Loop integrals and Baikov representations

Li Lin Yang
Zhejiang University

The first year of MITP

The first three years of the LHC, MITP 2013

Pre-MITP

Pre-MITP

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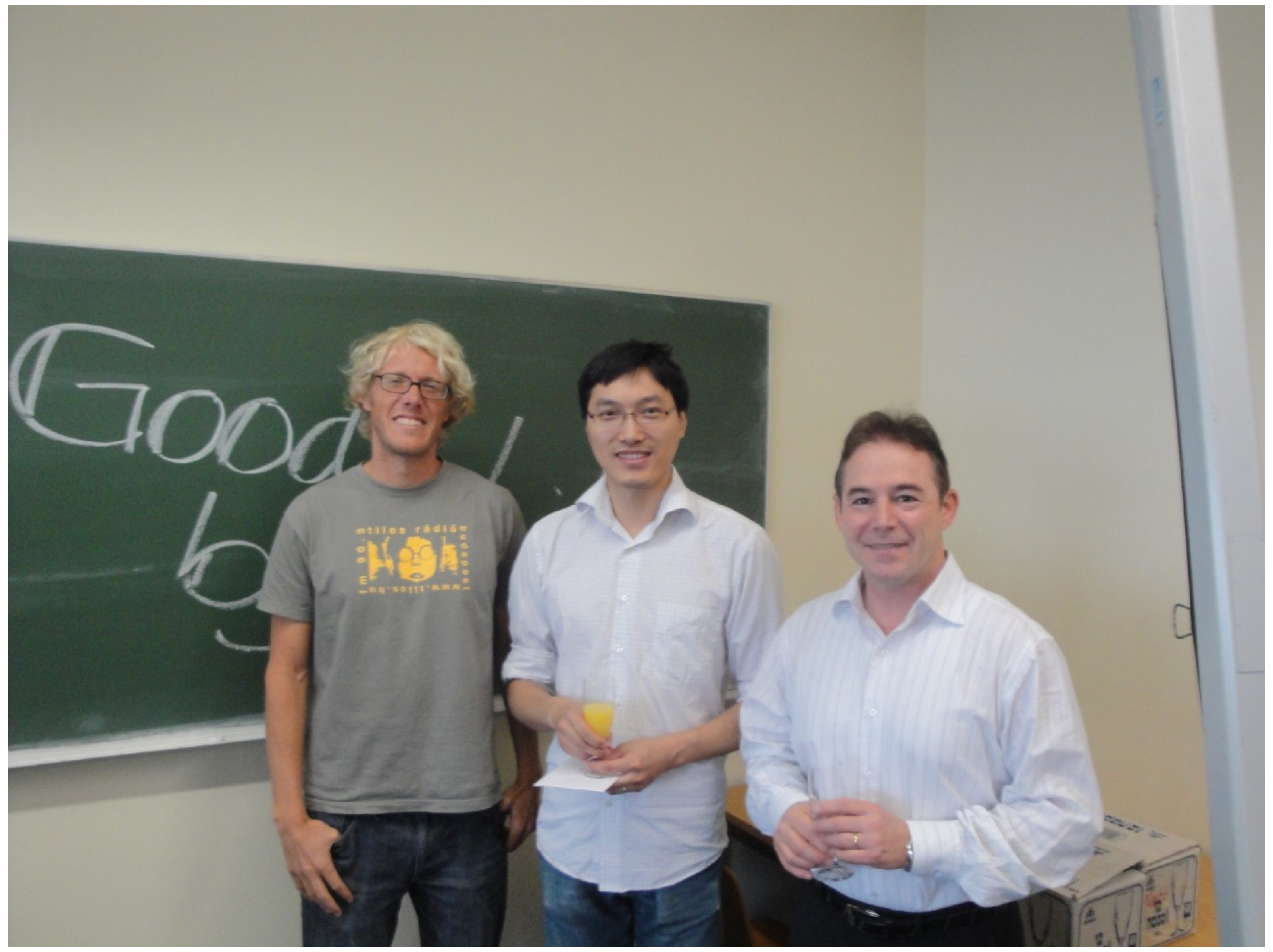

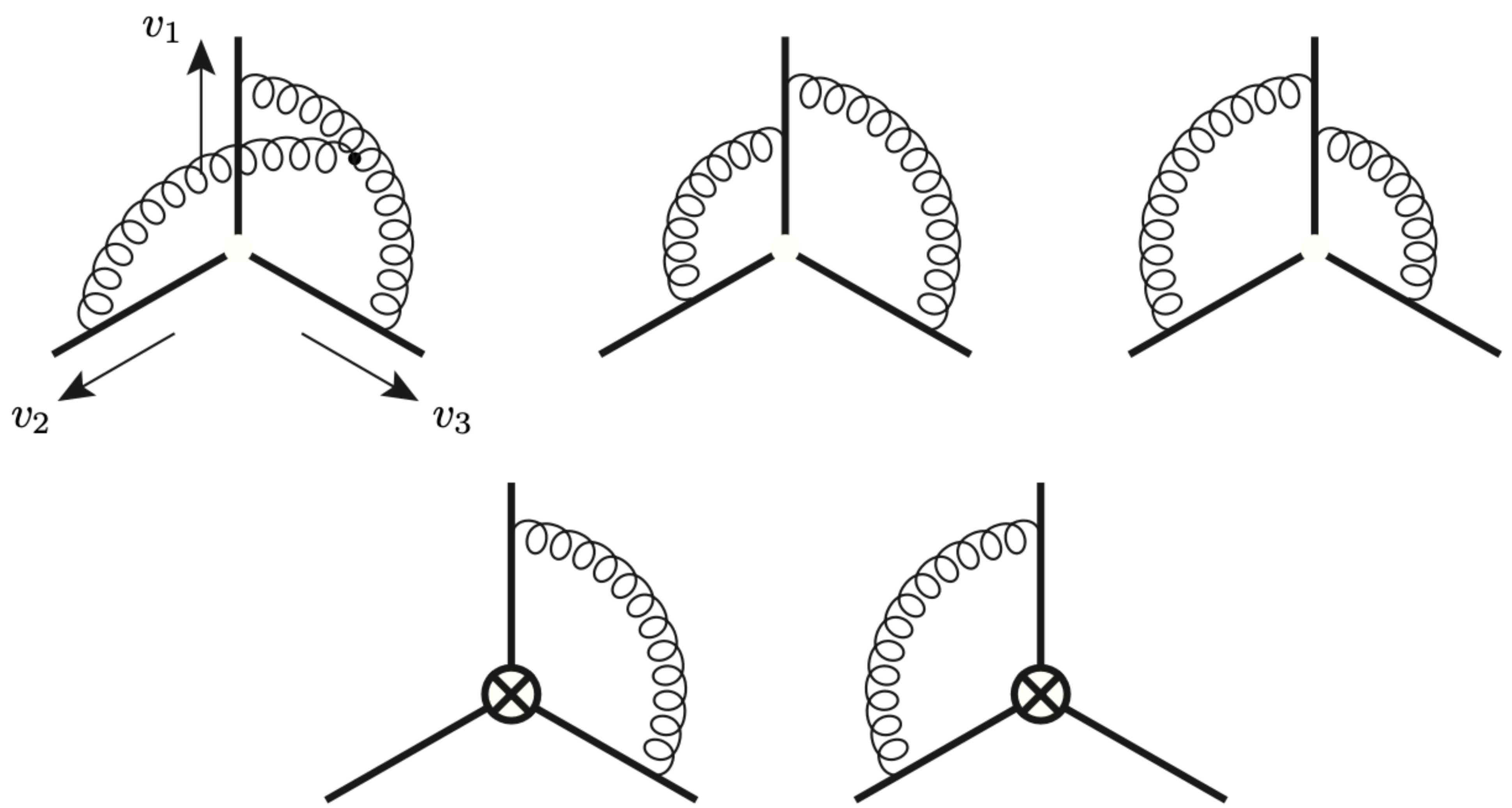

Ferroglia, Neubert, Pecjak, LLY (2009)

$$I(w_{12}, w_{23}, w_{31}) = 2(w_{23}\, w_{31} + w_{12})\, \frac{1}{(2\pi i)^5} \int_{-i\infty}^{+i\infty} \left[\prod_{i=1}^{5} dz_i \right] (2w_{23})^{2z_1 - 1} (2w_{31})^{2z_2 - 1} (2w_{12})^{2z_3}$$
$$\times \frac{\Gamma(1 - 2z_1)\, \Gamma(1 - 2z_2)}{\Gamma(z_1 + z_2 + z_3 + z_4 + z_5)}\, \Gamma(-2z_3)\, \Gamma(-z_4)\, \Gamma(z_1 + z_3)\, \Gamma(z_1 + z_5)\, \Gamma(z_2 - z_5)\, \Gamma(z_3 + z_5)$$
$$\times \Gamma(z_1 + z_2 + z_4)\, \Gamma(z_2 + z_3 + z_4)\, \Gamma(z_2 + z_4 + z_5)\, \Gamma(1 - z_2 - z_4 - z_5)\,. \tag{27}$$

Modern techniques for loop integrals

→ **Talk by Stefan Weinzierl**

IBP reduction → Differential Equations

Canonical basis → Canonical DEs → Solutions

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Try to understand (and possibly simplify) the procedure
using Baikov representations + intersection theory

The Baikov representations

Baikov (1996)

Change of variables from loop momenta to propagator denominators

$$\int \left[\prod_{i=1}^{L} \frac{d^d k_i}{i\pi^{d/2}}\right] \frac{1}{z_1^{a_1} z_2^{a_2} \cdots z_N^{a_N}}$$

$$z_m = \sum_{i,j} A_m^{ij} q_i \cdot q_j + f_m$$

$$\int_{\mathscr{C}} u_{\text{Std}}(z) \frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_N}{z_1^{a_1} \cdots z_N^{a_N}}$$ Standard Baikov rep.

$$\int_{\mathscr{C}} u_{\text{LBL}}(z) \frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{z_1^{a_1} \cdots z_n^{a_n}}$$ Loop-by-loop (LBL) Baikov rep.

Geometric formulation of IBP equivalence

Frellesvig et al. (2019)

Baikov integrals are special cases of generalized hypergeometric integrals

$$I = \int_{\mathscr{C}} u(z)\, \varphi(z) \longrightarrow n\text{-form}$$

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n-form

$$0 = \int_{\mathscr{C}} d(u(z)\xi(z)) = \int_{\mathscr{C}} u(z)\, \nabla_{\omega}\xi(z)$$

$(n-1)$-form

$$\nabla_{\omega} \equiv d + \omega\, \wedge$$

covariant derivative

$$\omega \equiv d \log u$$

connection

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$\nabla_\omega \equiv d + \omega\, \wedge$ covariant derivative

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$\varphi(z)$ and $\varphi(z) + \nabla_\omega \xi(z)$ are equivalent (in the sense of integration)

The equivalence classes form a vector space H^n_ω (the n-th twisted cohomology group)

$$\langle \varphi | \, : \, \varphi \sim \varphi + \nabla_\omega \xi$$

IBP reduction = vector decomposition

$\dim(H^n_\omega) = \nu = \#$ of master integrals with a given ω

A basis with ν vectors $\quad \{\langle e_1|, \langle e_2|, \ldots, \langle e_\nu|\}$

All vectors are linear combinations $\quad \langle \varphi| = \sum_{i=1}^{\nu} c_i \langle e_i|$

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$\#$ of proper critical points where $\omega_i \equiv \partial_i \log u = 0$ Lee, Pomeransky (2013)

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However, this dimension (when applied to the LBL Baikov representation) is often found to be bigger than the number of master Feynman integrals!

