Bottom-up modular invariance approach to flavour I

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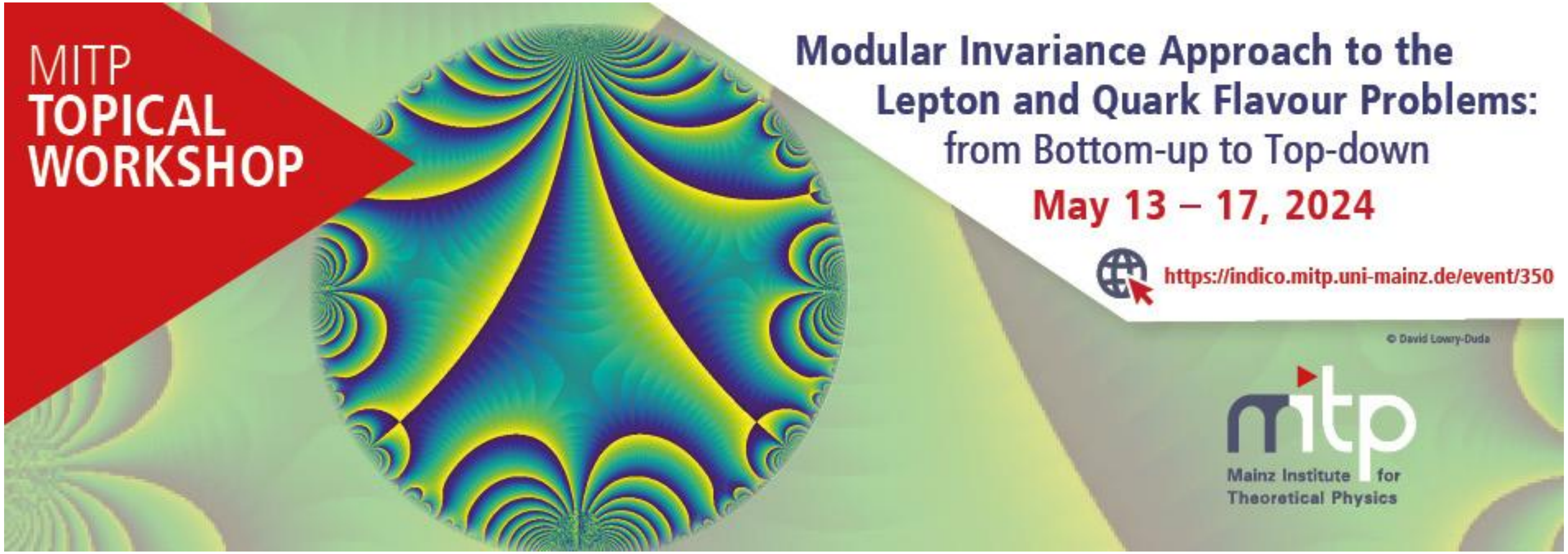
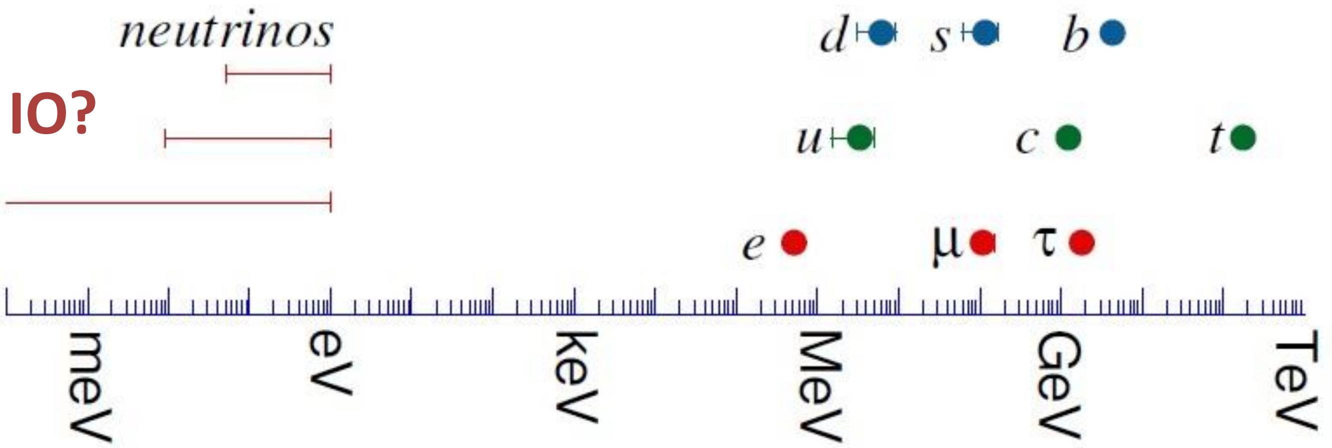


Flavor puzzles in SM

- Hierarchical masses, e.g. $\frac{m_t}{m_e} = \mathcal{O}(10^5)$

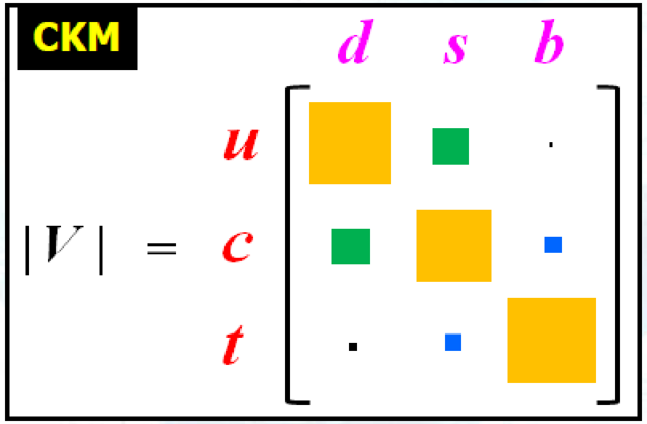
talk by Stephen King

NO or IO?

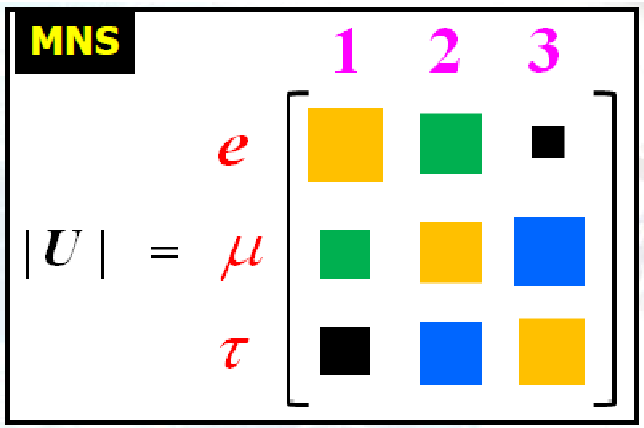


- Quark mixing vs lepton mixing

Quark mixings are small



Lepton mixings are large

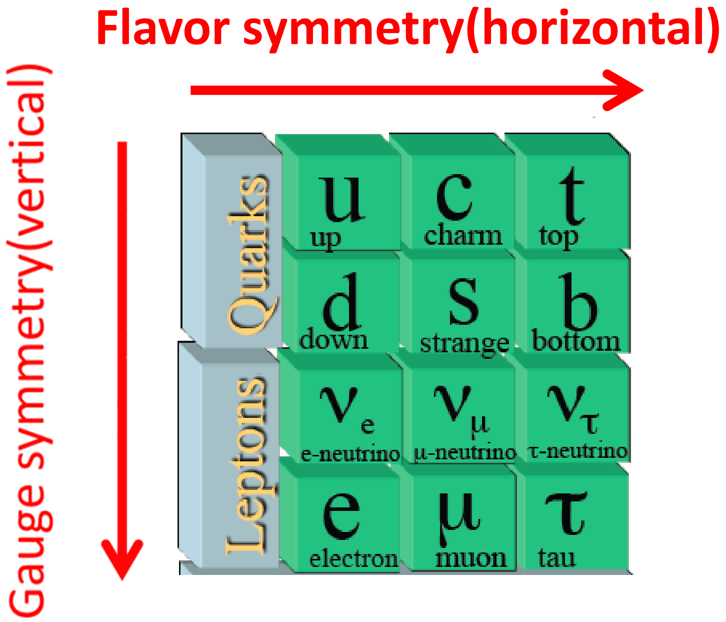


Quark and lepton mixing matrices have distinctive structures!

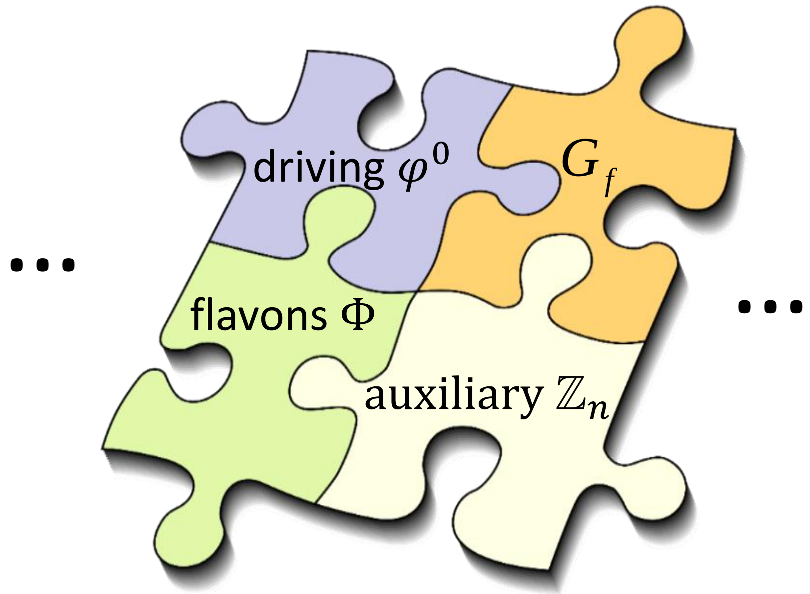
Flavor symmetry to flavor puzzle

The fundamental principle underlying the fermion masses and flavor mixing structure is unkonwn so far. Symmetry can help to reduce the number of free parameters in the Yukawa coupling.

➢ **Flavor symmetry:** relate three families 1st-family ⟺ 2nd-family ⟺ 3rd-family



talks by Stephen King,Myriam Mondragon



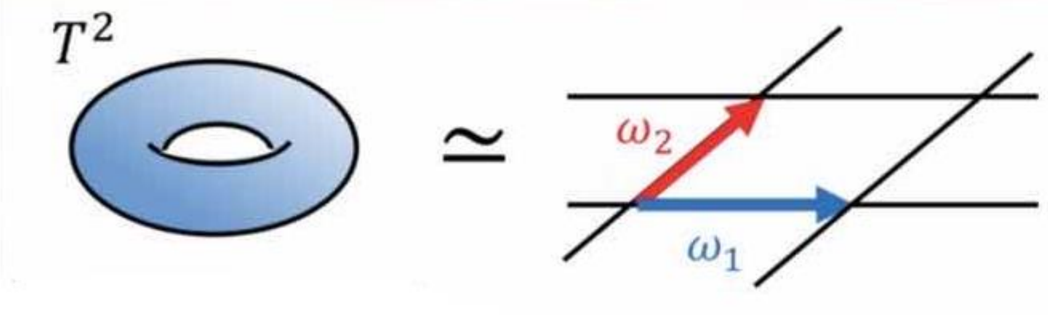
[reviews:Altarelli, Feruglio, 1002.0211; Ishimori, Kobayashi at al, 1003.3552; King, Luhn,1301.1340; King, Merle et al, 1402.4271; King,1701.04413; Xing,1909.09610; Feruglio, Romanino, 1912.06028; Almumin, Chen et al, 2204.08668; Ding,Valle, 2402.16963]

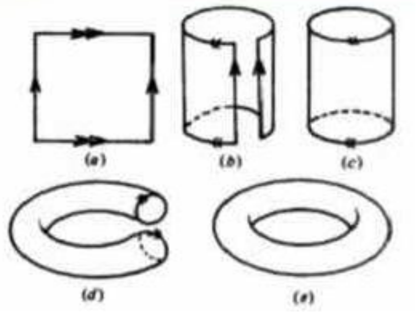
- To make the Lagrangian invariant under flavor symmetry, Higgs-like fields "flavons" Φ_e, Φ_ν are needed
- Structure of Yukawa couplings arises from the vacuum alignment of flavons
- Flavons and the vacuum alignment make the models **quite complicated**

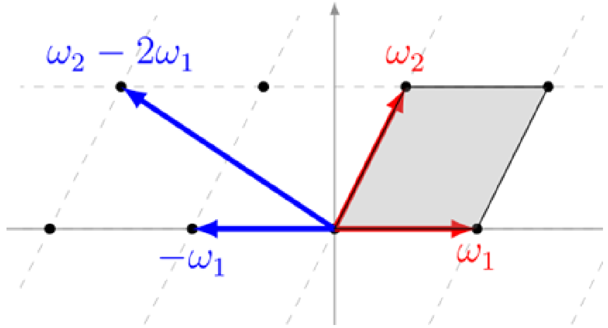
Modular symmetry

Modular symmetry $SL(2,Z)$ is the geometrical symmetry of the torus T^2

Torus $T^2 \cong \mathbb{C}/\Lambda_{(\omega_1,\omega_2)}$, lattice $\Lambda_{(\omega_1,\omega_2)} = n_1\omega_1 + n_2\omega_2$ [Feruglio, Romanino, 1912.06028]







Up to rotations and dilations, the shape of a torus is parameterized by $\tau = \omega_2/\omega_1, \mathrm{Im}\tau > 0$

The lattice (torus) is left **invariant** by $SL(2,\mathbb{Z})$ modular transformation

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \Longrightarrow \tau \to \gamma\tau = \frac{a\tau+b}{c\tau+d}$$

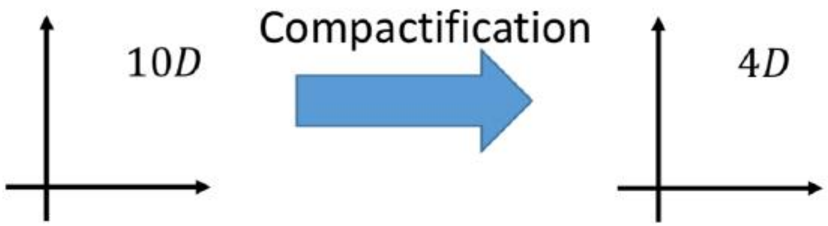
$$SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

modular generators:

$$S: \tau \to -\frac{1}{\tau}$$

$$T: \tau \to \tau + 1$$

Effective 4D theories are modular invariant:



torus compactification

$$S = \int d^4x d^6y \,\mathcal{L}_{10D} \Rightarrow \int d^4x \,\mathcal{L}_{\rm eff}(\varphi, \tau_i)$$

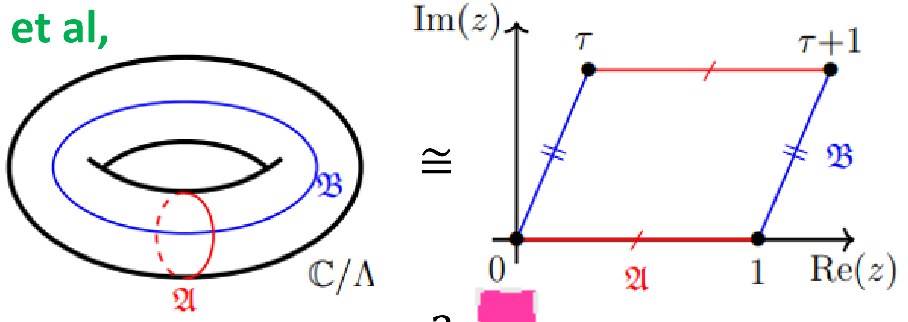
Modular invariant SUSY theory

- The field transformation

$$\psi \to (c\tau + d)^{-k} \rho(\gamma)\psi$$

weight $k \in \mathbb{Z}$ ρ is a unitary representation of Γ_N or Γ'_N

[Lauer, Mas, Nilles, 1989; Ferrara, Lust et al, 1989; Feruglio, 1706.08749]



$SL(2,Z)$ on torus T^2

- Superpotential

$$\mathcal{W} = \sum Y_{I_1 I_2 \ldots I_n}(\tau)\, \psi_{I_1}\psi_{I_2}\ldots\psi_{I_n}$$

Modular invariance requires

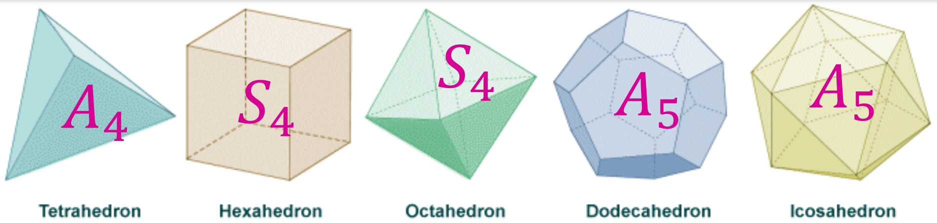
$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

$$k_Y = k_{I_1} + k_{I_2} + \ldots + k_{I_n}, \quad \rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \supset 1$$

- Yukawa couplings are modular forms $Y_{I_1 I_2 \ldots I_n}(\tau)$

- Γ_N and Γ'_N play the role of flavor symmetry groups

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 = T'$	$S'_4 \cong SL(2,\mathbb{Z}_4)$	$A'_5 \cong SL(2,\mathbb{Z}_5)$



finite modular groups

$$\begin{cases} \Gamma_N \equiv SL(2,\mathbb{Z}) / \pm\Gamma(N) \\ \Gamma'_N \equiv SL(2,\mathbb{Z}) / \Gamma(N) \end{cases}$$

- **Advantages**:

① No flavons (except τ) and complexity of vacuum alignment;

② No corrections from higher dimensional operators in SUSY limit;

③ significant reduction in the number of parameters

Modular invariant flavor models

Are neutrino masses modular forms?

