Flavour Physics with Lattice QCD+QED

Chris Sachrajda
University of Southampton

Pushing the Limits of Theoretical Physics

Celebrating the 10th Anniversary of MITP
&
Matthias @ 60

Mainz, May 8th - 12th 2023

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Selected Highlights

1. *Cancellation of Renormalon Ambiguities in the Heavy Quark Effective Theories*

M.Neubert and CTS, Nucl. Phys. B438 (1995) 235

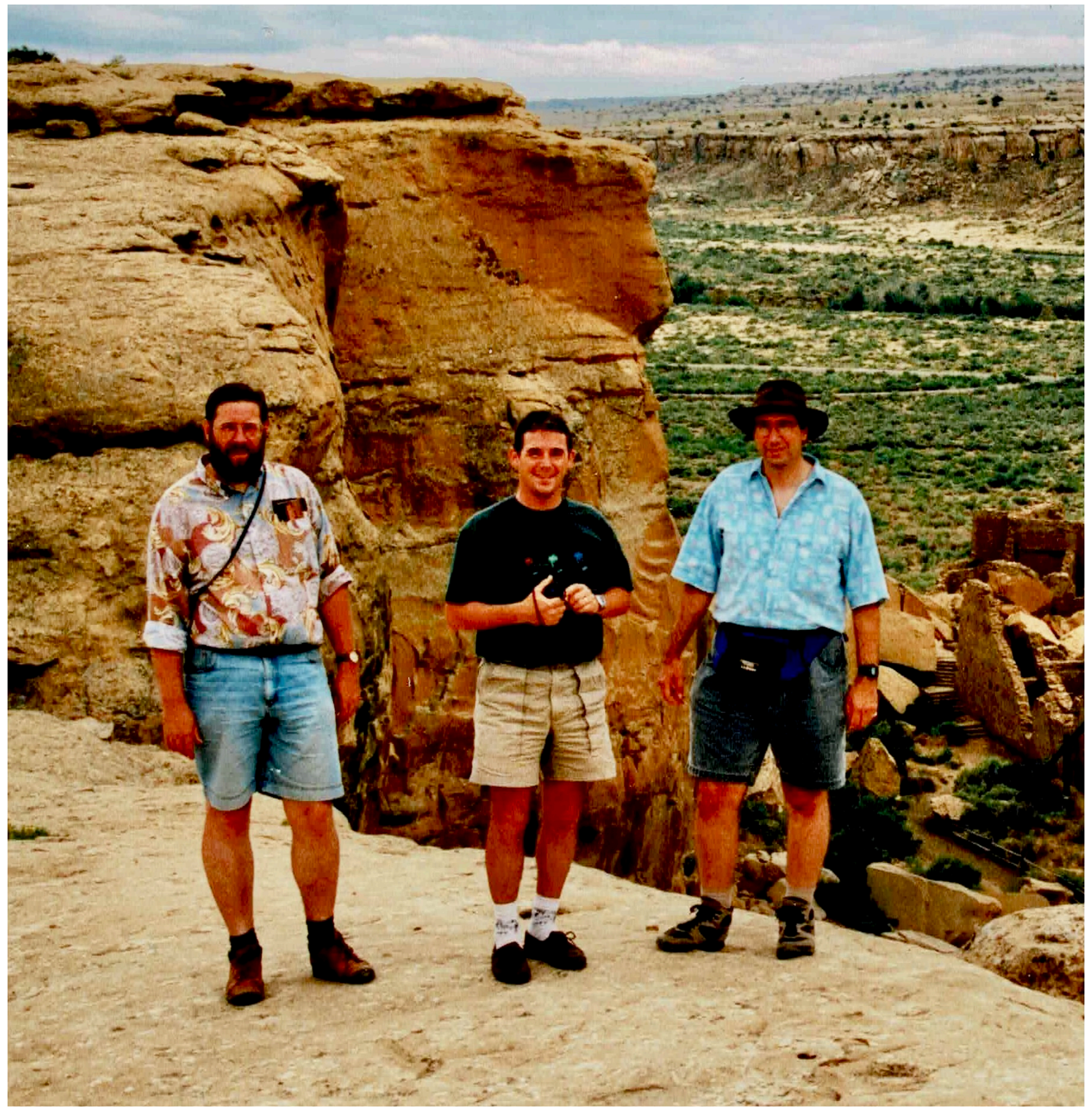
2. *Spectator Effects in inclusive decays of beauty hadrons*

M.Neubert and CTS, Nucl. Phys. B483 (1997) 339

3. QCD Factorisation in Charmless Two-Body B decays (BBNS)

M.Beneke, G.Buchalla, M.Neubert and CTS

- *QCD Factorisation for $B \to \pi\pi$ decays: Strong Phases and CP-violation in the heavy quark limit,* Phys. Rev. Lett. 83 (1999) 1914;
- *QCD Factorisation for exclusive nonleptonic B meson decays: General arguments and the case of heavy light final states,* Nucl. Phys. B591 (2000) 313;
- *QCD Factorization for $B \to \pi\pi, \pi K$ decays and extraction of Wolfenstein parameters,* Nucl. Phys. B606 (2001) 245
- *Penguins with charm and quark-hadron duality,* Eur. Phys.J. C61 (2009) 439



Courtesy of G.Martinelli

QED corrections to Weak Decay Amplitudes - Motivation

Quantity	Sec.	$N_f = 2+1+1$	Refs.	$N_f = 2+1$	Refs.	$N_f = 2$	Refs.
m_{ud}[MeV]	3.1.4	3.410(43)	[6, 7]	3.381(40)	[8–12]		
m_s[MeV]	3.1.4	93.40(57)	[6, 7, 13, 14]	92.2(1.0)	[8–11, 15]		
m_s/m_{ud}	3.1.5	27.23(10)	[7, 16, 17]	27.42(12)	[8–10, 15, 18]		
m_u[MeV]	3.1.6	2.14(8)	[6, 19]	2.27(9)	[20]		
m_d[MeV]	3.1.6	4.70(5)	[6, 19]	4.67(9)	[20]		
m_u/m_d	3.1.6	0.465(24)	[19, 21]	0.485(19)	[20]		
$\overline{m}_c$(3 GeV)[GeV]	3.2.2	0.988(11)	[6, 7, 14, 22, 23]	0.992(5)	[11, 24–26]		
m_c/m_s	3.2.3	11.768(34)	[6, 7, 14]	11.82(16)	[24, 27]		
$\overline{m}_b(\overline{m}_b)$[GeV]	3.3	4.203(11)	[6, 28–31]	4.171(20)	[11]		
$f_+(0)$	4.3	0.9698(17)	[32, 33]	0.9677(27)	[34, 35]	0.9560(57)(62)	[36]
$f_{K^\pm}/f_{\pi^\pm}$	4.3	1.1932(21)	[16, 37–39]	1.1917(37)	[8, 40–44]	1.205(18)	[45]
$f_{\pi^\pm}$[MeV]	4.6			130.2(8)	[8, 40, 41]		
$f_{K^\pm}$[MeV]	4.6	155.7(3)	[17, 37, 38]	155.7(7)	[8, 40, 41]	157.5(2.4)	[45]
Re(A_2)[GeV]	6.2			$1.50(4)(14)\times 10^{-8}$	[46]		
Im(A_2)[GeV]	6.2			$-8.34(1.03)\times 10^{-13}$	[46]		
$\hat{B}_K$	6.3	0.717(18)(16)	[47]	0.7625(97)	[8, 48–50]	0.727(22)(12)	[51]
B_2	6.4	0.46(1)(3)	[47]	0.502(14)	[50, 52]	0.47(2)(1)	[51]
B_3	6.4	0.79(2)(5)	[47]	0.766(32)	[50, 52]	0.78(4)(2)	[51]
B_4	6.4	0.78(2)(4)	[47]	0.926(19)	[50, 52]	0.76(2)(2)	[51]
B_5	6.4	0.49(3)(3)	[47]	0.720(38)	[50, 52]	0.58(2)(2)	[51]

Table 1: Summary of the main results of this review concerning quark masses, light-meson decay constants, and hadronic kaon-decay and kaon-mixing parameters. These are grouped in terms of N_f, the number of dynamical quark flavours in lattice simulations. Quark masses are given in the $\overline{\rm MS}$ scheme at running scale $\mu = 2\,$GeV or as indicated. BSM bag parameters $B_{2,3,4,5}$ are given in the $\overline{\rm MS}$ scheme at scale $\mu = 3\,$GeV. Further specifications of the quantities are given in the quoted sections. Results for $N_f = 2$ quark masses are unchanged since FLAG 16 [3], and are not included here. For each result we list the references that enter the FLAG average or estimate, and we stress again the importance of quoting these original works when referring to FLAG results. From the entries in this column one can also read off the number of results that enter our averages for each quantity. We emphasize that these numbers only give a very rough indication of how thoroughly the quantity in question has been explored on the lattice and recommend consulting the detailed tables and figures in the relevant section for more significant information and for explanations on the source of the quoted errors.