Generalized loop-by-loop Baikov representation

It turns out that the number of critical points actually counts the number of independent Baikov integrals of the form

$$\int_{\mathscr{C}} u_{\mathrm{LBL}}(z) \frac{\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}}$$

P_i are polynomial factors in u_{LBL}

We call these "generalized LBL Baikov integrals"

Chen, Jiang, Ma, Xu, LLY (2022)

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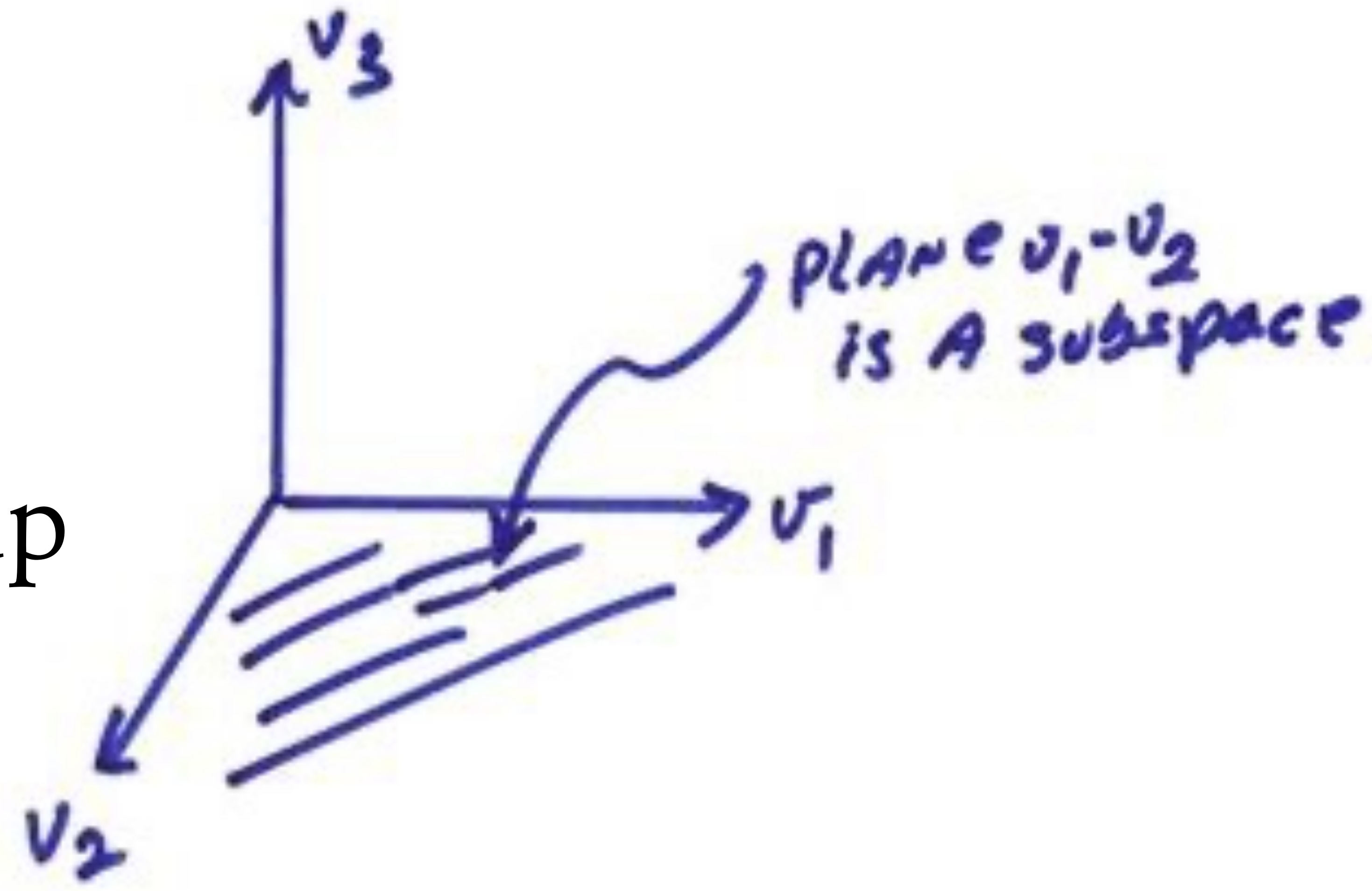
Not all of them correspond to Feynman integrals!

Feynman integrals live in a subspace of the cohomology group

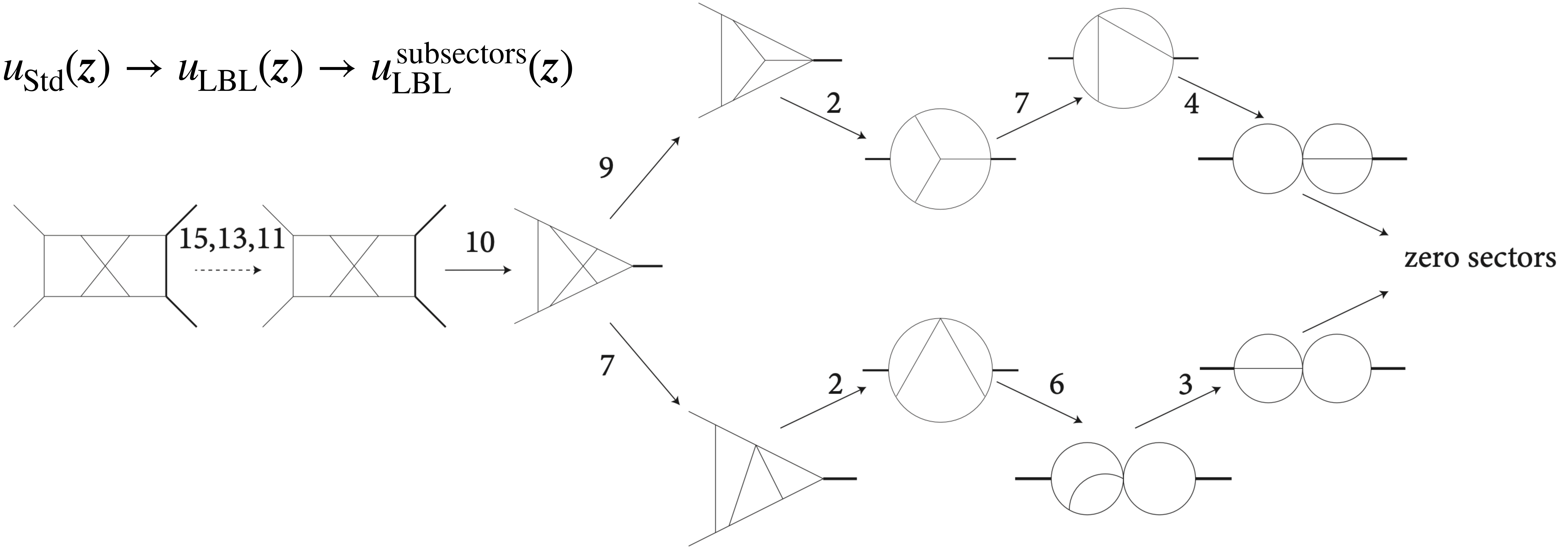


Recursive structure of generalized Baikov representations

Jiang, LLY (2023)

Baikov representations exhibit a recursive structure

$$u_{\mathrm{Std}}(z) \to u_{\mathrm{LBL}}(z) \to u_{\mathrm{LBL}}^{\mathrm{subsectors}}(z)$$



Helps to relate different sub-sectors within an integral family

Intersection numbers

To perform the vector decomposition, one introduces a dual space with elements

$$|\varphi_R\rangle \, : \, \varphi_R \sim \varphi_R + \nabla_{-\omega}\xi_R$$

The intersection numbers are "inner-products" between vectors and dual-vectors

$$\langle\varphi_L|\varphi_R\rangle_\omega = \frac{1}{(2\pi i)^n}\int \iota_\omega(\varphi_L)\wedge\varphi_R = \frac{1}{(2\pi i)^n}\int \varphi_L\wedge\iota_{-\omega}(\varphi_R)$$

Cho, Matsumoto (1995)
Frellesvig et al. (2019-2020)
Weinzierl (2020)

$$\langle e_i|d_j\rangle = \delta_{ij}$$

Orthonormal basis

$$\langle\varphi| = \sum_{i=1}^{\nu}\langle\varphi|d_i\rangle\langle e_i|$$

IBP reduction

Canonical DEs

I will only consider polylogarithmic integral families in this talk
(i.e., no higher genus geometries)

$$\mathrm{d}\vec{f}(\boldsymbol{x},\epsilon) = \epsilon \left(\sum_i \mathrm{dlog}(\alpha_i(\boldsymbol{x}))\, \boldsymbol{A}_i \right) \vec{f}(\boldsymbol{x},\epsilon)$$

Henn (2013)

How do we find a canonical basis?

How do we construct the coefficient matrix
(symbol letters and rational coefficients)?

Canonical bases from d-log integrands

Chen, Jiang, Xu, LLY (2020)
Chen, Jiang, Ma, Xu, LLY (2022)

The idea is simple: we look for integrands of the form

$$\int_{\mathscr{C}} u(z) \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}} = \int_{\mathscr{C}} \left[G(z)\right]^{\epsilon} \bigwedge_{j=1}^{n} d \log f_j(z)$$

Two simple building blocks

$$\mathrm{d}\log(z-c) = \frac{dz}{z-c}$$

$$\mathrm{d}\log(\tau[z, c; c_{\pm}]) = \frac{\sqrt{(c-c_+)(c-c_-)}dz}{(z-c)\sqrt{(z-c_+)(z-c_-)}}$$
$$\equiv \mathrm{d}\log \frac{\sqrt{c-c_+}\sqrt{z-c_-} + \sqrt{c-c_-}\sqrt{z-c_+}}{\sqrt{c-c_+}\sqrt{z-c_-} - \sqrt{c-c_-}\sqrt{z-c_+}}$$

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Projecting into the subspace of Feynman integrals

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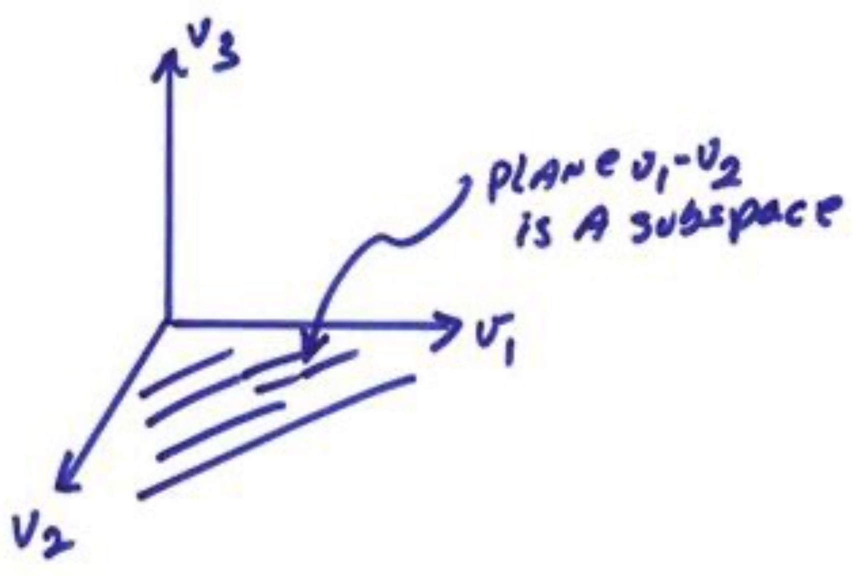

