^2
\end{aligned}
$$

$$\kappa^2 = 8\pi/M_P^2$$

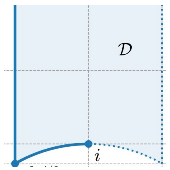
- Superpotential has modular weight $-\mathfrak{n} = -1, -2, \boxed{-3,} \ldots$

$$W(\tau) = \Lambda_W^3 \frac{\boxed{H(\tau)}}{\eta(\tau)^{2\mathfrak{n}}}$$

$$\mathfrak{n} = \kappa^2 \Lambda_K^2$$

- Simplified model, independent of the level *N*

q- and u-expansions of η



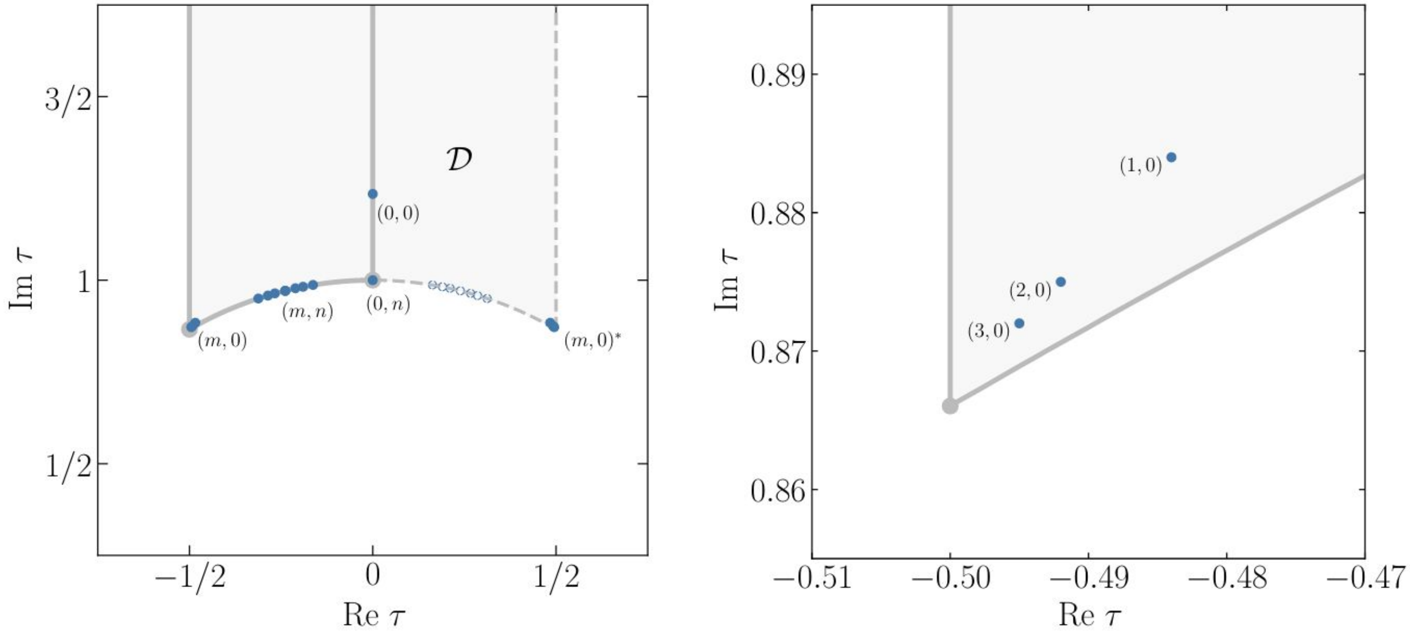
$$|q| \leq e^{-\sqrt{3}\pi} \simeq 0.004$$

$$\eta = q^{1/24} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2-n}{2}} = q^{1/24} \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \mathcal{O}(q^{22})\right)$$

$$u \equiv \frac{\tau - \omega}{\tau - \omega^2} \qquad \tilde{\eta}(u) \equiv \frac{\eta(u)}{\sqrt{1-u}}$$

$$u \xrightarrow{ST} \omega^2 u \qquad \tilde{\eta}(u) \xrightarrow{ST} \tilde{\eta}(u)$$

$$\begin{aligned} \tilde{\eta}(u) &\simeq e^{-i\pi/24} \left(0.800579 - 0.573569 u^3 - 0.780766 u^6 - 0.150007 u^9\right) + \mathcal{O}(u^{12}) \\ &\equiv e^{-i\pi/24} \left(\tilde{\eta}_0 + \tilde{\eta}_3 u^3 + \tilde{\eta}_6 u^6 + \tilde{\eta}_9 u^9\right) + \mathcal{O}(u^{12}) \,, \end{aligned}$$



$(0,0)$ is a single minimum at $\tau \simeq 1.2i$ on the imaginary axis, corresponding to the case $m = n = 0$;

$(0,n)$ is a single minimum at the symmetric point $\tau = i$ attained when $m = 0$, $n \neq 0$;

$(m,0)$ and $(m,0)^*$ are a pair of degenerate minima for each $m \neq 0$ and $n = 0$: $(m,0)$ is located in the vicinity of the left cusp $\tau = \omega$, approaching this symmetric point as m increases, while $(m,0)^*$ is its CP-conjugate;

(m,n) is a series of minima on the unit arc, corresponding to $m \neq 0$, $n \neq 0$; these minima shift towards $\tau = \omega$ ($\tau = i$) along the arc as m (n) grows.