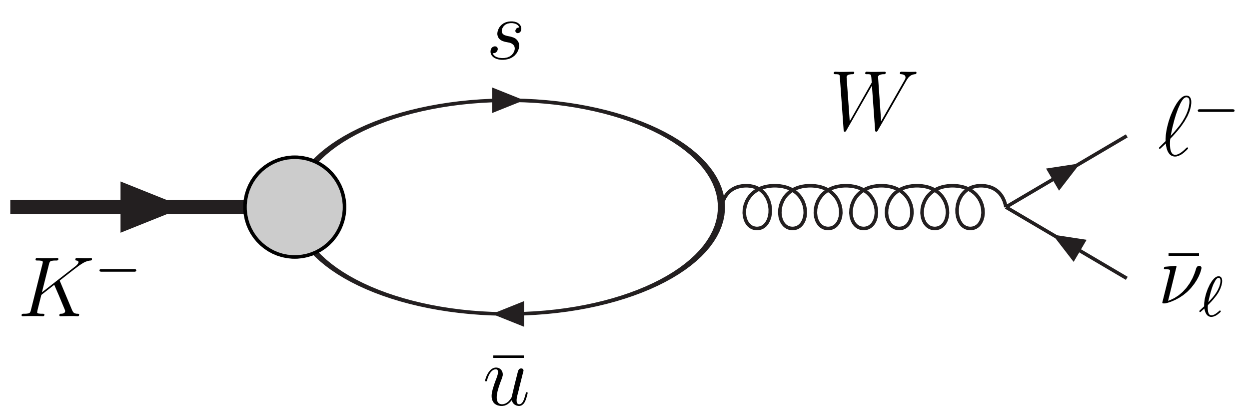
FLAG Review 2021, Y.Aoki et al., arXiv:2111.09849

- Lattice QCD results for some physical quantities are now so precise (sub percent) that QED corrections need to be included to make further progress.

- I shall use

$$f_K = 155.7(3)\ \text{MeV}$$

to illustrate our calculations.



$$\langle 0|A_\mu|K(p)\rangle = f_K p_\mu\,, \qquad r_\ell = m_\ell/m_K$$

$$\Gamma^{(0)} = \frac{G_F^2|V_{us}|^2 f_K^2}{8\pi} m_K^3 r_\ell^2 \left(1-r_\ell^2\right)^2$$

Computing QED Corrections to Weak Decay Amplitudes - The Framework

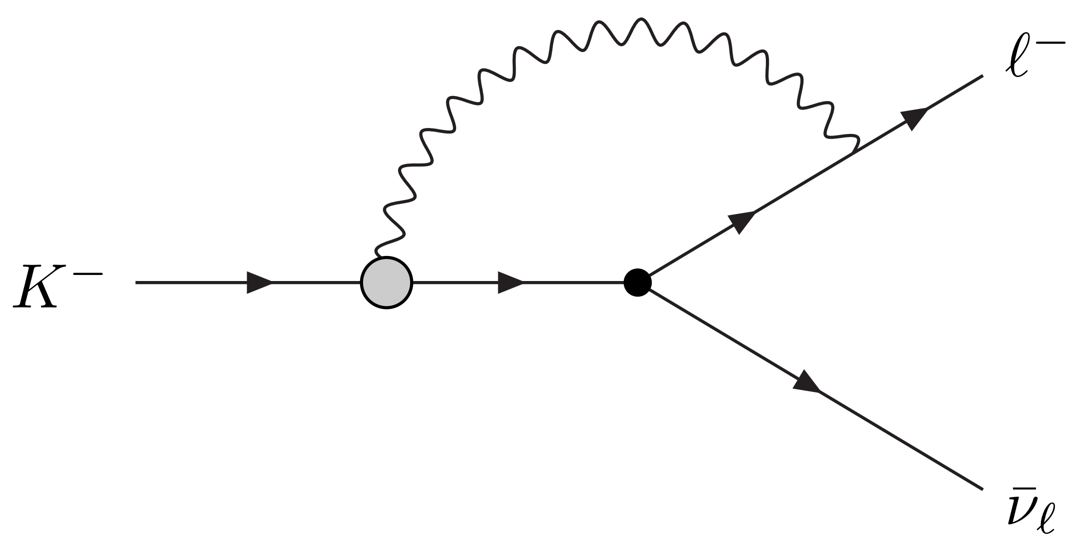
QED Corrections to Hadronic Processes in Lattice QCD

N.Carrasco, V.Lubicz, G.Martinelli, CTS, N.Tantalo, C.Tarantino and M.Testa

arXiv:1502.00257

- Our aim is to calculate Γ including $O(\alpha_{\rm em})$ corrections.

- f_K no longer contains all the QCD effects.



- Calculating electromagnetic corrections to decay amplitudes has the major complication, not present in computations of the spectrum,

  the presence of infrared divergences

- This implies that when studying such processes , the physical observable must include soft photons in the final state.

  F.Bloch and A.Nordsieck, PR 52 (1937) 54

$$\Gamma(K^- \to \ell^-\bar{\nu}_\ell(\gamma)) = \Gamma(K^- \to \ell^-\bar{\nu}_\ell) + \Gamma(K^- \to \ell^-\bar{\nu}_\ell\gamma) \equiv \Gamma_0 + \Gamma_1\,.$$

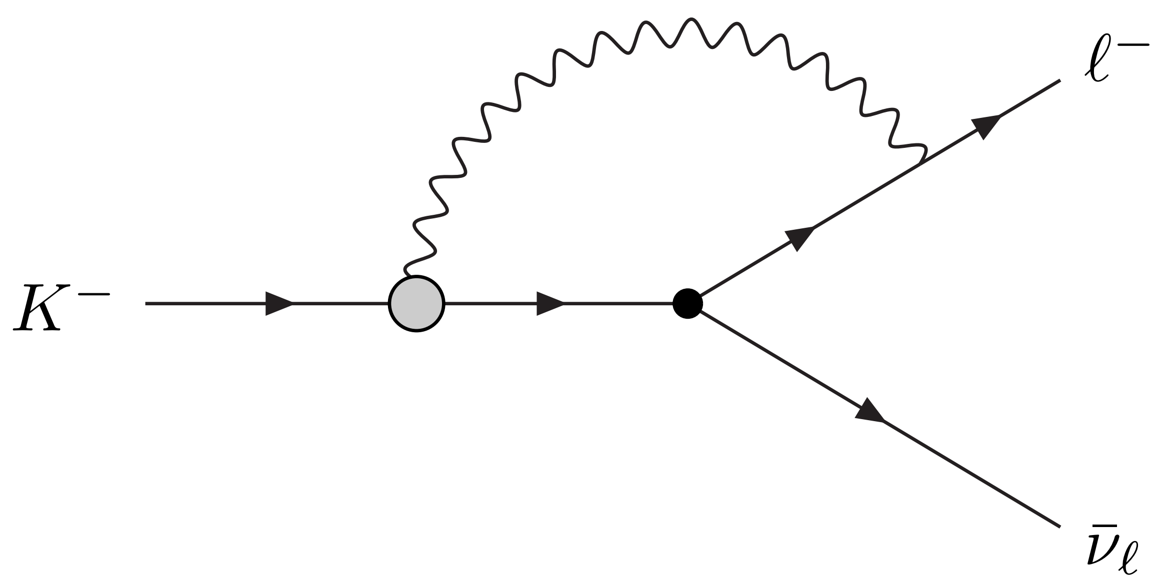
- The generic question is how best to combine this understanding with lattice calculations of non-perturbative hadronic effects.