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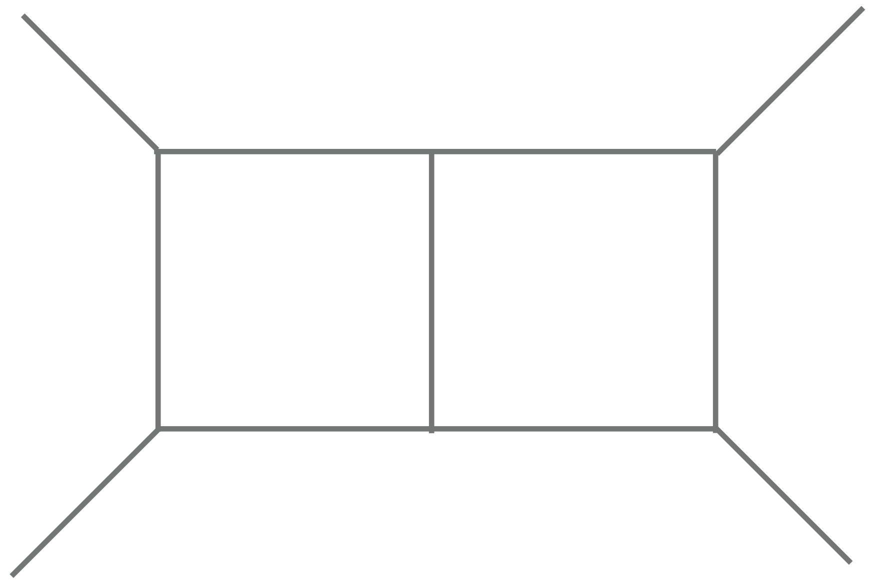
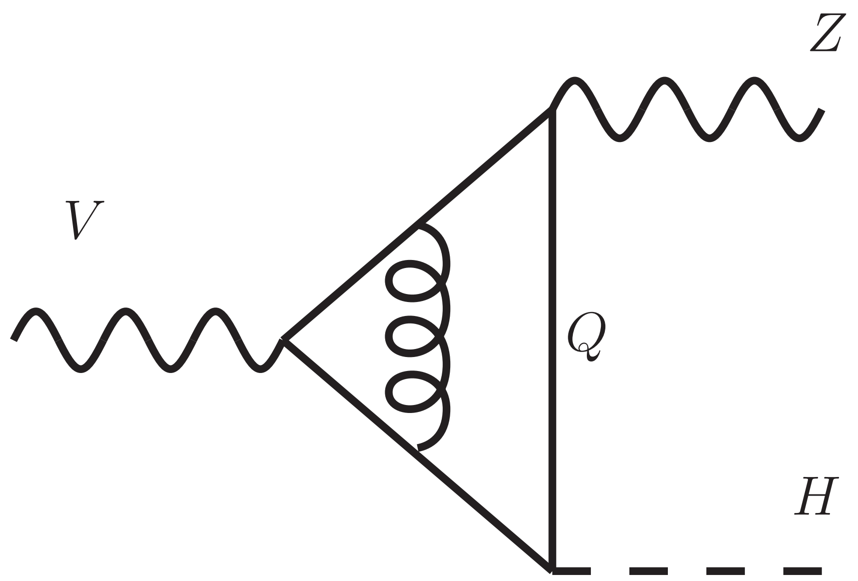
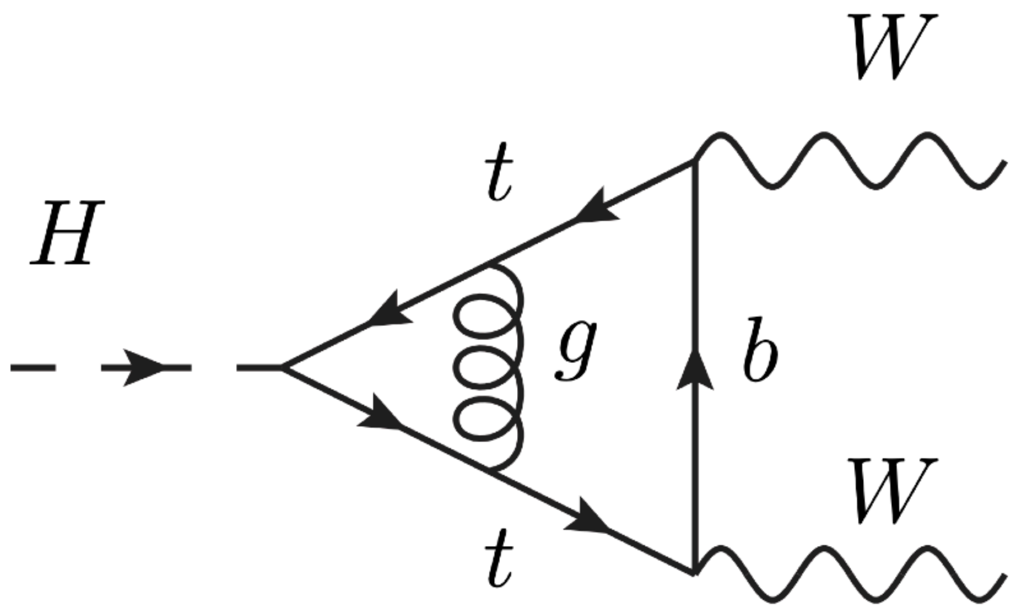
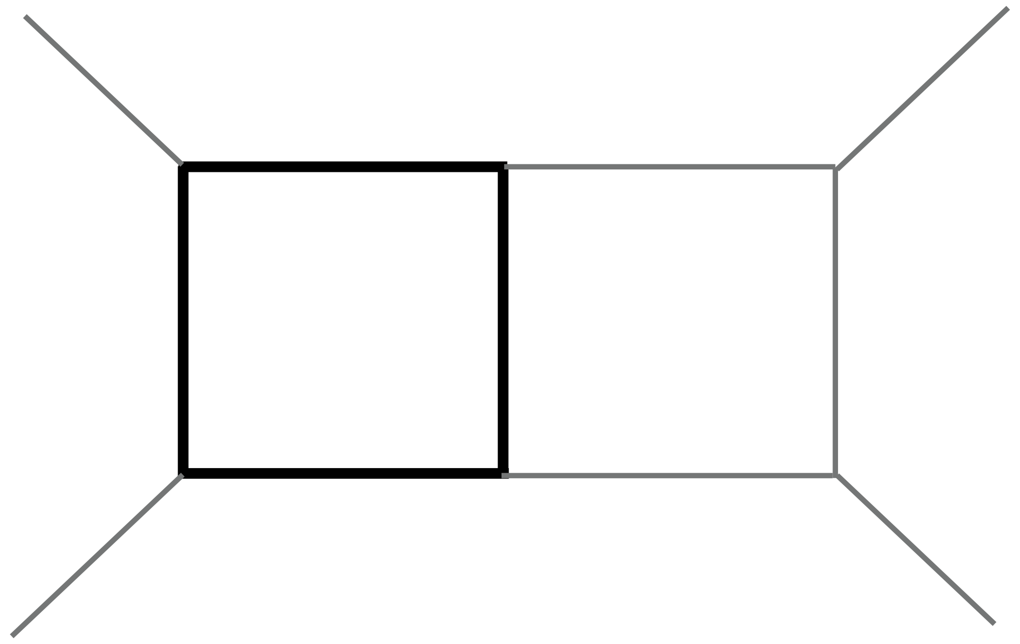
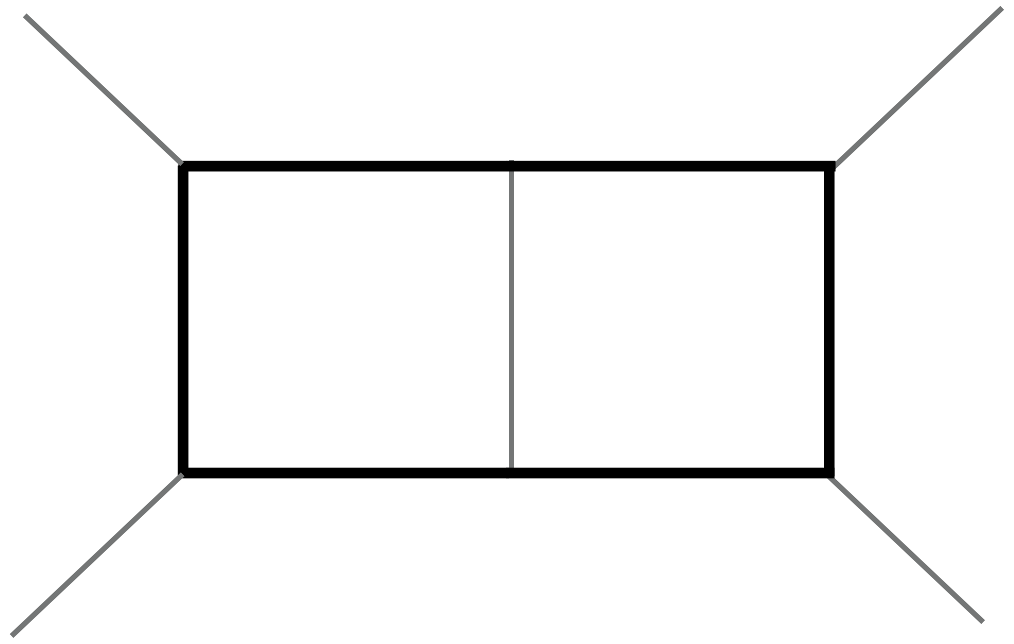
We worked out some simple and not-so-simple examples











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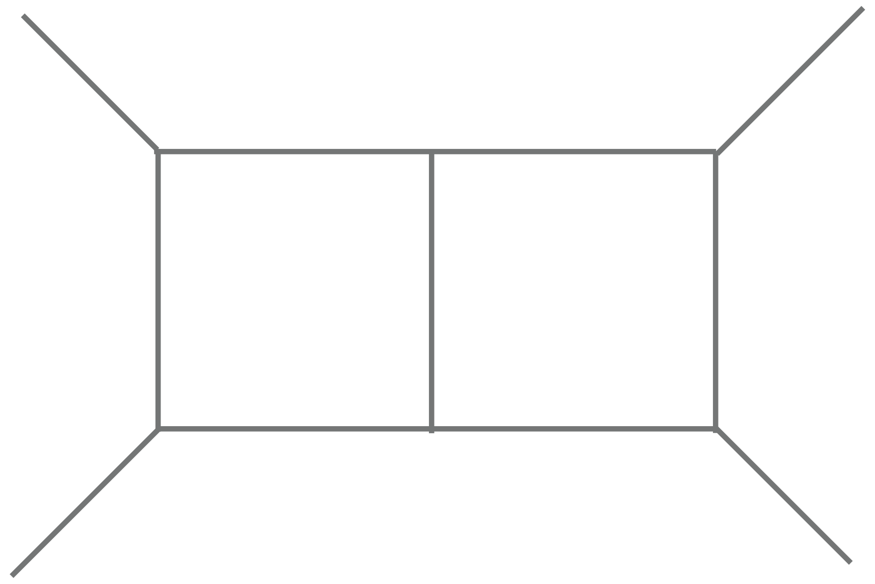