Ferruccio Feruglio (INFN, Padua and Padua U.) (Jun 27, 2017)

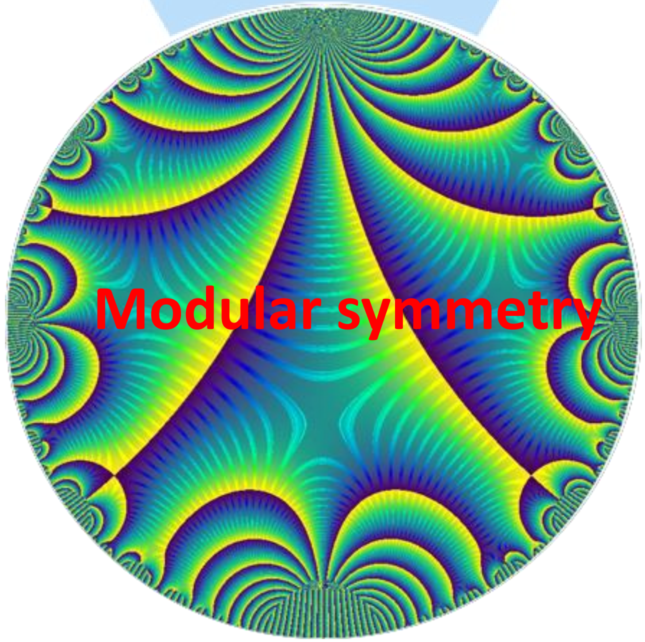
e-Print: 1706.08749 [hep-ph]

pdf DOI cite claim reference search 244 citations

➢ Bottom-up models for lepton and quark **[reviews: Kobayashi, Tanimoto, 2307.03384; Ding, King, arXiv:2311.09282]**

	Γ_N/Γ'_N	leptons alone	leptons & quarks	$SU(5)$	$SO(10)$
$N=2$	S_3	Kobayashi et al, 1803.10391...	—	Kobayashi et al, 1906.10341...	—
$N=3$	A_4	Feruglio, 1706.08749,1807.01125; Kobayashi, Tanaka, et al, 1803.10391; Kobayashi, Omoto, et al, 1808.03012...	Okada,Tanimoto,1905.13421; King, King, 2002.00969; Yao, Lu, Ding, 2012.13390...	Anda, King,Perdomo, 1812.05620; Chen, Ding, King,2101.12724...	Ding, King,Lu,2108.09655
	T'	Liu, Ding,1907.01488...	Lu, Liu, Ding,1912.07573...	—	—
$N=4$	S_4	Penedo,Petcov,1806.11040 ; Novichkov, Penedo et al,1811.04933...	Qu, Liu et al,2106.11659	Zhao, Zhang,2101.02266 ; Ding, King, Yao,2103.16311...	—
	S'_4	Novichkov,Penedo,Petcov,2006.03058...	Liu, Yao, Ding, 2006.10722...	—	—
$N=5$	A_5	Novichkov, Penedo et al,1812.02158; Ding, King, Liu, 1903.12588...	—	—	—
	A'_5	Wang, Yu, Zhou, 2010.10159 ...	Yao, Liu, Ding,2011.03501	—	—
$N=6$	Γ_6	—	—	Abe,Higaki et al, 2307.01419	—
	Γ'_6	Li,Liu,Ding,2108.02181	—	—	—
$N=7$	Γ_7	Ding, King et al, 2004.12662	—	—	—
	Γ'_7	—	—	—	—

General finite
modular groups



General finite modular groups: SL(2,$\mathbb{Z}$)/normal subgroups

Normal subgroups $\ker(\rho)$			Finite modular groups $\Gamma/\ker(\rho) \cong \mathrm{Im}(\rho)$	
Index	Label	Additional relators	Group structure	GAP Id
6	$\Gamma(2)$	T^2	S_3	[6, 1]
12	–	S^2T^2	$Z_3 \rtimes Z_4 \cong 2D_3$	[12, 1]
	$\pm\Gamma(3)$	S^2, T^3	A_4	[12, 3]
18	–	$ST^{-2}ST^2$	$S_3 \times Z_3$	[18, 3]
24	$\Gamma(3)$	T^3	T'	[24, 3]
	–	S^2T^3		
	$\pm\Gamma(4)$	S^2, T^4	S_4	[24, 12]
	–	$S^2, (ST^{-1}ST)^2$	$A_4 \times Z_2$	[24, 13]
36	–	$S^3T^{-2}ST^2$	$(Z_3 \rtimes Z_4) \times Z_3$	[36, 6]
42	–	$T^6, (ST^{-1}S)^2TST^{-1}ST^2$	$Z_7 \rtimes Z_6$	[42, 1]
	–	$T^6, ST^{-1}ST(ST^{-1}S)^2T^2$		
48	–	S^2T^4	$2O$	[48, 28]
	–	T^8, ST^4ST^{-4}	$GL(2,3)$	[48, 29]
	$\Gamma(4)$	T^4	$A_4 \rtimes Z_4 \cong S_4'$	[48, 30]
	–	$(ST^{-1}ST)^2$	$A_4 \times Z_4$	[48, 31]
	–	$S^2(ST^{-1}ST)^2$	$T' \times Z_2$	[48, 32]
	–	T^{12}, ST^3ST^{-3}	$((Z_4 \times Z_2) \rtimes Z_2) \rtimes Z_3$	[48, 33]
54	–	$T^6, (ST^{-1}ST)^3$	$(Z_3 \times Z_3) \rtimes Z_6$	[54, 5]
60	$\pm\Gamma(5)$	S^2, T^5	A_5	[60, 5]
72	–	T^{12}, ST^4ST^{-4}	$S_4 \times Z_3$	[72, 42]
	$\pm\Gamma(6)$	$S^2, T^6, (ST^{-1}STST^{-1}S)^2T^2$	$A_4 \times S_3$	[72, 44]

[Liu,Ding, 2112.14761; Ding, Liu, Lu, Weng, 2307.14926; Arriaga-Osante, Liu, Ramos-Sanchez, 2311.10136]

The flavor symmetry can be general finite modular groups rather than Γ_N and Γ'_N.

Minimal modular lepton model

Modular symmetry allows to construct quite predictive lepton models. The modular flavor symmetry is modular binary octahedral group $2O$ which is the Shur double cover of S_4

	L	$E_D^c = (e^c, \mu^c)$	τ^c	N^c	$H_{u,d}$
$2O$	$\mathbf{3}$	$\widehat{\mathbf{2}}'$	$\mathbf{1}'$	$\mathbf{3}$	$\mathbf{1}$
k_I	-1	6	5	1	0

[Ding, Liu, Lu, Weng, 2307.14926]

- **Charged leptons**

$$\mathcal{W}_E = \alpha \left(E_D^c L Y^{(5)}_{\widehat{\mathbf{2}}'}\right)_{\mathbf{1}} H_d + \beta \left(E_D^c L Y^{(5)}_{\widehat{\mathbf{4}}}\right)_{\mathbf{1}} H_d + \gamma \left(\tau^c L Y^{(4)}_{\mathbf{3}'}\right)_{\mathbf{1}} H_d$$

$$M_E = \begin{pmatrix} -\alpha Y^{(5)}_{\widehat{\mathbf{2}}',2} - \sqrt{2}\beta Y^{(5)}_{\widehat{\mathbf{4}},3} & \sqrt{3}\beta Y^{(5)}_{\widehat{\mathbf{4}},1} & \sqrt{2}\alpha Y^{(5)}_{\widehat{\mathbf{2}}',1} + \beta Y^{(5)}_{\widehat{\mathbf{4}},4} \\ -\alpha Y^{(5)}_{\widehat{\mathbf{2}}',1} + \sqrt{2}\beta Y^{(5)}_{\widehat{\mathbf{4}},4} & -\sqrt{2}\alpha Y^{(5)}_{\widehat{\mathbf{2}}',2} + \beta Y^{(5)}_{\widehat{\mathbf{4}},3} & -\sqrt{3}\beta Y^{(5)}_{\widehat{\mathbf{4}},2} \\ \gamma Y^{(4)}_{\mathbf{3}',1} & \gamma Y^{(4)}_{\mathbf{3}',3} & \gamma Y^{(4)}_{\mathbf{3}',2} \end{pmatrix} v_d$$

- **Neutrino mass : seesaw mechanism**

$$\mathcal{W}_\nu = g H_u (N^c L)_{\mathbf{1}} + \Lambda \left(N^c N^c Y^{(2)}_{\mathbf{2}}\right)_{\mathbf{1}}$$

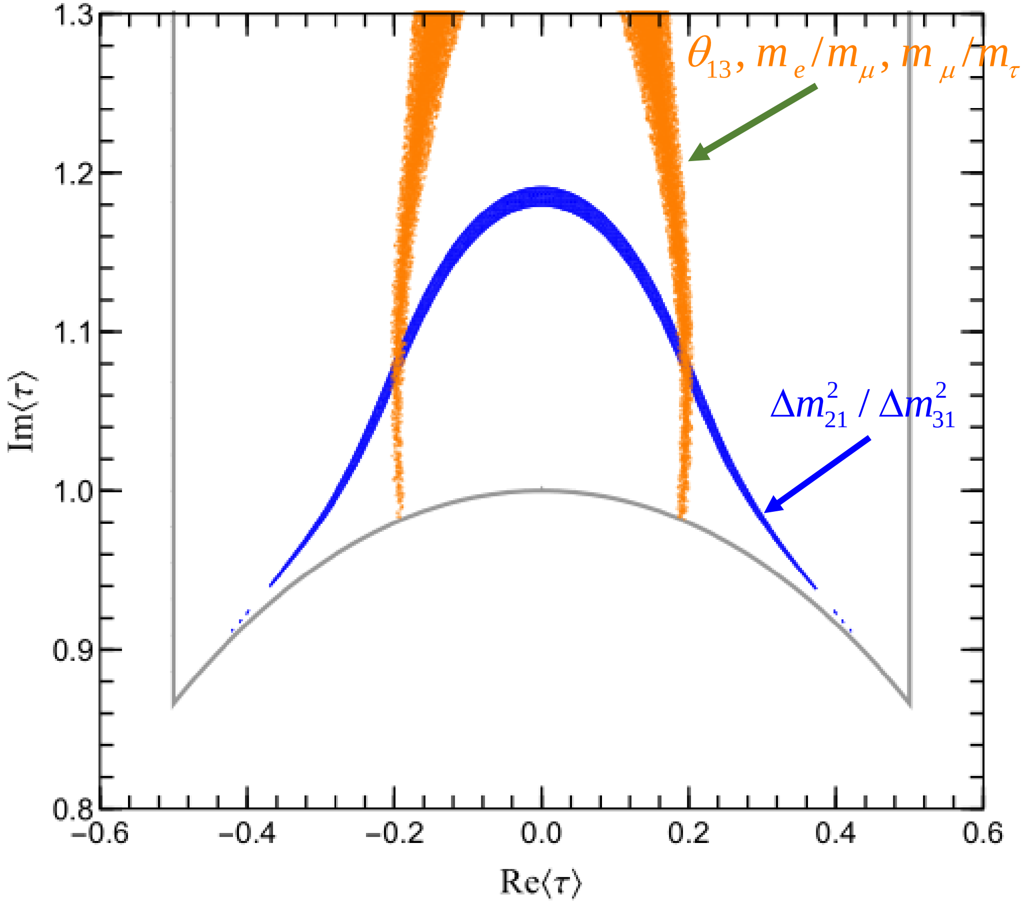
Minimal #p: $\alpha, \beta, \gamma, g^2/\Lambda$

$$M_D = g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u\,, \quad M_N = \begin{pmatrix} -2Y^{(2)}_{\mathbf{2},1} & 0 & 0 \\ 0 & \sqrt{3}Y^{(2)}_{\mathbf{2},2} & Y^{(2)}_{\mathbf{2},1} \\ 0 & Y^{(2)}_{\mathbf{2},1} & \sqrt{3}Y^{(2)}_{\mathbf{2},2} \end{pmatrix} \Lambda$$

Light neutrino mass

$$m_1 = \frac{1}{|2Y^{(2)}_{\mathbf{2},1}|}\frac{g^2v_u^2}{\Lambda}\,, \quad m_2 = \frac{1}{|Y^{(2)}_{\mathbf{2},1} - \sqrt{3}Y^{(2)}_{\mathbf{2},2}|}\frac{g^2v_u^2}{\Lambda}\,, \quad m_3 = \frac{1}{|Y^{(2)}_{\mathbf{2},1} + \sqrt{3}Y^{(2)}_{\mathbf{2},2}|}\frac{g^2v_u^2}{\Lambda}$$

only depends on modulus τ up to overall scale



Neutrino mass spectrum is normal ordering

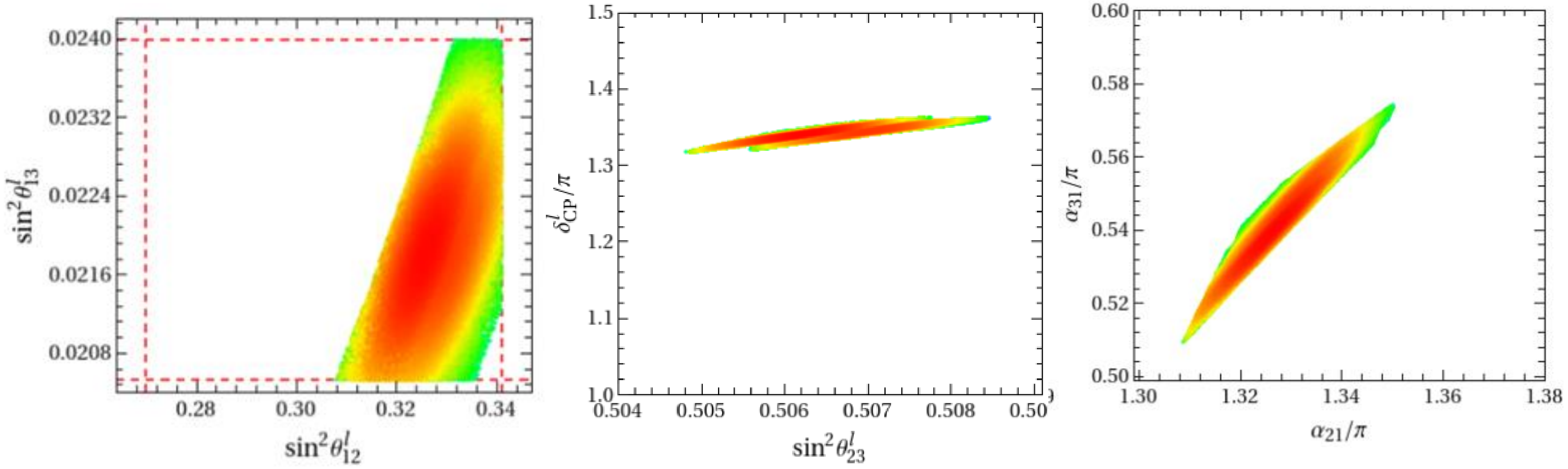
Minimal: only 4 real couplings plus modulus τ can explain 12 observables

$$\langle\tau\rangle = -0.1921+1.0854i,\ \ \beta/\alpha = 0.7159,\ \ \gamma/\alpha = 87.4471,$$

$$\alpha v_d = 0.02881\,\mathrm{MeV},\ \ g^2 v_u^2/\Lambda = 71.8888\,\mathrm{meV}$$