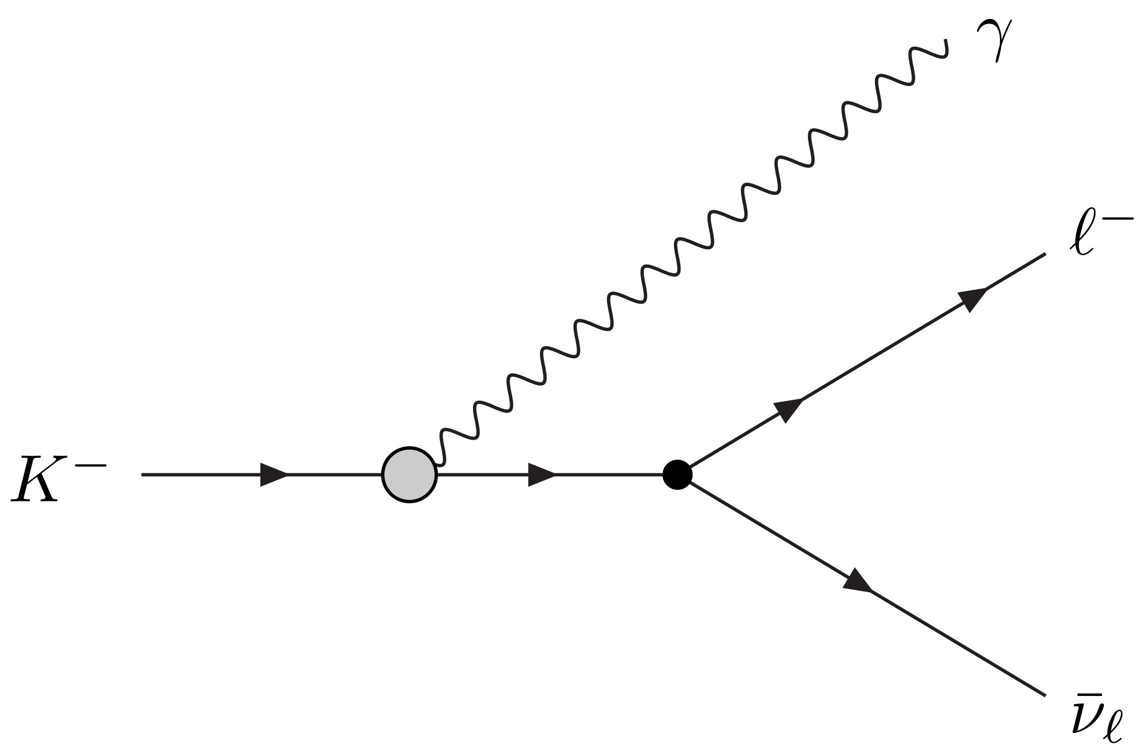
- Our proposal is to separate $\Gamma_0 + \Gamma_1$ into terms each of which is infrared convergent

$$\Gamma(\Delta E_\gamma) = \Gamma_0 + \Gamma_1(\Delta E_\gamma) = \Gamma_0 + \int_0^{2\Delta E_\gamma/m_P} dx_\gamma \, \frac{d\Gamma_1}{dx_\gamma}$$

$$= \lim_{L\to\infty}\left[\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)\right] + \lim_{\mu_\gamma\to 0}\left[\Gamma_0^{\mathrm{pt}}(\mu_\gamma) + \Gamma_1^{\mathrm{pt}}(\Delta E_\gamma, \mu_\gamma)\right] + \Gamma_1^{\mathrm{SD}}(\Delta E_\gamma) + \Gamma_1^{\mathrm{INT}}(\Delta E_\gamma)\,.$$

- $x_\gamma = 2E_\gamma/m_K$ in the rest frame of the kaon
- pt = "point like", SD = "Structure Dependent" and "INT" is the interference between pt and SD
- "pt" contributions can be calculated in perturbation theory, whereas $\Gamma_0(L)$ and (for large ΔE_γ) Γ_1^{SD} and Γ_1^{INT} need to be computed non perturbatively.





Issues not discussed here

- When including QED, questions such as "**What is QCD?**" or equivalently "**How large are the electromagnetic corrections?**" are convention dependent due to the electromagnetic shift in the quark masses.

Light-meson leptonic decay rates in lattice QCD+QED
M.Di Carlo, D Giusti, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo , S.Simula and N.Tantalo, arXiv:1904.08731

- Definition of G_F at $O(\alpha_{\rm em})$. This must be consistent with the procedure being used.

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3}\left[1 - \frac{8m_e^2}{m_\mu^2}\right]\left[1 + \frac{\alpha_{\rm em}}{2\pi}\left(\frac{25}{4} - \pi^2\right)\right]$$

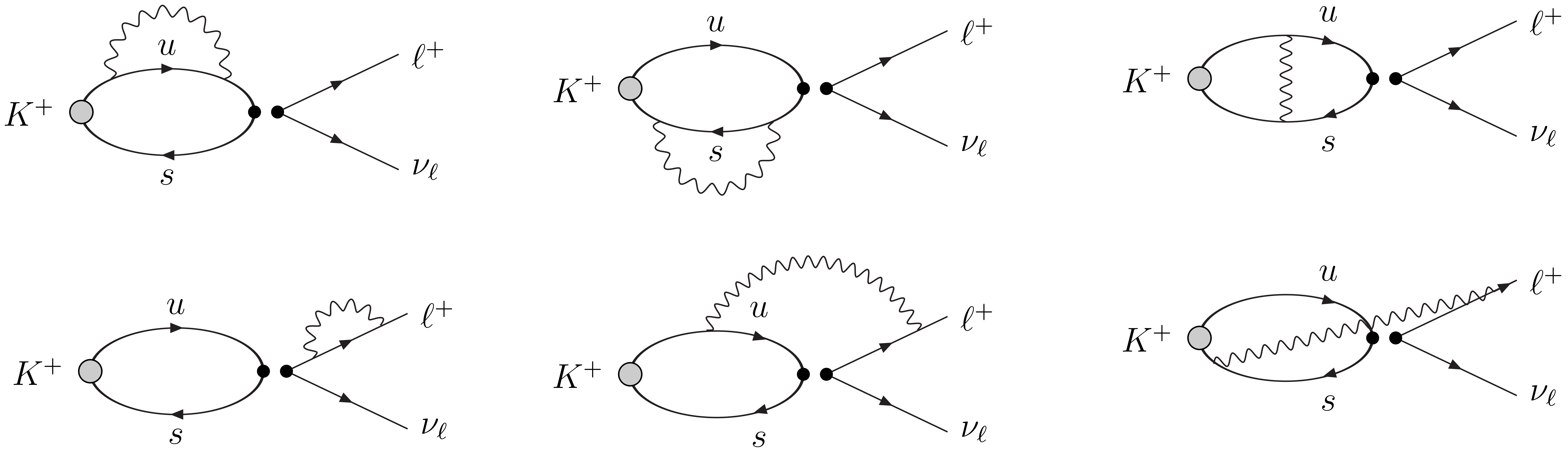
- Renormalization of the lattice operators including $O(\alpha_{\rm em})$ effects.

Non-perturbative renormalization in QCD+QED and its application to weak decays
M.Di Carlo, D Giusti, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo , S.Simula and N.Tantalo, arXiv:1911.00938

- Perturbative evaluation of $\Gamma_0^{\rm pt} + \Gamma_1^{\rm pt}(\Delta E_\gamma)$.

QED Corrections to Hadronic Processes in Lattice QCD
N.Carrasco, V.Lubicz, G.Martinelli, CTS, N.Tantalo, C.Tarantino and M.Testa, arXiv:1502.00257

- Evaluation of the diagrams.



+ disconnected diagrams + real photon emission

Finite-Volume Corrections

Finite-Volume QED corrections to decay amplitudes in lattice QCD
V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula and N.Tantalo, arXiv:1611.08497

- The photon is massless $\Rightarrow$ difficulties in a finite volume.

- We have implemented the framework in QED$_\mathrm{L}$ in which $A_\mu(\vec{k}=0,k_4)=0$ for all k_4.
  M.Hayakawa and S.Uno arXiv:0804.2044

  - Transfer matrix exists but locality is broken
  - $L\to\infty$ limit should be taken first

- Evaluation of FV effects is based on the Poisson Summation formula, e.g. in one dimension

$$\frac{1}{L}\sum_{n=-\infty}^{\infty} f(p_n^2)=\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,f(p^2)+\sum_{n\neq 0}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,f(p^2)\,e^{inpL}\,.$$

  - For decay constants, form factors etc the FV effects fall exponentially, typically $\propto \exp[-cm_\pi L]$.
  - This is not the case when $f(p^2)$ has a singularity.