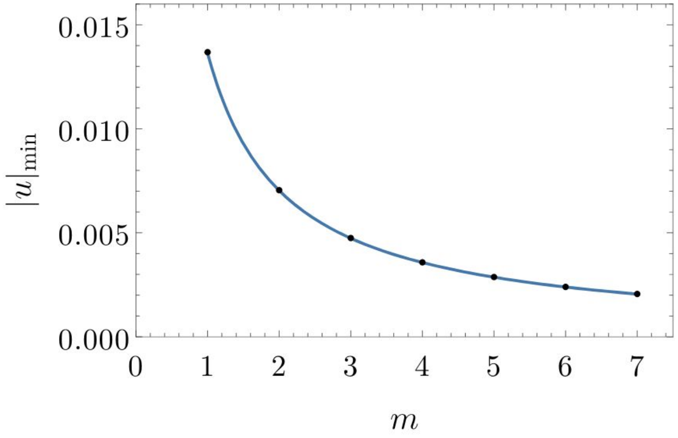
The (*m*,0) family of potentials

- u-expand (m,0) potentials to analyse them near the left cusp

$$V_{m,0} = \Lambda_V^4 \, \frac{1728^m}{\sqrt{3}\,\tilde{\eta}_0^{12}} \left\{ -1 - 2\,|u|^2 + \left(A_m^2 - 3\right) |u|^4 \right\} + \mathcal{O}(|u|^6)$$

$$A_m \equiv \frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}\,m + \frac{6\,|\tilde{\eta}_3|}{\tilde{\eta}_0}$$
$$\simeq 68.78\,m + 4.30$$

- "Mexican"-hat potential
  (cusp is a maximum!)



$$|u|_{\text{min}} \simeq \left(A_m^2 - 3\right)^{-1/2}$$
$$\simeq A_m^{-1} = \frac{0.0145}{\boxed{m} + 0.0625}$$

The (*m*,0) family of potentials

(phase dependence)

$$u = |u| e^{i\phi}$$

- u-expanding to higher order shows dependence on $\phi \in [-\pi/3, 0]$

$$\boxed{\begin{aligned} V_{m,0} \propto & -1 - 2\,|u|^2 + \left(A_m^2 - 3\right)|u|^4 + \left(-4 + 2A_m^2 + B_m^2 \cos 6\phi\right)|u|^6 \\ & + 2A_m B_m^2 \cos 3\phi\,|u|^7 + \left(-5 + 3A_m^2 + 2B_m^2 \cos 6\phi\right)|u|^8 + \mathcal{O}(|u|^9) \end{aligned}}$$

$$\begin{aligned} B_m^2 &\equiv \frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}\,m\left[\frac{864\,|\tilde{\eta}_3|^3}{\pi^6\,\tilde{\eta}_0^{27}}\,(m-2) + \frac{3\left(31\,\tilde{\eta}_3^2 - 10\tilde{\eta}_0\tilde{\eta}_6\right)}{\tilde{\eta}_0|\tilde{\eta}_3|}\right] + \frac{6\left(7\tilde{\eta}_3^2 - 2\tilde{\eta}_0\tilde{\eta}_6\right)}{\tilde{\eta}_0^2} \\ &\simeq 4730.60\,m^2 - 2069.73\,m + 33.26\,. \end{aligned}$$

- Phase of u mostly determined by $|u|^6$ and $|u|^7$ terms

$$\boxed{\phi_{\text{min}} \simeq -\frac{2\pi}{9} = -40^\circ}$$

The global SUSY limit (a comment)

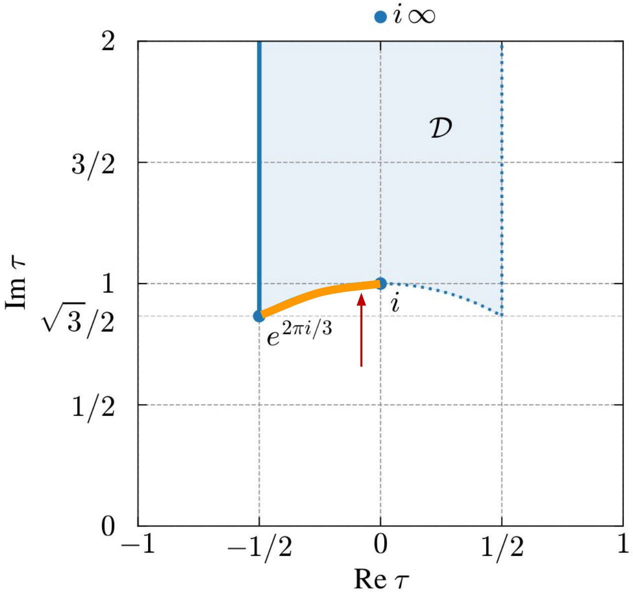
$$K(\tau,\overline{\tau}) = -\Lambda_K^2 \log(2\ \mathrm{Im}\,\tau)$$

$$\kappa^2 = 8\pi/M_P^2$$

$$\begin{tikzcd}
\boxed{\mathfrak{n} = \kappa^2\Lambda_K^2 \to 0} \arrow[d, "W(\tau) = \Lambda_W^3 H(\tau)"'] \\
\boxed{V(\tau,\overline{\tau}) = \frac{4\Lambda_W^6}{\Lambda_K^2}(\mathrm{Im}\,\tau)^2\left|H'(\tau)\right|^2}
\end{tikzcd}$$

$$H(\tau) = (j(\tau) - 1728)^{m/2}\, j(\tau)^{n/3}\, \mathcal{P}\left(j(\tau)\right)$$

- Global minima are zeros of *H*’
- non-trivial $\mathcal{P}(j)$ can be engineered to produce minima at arbitrary points in the fundamental domain



No, there is no tuning in choosing this form of the superpotential (arguably)

$$H(\tau) \propto (J(\tau) - 1)^{m/2}$$

Subset of all possible $H(\tau)$ which vanish only at the symmetric point $\tau=i$ (itself distinguished by modular symmetry)

$$J(\tau) \equiv j(\tau)/1728$$