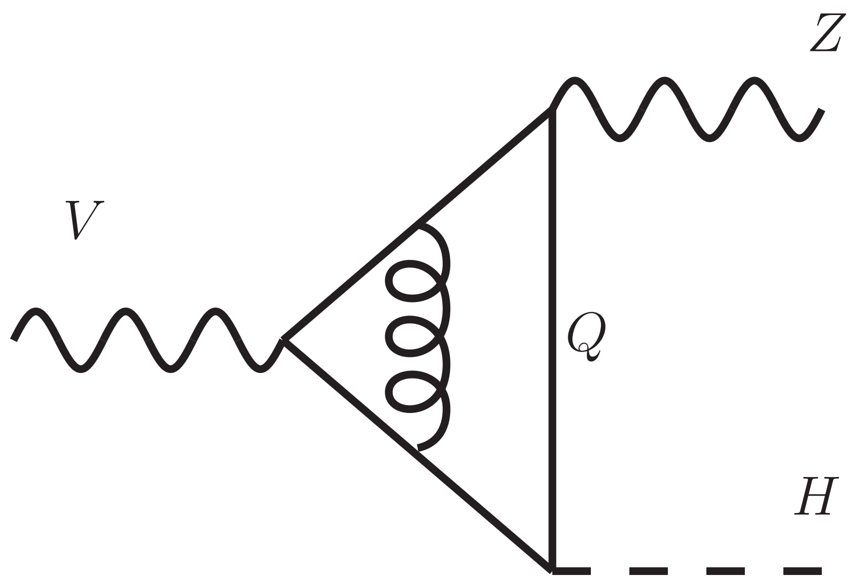

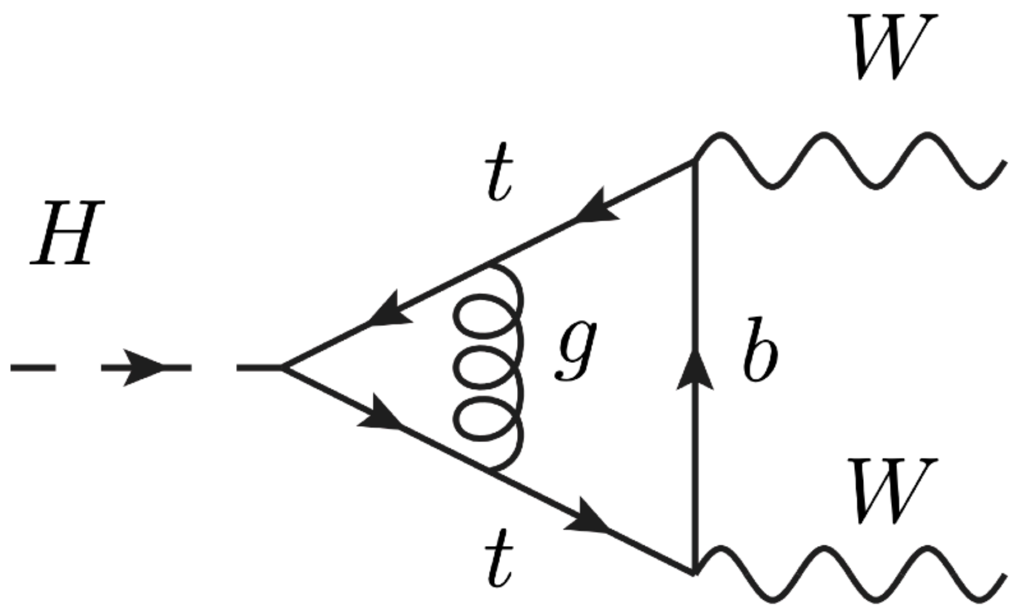

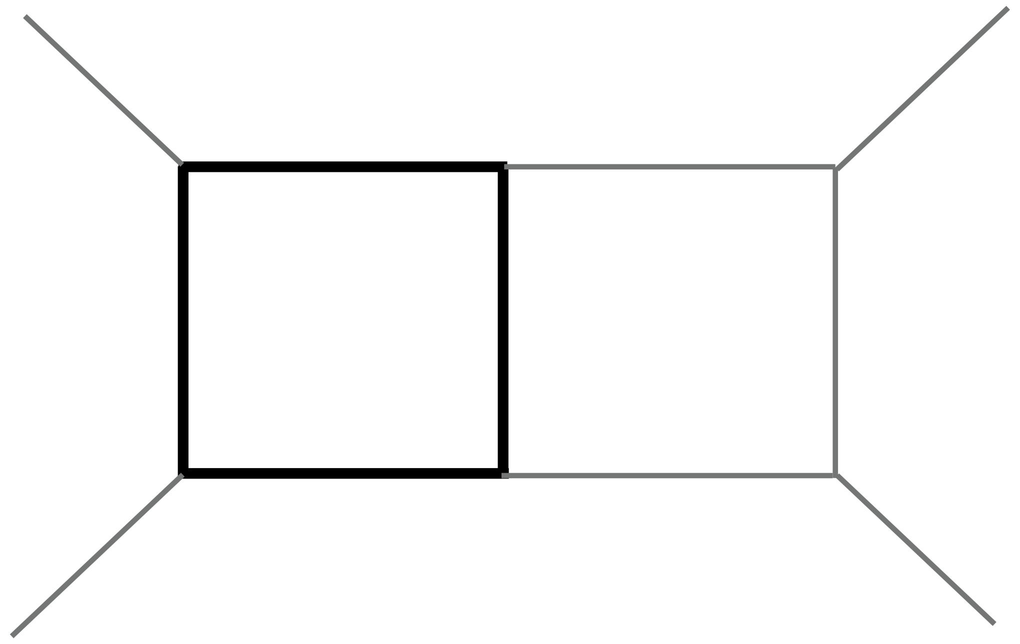

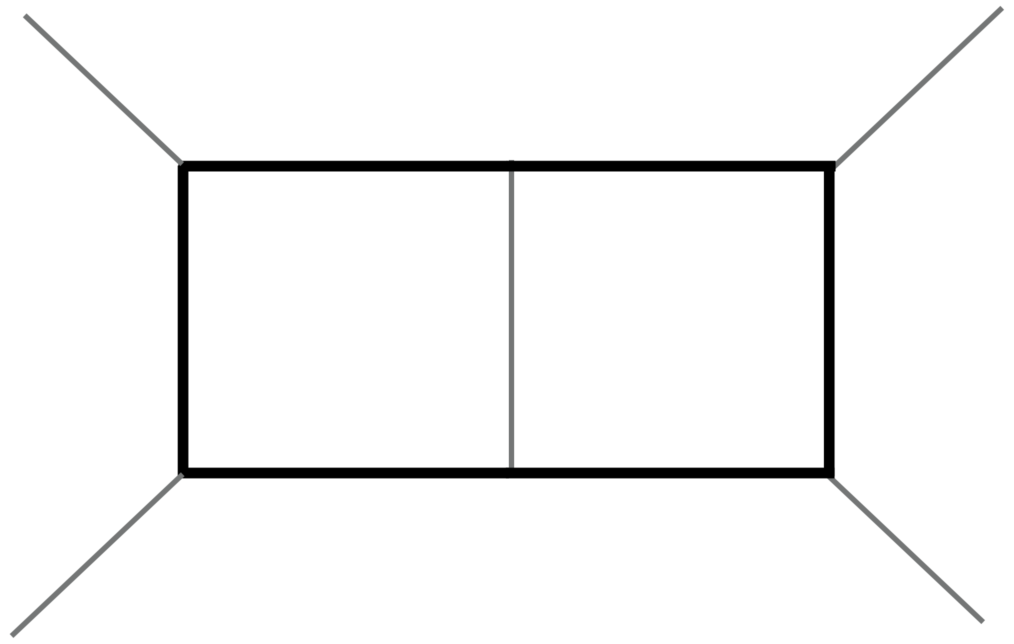

For complicated cases it is necessary to perform non-trivial variable changes first!

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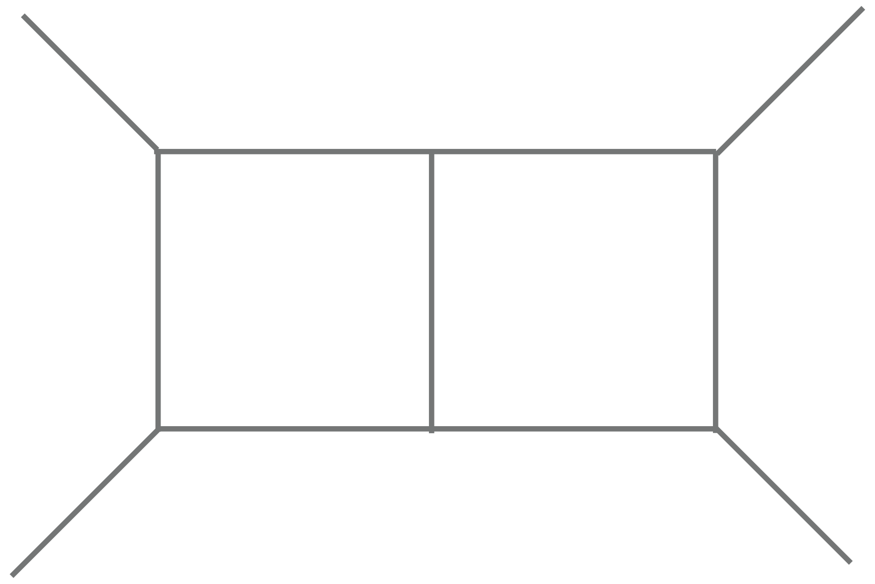