- In the presence of a photon, if the integrand/summand $\to \dfrac{1}{(k^2)^{\frac{n}{2}}}$ as $k \to 0$ then we have the scaling law:

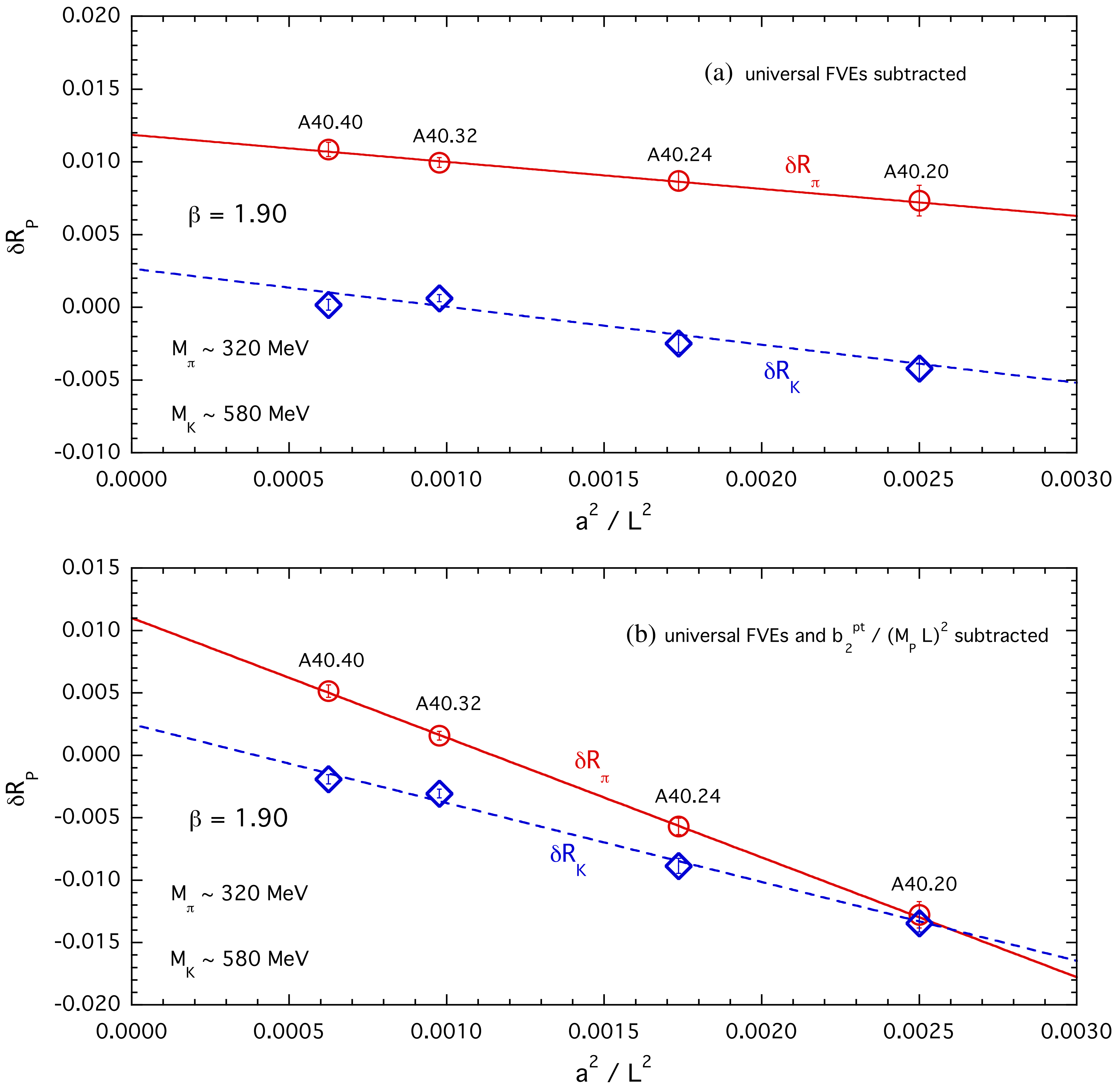
$$\xi' = \int \frac{dk_0}{(2\pi)} \left(\frac{1}{L^3} \sum_{\vec{k} \neq 0} - \int \frac{d^3k}{(2\pi)^3} \right) \frac{1}{(k^2)^{\frac{n}{2}}} = O\left(\frac{1}{L^{4-n}} \right)$$

- For the spectrum $n = 3$ and the leading FV corrections are $O(1/L)$.
- For decay amplitudes $n = 4$ and we have the form: $\Gamma_0^{\mathrm{pt}}(L) = C_0(r_\ell) + \tilde{C}_0(r_\ell)\log\left(m_K L\right) + \dfrac{C_1(r_\ell)}{m_K L} + \ldots$

where $r_\ell = m_\ell / m_K$.

- The exhibited L-dependent terms are *universal*, i.e. independent of the structure of the meson!
  - We have evaluated these coefficients.

- The leading structure-dependent FV effects in $\Gamma_0 - \Gamma_0^{\mathrm{pt}}$ are of $O(1/L^2)$.



- Finite-volume behaviour of 4-points, obtained at the same value of β and quark masses using ETMC twisted mass ensembles.
  - The universal $O(1/L)$ terms have been subtracted.
  - The leading SD finite-volume terms appear to be of $O(1/L^2)$ as expected.

- However, it has recently been shown that the point-like $O(1/L^3)$ terms are not negligible together with an argument that the SD $O(1/L^2)$ terms are very small.

M.Di Carlo, M.T.Hansen, A.Portelli and N.Hermansson -Truedsson
Phys. Rev. D105 (2022) 074509

To be investigated further

Light-meson leptonic decay rates in lattice QCD+QED
M.Di Carlo, D Giusti, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo , S.Simula and N.Tantalo, arXiv:1904.08731

QED Corrections to V_{us}

- Writing

$$\frac{\Gamma(K_{\mu 2})}{\Gamma(\pi_{\mu 2})} = \left| \frac{V_{us}}{V_{ud}} \frac{f_K^{(0)}}{f_\pi^{(0)}} \right|^2 \frac{m_\pi^3}{m_K^3} \left(\frac{m_K^2 - m_\mu^2}{m_\pi^2 - m_\mu^2} \right)^2 \left(1 + \delta R_{K\pi}\right),$$

where $m_{K,\pi}$ are the physical masses, using numerous twisted mass ensembles we find

$$\delta R_{K\pi} = -0.0126(14) \qquad \left[\delta R_\pi = +0.0153(19),\ \delta R_K = +0.0024(10)\right]$$

- $f_P^{(0)}$ are the decay constants obtained in iso-symmetric QCD with the renormalized $\overline{\text{MS}}$ masses and coupling equal to those in the full QCD+QED theory extrapolated to infinite volume and to the continuum limit.

  - Using ChPT, $\delta R_\pi = +0.0176(21),\ \delta R_K = +0.0064(24).$ PDG(2018)
  - Boyle et al. $\delta R_{K\pi} = -0.0086(3)_{\text{stat}}\,(^{+11}_{-4})_{\text{fit}}\,(5)_{\text{disc}}\,(5)_{\text{quench}}\,(39)_{\text{FV}}$ arXiv:2211.12865

QED Corrections to V_{us} (cont.)

- We obtained

$$\left|\frac{V_{us}}{V_{ud}}\right| = 0.23135(46)\,.$$

- Taking $V_{ud} = 0.97420(21)$ (J.Hardy and I.S.Towner, CKM(2016) 028) $\Rightarrow$ $V_{us} = 0.22538(46)$ and with $|V_{ub}| = 0.00413(49)$,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99988(46)\,.$$

- However, taking $|V_{ud}| = 0.97370(14)$ (C.Y.Seng et al., arXiv:1807.10197), $|V_{us}| = 0.22526(46)$,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99885(34)\,.$$

- The latest PDG value is $V_{ud} = 0.97373(31)$, which is the average of the 15 most precise determinations and with a more conservative error. (Unitarity within a little more than 1σ.)

Infinite-volume reconstruction

- IVR is an idea by Xu Feng and Luchang Jin originally introduced to avoid non-exponential FV effects in calculations of QED corrections to the spectrum. X.Feng and L.Jin, arXiv:1812.09817

- We have been extending the technique to QED corrections to leptonic decay amplitudes.