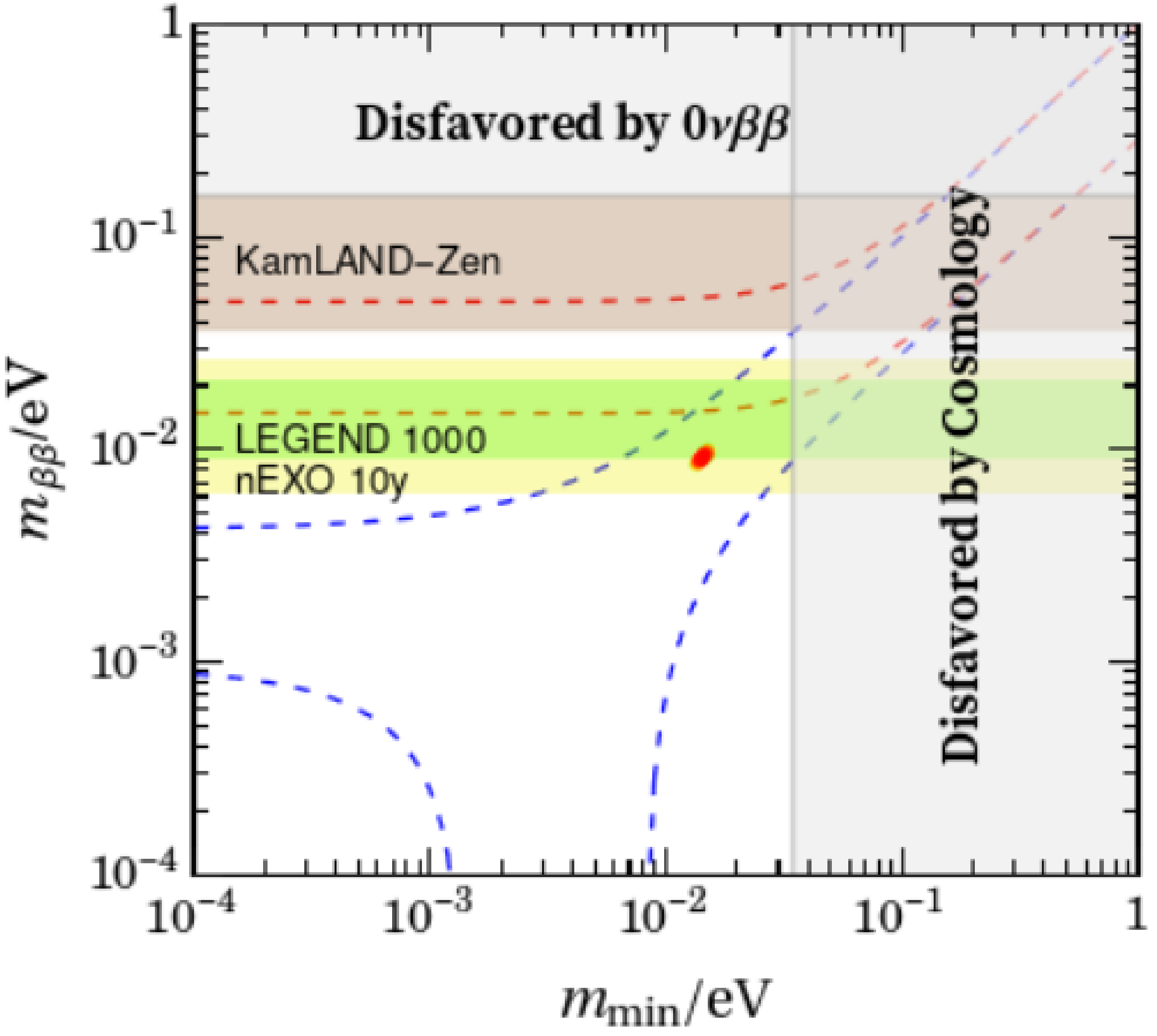
***τ* is the unique source breaking both modular and CP symmetries.** All observables are within the 3σ regions

$$\sin^2\theta_{12} = 0.3261,\ \sin^2\theta_{13} = 0.02182,\ \sin^2\theta_{23} = 0.5063,\ \delta_{CP} = 1.34\pi,$$
$$\alpha_{21} = 1.3268\pi,\ \alpha_{31} = 0.5401\pi,\ m_e/m_\mu = 0.004737,\ m_\mu/m_\tau = 0.05876,$$
$$m_1 = 14.27\,\mathrm{meV},\ \ m_2 = 16.67\,\mathrm{meV},\ \ m_3 = 51.64\,\mathrm{meV}$$



The effective neutrino masses:

$$m_{\beta} = 16.76\ \text{meV},\ \ m_{\beta\beta} = 9.17\ \text{meV}$$



within the sensitivity of future 0νββ experiments

Extension to quark sector

	$Q_D=(Q_1,Q_2)$	Q_3	$U_D^c=(u^c,c^c)$	t^c	$D_D^c=(d^c,s^c)$	b^c
$2O$	$\mathbf{2}$	$\mathbf{1}'$	$\widehat{\mathbf{2}}'$	$\mathbf{1}'$	$\mathbf{2}$	$\mathbf{1}'$
k_I	k_{Q_D}	k_{Q_D}	$3-k_{Q_D}$	$6-k_{Q_D}$	$6-k_{Q_D}$	$-k_{Q_D}$

$$M_u=\begin{pmatrix}\alpha_u Y^{(3)}_{\hat{\mathbf{4}},3} & -\alpha_u Y^{(3)}_{\hat{\mathbf{4}},2} & 0\\ \alpha_u Y^{(3)}_{\hat{\mathbf{4}},4} & \alpha_u Y^{(3)}_{\hat{\mathbf{4}},1} & 0\\ -\beta_u Y^{(6)}_{\mathbf{2},2} & \beta_u Y^{(6)}_{\mathbf{2},1} & \gamma_u Y^{(6)}_{\mathbf{1}}\end{pmatrix} v_u\,,$$

$$M_d=\begin{pmatrix}\alpha_d Y^{(6)}_{\mathbf{1}}-\gamma_d Y^{(6)}_{\mathbf{2},1} & \beta_d Y^{(6)}_{\mathbf{1}'}+\gamma_d Y^{(6)}_{\mathbf{2},2} & -\delta_d Y^{(6)}_{\mathbf{2},2}\\ \gamma_d Y^{(6)}_{\mathbf{2},2}-\beta_d Y^{(6)}_{\mathbf{1}'} & \alpha_d Y^{(6)}_{\mathbf{1}}+\gamma_d Y^{(6)}_{\mathbf{2},1} & \delta_d Y^{(6)}_{\mathbf{2},1}\\ 0 & 0 & \varepsilon_d\end{pmatrix} v_d$$

The complex modulus τ is common in both quark and lepton sectors, and its value is fixed by the lepton parameters

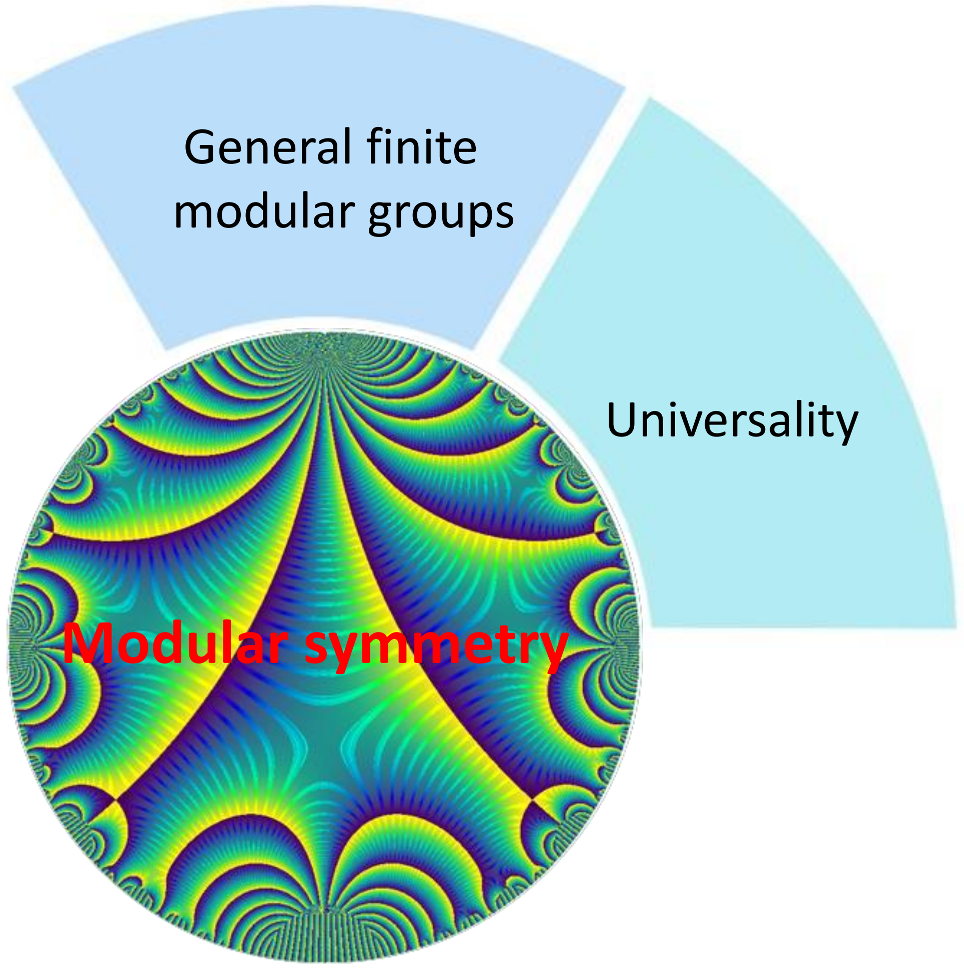
$$\langle\tau\rangle=-0.1946+1.0799i$$

The quark masses and CKM mixing parameters can be well accommodated with $\chi_q^2=6.4$:

$$\theta_{12}^q=0.229,\quad \theta_{13}^q=0.00393,\quad \theta_{23}^q=0.0388,\quad \delta_{CP}^q=61.27^\circ,$$

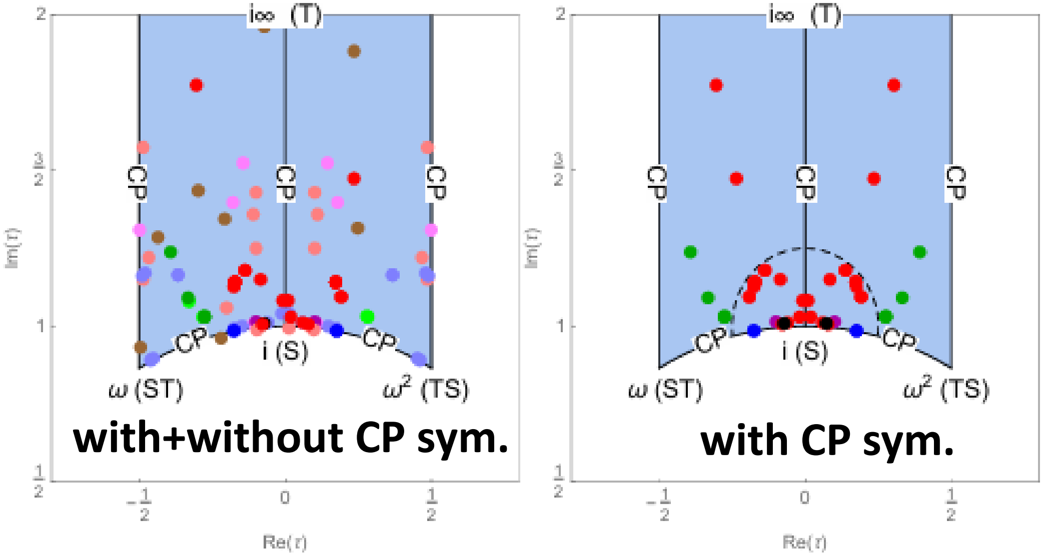
$$m_u/m_c=0.00243,\quad m_c/m_t=0.00245,\quad m_d/m_s=0.0510,\quad m_s/m_b=0.0234$$

- The model uses **14** parameters to describe the masses and mixing of both quark and lepton sectors: **12** masses+**6** mixing angles+**4** CP phases.


General finite
modular groups
Universality
Modular symmetry

Universality of modular invariant models around fixed points

Many modular invariant bottom-up models differ in the level N, the representation ρ_I and weight k_I assignments of matter fields.



[Feruglio, 2211.00659,2302.11580]

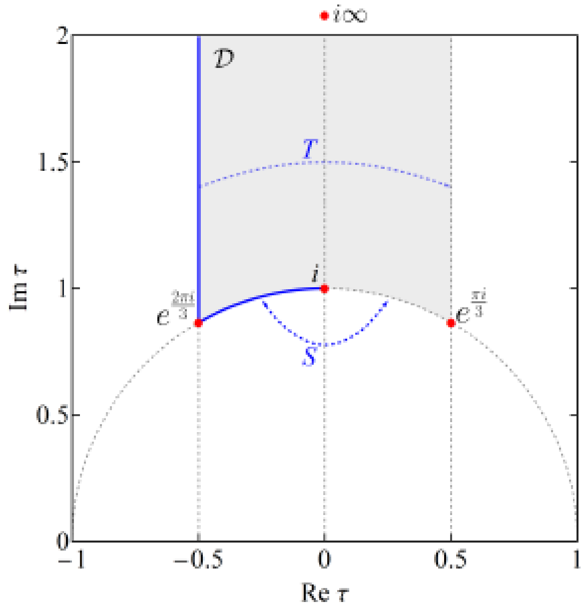
- At certain values of τ, modular symmetry is partially broken and some residual symmetry remains.

➢ **There are only 3 inequivalent modular fixed points in the fundamental domain**

$$\gamma_0 \tau_0 = \tau_0 \quad \Rightarrow \quad \tau_0 = i,\ e^{2\pi i/3},\ i\infty$$

Fixed points of modular symmetry:

τ_0	γ_0	invariant under	Residual sym. G_0
i	S	$\tau \xrightarrow{S} -\frac{1}{\tau}$	$\mathbb{Z}_4^S$
$e^{2\pi i/3}$	ST	$\tau \xrightarrow{ST} -\frac{1}{\tau+1}$	$\mathbb{Z}_3^{ST}$
$i\infty$	T	$\tau \xrightarrow{T} \tau+1$	$\mathbb{Z}_N^T$



- Linearization of the residual transformation γ_0

$$\Omega(\gamma_0) \equiv (c_0\tau_0 + d_0)^{-k}\rho(\gamma_0)$$

$$\left.\begin{array}{l} \tau_0 = i, e^{2\pi i/3}:\ u = \dfrac{\tau-\tau_0}{\tau-\tau_0^*}, \Phi = (1-u)^k\psi \\ \tau_0 = i\infty:\ u = e^{\frac{2\pi i\tau}{N}}, \Phi = \psi \end{array}\right\} \Longrightarrow \begin{cases} u \xrightarrow{\gamma_0} \dfrac{a_0 - c_0\tau_0}{a_0 - c_0\tau_0^*} u, & u \xrightarrow{CP} u^* \\ \Phi \xrightarrow{\gamma_0} \Omega(\gamma_0)\Phi, & \Phi \xrightarrow{CP} \Phi^* \end{cases}$$

The residual symmetry acts as traditional flavor abelian symmetry broken by the VEV of u

- Fermion mass terms in terms of new variable **see talk by Joao Penedo**

$$\mathcal{L}_m = \Phi_i^c m_{ij}(u,u^*)\Phi_j \Longrightarrow \begin{cases} m(\gamma_0 u, \gamma_0 u^*) = \Omega^{c*} m(u,u^*)\Omega \\ m(u^*,u) = m^*(u,u^*) \end{cases}$$

modular invariance under residual γ_0 and CP

Expanding $m_{ij}(\tau)$ in powers of u in the vicinity of τ_0

$$m_{ij}(u,u^*) = m_0\left[x_{ij}^0 + x_{ij}^{10}u + x_{ij}^{01}u^* + x_{ij}^{20}u^2 + x_{ij}^{11}uu^* + x_{ij}^{02}u^{*2} + \cdots\right]$$

- The coefficients $x_{ij}^0, x_{ij}^{10}, x_{ij}^{01}$, … are assumed to be of order one, normalization of modular form is relevant. **[Petcov,2311.04185]**
- Invariance under the residual symmetry γ_0 allows to fix the pattern of mass matrix
- Only need $\Omega(\gamma_0)$ of LH leptons which are assumed to be irreducible triplet of Γ_N'
- **Universality:** independent of level , the modular weight of matter fields and the kinetic terms