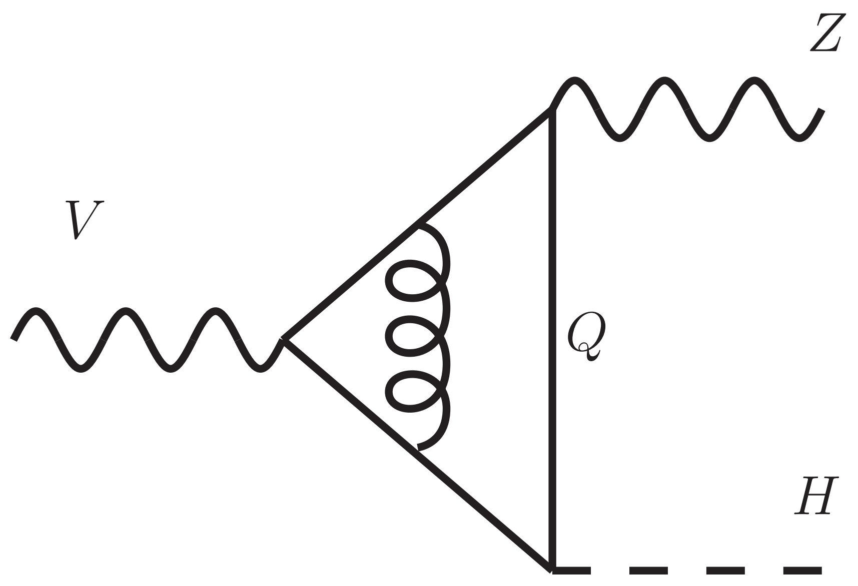

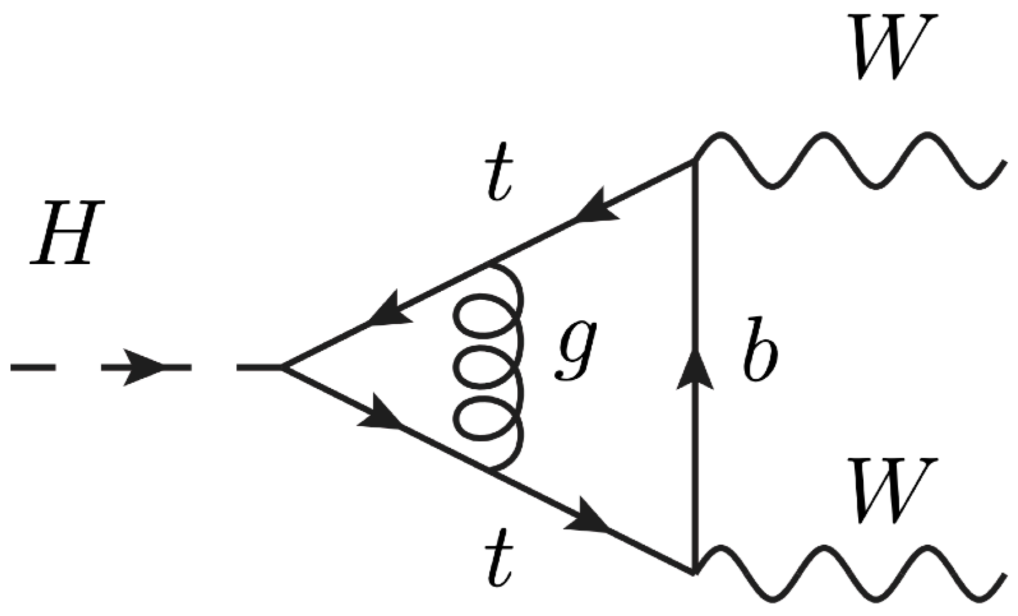

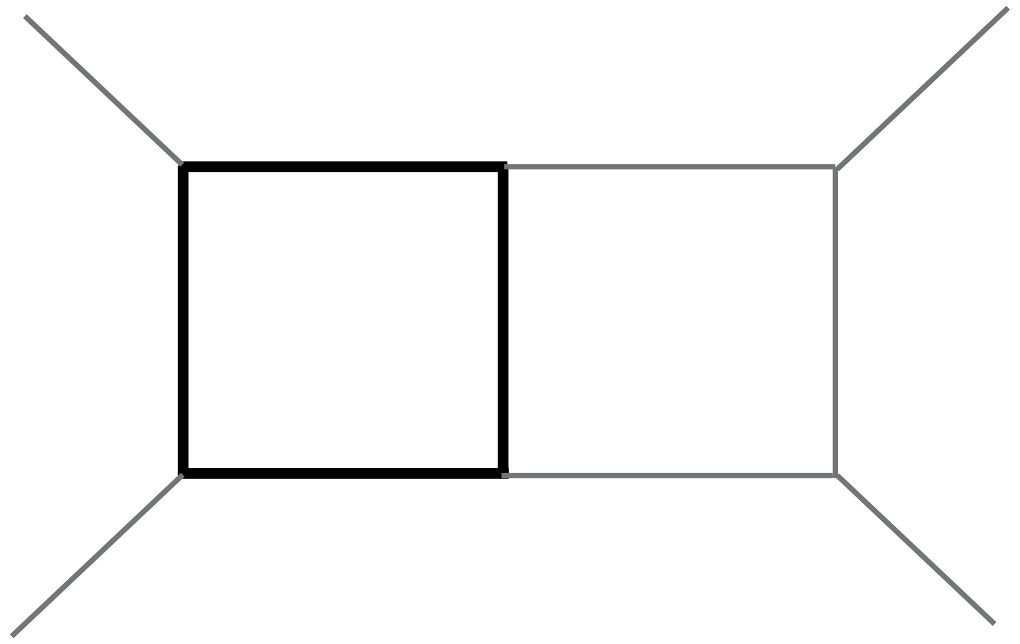

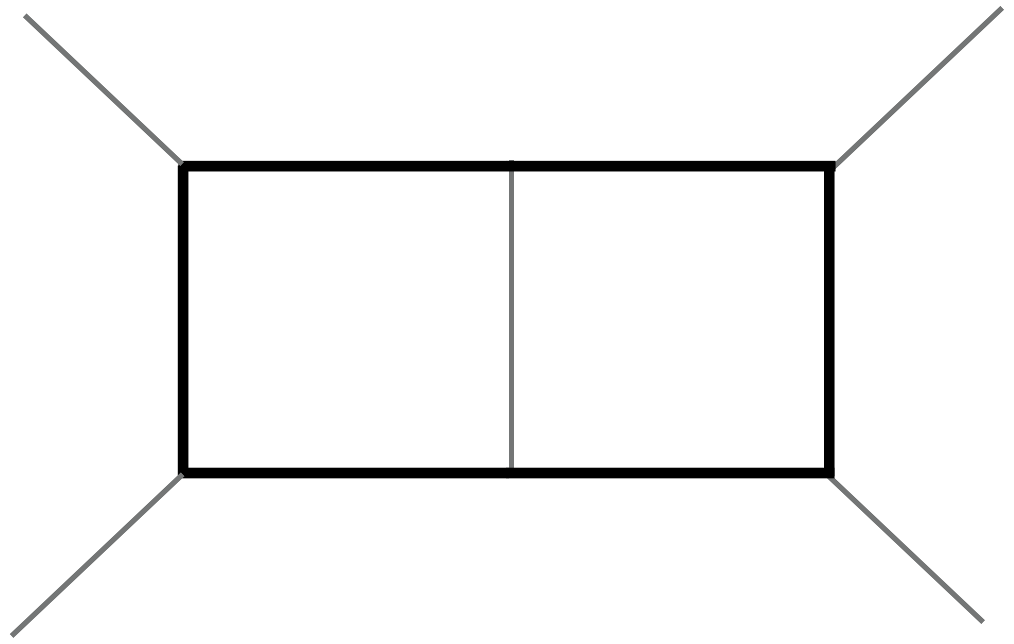

For complicated cases it is necessary to perform non-trivial variable changes first!

How to find the suitable variable changes systematically?

How do we know that the DEs of d-log integrals are canonical in general?

How to easily construct the symbol letters and rational coefficients in the DEs?

One-loop symbol letters

Abreu et al. (2017)
Chen, Ma, LLY (2022)
Jiang, LLY (2023)

One-loop symbol letters can be generically constructed…

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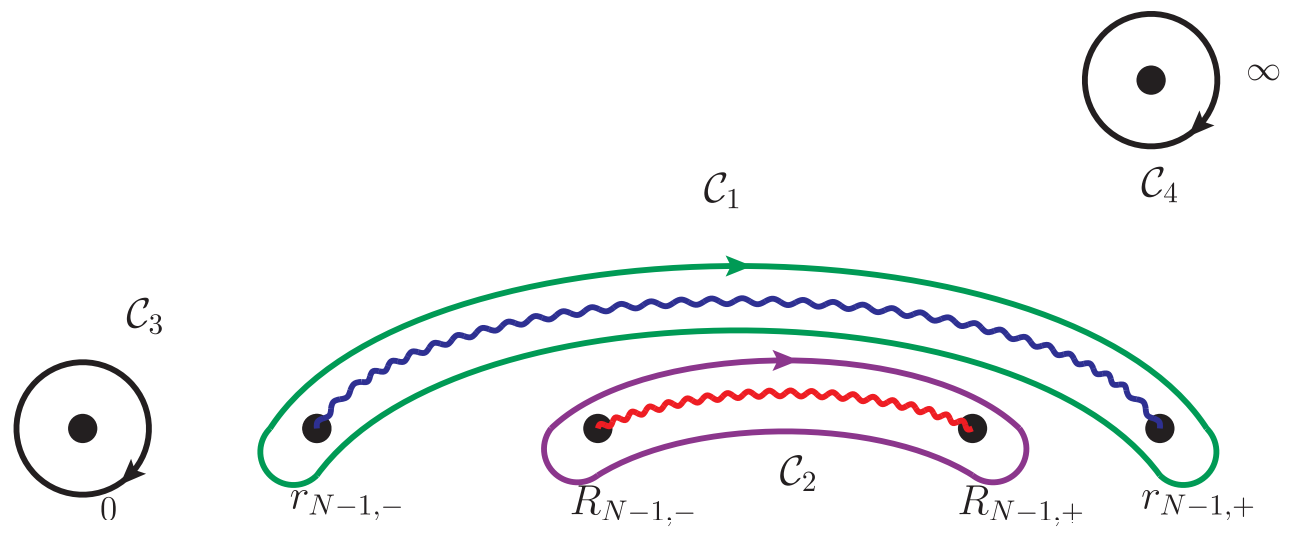