N.H.Christ, X.Feng, L.Jin and CTS, PoS **LATTICE2019** (2020), 259

Radiative Corrections to Leptonic Decays using Infinite Volume Reconstruction

N.H.Christ, X.Feng, L.Jin, CTS and T.Wang, arXiv:2304.08026

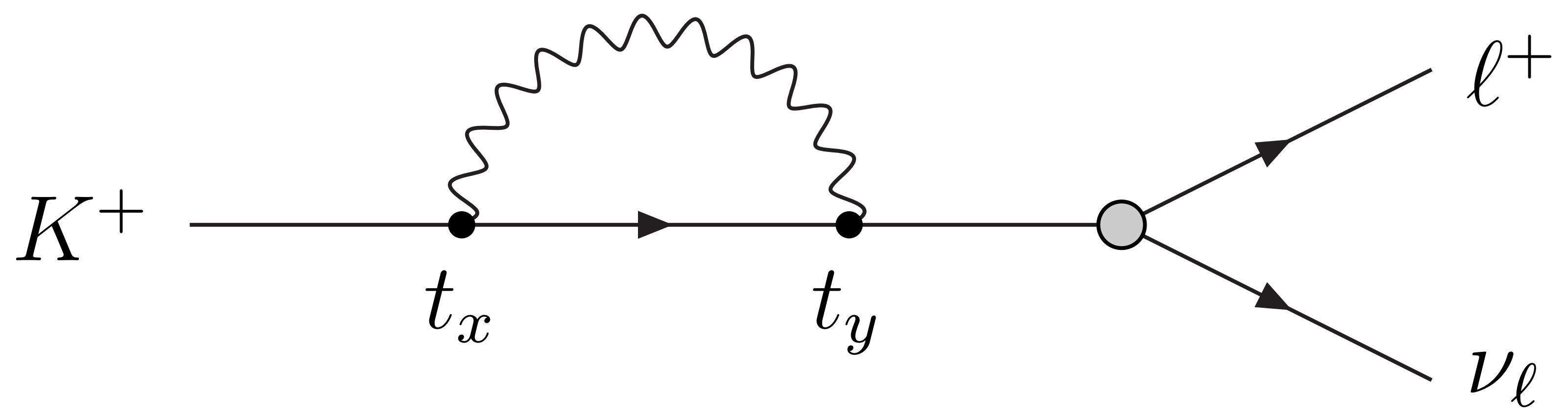
- Two attractive features of IVR for leptonic decays:
    1. Infrared divergences cancel analytically;
    2. Finite-volume corrections are exponentially small in the volume.

- The difficulties: large distance behaviour of correlation functions is generically of the form $e^{-m\sqrt{x^2+t^2}}$
- We use the large time behaviour to isolate the state we are interested in.
- For $t \gg x$, for example, $e^{-m\sqrt{x^2+t^2}} \simeq e^{-mt}\left(1+\dfrac{x^2}{2t^2}\right)$ and the coefficient of e^{-mt} has large FV corrections.

- Numerical computations of leptonic decay rates are under way.

Illustration of Infinite-Volume Reconstruction

- For illustration consider the following diagram which contributes both to the electromagnetic mass-shift and to the wave function renormalisation of the kaon:



- For large $|t_y - t_x|$, $|t_y - t_x| > t_s$ say, the only state propagating between the two currents is $|K^+\gamma\rangle$.
- As will be demonstrated on the following slide, it is then sufficient to evaluate the correlation functions with $|t_y - t_x| \leq t_s$ and avoid non-exponential FV effects. For example consider:

$$H_2(z, t_z) = \langle K^+(\vec{0}\,)|T[J^\mu(\vec{z}, t_z)J^\nu(0)]|K^+(\vec{0}\,)\rangle$$

where J^μ and J^ν are electromagnetic currents.

Demonstration

$$H_2(z, t_z) = \langle K^+(\vec{0}\,)|T[J^\mu(\vec{z}, t_z)J^\nu(0)]|K^+(\vec{0}\,)\rangle$$

$$H_2(\vec{z}, t_s) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\langle K^+(\vec{0}\,)|J^\mu(\vec{z}, t_s)|K^+(\vec{p})\rangle\,\langle K^+(\vec{p})|J^\nu(0)|K^+(\vec{0}\,)\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}e^{i\vec{p}\cdot\vec{z}}\,e^{-(E_p-m_K)t_s}\langle K^+(\vec{0}\,)|J^\mu(0)|K^+(\vec{p})\rangle\,\langle K^+(\vec{p})|J^\nu(0)|K^+(\vec{0}\,)\rangle$$

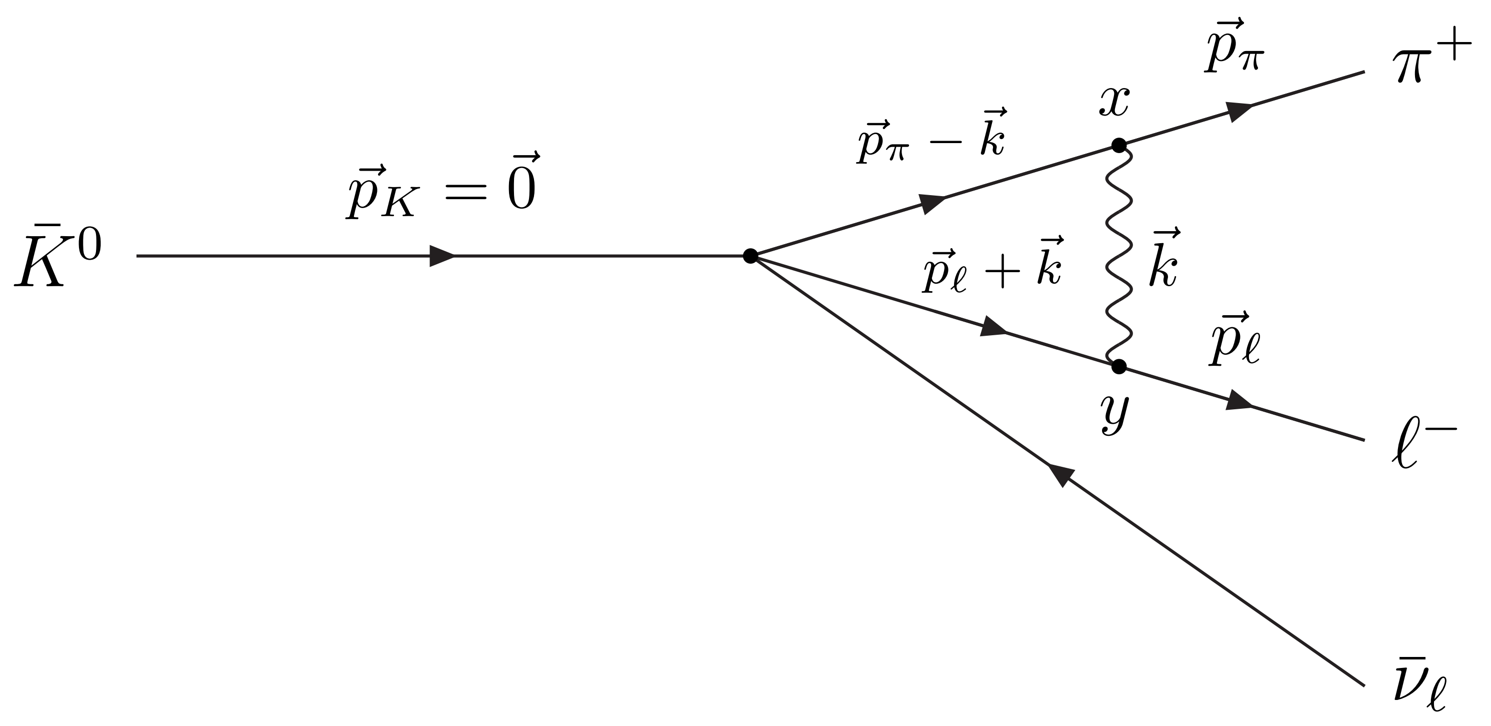
Performing the inverse Fourier transform:

$$\frac{1}{2E_p}\langle K^+(\vec{0}\,)|J^\mu(0)|K^+(\vec{p})\rangle\langle K^+(\vec{p})|J^\nu(0)|K^+(\vec{0}\,)\rangle = e^{(E_p-m_K)t_s}\int d^3z\, e^{-i\vec{p}\cdot\vec{z}}H_2(\vec{z}, t_s)$$

So that finally:

$$H_2(z, t_z)|_{tz>ts} \quad = \int \frac{d^3p}{(2\pi)^3}\int d^3z' H_2(\vec{z}\,', t_s)e^{-(E_p-m_K)(t_z-t_s)}\,e^{i\vec{p}\cdot(\vec{z}-\vec{z}\,')} \qquad \left(E_p = \sqrt{\vec{p}^2 + m_K^2}\right)$$