➢ The vicinity of $\tau = i$ is preferred by neutrino oscillation **[Feruglio, 2302.11580]**

τ		mass ordering	$\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}}$	$\sin^2\theta_{12}$	$\sin^2\theta_{13}$	$\sin^2\theta_{23}$
$\approx i$	k_S even $m_\nu(0,0)$ regular	NO/IO	$\mathcal{O}(1)$	$\mathcal{O}(x^2)$	$\mathcal{O}(x^2)$	$\mathcal{O}(1)$
$\approx i$	k_S odd $m_\nu(0,0)$ regular	IO	$\mathcal{O}(x)$	$\frac{1}{2}(1+\mathcal{O}(x))$	$\mathcal{O}(x^2)$	$\mathcal{O}(1)$
$\approx i$	k_S odd $m_\nu(0,0)$ singular	NO	$\mathcal{O}(x^3)$	$\frac{1}{2}(1+\mathcal{O}(x))$	$\mathcal{O}(x^2)$	$\mathcal{O}(1)$
$\approx \omega$		NO/IO	$\mathcal{O}(x)$	$\frac{1}{2}(1+\mathcal{O}(x))$	$\mathcal{O}(x^2)$	$\mathcal{O}(x^2)$

$$x = |u|$$

$$m_\nu^{-1} = m_{0\nu}^{-1}\begin{pmatrix} x_{11}\, x & x^0_{12} & x^0_{13} \\ \cdot & x_{22}\, x & x_{23}\, x \\ \cdot & \cdot & x_{33}\, x \end{pmatrix} + \mathcal{O}(x^2)$$

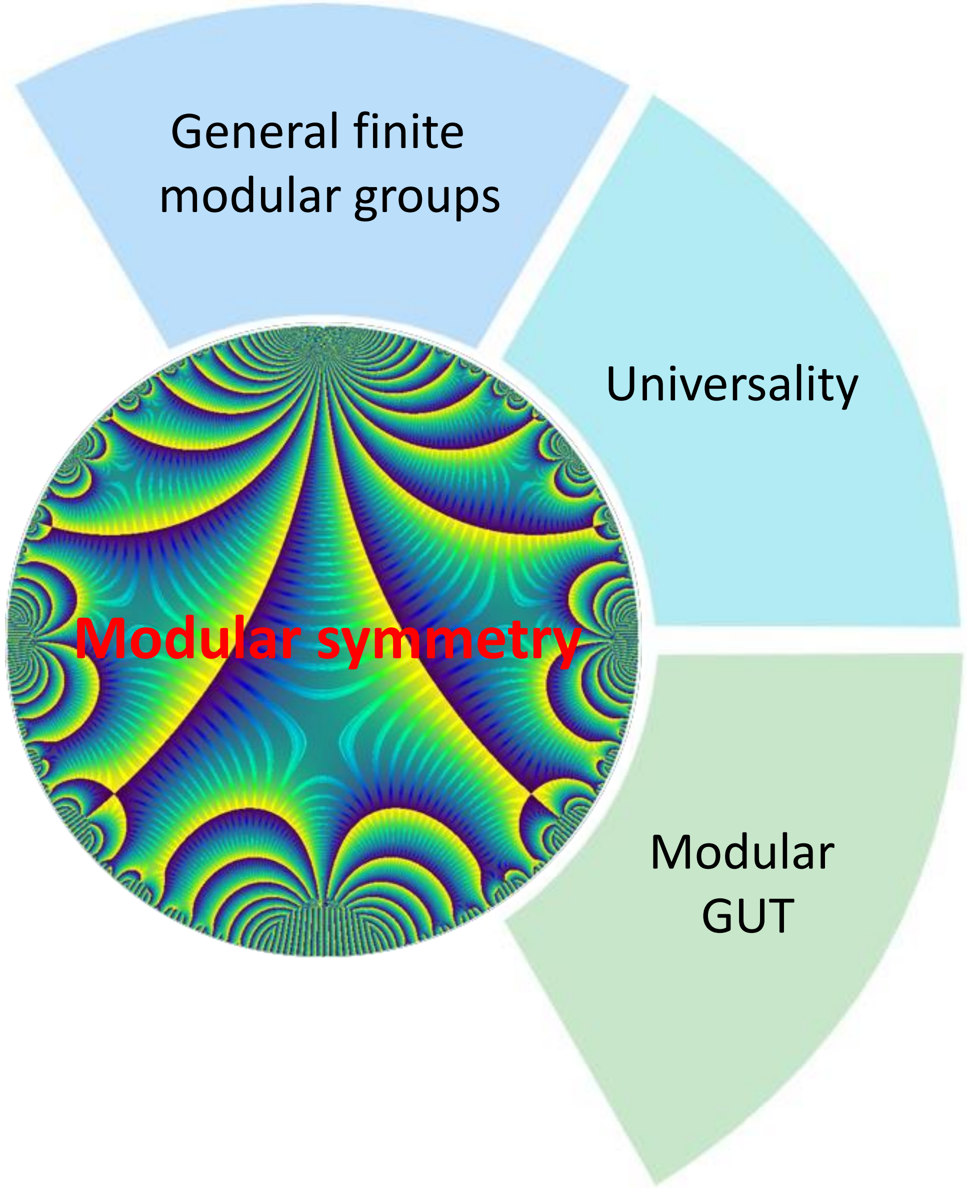
[Ding, Feruglio, Liu, 2402.14915]

The same universal behavior in Symplectic modular invariant theories near the fixed points.

- Charged lepton mass hierarchies require tuning the order one coefficients for $\tau \approx i$, they can be naturally generated for $\tau \approx e^{2\pi i/3}, i\infty$, large lepton angles necessitates singlet assignment of LH leptons, but predictivity is reduced somewhat. **[Novichkov, Penedo, Petcov, 2102.07488]**

- Predictive model for both charged lepton mass hierarchies and neutrino mixing without fine-tuning? both quark and lepton sectors without fine-tuning?

- Critical behavior similar to phase transition or cosmological evolution? **[Feruglio, 2302.11580]**

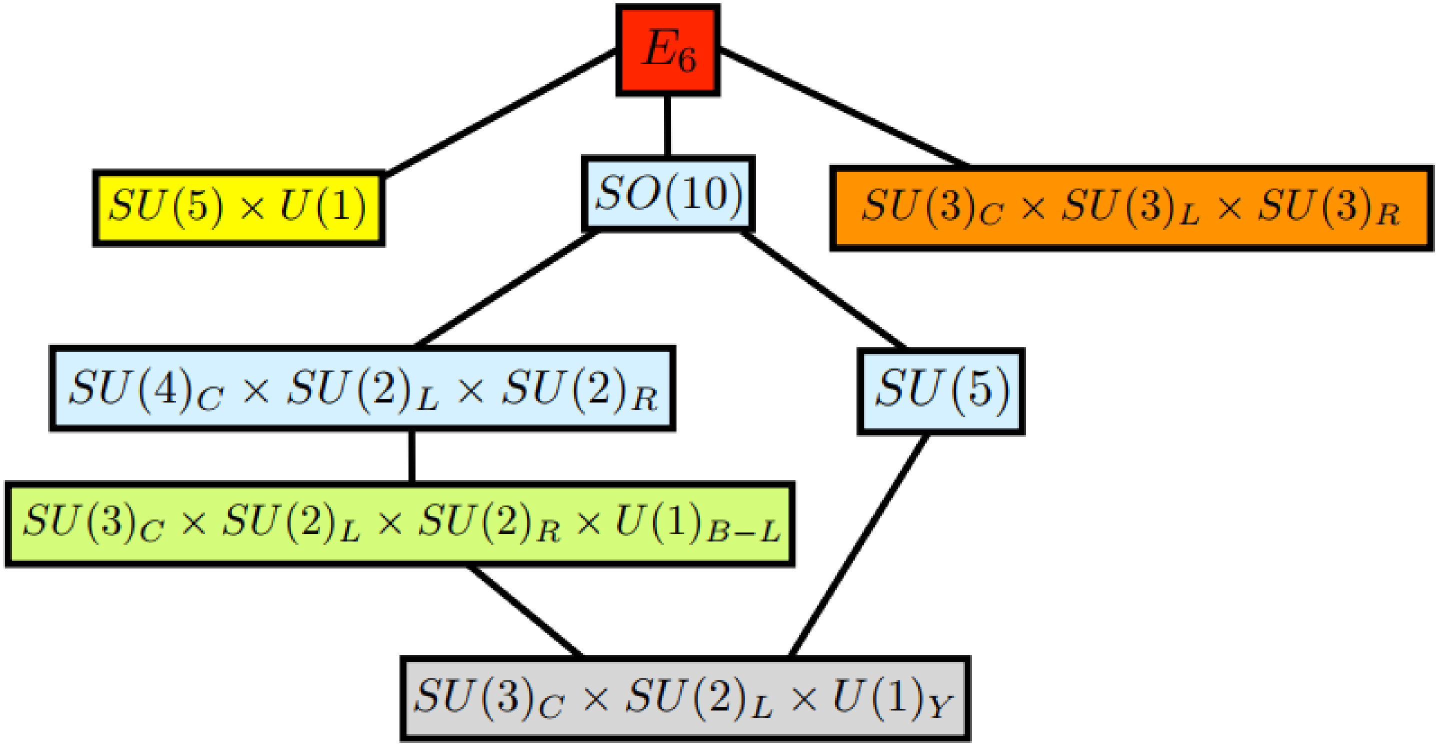

General finite modular groups
Universality
Modular symmetry
Modular GUT

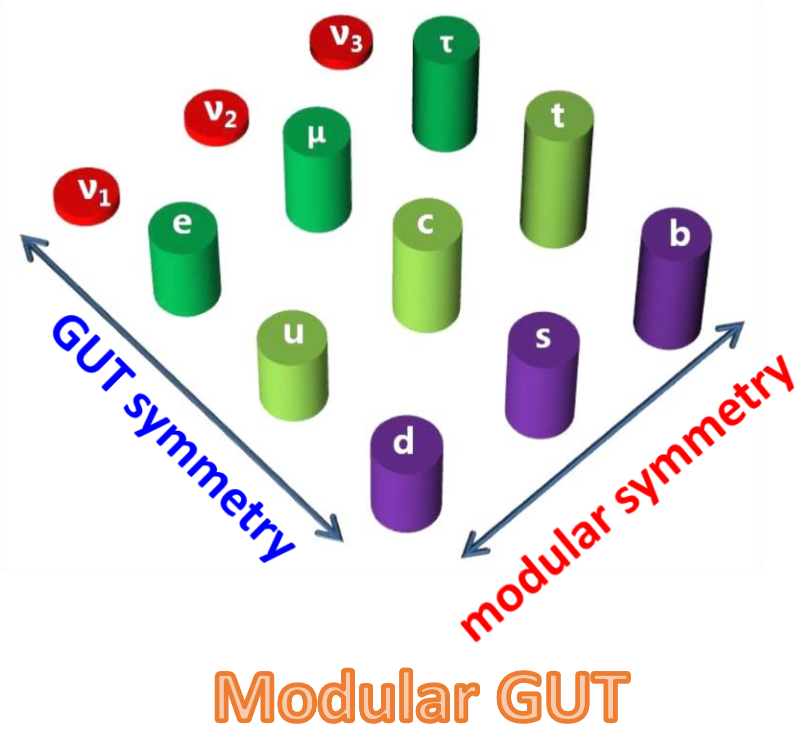
Modular GUTs

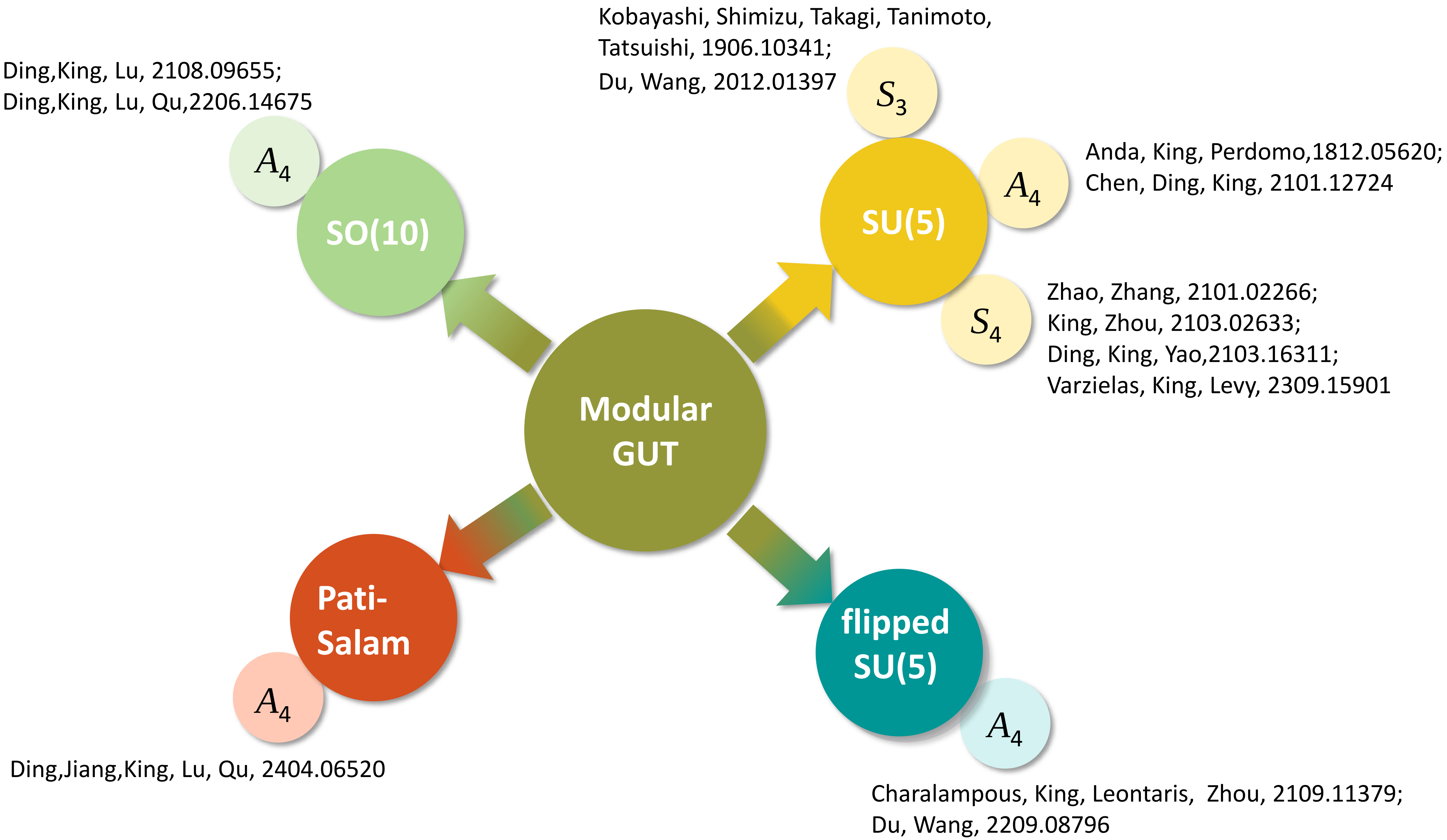
The symmetry group of modular GUT is $G_{\text{GUT}} \times \Gamma_N$ or $G_{\text{GUT}} \times \Gamma'_N$

- **GUTs:** connecting quarks and leptons leptons ⟺ quarks
- **Modular symmetry:** relating three families and Yukawa couplings are modular forms

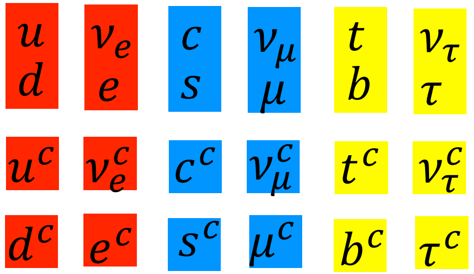
➢ Candidates of GUT gauge group G_{GUT}



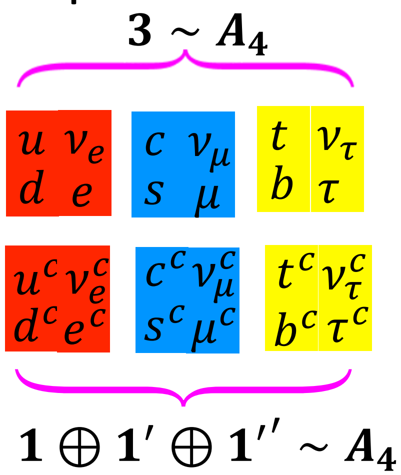



Ding,King, Lu, 2108.09655;
Ding,King, Lu, Qu,2206.14675
A_4
SO(10)
Kobayashi, Shimizu, Takagi, Tanimoto, Tatsuishi, 1906.10341;
Du, Wang, 2012.01397
S_3
SU(5)
A_4
Anda, King, Perdomo,1812.05620;
Chen, Ding, King, 2101.12724
S_4
Zhao, Zhang, 2101.02266;
King, Zhou, 2103.02633;
Ding, King, Yao,2103.16311;
Varzielas, King, Levy, 2309.15901
Modular GUT
Pati-Salam
A_4
Ding,Jiang,King, Lu, Qu, 2404.06520
flipped SU(5)
A_4
Charalampous, King, Leontaris, Zhou, 2109.11379;
Du, Wang, 2209.08796

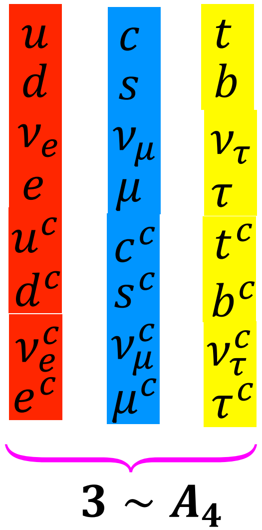
- GUTs unify the quark and leptons fields into few multiplets, the possible transformations of matter fields under modular symmetry is quite limited.