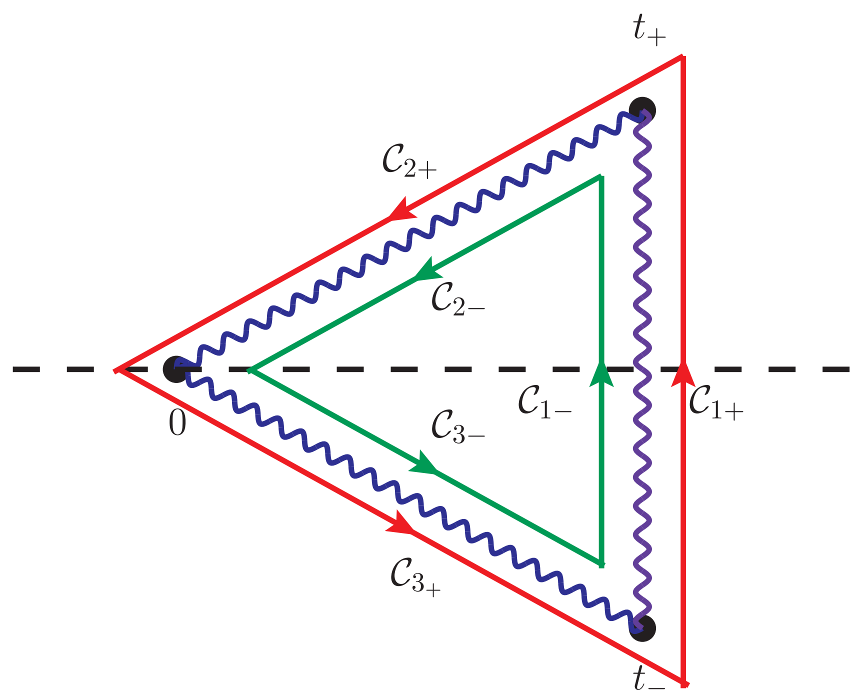

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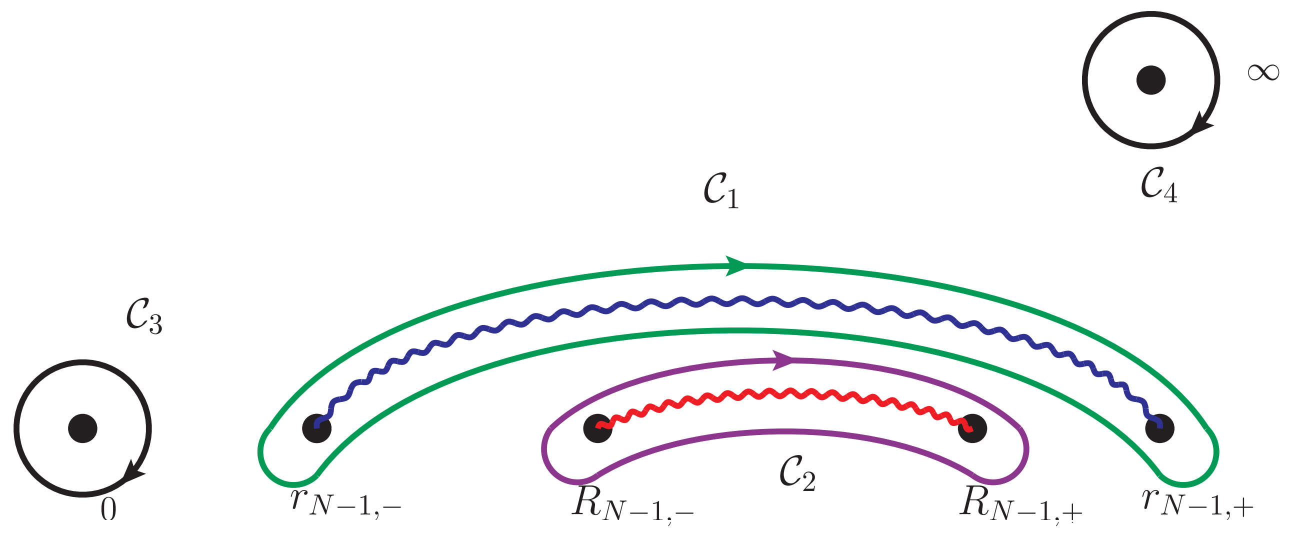
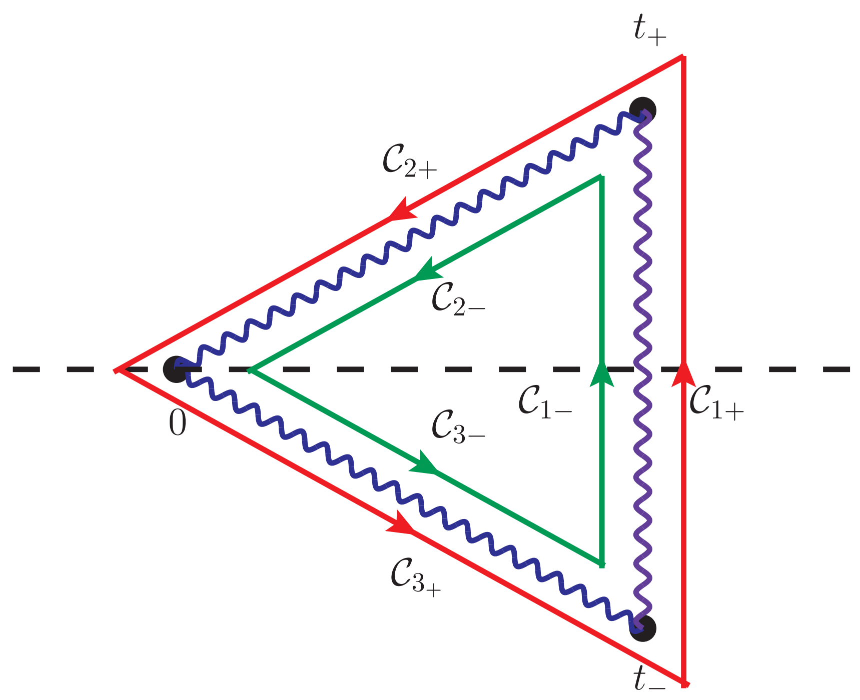
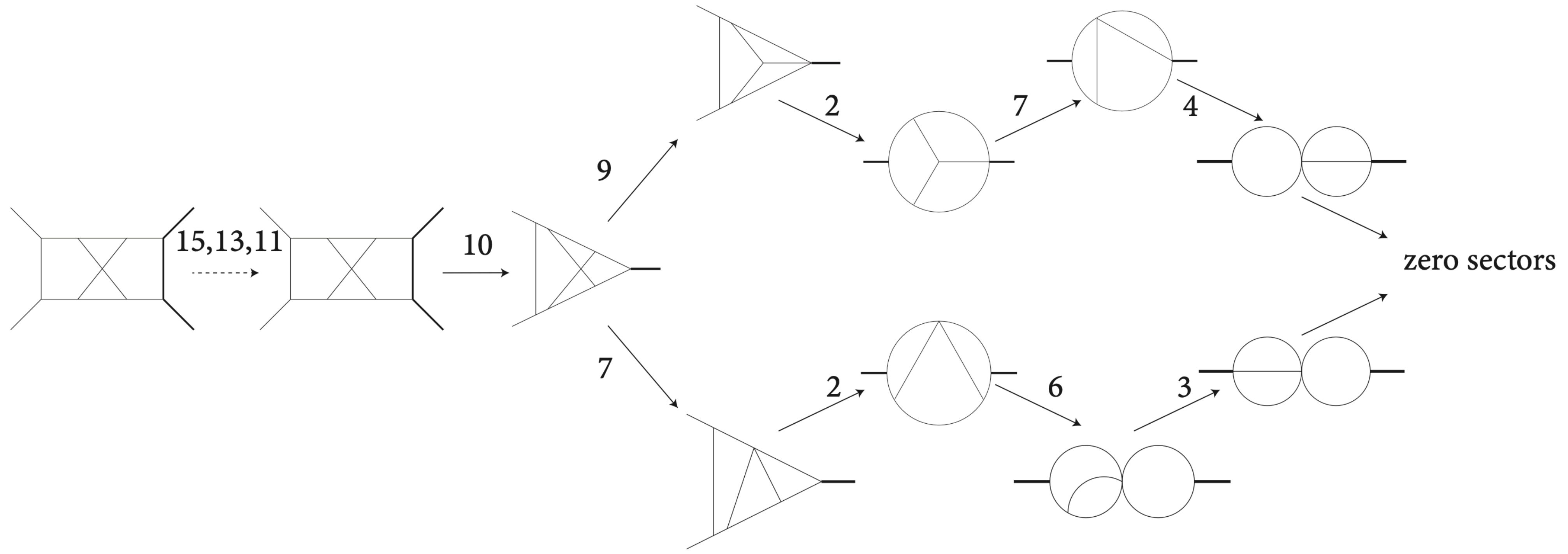




Or directly from the recursive structure



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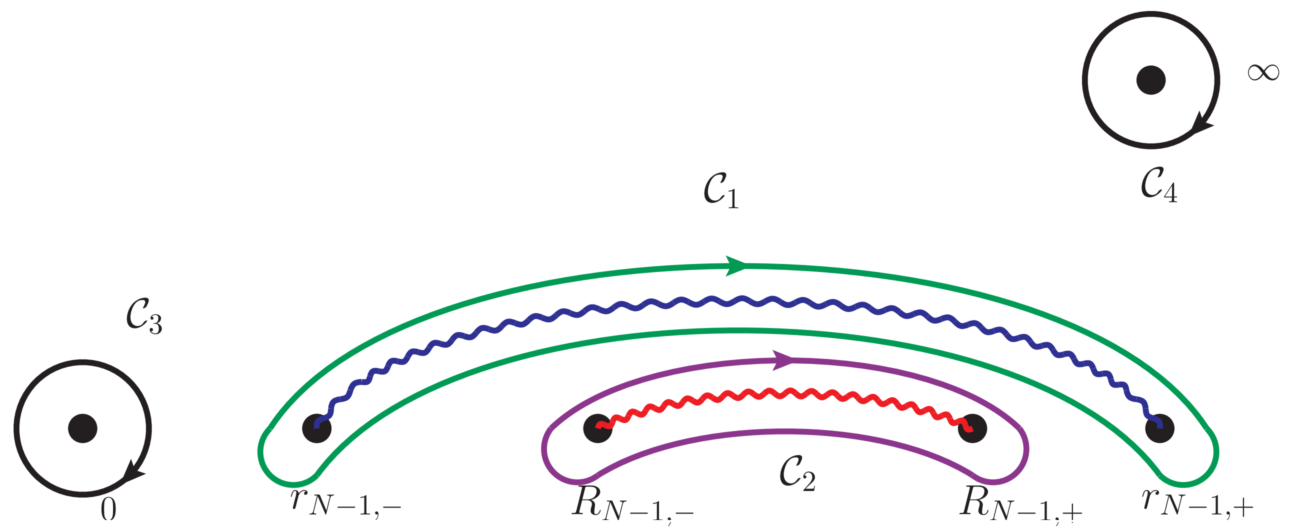

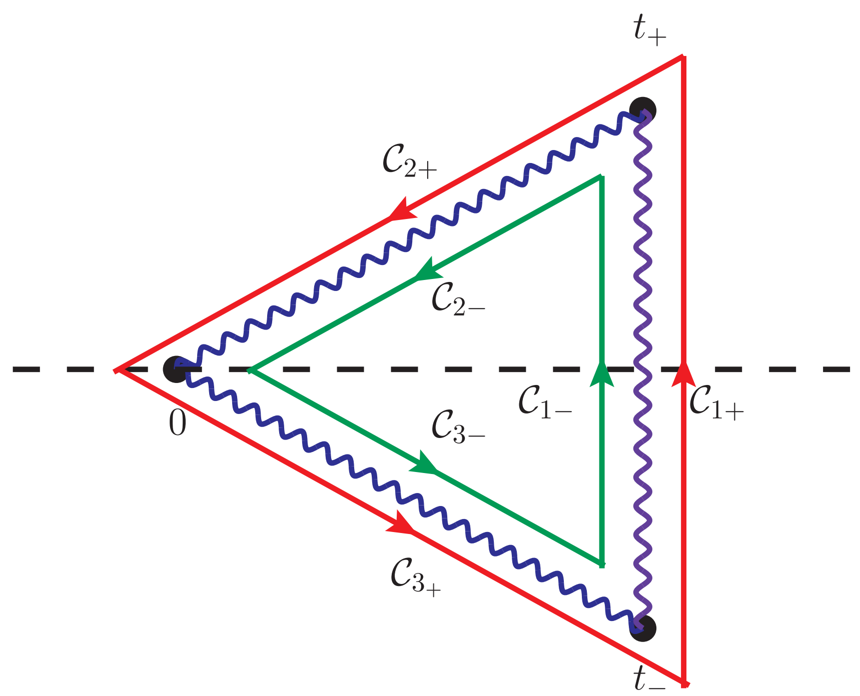

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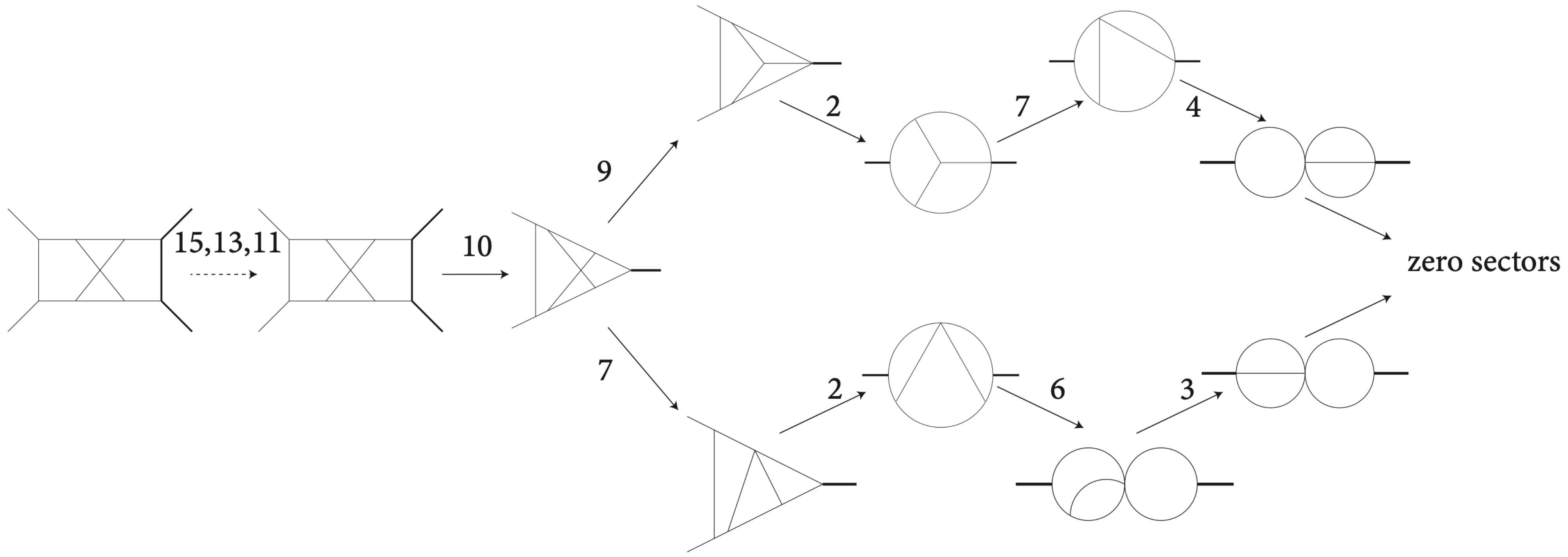