IVR and $K_{\ell 3}$ Decays



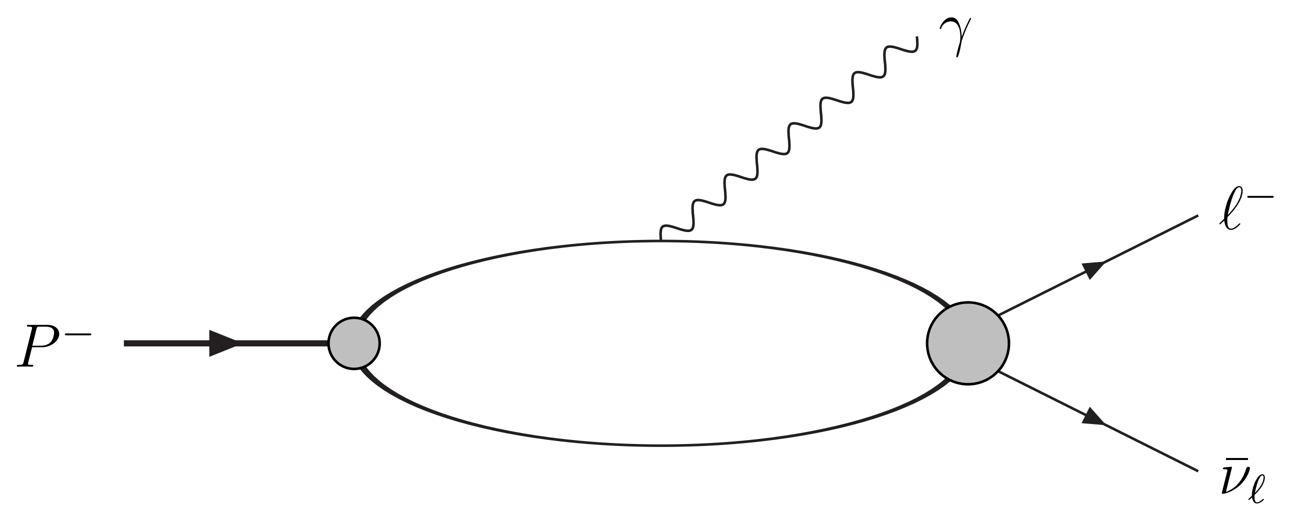
- For $K_{\ell 3}$ decays, the amplitude contains an imaginary part which creates the following difficulties in a conventional lattice calculation:

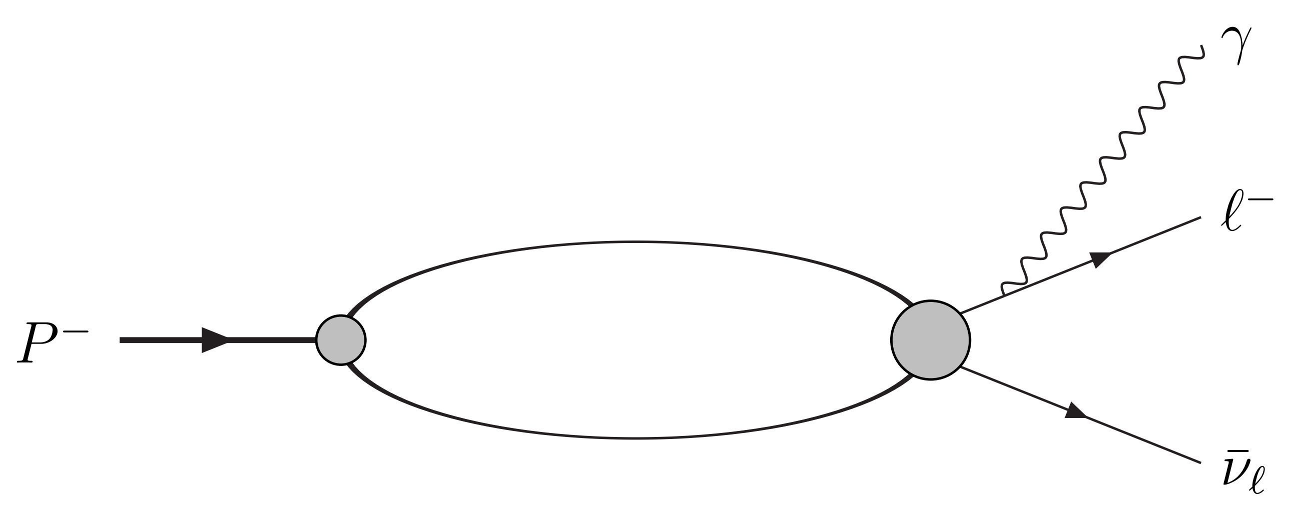
- In a finite Euclidean space, the presence of internal $\pi\ell$ states with energies lower that those of the external $\pi\ell$ pair $\Rightarrow$ the corresponding contributions grow exponentially with x_4 relative to the pure QCD diagram leading to dominance by unphysical terms.

- In a finite volume Euclidean calculation the imaginary contribution to the amplitude is missed and the approximation of the principle part by a discrete sum introduced potentially large finite-volume corrections.

- In an appendix to our paper we sketch how these difficulties are overcome by using IVR.

$P \to \ell \nu_\ell \gamma$ radiative decays - the form factors





Non-perturbative contribution to $P \to \ell \bar{\nu}_\ell \gamma$ is encoded in:

$$H_W^{\alpha r}(k, \vec{p}) = \epsilon_\mu^r(k)\, H_W^{\alpha\mu}(k, \vec{p}) = \epsilon_\mu^r(k) \int d^4y\, e^{ik\cdot y}\, \mathrm{T}\, \langle 0 |\, j_W^\alpha(0)\, j_{\mathrm{em}}^\mu(y)\, | P(\vec{p}\,) \rangle$$

$$= \epsilon_\mu^r(k) \Bigg\{ \frac{H_1}{m_P} \left[k^2 g^{\mu\alpha} - k^\mu k^\alpha\right] + \frac{H_2}{m_P} \frac{\left[(p\cdot k - k^2)k^\mu - k^2(p-k)^\mu\right](p-k)^\alpha}{(p-k)^2 - m_P^2}$$

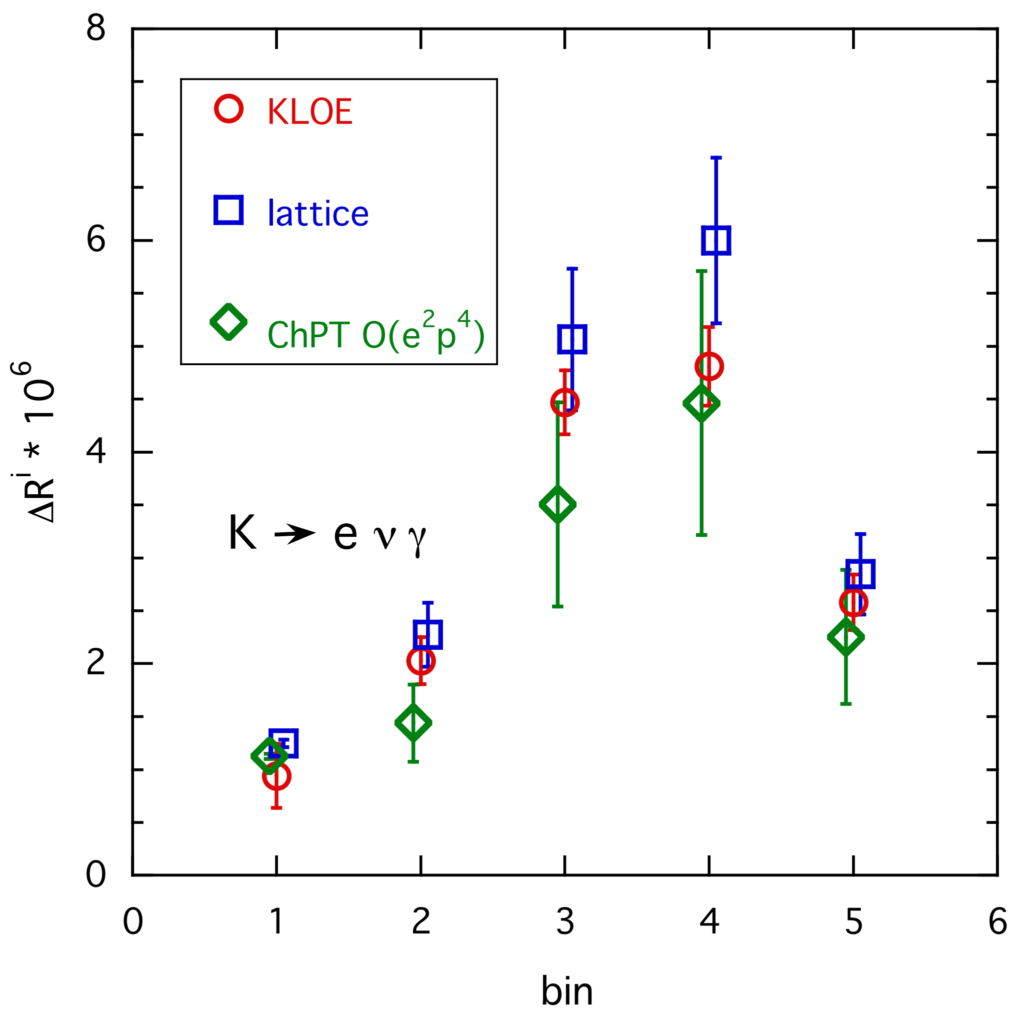
$$-i\frac{F_V}{m_P}\varepsilon^{\mu\alpha\gamma\beta} k_\gamma p_\beta + \frac{F_A}{m_P}\left[(p\cdot k - k^2)g^{\mu\alpha} - (p-k)^\mu k^\alpha\right] + f_P \left[g^{\mu\alpha} - \frac{(2p-k)^\mu (p-k)^\alpha}{(p-k)^2 - m_P^2} \right] \Bigg\}$$