SM: $3 \times 6 = 18$ multiplets



Pati-Salam: $3 \times 2 = 6$ multiplets



SO(10): $3 \times 1 = 3$ multiplets

- The modular symmetry can **strongly constrain** the GUT Yukawa couplings and thus enhance the predictive power, but the complexity of GUT breaking to SM remains

SO(10) fermion mass matrices:

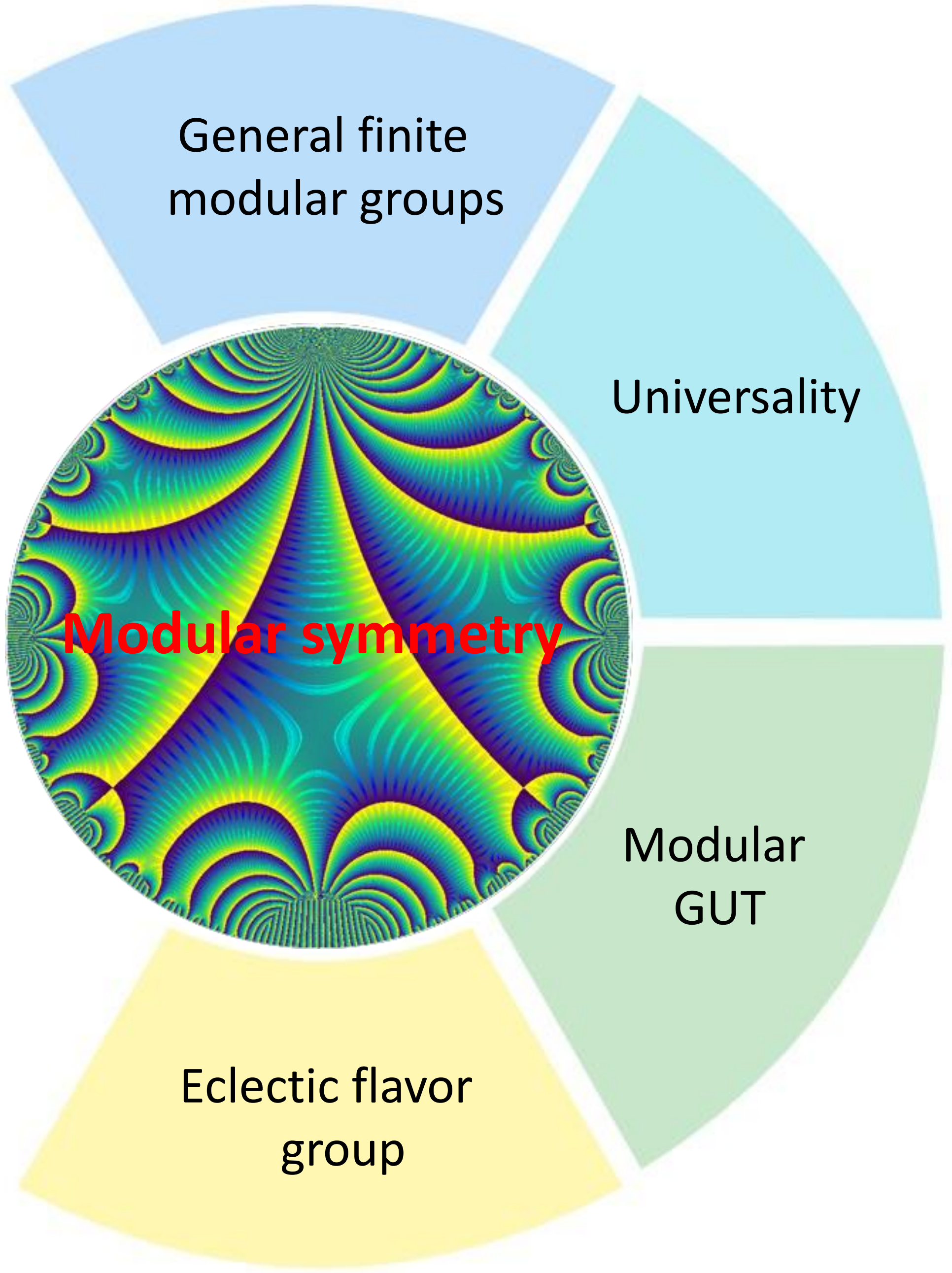
$$M_u = \left(\mathcal{Y}^{10} + r_2 \mathcal{Y}^{\overline{126}} + r_3 \mathcal{Y}^{120}\right) v_u, \quad M_d = r_1 \left(\mathcal{Y}^{10} + \mathcal{Y}^{\overline{126}} + \mathcal{Y}^{120}\right) v_d ,$$

$$M_\ell = r_1 \left(\mathcal{Y}^{10} - 3\mathcal{Y}^{\overline{126}} + c_e \mathcal{Y}^{120}\right) v_d , \quad M_{\nu_D} = \left(\mathcal{Y}^{10} - 3r_2 \mathcal{Y}^{\overline{126}} + c_\nu \mathcal{Y}^{120}\right) v_u ,$$

$$M_{\nu_R} = v_R \mathcal{Y}^{\overline{126}}, \quad M_{\nu_L} = v_L \mathcal{Y}^{\overline{126}} .$$

$r_{1,2,3}$ and $c_{e,\mu}$: mixture of SO(10) Higgs multiplets into SM Higgs

- Modular GUT for flavor puzzle so far, the interplay of modular symmetry with proton decay, GW ?


General finite modular groups
Universality
Modular symmetry
Modular GUT
Eclectic flavor group

Kähler potential problem in modular symmetry

- Kähler potential potential not fixed by modular flavor symmetry

$$\mathcal{K} = (-i\tau + i\bar{\tau})^{-k_\psi} \left(\psi^\dagger \psi\right)_{\mathbf{1}} + \sum_{n, \boldsymbol{r}_1, \boldsymbol{r}_2} c^{(n, \boldsymbol{r}_1, \boldsymbol{r}_2)} (-i\tau + i\bar{\tau})^{-k_\psi + n} \left(\psi^\dagger Y_{\boldsymbol{r}_1}^{(n)\dagger} Y_{\boldsymbol{r}_2}^{(n)} \psi\right)_{\mathbf{1}}$$

minimal Kähler potential — non-canonical terms

many non-canonical terms on the same footing as the minimal Kähler potential

- Modifying Kähler metric and kinetic terms [Chen, Ramon-Sanchez, Ratz, 1909.06910]

$$\mathcal{K}_\psi^{ij} = \frac{\partial^2 \mathcal{K}}{\partial \psi_i^\dagger \partial \psi_j} = \langle -i\tau + i\bar{\tau} \rangle^{-k} \delta^{ij} + \Delta\mathcal{K}_\psi^{ij}$$

flavor universal ($\langle -i\tau + i\bar{\tau} \rangle^{-k} \delta^{ij}$) — flavor off-diagonal ($\Delta\mathcal{K}_\psi^{ij}$)

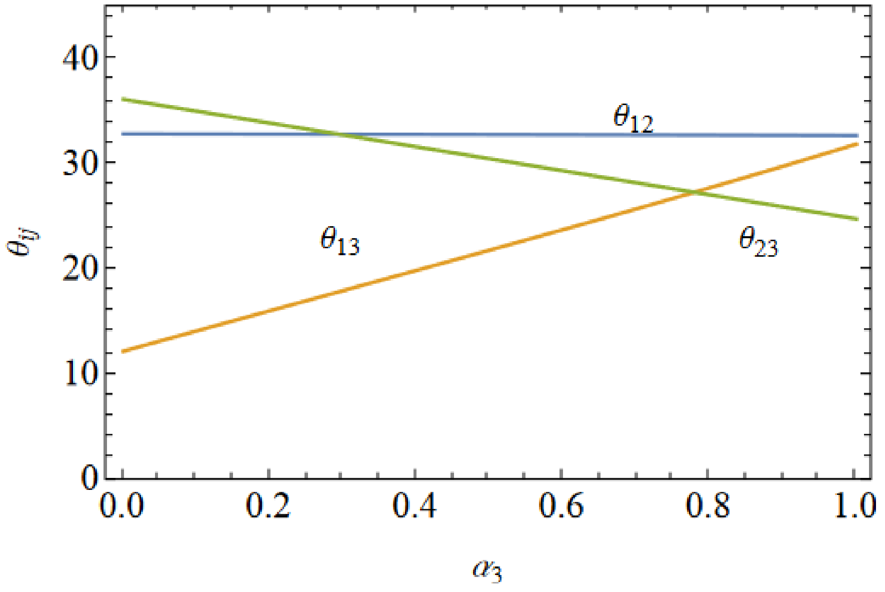
- Go to canonical basis

$$\mathcal{K}_\psi(\tau, \tau^*)\mathcal{K} = Z_\psi^{-1\dagger}(\tau, \tau^*)\ Z_\psi^{-1}(\tau, \tau^*)$$

$$\mathcal{K}_{\psi^c}(\tau, \tau^*)\mathcal{K} = Z_{\psi^c}^{-1\dagger}(\tau, \tau^*)\ Z_{\psi^c}^{-1}(\tau, \tau^*)$$

$$m(\tau) \to Z_{\psi^c}^T(\tau, \tau^*) m(\tau) Z_\psi(\tau, \tau^*)$$

➜ **sizable corrections to mixing parameters**



Eclectic flavor group as a solution to Kähler potential problem

- Modular flavor symmetries from top-down approach (orbifold string compactification) gives
  - Normal symmetries of extra dimensions ➜traditional flavor symmetries
  - String duality transformations ➜modular flavor symmetries
  - The multiplicative closure of these groups is defined as the eclectic flavor group

[Nilles, Ramos-Sanchez, Vaudrevange, 2001.01736; 2004.05200]

$$G_{\text{eclectic}} = G_{\text{traditional}} \cup G_{\text{modular}}$$

see talk by Saul Ramos-Sanchez

- Traditional flavor symmetry vs. modular symmetry transformations

modular: $\tau \xrightarrow{\gamma} \gamma\tau = \dfrac{a\tau + b}{c\tau + d}, \quad \psi \xrightarrow{\gamma} (c\tau + d)^{-k}\rho(\gamma)\psi, \quad \gamma \in SL(2,\mathbb{Z})$

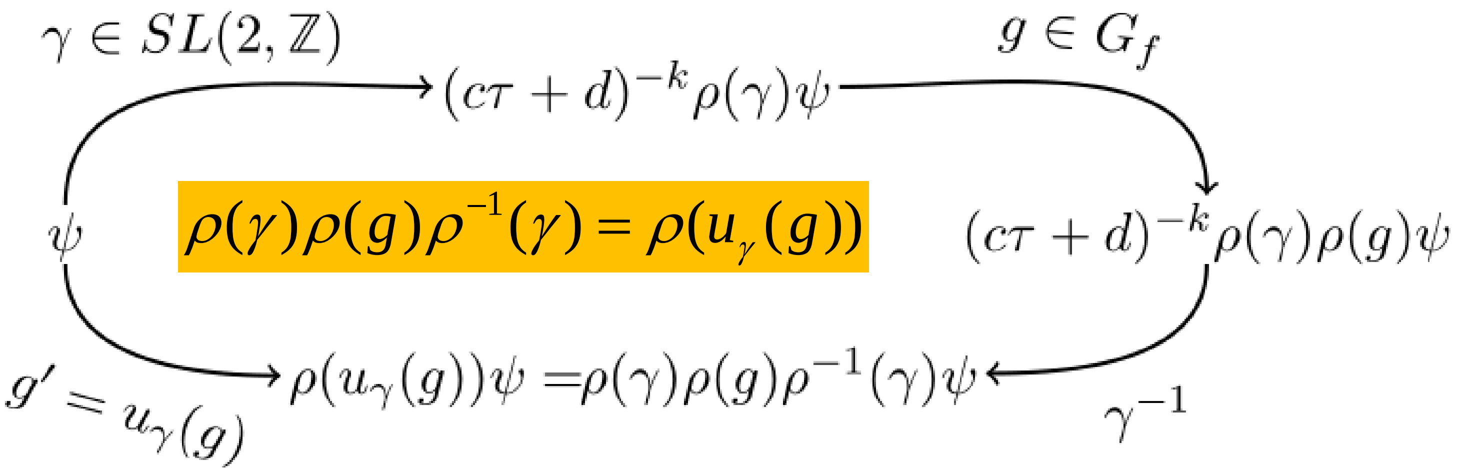
flavor: $\tau \xrightarrow{g} \tau, \quad \psi \xrightarrow{g} \rho(g)\psi, \quad g \in G_f$

τ distinguishes flavor symmetry from modular symmetry

	Traditional flavor symmetry	Modular symmetry	Eclectic flavor Group
Kahler potential	2 SILVER	3 BRONZE	1 GOLD
Superpotential	3 BRONZE	2 SILVER	1 GOLD
Vacuum problem	2 SILVER	1 GOLD	3 BRONZE

- The matter fields transform nontrivially under both flavor and modular symmetries, the Kähler potential is constrained
- Enhanced flavor symmetry at the modular symmetry fixed points $\tau = i, e^{2\pi \mathrm{i}/3}, i\infty$
- **The notorious flavons reoccur to break flavor symmetry**

- Consistency condition [Nilles, Ramos-Sanchez, Vaudrevange, 2001.01736]



$$\begin{aligned} \gamma &\to u_\gamma \\ S &\to u_S \\ T &\to u_T \\ S^{N_s} = 1 &\to (u_S)^{N_s} = 1 \\ T^N = 1 &\to (u_T)^{N_T} = 1 \\ (ST)^3 = 1 &\to (u_S \circ u_T)^3 = 1 \\ S^2 T = TS^2 &\to (u_S)^2 \circ u_T = u_T \circ (u_S)^2 \end{aligned}$$

