SCATTERING AMPLITUDES IN THE HIGH-ENERGY LIMIT

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INFN - University of Torino

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Fellini
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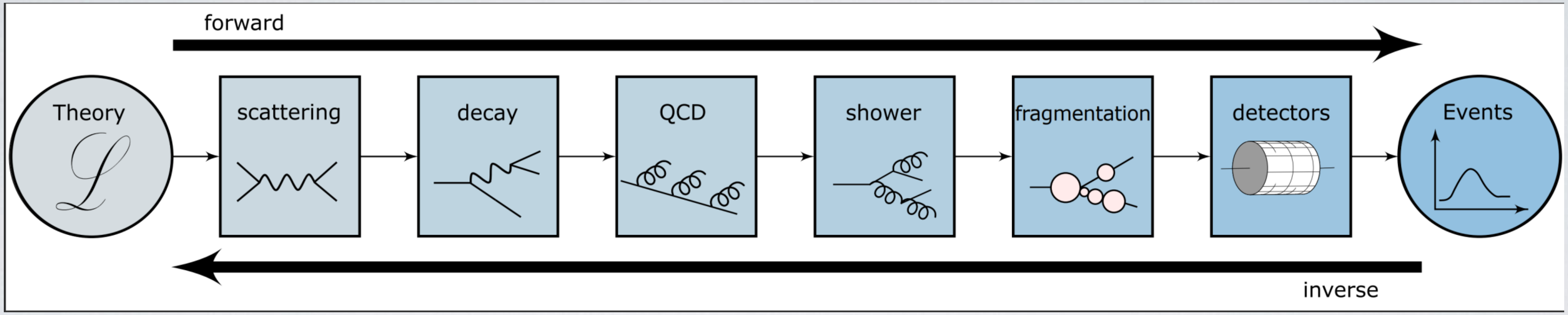
OUTLINE

- **Factorisation of amplitudes in the high-energy limit**
- **Scattering amplitudes by iterated solution of the BFKL equation**
- **The two-Reggeon cut: imaginary amplitude**
- **The three-Reggeon cut: real amplitude**

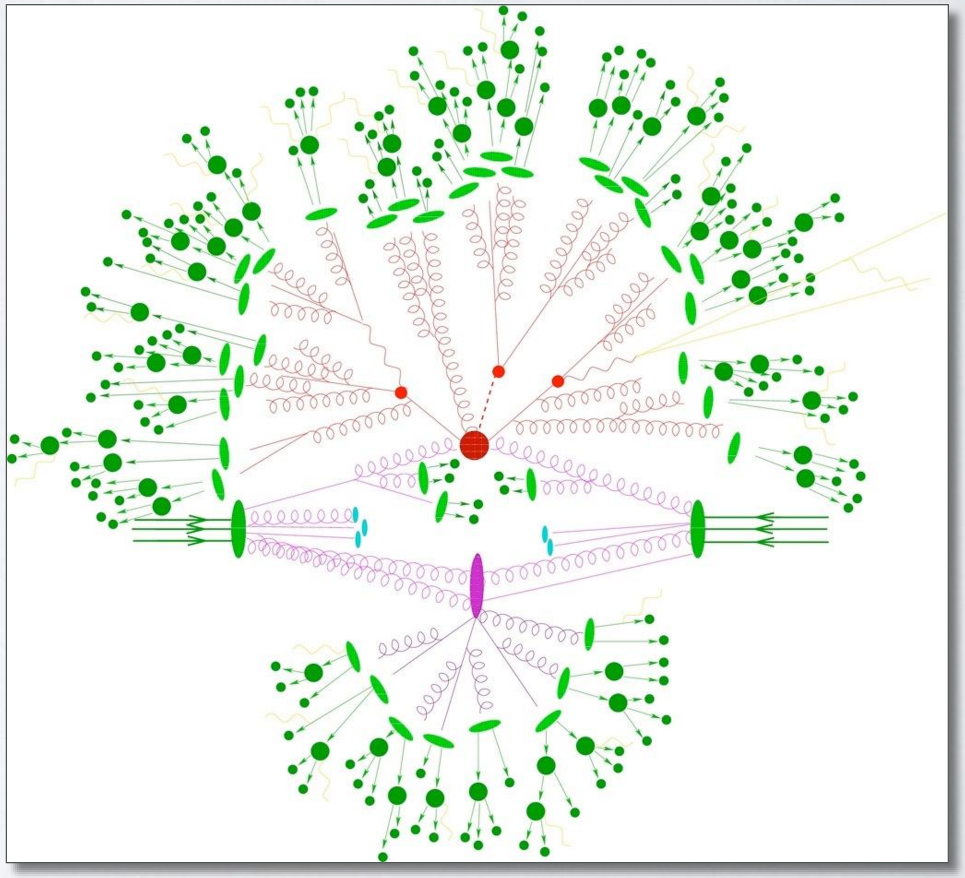
- *JHEP 1706 (2017) 016, [arXiv:1701.05241], with S. Caron-Huot and E. Gardi,*
- *JHEP 1803 (2018) 098, [arXiv:1711.04850], with S. Caron-Huot, E. Gardi, and J. Reichel,*
- *JHEP 08 (2020) 116, [arXiv:2006.01267], with S. Caron-Huot, E. Gardi and J. Reichel,*
- *JHEP 03 (2022), 053, with G. Falcioni, E. Gardi, N. Maher and C. Milloy,*
- *Phys. Rev. Lett. 128, (2022) no.13, with G. Falcioni, E. Gardi, N. Maher, C. Milloy.*