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- Easy to compute in terms of Gram determinants or minors of a single matrix

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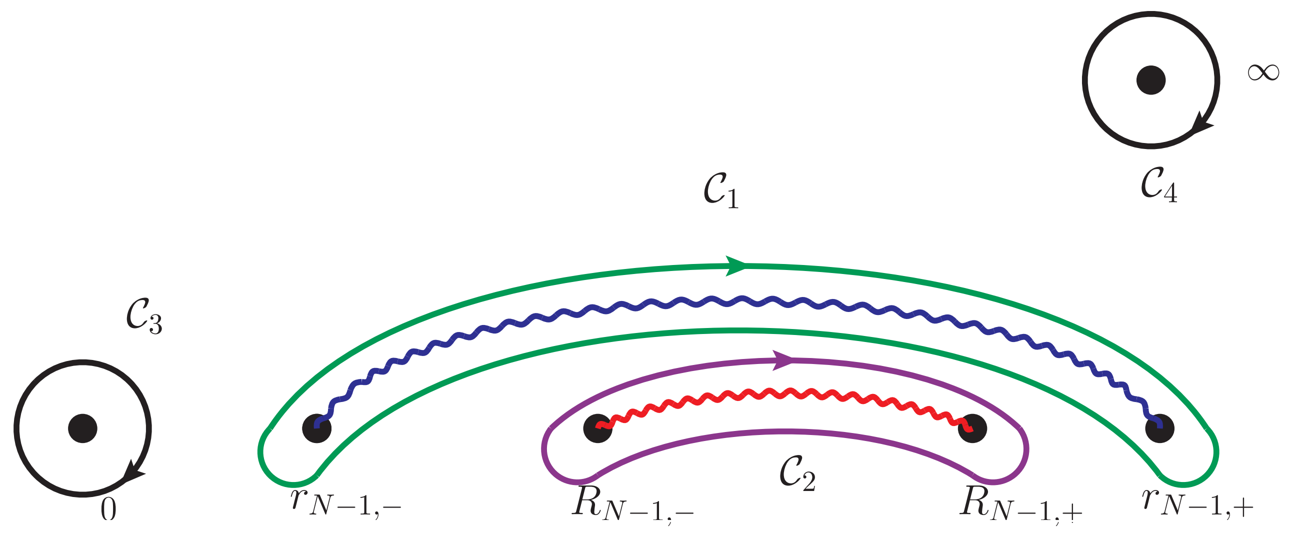

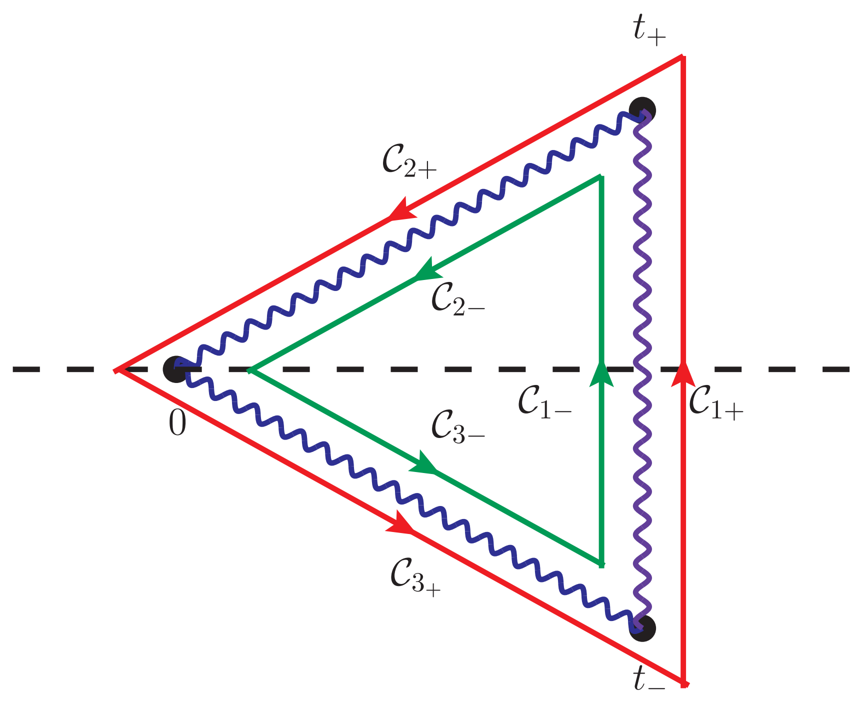

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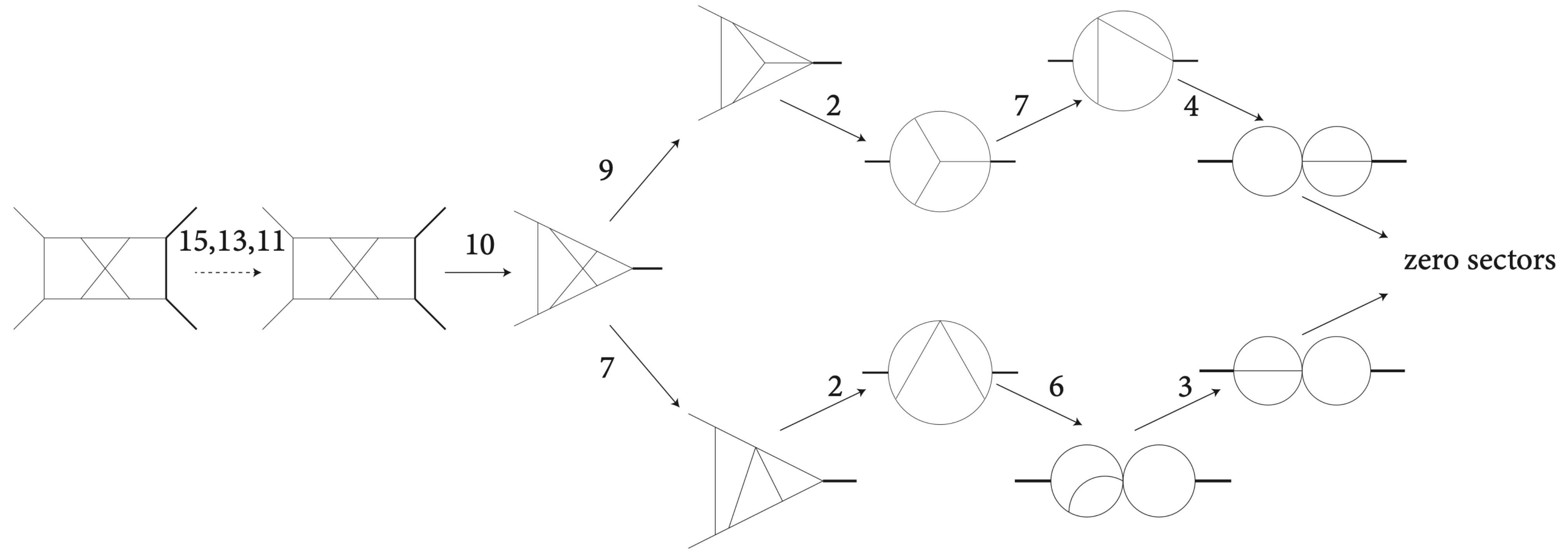

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Not easy to extend to higher loops…

Differential equations from intersection theory

$$\langle \dot{\varphi}_I | \equiv \hat{\mathrm{d}} \langle \varphi_I | = \left(\hat{\mathrm{d}}\Omega\right)_{IJ} \langle \varphi_J |$$

$$\left(\hat{\mathrm{d}}\Omega\right)_{IK} = \langle \dot{\varphi}_I | \varphi_J \rangle \left(\eta^{-1}\right)_{JK}$$

$$\eta_{IJ} = \langle \varphi_I | \varphi_J \rangle$$

The evaluation of intersection numbers is deeply related to the multivariable poles of the integrands (which are determined by the polynomial factors in the *u*-function)

$$I = \int_{\mathscr{C}} u(z)\, \varphi(z) \qquad\qquad u = \prod_{\boldsymbol{i}} \left[P_i(\boldsymbol{z})\right]^{\beta_i}$$

$$\langle \varphi_L | \varphi_R \rangle = \sum_{\boldsymbol{p}} \mathrm{Res}_{\boldsymbol{z}=\boldsymbol{p}} \left(\psi_L \hat{\varphi}_R\right)$$

Chestnov et al. (2022)

$$\nabla_n \cdots \nabla_1 \psi_L = \varphi_L$$

The multivariable poles

The construction of d-log bases is also deeply related to the poles from the *u*-function

$$\mathrm{d}\log(z-c) = \frac{dz}{z-c}$$

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And for d-log integrands, there are only simple poles!

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And for d-log integrands, there are only simple poles!