$N_s = 4\ (2)$ for $\Gamma'_N\ (\Gamma_N)$

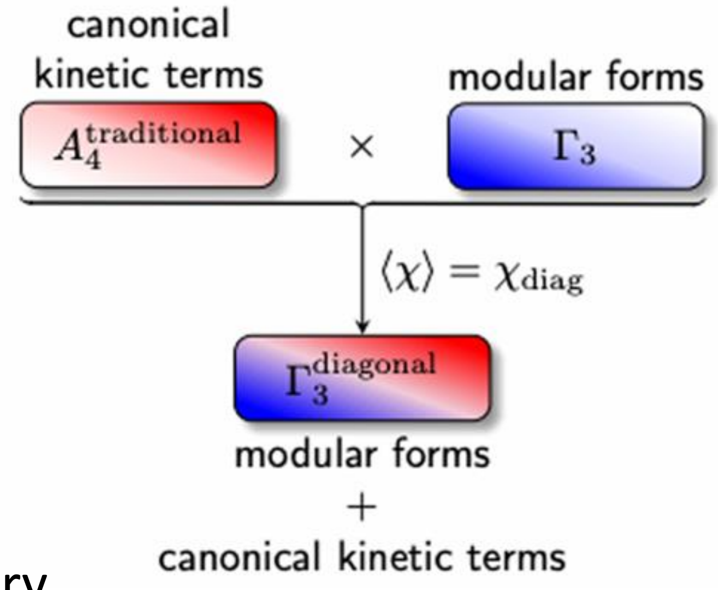
Finite modular group $\Gamma'_N\ (\Gamma_N)$ must be a subgroup of the outer automorphism group of G_f

$$G_{\text{eclectic}} \cong G_f \rtimes \Gamma'_N\ (G_f \rtimes \Gamma_N)$$

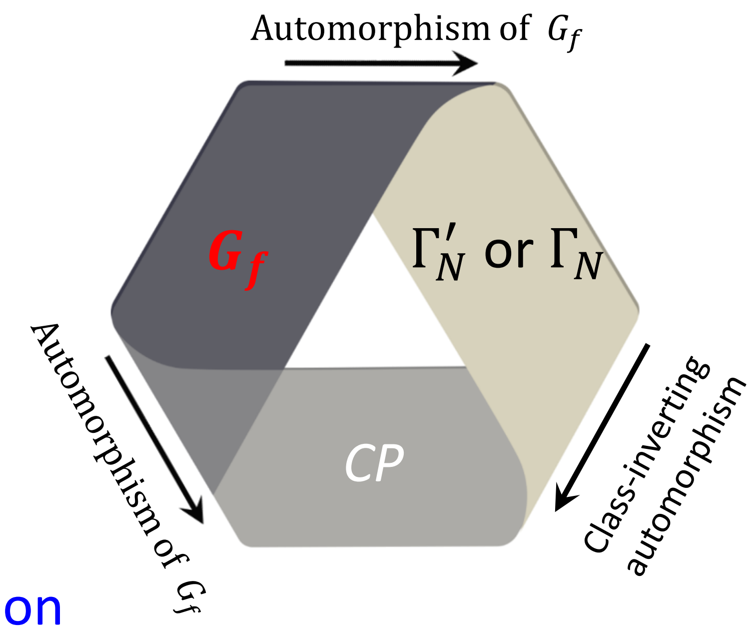
- Quasi-eclectic flavor symmetry: $u_\gamma(g) = g$ [Chen, Perez, Hamud, Sanchez, Ratz,Shukla, 2108.02240]

$$G_{\text{Quasi-eclectic}} \cong G_f \times \Gamma'_N\ (G_f \times \Gamma_N)$$

- Flavor symmetry and modular symmetry are commutable
- No constraint on the choice of G_f and $\Gamma'_N\ (\Gamma_N)$
- Non-canonical Kähler potentials are suppressed by flavor symmetry



- Eclectic flavor group can combine with CP: **unification of flavor, CP and modular symmetries**



CP transformation: $\psi(x) \xrightarrow{\mathcal{CP}} X_{\boldsymbol{r}}\,\bar{\psi}(x_{\mathcal{P}})$

$$X_{\boldsymbol{r}}\,\rho^*_{\boldsymbol{r}}(g)\,X^{-1}_{\boldsymbol{r}} = \rho_{\boldsymbol{r}}(g')\,,$$
$$X_{\boldsymbol{r}}\,\rho^*_{\boldsymbol{r}}(S)\,X^{-1}_{\boldsymbol{r}} = \rho_{\boldsymbol{r}}(g_1 S^{-1})\,,$$
$$X_{\boldsymbol{r}}\,\rho^*_{\boldsymbol{r}}(T)\,X^{-1}_{\boldsymbol{r}} = \rho_{\boldsymbol{r}}(g_2 T^{-1})\,, \quad g', g_1, g_2 \in G_f$$

- Possible eclectic flavor groups: a few G_f suitable to eclectic extension

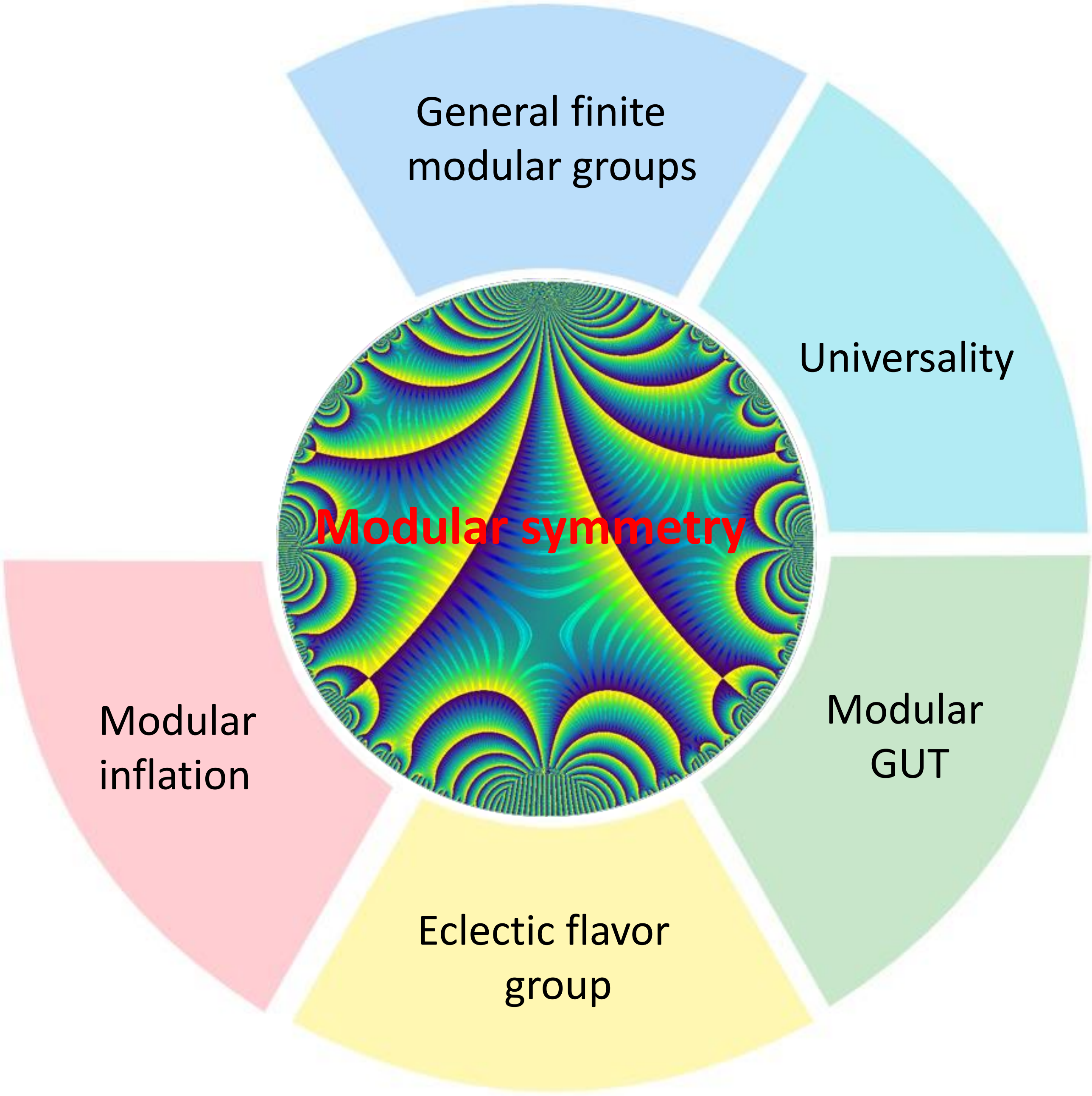
[Nilles,Ramos-Sanchez, Vaudrevange, et al, 2001.01736]

flavor group $\mathcal{G}_{\rm fl}$	GAP ID	Aut($\mathcal{G}_{\rm fl}$)	finite modular groups		eclectic flavor group
Q_8	[8, 4]	S_4	without $\mathcal{CP}$	S_3	GL(2, 3)
			with $\mathcal{CP}$	–	–
$\mathbb{Z}_3 \times \mathbb{Z}_3$	[9, 2]	GL(2, 3)	without $\mathcal{CP}$	S_3	$\Delta(54)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$	[108, 17]
A_4	[12, 3]	S_4	without $\mathcal{CP}$	S_3	S_4
				S_4	S_4
			with $\mathcal{CP}$	–	–
T'	[24, 3]	S_4	without $\mathcal{CP}$	S_3	GL(2, 3)
			with $\mathcal{CP}$	–	–
$\Delta(27)$	[27, 3]	[432, 734]	without $\mathcal{CP}$	S_3	$\Delta(54)$
				T'	$\Omega(1)$
			with $\mathcal{CP}$	$S_3 \times \mathbb{Z}_2$	[108, 17]
				GL(2, 3)	[1296, 2891]
$\Delta(54)$	[54, 8]	[432, 734]	without $\mathcal{CP}$	T'	$\Omega(1)$
			with $\mathcal{CP}$	GL(2, 3)	[1296, 2891]

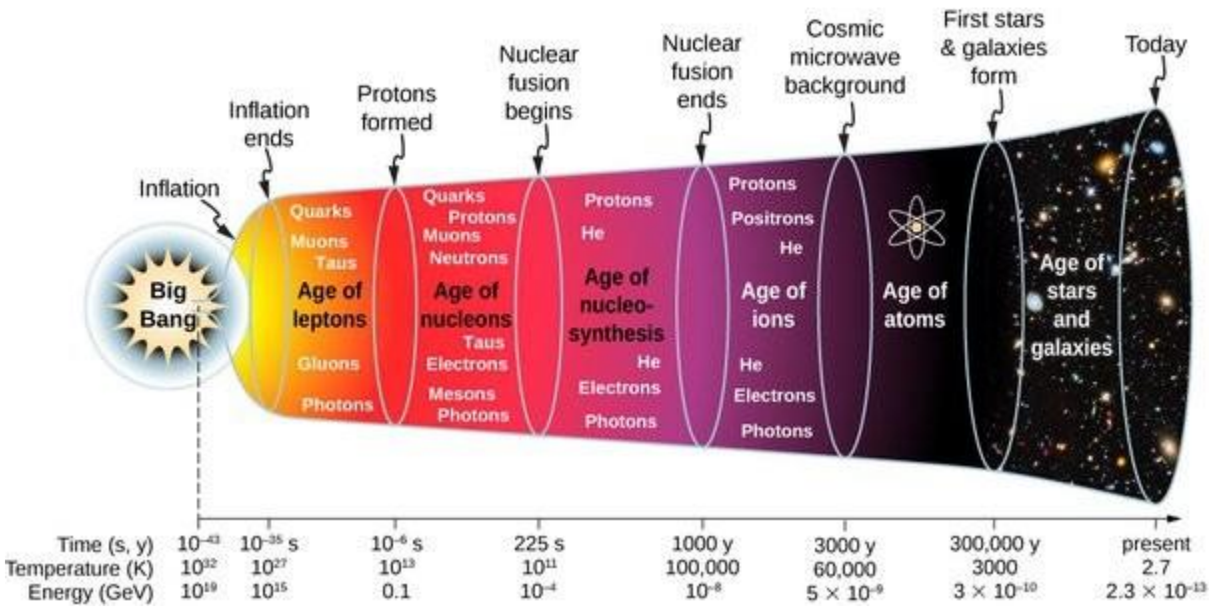
[Ding, Li, Lu,2405.xxxxx]

[Ding, Li, 2308.16901]
[Ding,King, Li, Liu,Lu, 2303.02071]

[Baur, Nilles et al,2207.10677]


General finite modular groups
Universality
Modular symmetry
Modular GUT
Eclectic flavor group
Modular inflation

Modular inflation



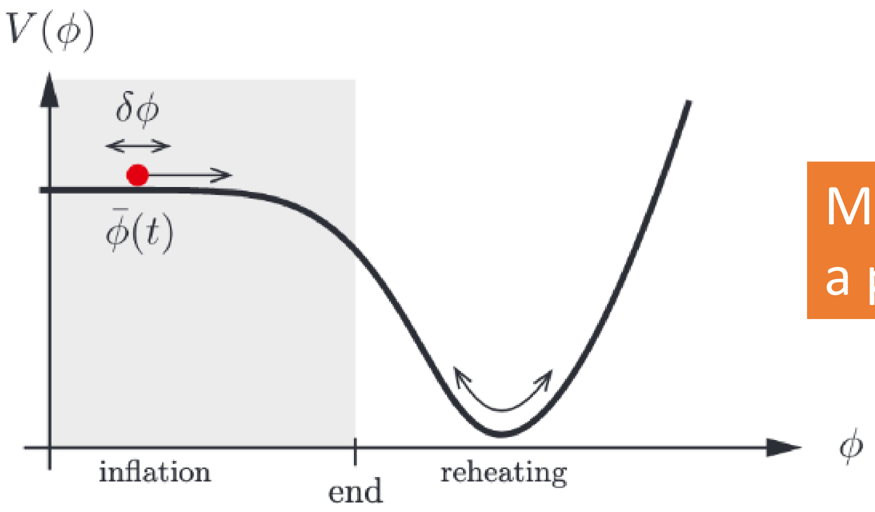
Accelerated expansion of the early universe favored by cosmic microwave background (CMB) observations

$$\ln\left(10^{10}A_s\right) = 3.044 \pm 0.014\,(68\%\mathrm{CL})\,,$$
$$n_s = 0.9649 \pm 0.0042\,(68\%\mathrm{CL})\,,$$
$$r < 0.036\,(95\%\mathrm{CL})\,.$$

A_s: scalar amplitude
n_s: spectral index
r: tensor-to-scalar ratio

[Planck Collaboration,1807.06211]

- Inflation can be realized by a slow-rolling scalar field (inflaton)



SM cannot explain inflation!

Modular inflation: modulus τ plays the role of inflaton, a plateau in the scalar potential is necessary

Modular invariant scalar potential of τ

- The modular invariant effective action in SUGRA for $\tau + S$ (dilaton)

[Cvetic et al., Nucl. Phys. B 361 (1991); Leedom, Righi, Westphal,2212.03876]

$$\mathcal{G}(\tau,\bar{\tau},S,\bar{S}) = \kappa^2\mathcal{K}(\tau,\bar{\tau},S,\bar{S}) + \ln|\kappa^3\mathcal{W}(\tau,S)|^2$$

$$\kappa = \sqrt{8\pi G_N}$$

Kähler potential:
$$\kappa^2\mathcal{K}(\tau,\bar{\tau},S,\bar{S}) = K(S,\bar{S}) - 3\ln\left[-i(\tau-\bar{\tau})\right],$$
$$K(S,\bar{S}) = -\ln(S+\bar{S}) + \delta k(S,\bar{S})$$

$j(\tau)$: j-invariant function

Superpotential:
$$\mathcal{W}(S,\tau) = \Lambda_W^3\frac{\Omega(S)H(\tau)}{\eta^6(\tau)}, \quad H(\tau) = (j(\tau)-1728)^{m/2}j(\tau)^{n/3}\mathcal{P}(j(\tau))$$

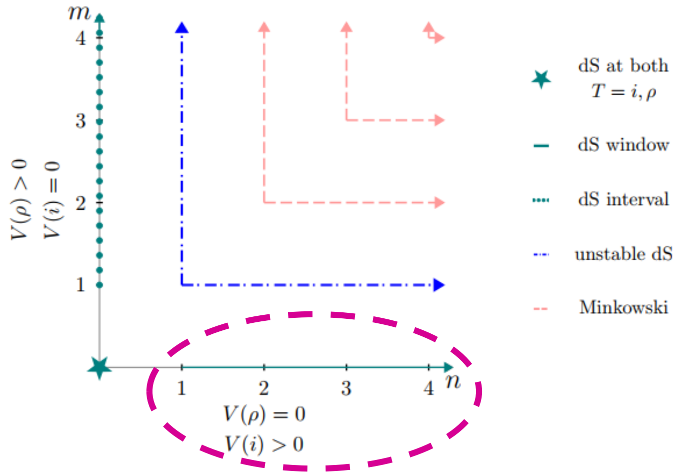
$m, n = 0,1,2,\ldots$

$\Omega(S)$ is technically arbitrary, $\mathcal{P}(j)$ is an arbitrary polynomial function of $j(\tau)$

$$V(\tau,S) = \Lambda^4 e^{K(S,\bar{S})}Z(\tau,\bar{\tau})|\Omega(S)|^2\left[(A(S,\bar{S})-3)|H(\tau)|^2 + \hat{V}(\tau,\bar{\tau})\right]$$

- Vacuum structure at fixed points $\tau = i,\ e^{2\pi i/3} \equiv \rho$

see talk by Nicole Righi



[Leedom, Righi,Westphal,2212.03876]