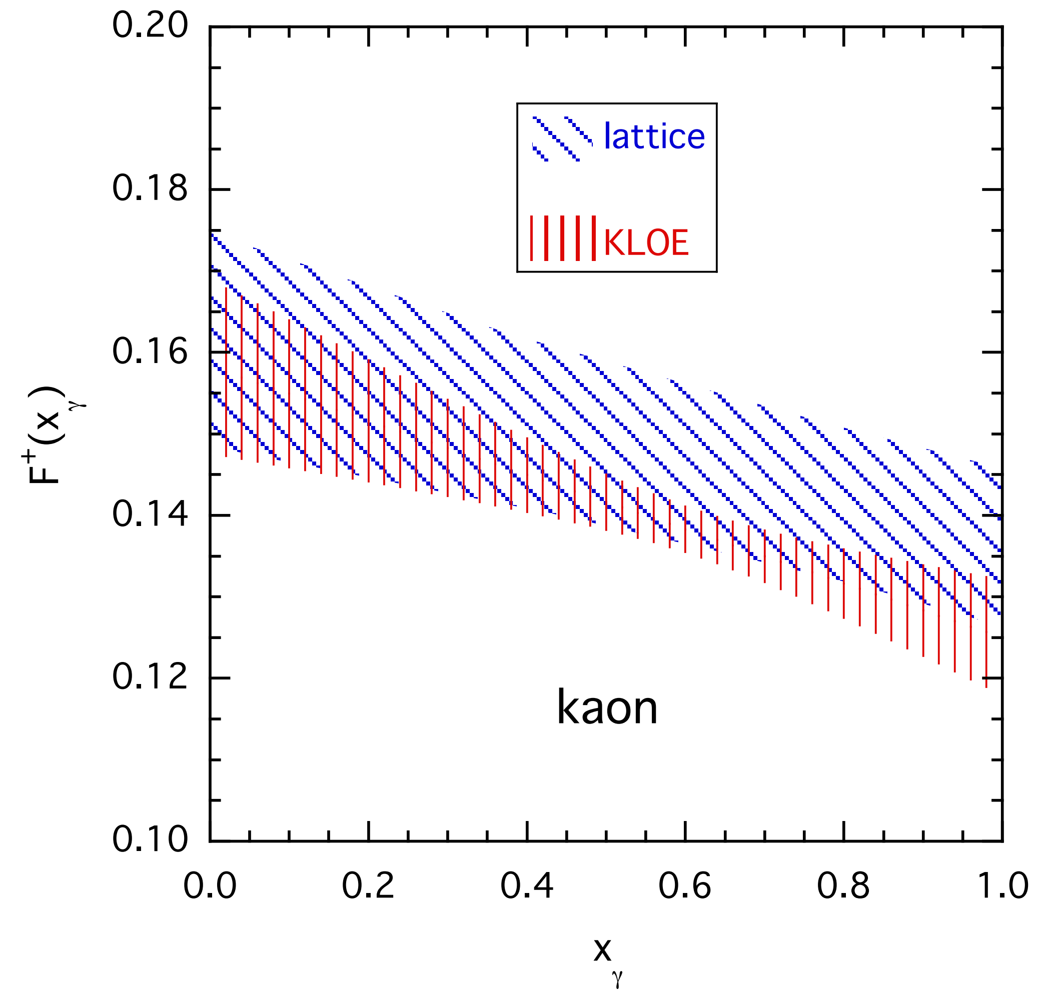
- For decays into a real photon, $k^2 = 0$ and $\varepsilon \cdot k = 0$, only the decay constant f_p and the vector and axial form factors $F_V(x_\gamma)$ and $F_A(x_\gamma)$ are needed to specify the amplitude ($x_\gamma = 2p\cdot k/m_P^2,\ 0 < x_\gamma < 1 - m_\ell^2/m_P^2$).

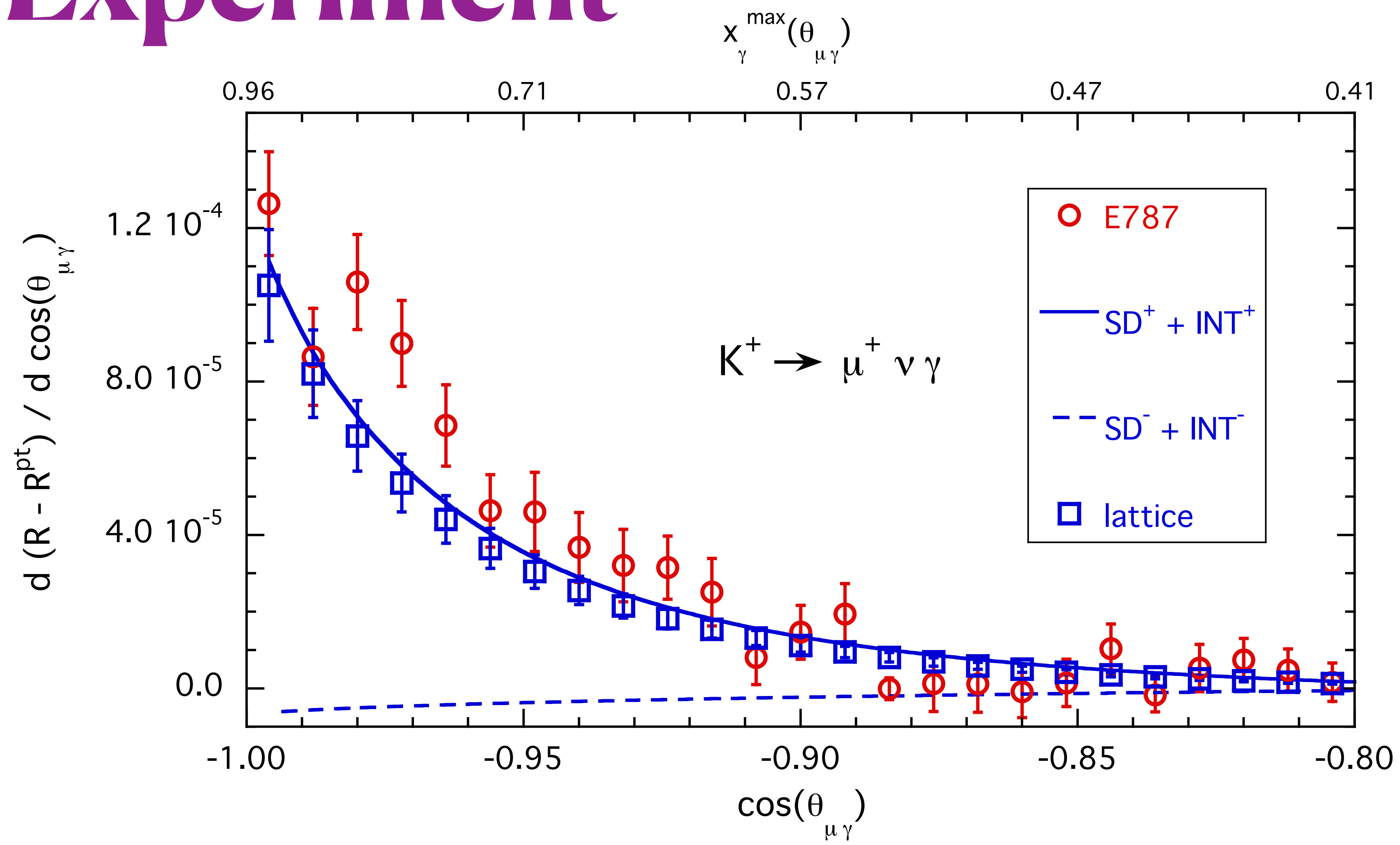
- In phenomenology $F^\pm \equiv F_V \pm F_A$ are more natural combinations.