A complication: the poles can be non-factorized and/or degenerate, e.g.:

$$u = z_1^{\beta_1} z_2^{\beta_2} (z_1+z_2)^{\beta_3}$$

Factorization transformations

Chen, Feng, LLY (2023)

It is possible to perform variable changes (in the spirit of sector decomposition) to factorize the non-factorized poles, such that

$$u(z)\Big|_{z\to p} \xrightarrow{\;z \to \boldsymbol{x}^{(\alpha)}\;} u(\boldsymbol{x}^{(\alpha)})\big|_{\boldsymbol{x}^{(\alpha)}\to\boldsymbol{\rho}^{(\alpha)}} = \bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)}) \prod_i \left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}$$

Different (α) labels different variable changes

Non-vanishing

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Different (α) labels different variable changes

$$u = z_1^{\beta_1} z_2^{\beta_2} (z_1 + z_2)^{\beta_3} \xrightarrow{\substack{z_1 = x_1 \\ z_2 = x_1(x_2 - 1)}} u = x_1^{\beta_1+\beta_2+\beta_3} x_2^{\beta_3} (x_2 - 1)^{\beta_2}$$

Canonical DEs from factorized poles

Chen, Feng, LLY (2023)

$$\langle\dot{\varphi}_I| \equiv \hat{\mathrm{d}}\,\langle\varphi_I| = \left(\hat{\mathrm{d}}\Omega\right)_{IJ}\langle\varphi_J|$$

$$\left(\hat{\mathrm{d}}\Omega\right)_{IK} = \langle\dot{\varphi}_I|\varphi_J\rangle\left(\eta^{-1}\right)_{JK}$$

$$\varphi^{(\boldsymbol{b})} = C^{(\boldsymbol{b})}\bigwedge_i\left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{b_i}\mathrm{d}x_i^{(\alpha)}$$

$$u(\boldsymbol{x}^{(\alpha)})\big|_{\boldsymbol{x}^{(\alpha)}\to\boldsymbol{\rho}^{(\alpha)}} = \bar{u}_\alpha(\boldsymbol{\rho}^{(\alpha)})\prod_i\left[x_i^{(\alpha)} - \rho_i^{(\alpha)}\right]^{\gamma_i^{(\alpha)}}$$

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$$b_{I,k} + b_{J,k} = -1$$

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(Selection rules for non-zero entries)

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(Selection rules for non-zero entries)

By-product: one can show that the coefficient matrix is indeed proportional to ϵ

Canonical DEs from factorized poles

Chen, Feng, LLY (2023)

It turns out the entry with an integration can be written as d-logs by studying the univariate intersection numbers after taking the residues of the $(n-1)$-variable poles

$$-\frac{\gamma_k^{(\alpha)}}{\boldsymbol{\gamma}^{(\alpha)}}\,\hat{\mathrm{d}}\int C_I^{(\boldsymbol{b}_I)} C_J^{(\boldsymbol{b}_J)}\,\hat{\mathrm{d}}\rho_k^{(\alpha)}$$

$$\langle\dot{\varphi}_I|\varphi_I\rangle = \sum_{\alpha\neq I}\frac{\gamma^{(\alpha)}}{\gamma^{(I)}}\,\hat{\mathrm{d}}\log(c_I-c_\alpha) + \eta_{II}\beta_0\,\hat{\mathrm{d}}\log P_0$$

$$\langle\dot{\varphi}_I|\varphi_J\rangle = -\hat{\mathrm{d}}\log(c_I-c_J) + \eta_{IJ}\beta_0\,\hat{\mathrm{d}}\log P_0\,,$$

$$\begin{aligned}\langle\dot{\varphi}_I|\varphi_I\rangle &= \frac{1}{\gamma^{(I)}}\,\hat{\mathrm{d}}\log(\bar{u}_I(c_I)) - \hat{\mathrm{d}}\log(c_+-c_-)\\ &\quad + \hat{\mathrm{d}}\log(c_I-c_+) + \hat{\mathrm{d}}\log(c_I-c_-)\,,\\ \langle\dot{\varphi}_I|\varphi_J\rangle &= \langle\dot{\varphi}_J|\varphi_I\rangle = -\hat{\mathrm{d}}\log\tau[c_I,c_J;c_\pm]\,.\end{aligned}$$

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$$+\,\hat{\mathrm{d}}\log(c_I-c_+)+\hat{\mathrm{d}}\log(c_I-c_-)\,,$$

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Purely algebraic method to determine the symbol letters and the rational coefficients in the canonical DEs!

A simple example

$$z_1 = l_1^2 - m^2\,,\; z_2 = (l_2 - p)^2 - m^2\,,\; z_3 = (l_1 - l_2)^2$$
$$z_4 = l_2^2\,, \quad z_5 = (l_1 - p)^2\,, \quad p^2 = s\,.$$

(

Cut on z_1, z_2, z_3 $\qquad u = z_4^{\delta_1} z_5^{\delta_5} \left[\mathcal{G}(z_4, z_5)\right]^{-\epsilon}$

$$\mathcal{G} \equiv \mathrm{G}(l_1, l_2, p)\Big|_{z_1=z_2=z_3=0} = -2m^6 + m^4(s + z_4 + z_5)$$
$$+ m^2(2z_4 z_5 - s z_4 - s z_5) + z_4 z_5(s - z_4 - z_5)\,. \quad (24)$$

The poles are given by $\qquad \boldsymbol{p} \in \left\{(0,0), (m^2, m^2), (\infty, 0), (0, \infty), (\infty, \infty)\right\}$

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The poles are given by $\qquad \boldsymbol{p} \in \{(0,0), (m^2, m^2), (\infty, 0), (0, \infty), (\infty, \infty)\}$

$t_4 = 1/z_4 \qquad u = t_4^{2\epsilon - \delta_1} z_5^{\delta_5} \left[\mathcal{G}_{\infty 0}(z_4, z_5)\right]^{-\epsilon}$

$$\mathcal{G}_{\infty 0} \equiv t_4^2\, \mathcal{G}(1/t_4, z_5) \equiv t_4[r_+(t_4) - z_5][z_5 - r_-(t_4))]$$

Degenerate pole since $\quad z_5 - r_-(t_4) = z_5 - m^2(m^2 - s)t_4 + \mathcal{O}(t_4^2)$

A simple example

3 different variable changes corresponding to 3 ways to combine the vanishing factors

$$\boldsymbol{x}^{(4)} : (\{t_4\}, \{z_5, z_5 - r_-(t_4)\})$$
$$\boldsymbol{x}^{(5)} : (\{t_4, z_5 - r_-(t_4)\}, \{z_5\})$$
$$\boldsymbol{x}^{(6)} : (\{z_5 - r_-(t_4)\}, \{t_4, z_5\})$$

$$t_4 = x_1^{(5)}\,, \quad z_5 = x_1^{(5)} x_2^{(5)}$$

$$u \to \left[m^2(m^2 - s)\right]^{-\epsilon} \left(x_1^{(5)}\right)^{\epsilon - \delta_1 + \delta_2} \left(x_2^{(5)}\right)^{\delta_2}$$

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$$u \to \left[m^2(m^2 - s)\right]^{-\epsilon} \left(x_1^{(5)}\right)^{\epsilon - \delta_1 + \delta_2} \left(x_2^{(5)}\right)^{\delta_2}$$

$$\varphi_1 = \frac{\mathrm{d}z_4 \mathrm{d}z_5}{z_4 z_5}, \quad \varphi_2 = \frac{\sqrt{s(s-4m^2)}}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5,$$
$$\varphi_3 = \frac{z_4 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5, \quad \varphi_4 = \frac{z_5 - m^2}{\mathcal{G}} \mathrm{d}z_4 \mathrm{d}z_5$$

$$\varphi_1^{(-1,-1)} = \frac{\mathrm{d}x_1^{(5)} \mathrm{d}x_2^{(5)}}{x_1^{(5)} x_2^{(5)}}, \quad \varphi_3^{(-1,0)} = \frac{\mathrm{d}x_1^{(5)} \mathrm{d}x_2^{(5)}}{x_1^{(5)} \left[m^2(m^2 - s)\right]}$$

$$\langle \dot{\varphi}_1 | \bar{\varphi}_3 \rangle \longrightarrow \hat{d} \log[m^2(m^2 - s)]$$

Newton polytopes

It is interesting to note that in this simple example, the symbol letters are related to the coefficients at the vertices of the degenerate facet of the Newton polytope associated with the polynomial factor in u

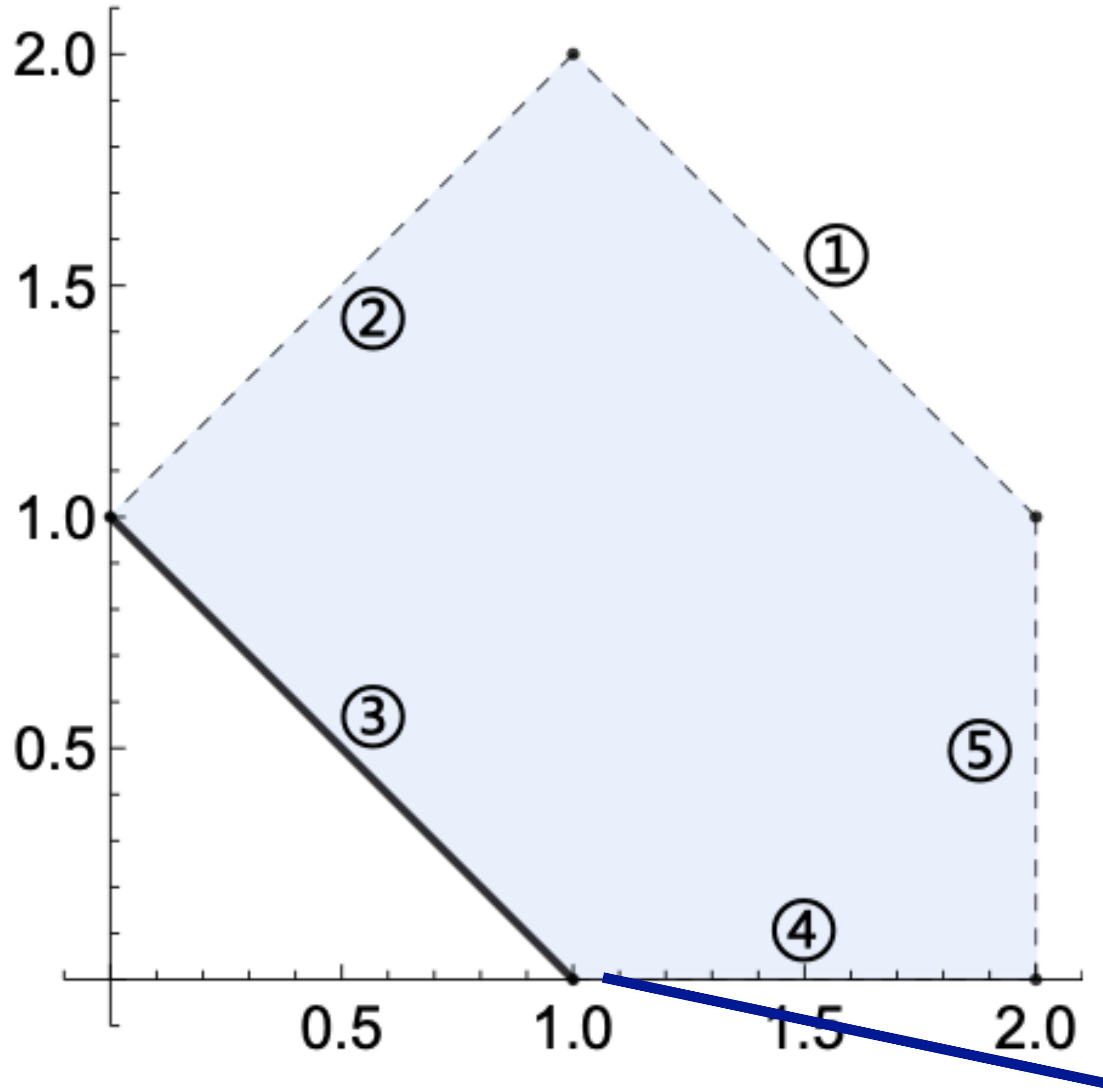

FIG. 1. The Newton polytope of $\mathcal{G}_{\infty 0}$. Horizontal and vertical axis are the power of t_4 and z_5. The solid line represents the zero facet of $(\infty, 0)$.

$m^2(m^2 - s)$

Summary and outlook

- Some interesting observations on the construction of canonical bases and the determination of symbol letters
- Outlook:
  - Generic algorithm for factorization transformations
  - Using factorization transformations for d-log constructions
  - Deeper relations to the geometry of Newton polytopes
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Happy 10th anniversary for MITP!

Happy 60th birthday for Matthias!