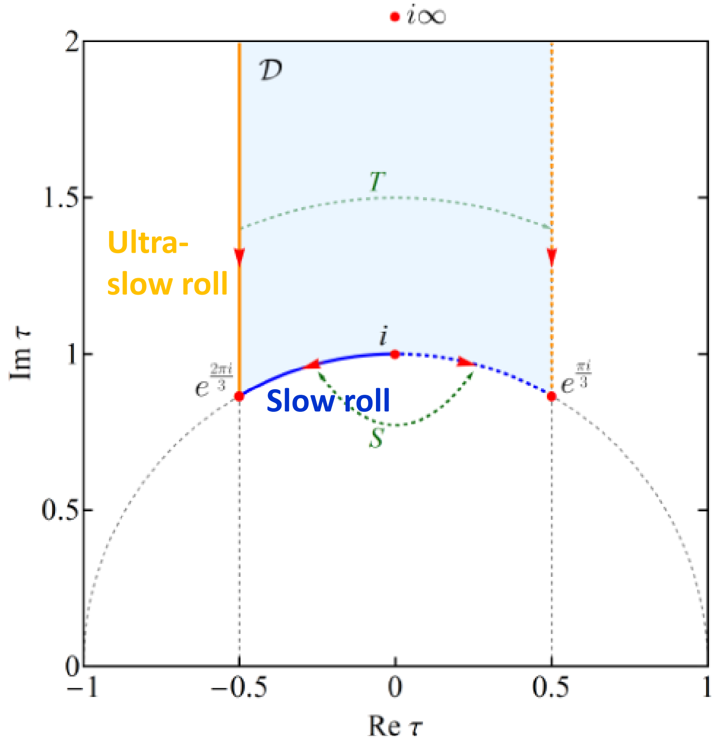
τ rolls toward the fixed points

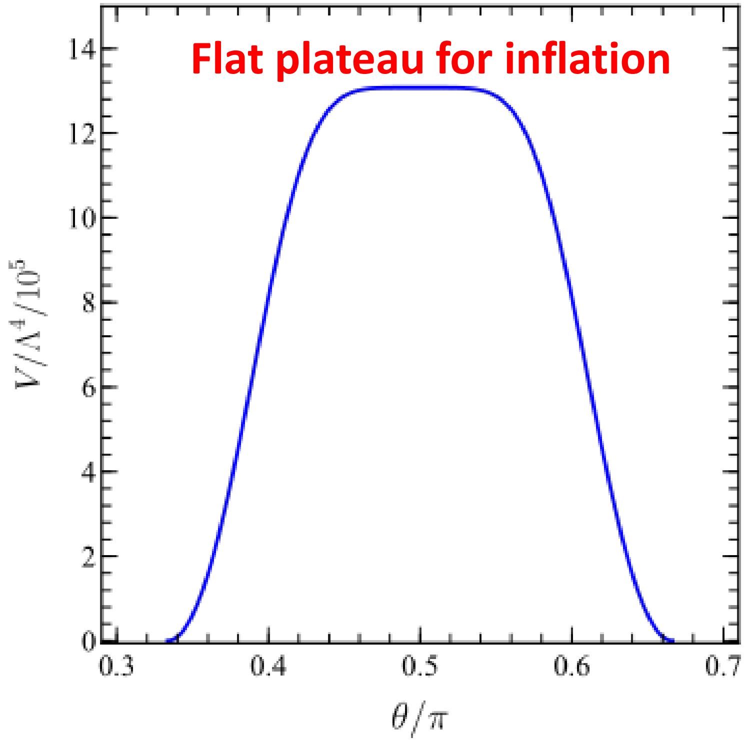
➢ Slow roll along the unit arc $\tau = e^{i\theta}$ from $\tau = i$ to $\tau = e^{2\pi i/3}$

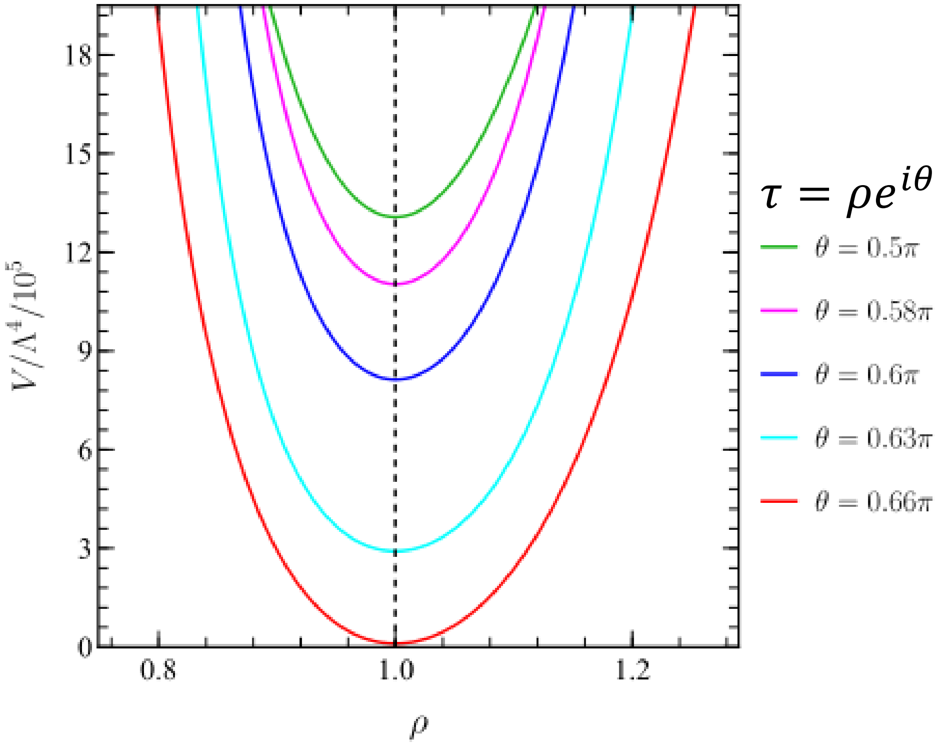
Hilltop-like inflation: $V(\phi) = V_0\left[1 - C_2\phi^2 - C_4\phi^4 - C_6\phi^6 + \ldots\right]$

Invariant under: $\phi \xrightarrow{S} -\phi$ $\quad \phi = \sqrt{3/2}M_{\rm Pl}\ln(\tan(\theta/2))$

- For the power exponents $m = 0,\ n \geq 2,\ V(i) > 0, V\left(e^{2\pi i/3}\right) = 0$, the scalar potential reaches minimum at the unit arc along the radial direction, can only roll along angular direction



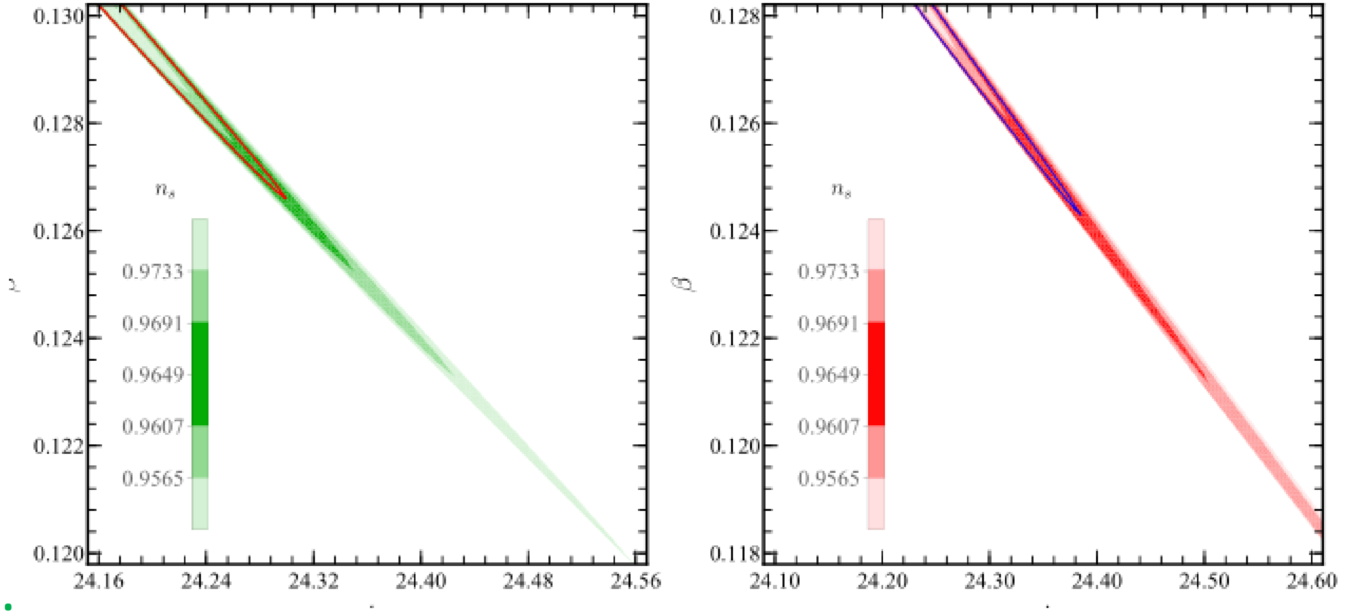




[Ding, Jiang, Zhao, 2405.06497; King, Wang, 2405.xxxxx]

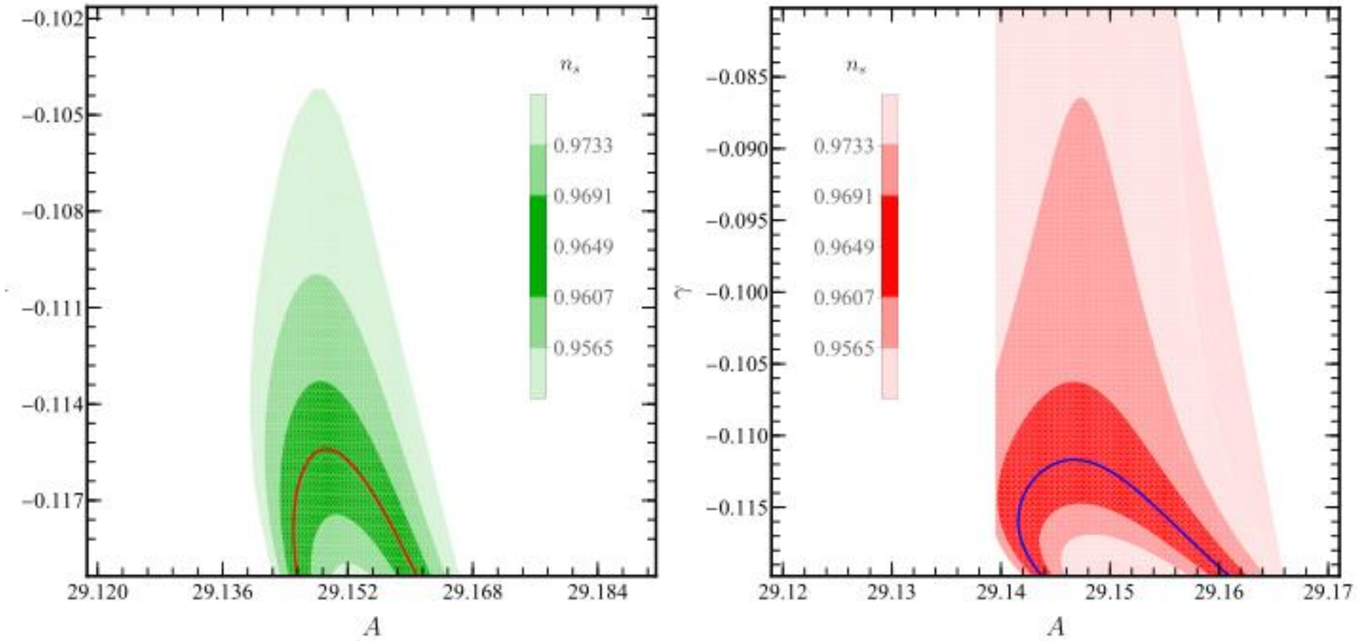
- Successful inflation can be reproduced, the tensor-to-scalar ratio $r < 10^{-6}$

(a) $\mathcal{P}(j) = 1 + \beta(1 - j/1728)$

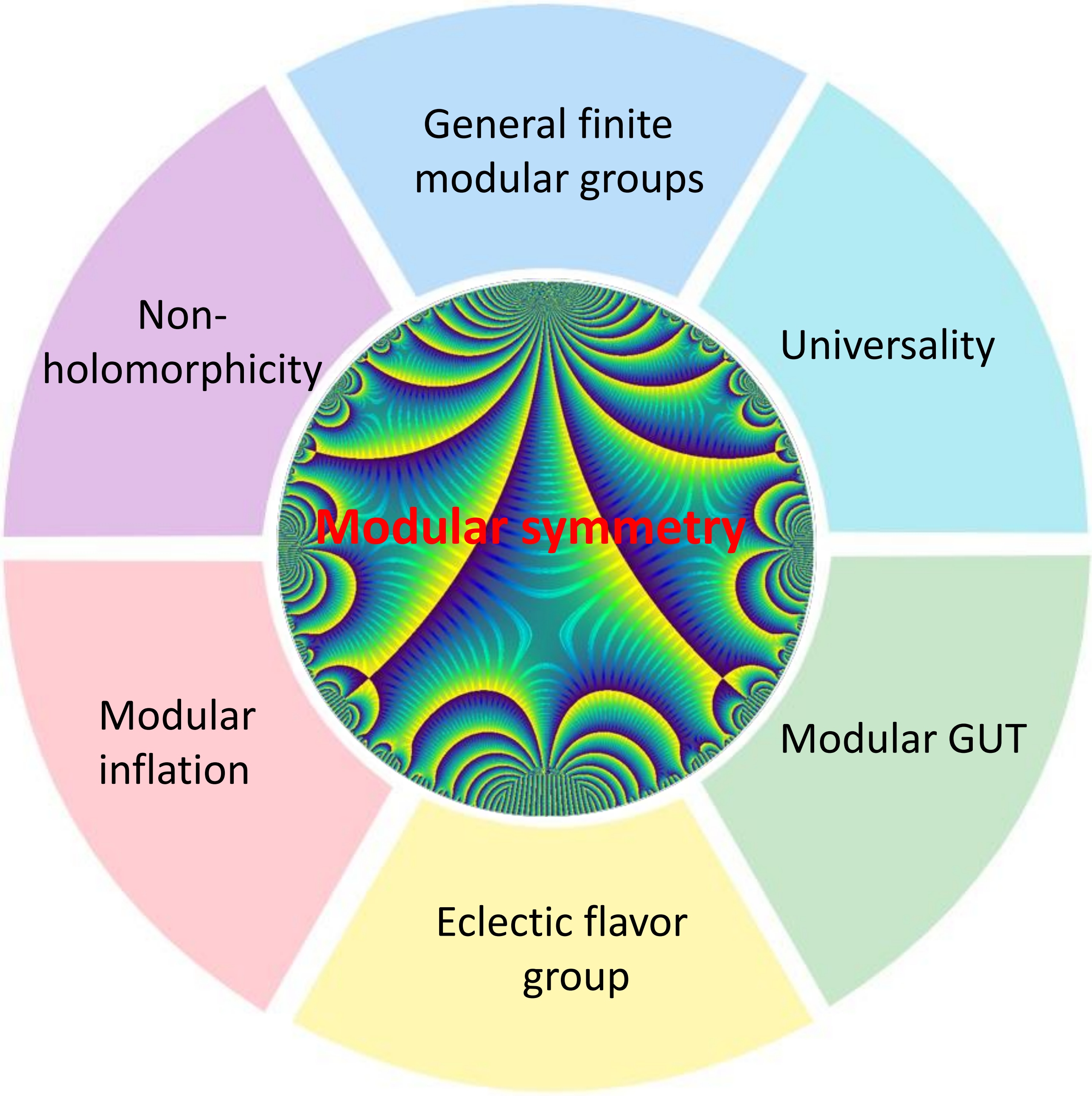


[Ding, Jiang, Zhao, 2405.06497; King, Wang, 2405.xxxxx]

(b) $\mathcal{P}(j) = 1 + \gamma\,(1 - j/1728)^2$



[other scenarios of modular inflation, see Gunji, Ishiwata, Yoshida, 2208.10086; Abe, Higaki, Kaneko, Kobayashi, Otsuka, 2303.02947...]