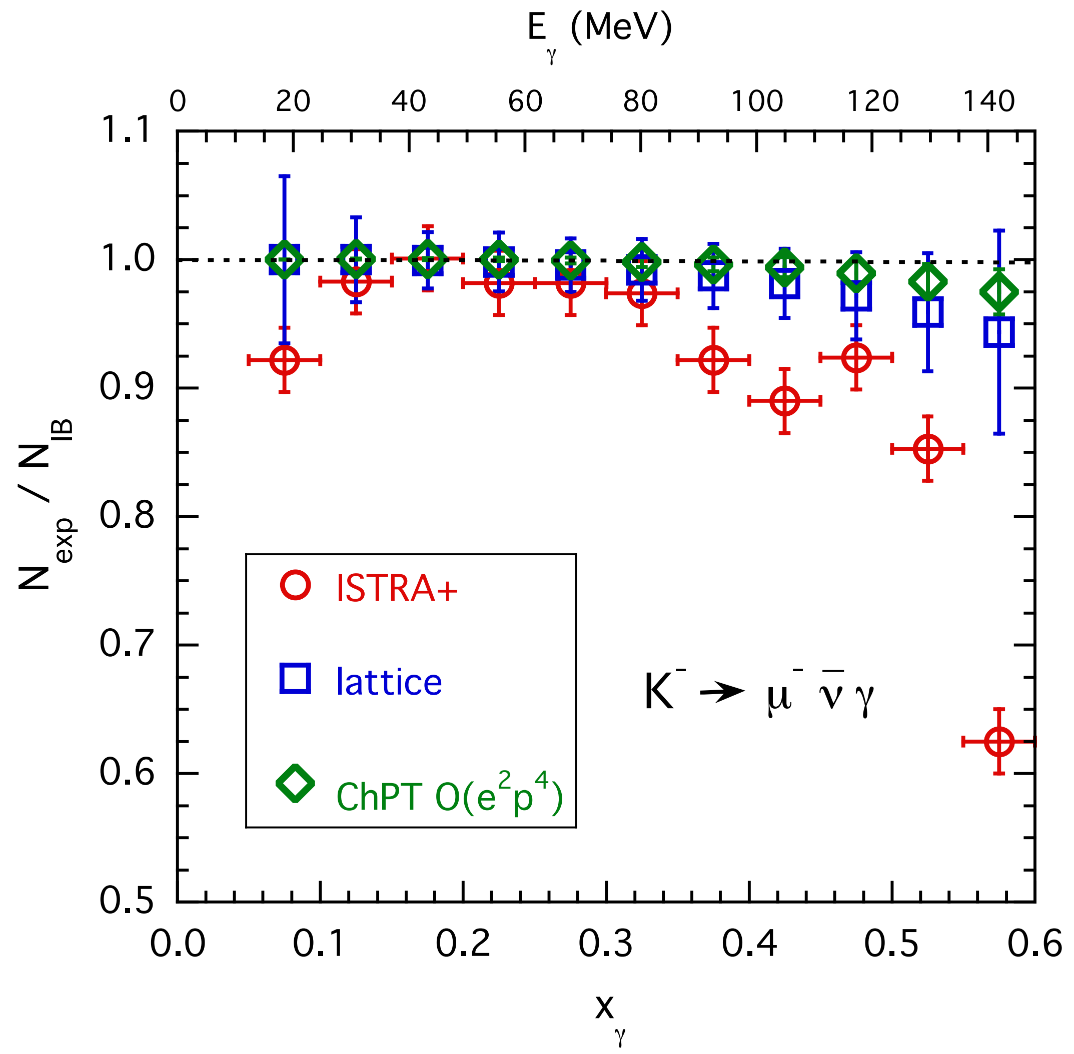
- We have computed $F_V(x_\gamma)$ and $F_A(x_\gamma)$ for $\pi, K, D_{(s)}$ mesons (and $H_{1,2}$ in an exploratory simulation for $K \to \pi \ell \nu_\ell \ell'^+ \ell'^-$ decays).

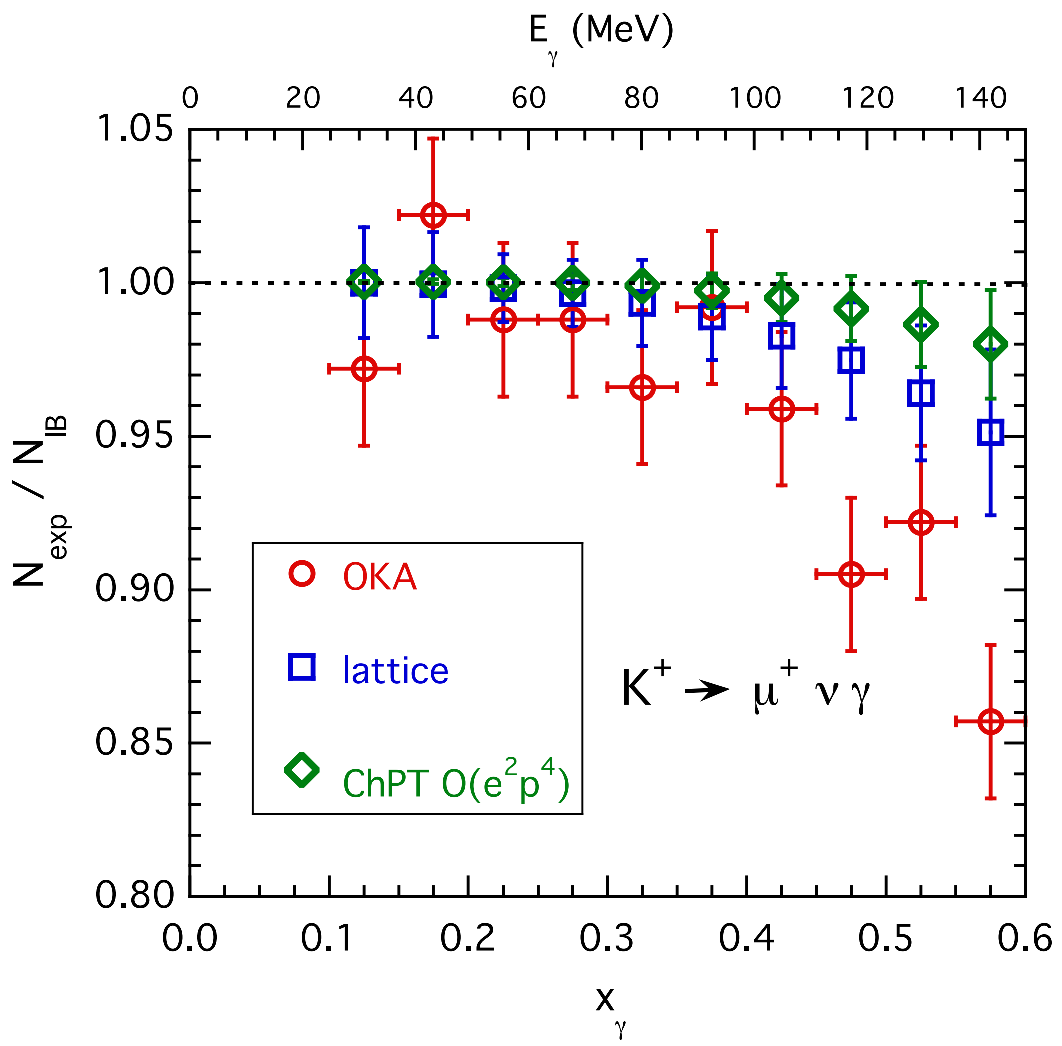
Comparison with Experiment











- Good Agreement with KLOE
- Significant tensions with $K \to \mu\nu_\mu\gamma$ experiments
- Unable to find a set of phenomenological form factors to account for all the data.
- NA62 will soon have the most precise results for $K \to e\nu_e\gamma$ decay rates.
- Is it conceivable that we have LFU-violation here also ?

A.Desiderio, R.Frezzotti, M.Garofalo, D.Giusti, M.Hansen, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula and N.Tantalo. arXiv:2006.05358
R.Frezzotti, M.Garofalo, V.Lubicz, G.Martinelli, CTS, F.Sanfilippo, S.Simula and N.Tantalo, arXiv:2012.02120

Conclusions and Prospects

- Electromagnetic corrections (and strong isospin breaking corrections) to leptonic decays are well underway. Precision will naturally improve with time.

    1. Currently the renormalization is of $O(\alpha_{\rm em}\,\alpha_s(M_W))$.
    2. Disconnected diagrams have not been evaluated (electroquenched appx.)

- IVR most likely to enable the computation of $K_{\ell 3}$ decay amplitudes.

    * Not generalisable (at present ?) to semileptonic decays of heavy mesons.

Congratulations to the MITP and to Matthias on their special jubilees and best wishes for many more fruitful years!