Modular symmetry
General finite modular groups
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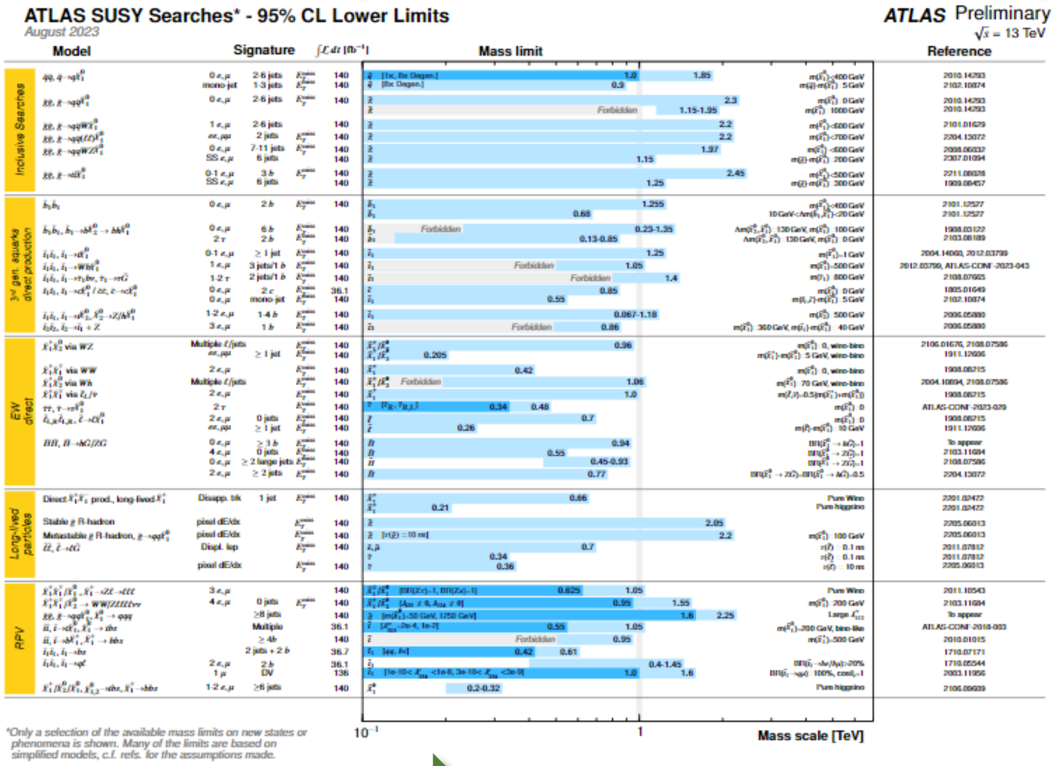
Modular inflation
Non-holomorphicity

Non-holomorphic modular flavor symmetry

- Modular symmetry requires SUSY to protect holomorphicity of modular form

SUSY → holomorphicity → $Y(\tau)$: Yukawa couplings=modular form of level N and weight k_Y



A finite number of independent modular forms at weight k_Y

[PDG,2023]

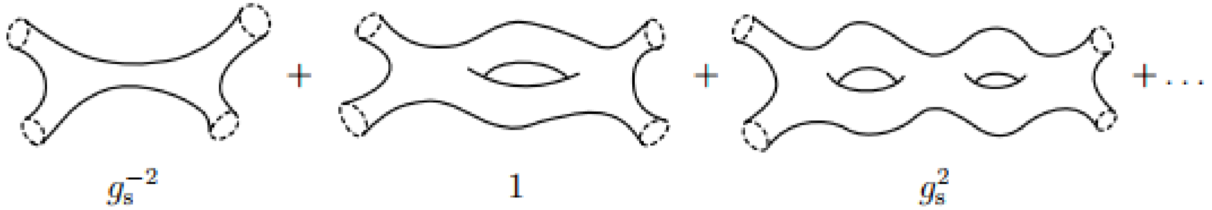
low energy SUSY is not observed so far!

Non-SUSY → ??? → $Y(\tau,\bar{\tau})$: non-holomorphic Yukawa couplings

A finite choice of $Y(\tau,\bar{\tau})$ to have prediction power

Harmonic (Laplacian) condition replacing holomorphicity

- From top-down, the effective interactions in the low energy expansion of the four-graviton amplitude are **non-holomorphic automorphic** functions satisfying Laplace eigenvalue equations.



[Green, Gutperle, hep-th/9701093; Green, Russo, Vanhove,1001.2535; Hoker,Kaidi,2208.07242]

- The bottom-up approach based on automorphic forms **[Ding, Feruglio, Liu, 2010.07952]**
- Automorphic forms coincide with the harmonic Maass forms for single modulus τ

- Harmonic Maass forms of level N and weight k

① **modularity:** $Y(\gamma\tau) = (c\tau + d)^k Y(\tau), \quad \gamma \in \Gamma(N)$

② **harmonic condition:** $\Delta_k Y(\tau) = 0, \quad \Delta_k \equiv -4y^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} + 2iky\frac{\partial}{\partial\bar{\tau}}$

③ **growth condition:** $Y(\tau) = \mathcal{O}(y^\alpha), \quad y \to +\infty$

- Fourier expansion:

$$Y(\tau) = \underbrace{\sum_{n\in\frac{1}{N}\mathbb{Z}^+} c_n^+ q^n}_{\text{holomorphic}} + \underbrace{c_0^- y^{1-k} + \sum_{n\in\frac{1}{N}\mathbb{Z}^-} c_n^- \Gamma(1-k, -4\pi n y) q^n}_{\text{non-holomorphic}}$$

[Qu, Ding, in progress]

- Transformation under modular group: $Y_i(\gamma\tau) = (c\tau + d)^k \rho_{ij}(\gamma)\, Y_j(\tau), \quad \gamma \in SL(2,\mathbb{Z})$

$\rho(\gamma)$ is unitary representation of $\Gamma'_N = \Gamma/\Gamma(N)$ for odd k and $\Gamma_N = \Gamma/\pm\Gamma(N)$ for even k

Harmonic Maass forms of level 3

- Weight k harmonic Maass forms $Y_r^{(k)}(\tau)$ related with modular forms $Y_r^{(2-k)}(\tau)$ of weight $2-k$

$$2iy^k \frac{\partial}{\partial\tau}\overline{Y_{\boldsymbol{r}}^{(k)}(\tau)} \propto Y_{\boldsymbol{r}}^{(2-k)}(\tau), \qquad \left(\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\right)^{1-k} Y_{\boldsymbol{r}}^{(k)}(\tau) \propto Y_{\boldsymbol{r}}^{(2-k)}(\tau)$$

- weight 2 modular forms transforming as an A_4 triplet 3 [Feruglio, 1706.08749; Liu, Ding, 1907.01488]

$$Y_{\mathbf{3}}^{(2)}(\tau) \equiv \begin{pmatrix} \varepsilon^2(\tau) \\ \sqrt{2}\,\vartheta(\tau)\varepsilon(\tau) \\ -\vartheta^2(\tau) \end{pmatrix} = \begin{pmatrix} 1+12q+36q^2+\dots \\ -6q^{1/3}\left(1+7q+8q^2+\dots\right) \\ -18q^{2/3}\left(1+2q+5q^2+\dots\right) \end{pmatrix}$$

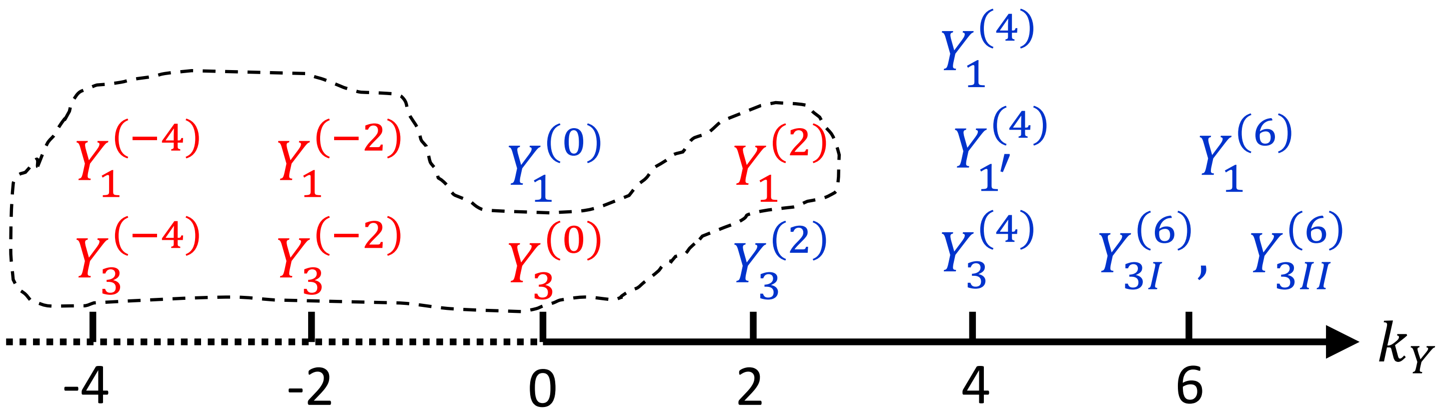
$$\vartheta(\tau) = 3\sqrt{2}\,\frac{\eta^3(3\tau)}{\eta(\tau)}\,, \qquad \varepsilon(\tau) = -\frac{3\eta^3(3\tau)+\eta^3(\tau/3)}{\eta(\tau)}$$

Lifted to weight zero harmonic Maass forms

$$Y_{\mathbf{3}}^{(0)}(\tau) = \begin{pmatrix} -\left(0.7868+\frac{3q}{\pi}+\frac{9q^2}{2\pi}+\dots\right)+\left(y-\frac{3\,e^{-4\pi y}}{\pi q}-\frac{9\,e^{-8\pi y}}{2\pi q^2}+\cdots\right) \\ \frac{9q^{1/3}}{2\pi}\left(1+\frac{7q}{4}+\frac{8q^2}{7}+\cdots\right)+\frac{27q^{1/3}e^{\pi y/3}}{\pi}\left(\frac{e^{-3\pi y}}{4q}+\frac{e^{-7\pi y}}{5q^2}+\cdots\right) \\ \frac{27q^{2/3}}{\pi}\left(\frac{1}{4}+\frac{q}{5}+\frac{5q^2}{16}+\cdots\right)+\frac{9\,q^{2/3}e^{2\pi y/3}}{2\pi}\left(\frac{e^{-2\pi y}}{q}+\frac{7\,e^{-6\pi y}}{4q^2}+\cdots\right) \end{pmatrix}$$

[Qu, Ding, in progress]

- Harmonic Maass forms coincide with modular forms for weight $k > 3$, negative weight Maass forms and phenomenological implications in mass hierarchies and strong CP problem?



- **General comments and questions**
  - Origin of harmonic condition and the possible symmetry related?
  - Possible top-down connection?
  - Are harmonic Maass forms applicable to flavor puzzle?
  - ……

An example model based on A_4 modular symmetry

- field content and assignment

[Qu, Ding, in progress]

	L	e^c	μ^c	τ^c	H
$SU(2)_L \times U(1)_Y$	$(\mathbf{2}, -1/2)$	$(\mathbf{1}, 1)$			$(\mathbf{2}, 1/2)$
A_4	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}''$	$\mathbf{1}'$	$\mathbf{1}$
k_I	-2	0	2	2	0

- Neutrino masses arise from Weinberg operator (two-component formalism for fermions)

$$
\begin{aligned}
-\mathcal{L}_e &= \alpha(H^\dagger e^c L Y_{\mathbf{3}}^{(-2)})_{\mathbf{1}} + \beta(H^\dagger \mu^c L Y_{\mathbf{3}}^{(0)})_{\mathbf{1}} + \gamma(H^\dagger \tau^c L Y_{\mathbf{3}}^{(0)})_{\mathbf{1}} + \text{h.c.}\,, \\
-\mathcal{L}_\nu &= \frac{1}{2\Lambda}(LLH^2 Y_{\mathbf{3}}^{(-4)})_{\mathbf{1}} + \frac{g}{2\Lambda}(LLH^2 Y_{\mathbf{1}}^{(-4)})_{\mathbf{1}} + \text{h.c.}\,,
\end{aligned}
$$

Lepton mass matrices

$$
M_e = \begin{pmatrix} \alpha Y_{\mathbf{3},1}^{(-2)} & \alpha Y_{\mathbf{3},3}^{(-2)} & \alpha Y_{\mathbf{3},2}^{(-2)} \\ \beta Y_{\mathbf{3},2}^{(0)} & \beta Y_{\mathbf{3},1}^{(0)} & \beta Y_{\mathbf{3},3}^{(0)} \\ \gamma Y_{\mathbf{3},3}^{(0)} & \gamma Y_{\mathbf{3},2}^{(0)} & \gamma Y_{\mathbf{3},1}^{(0)} \end{pmatrix} v\,, \qquad
M_\nu = \begin{pmatrix} 2Y_{\mathbf{3},1}^{(-4)} + gY_{\mathbf{1}}^{(-4)} & -Y_{\mathbf{3},3}^{(-4)} & -Y_{\mathbf{3},2}^{(-4)} \\ -Y_{\mathbf{3},3}^{(-4)} & 2Y_{\mathbf{3},2}^{(-4)} & -Y_{\mathbf{3},1}^{(-4)} + gY_{\mathbf{1}}^{(-4)} \\ -Y_{\mathbf{3},2}^{(-4)} & -Y_{\mathbf{3},1}^{(-4)} + gY_{\mathbf{1}}^{(-4)} & 2Y_{\mathbf{3},3}^{(-4)} \end{pmatrix} \frac{v^2}{\Lambda}
$$

Adjusting α, β, γ for electron, muon and tau masses

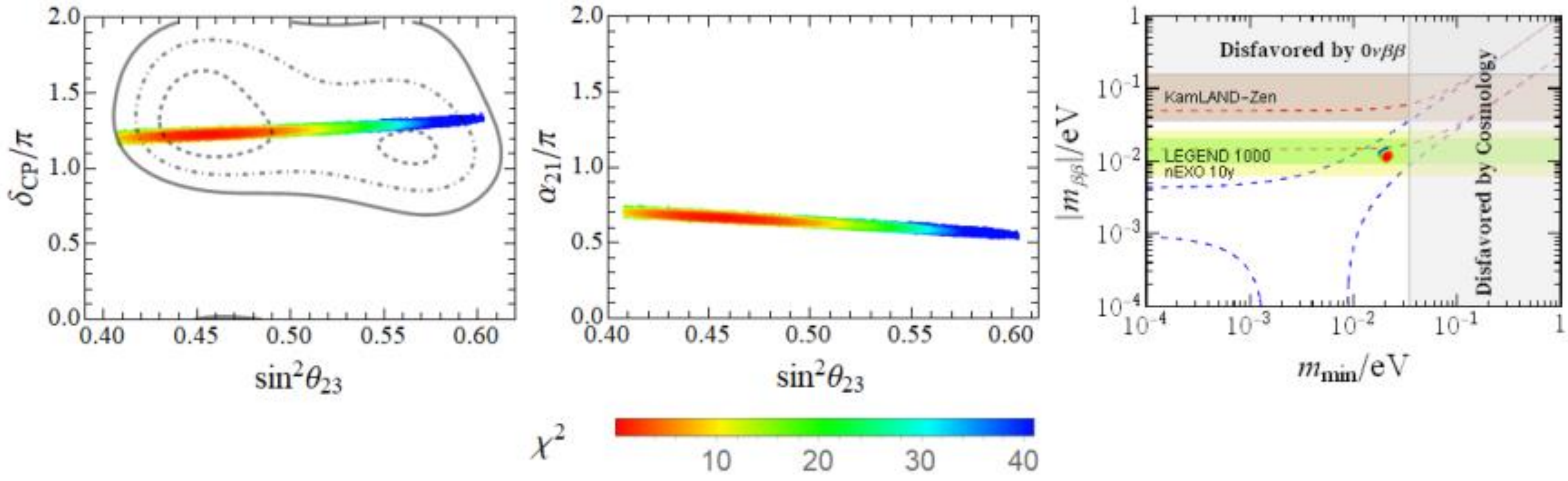
Neutrino masses and mixing described by three parameters: $g, \frac{v^2}{\Lambda}, \tau$

➢ Best agreement with data is achieved at

$$\beta/\alpha = 125.743\,,\ \gamma/\alpha = 1574.04\,,\ \alpha v = 3.62033\,\mathrm{MeV}\,,$$
$$g = 0.753046 + 0.385855i\,,\ \langle\tau\rangle = 0.369080 + 0.961090i\,,\ v^2/\Lambda = 97.3898\,\mathrm{meV}$$

Predictions for lepton masses and mixing parameters

$$m_e/m_\mu = 0.004737\,,\quad m_\mu/m_\tau = 0.05868\,,\quad m_e = 0.46965\mathrm{MeV}\,,$$
$$m_1 = 21.0918\,\mathrm{meV}\,,\quad m_2 = 22.7808\,\mathrm{meV}\,,\quad m_3 = 54.3128\,\mathrm{meV}\,,$$
$$\sin^2\theta_{12} = 0.3070\,,\quad \sin^2\theta_{13} = 0.02224\,,\quad \sin^2\theta_{23} = 0.4541\,,$$
$$\delta_{CP} = 1.2252\pi\,,\quad \alpha_{21} = 0.6667\pi\,,\quad \alpha_{31} = 1.3855\pi\,,\quad \chi^2_{\min} \approx 0$$



Summary

Modular flavor symmetry is a very interesting approach to address flavor puzzle of SM, closely related to top-down. It was proposed for flavor, but not only for flavor. There are still many aspects needing clarification and development toward theory of flavor.

LET'S
DO
IT
TOGETHER!