PRECISION FOR COLLIDER PHENOMENOLOGY



(Figure from 2203.07460)

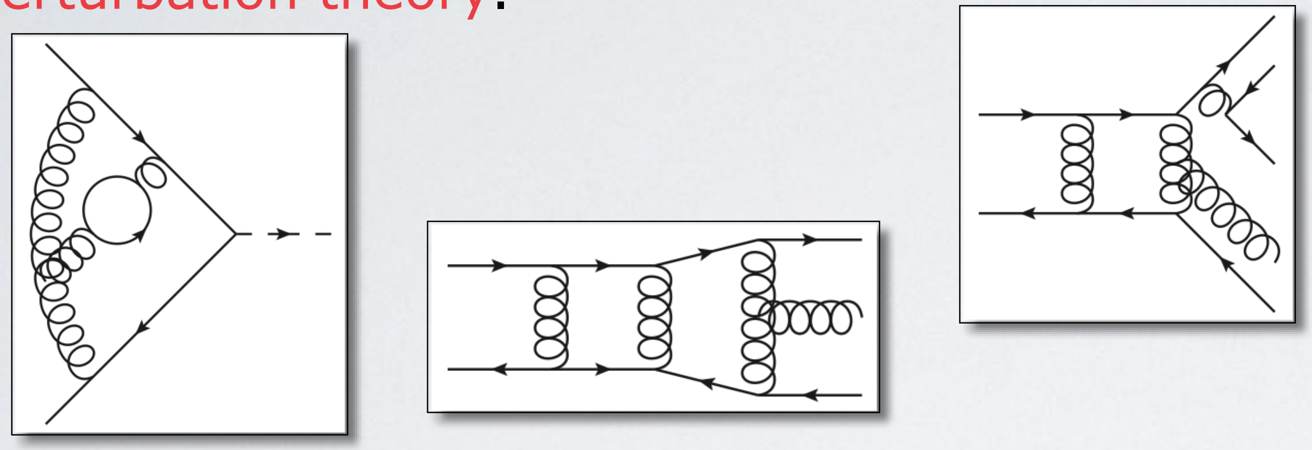
- Here we focus (mostly) on the first step in this chain: the perturbative calculation of hard scattering kernels. This task alone involves an incredible amount of work:



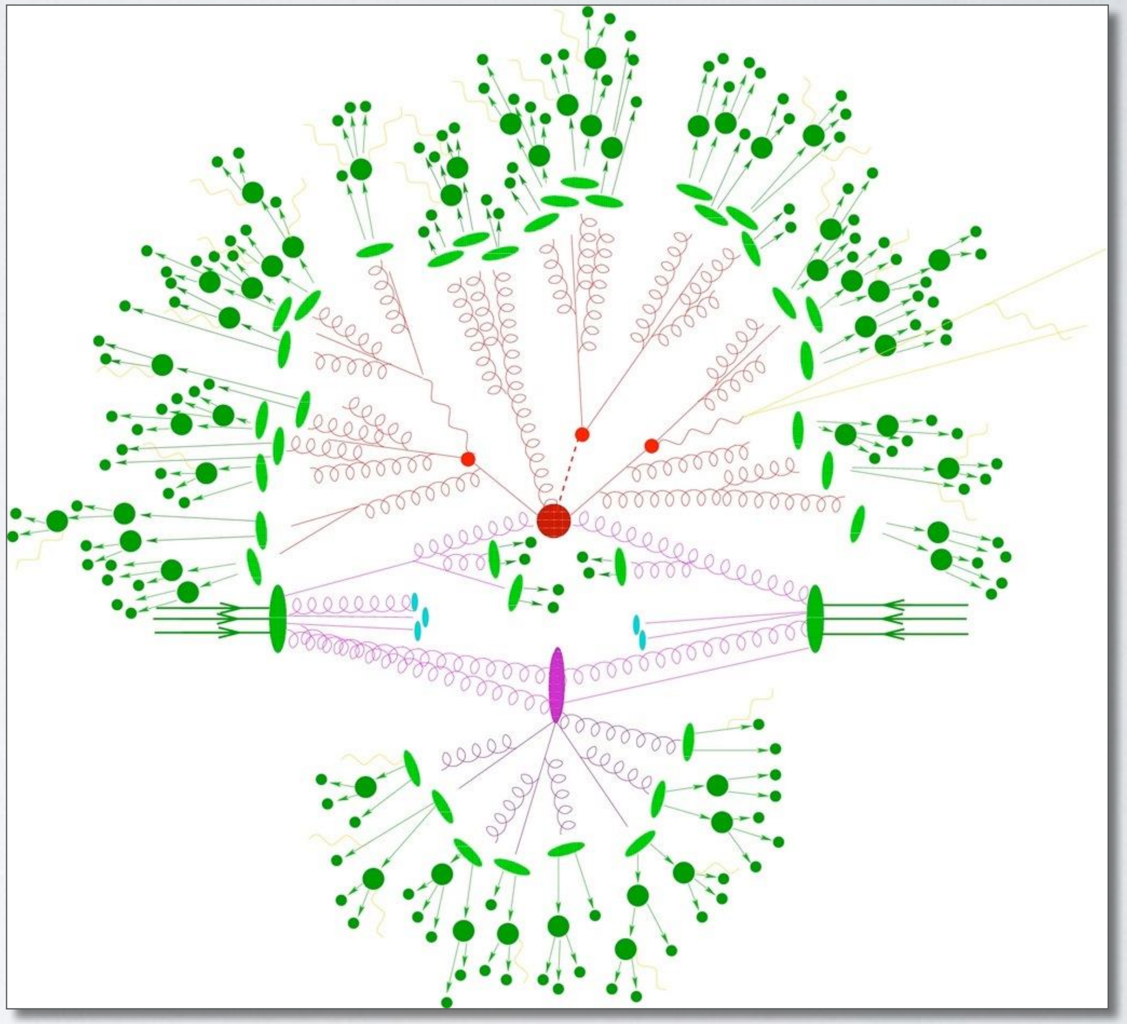
- QCD corrections
- Mixed QCD-EW correction
- Multi-loop and multi-leg processes
- Large logarithms
- SM vs SMEFT
- …

PRECISION FOR COLLIDER PHENOMENOLOGY

- Hard scattering processes are calculated in perturbation theory.



- Going beyond NNLO and N3LO is difficult, yet necessary to match the precision of current and forthcoming experiments!
- Loop and phase space integrals:
    - Analytic vs numerical evaluation
    - Space of functions
    - Infrared divergences
    - Large logarithms



PRECISION FOR COLLIDER PHENOMENOLOGY

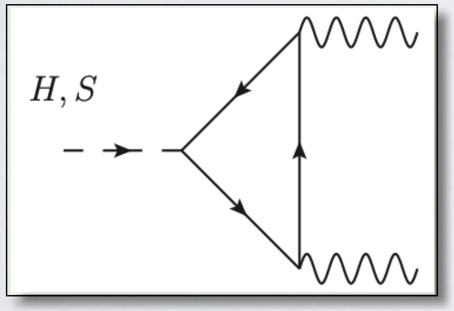
- The presence of largely different scales gives rise to large logarithms:

$$d\sigma \sim 1+\alpha_s(L^2+L+1)+\alpha_s^2(L^4+L^3+L^2+L+1)+\dots$$

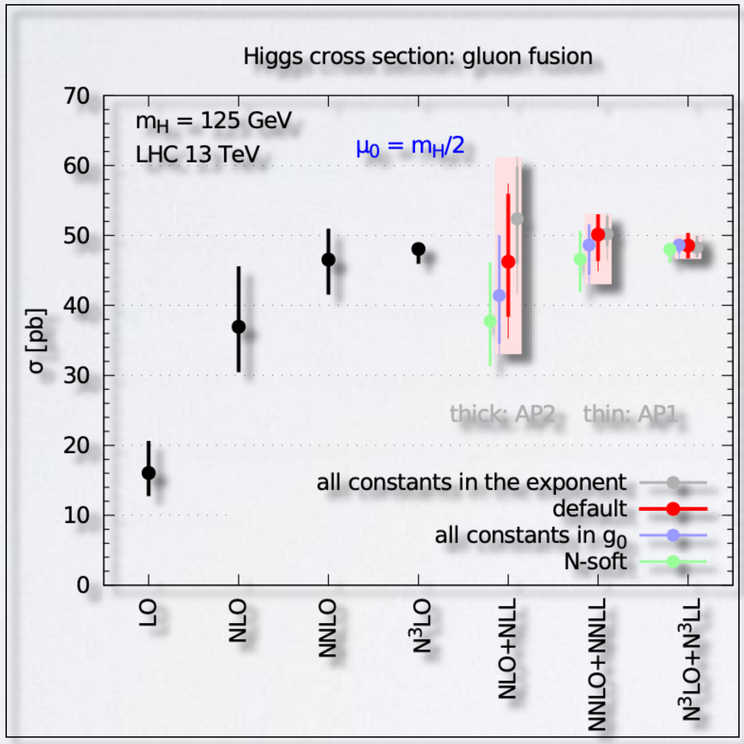
or

$$d\sigma \sim 1+\alpha_s(L+1)+\alpha_s^2(L^2+L+1)+\dots$$

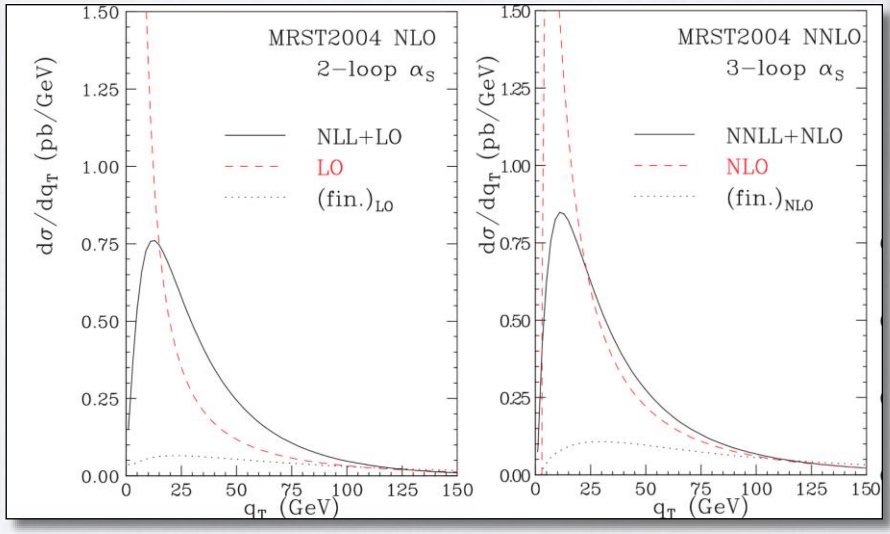
$$\sim \log^2 \frac{m_H^2}{m_b^2}$$



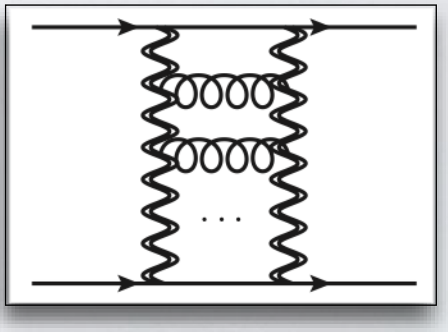
$$\sim \log^2(1-z)$$



$$\sim \log \frac{m_H^2}{p_T^2}$$



$$\sim \log \frac{s}{-t}$$



- Determine logarithms to all order:
  - → improve the convergence of the perturbative series;
  - → study perturbative corrections in a simplified kinematic regime!

PRECISION FOR COLLIDER PHENOMENOLOGY

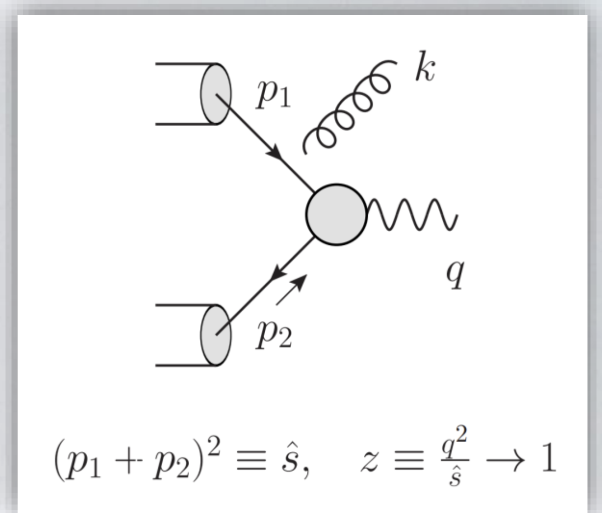
- Back in 2012 two towers of logarithms caught my attention:

→ Threshold logarithms at NLP:

$$\Delta_{ab}(z) = \sum_{n=0}^{\infty} \alpha_s^n \Big[c_n \delta(1-z)$$
$$+ \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z} \right]_+ \right.$$
$$\left. + d_{nm} \ln^m(1-z) \right) + \ldots \Big].$$

LP terms

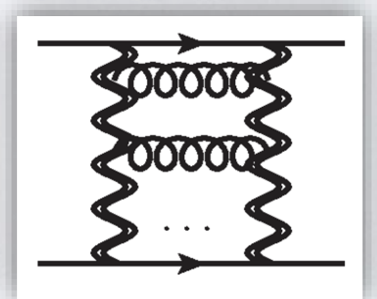
NLP terms



$(p_1+p_2)^2 \equiv \hat{s}, \quad z \equiv \frac{q^2}{\hat{s}} \to 1$

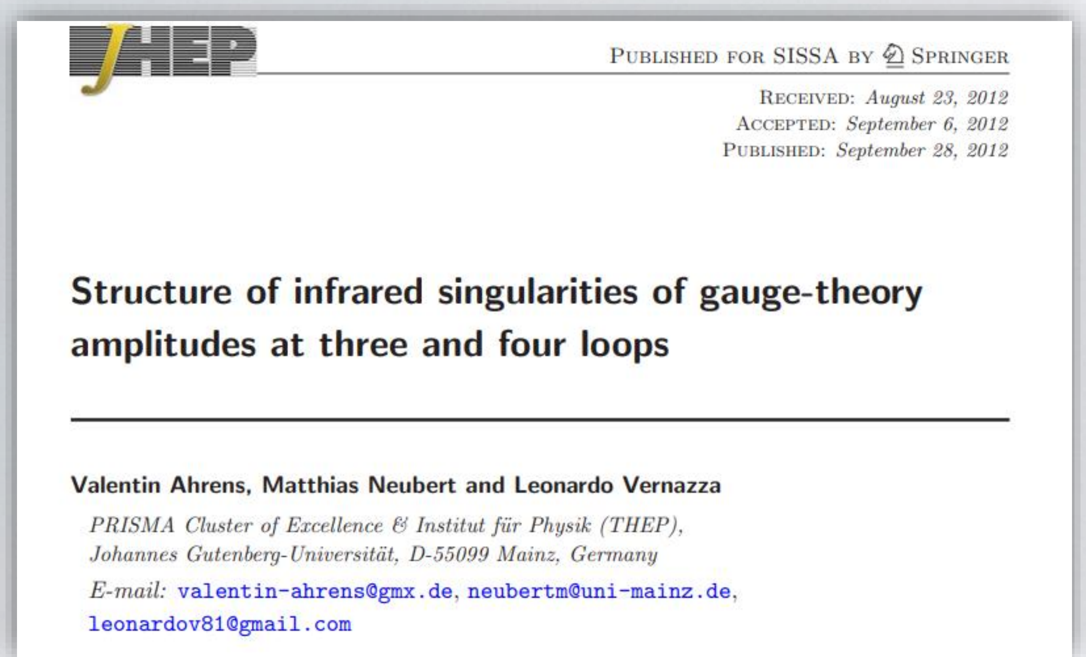
→ High-energy logarithms:

$$\mathcal{M}_{ij\to ij} = \mathcal{M}^{(0)} + \underbrace{\begin{array}{c} \frac{\alpha_s}{\pi}\log\frac{s}{-t}\mathcal{M}^{(1,1)} \\ + \left(\frac{\alpha_s}{\pi}\right)^2 \log^2\frac{s}{-t}\mathcal{M}^{(2,2)} \end{array}}_{\text{LL}} + \underbrace{\begin{array}{c} \frac{\alpha_s}{\pi}\mathcal{M}^{(1,0)} \\ \left(\frac{\alpha_s}{\pi}\right)^2 \log\frac{s}{-t}\mathcal{M}^{(2,1)} \end{array}}_{\text{NLL}} + \underbrace{\left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{M}^{(2,0)}}_{\text{NNLL}} + \ldots$$



HAPPY BIRTHDAY MATTHIAS AND MITP!

Indeed, the first contact with high-energy logarithms was in a paper with Matthias!



JHEP
PUBLISHED FOR SISSA BY SPRINGER
RECEIVED: *August 23, 2012*
ACCEPTED: *September 6, 2012*
PUBLISHED: *September 28, 2012*

Structure of infrared singularities of gauge-theory amplitudes at three and four loops

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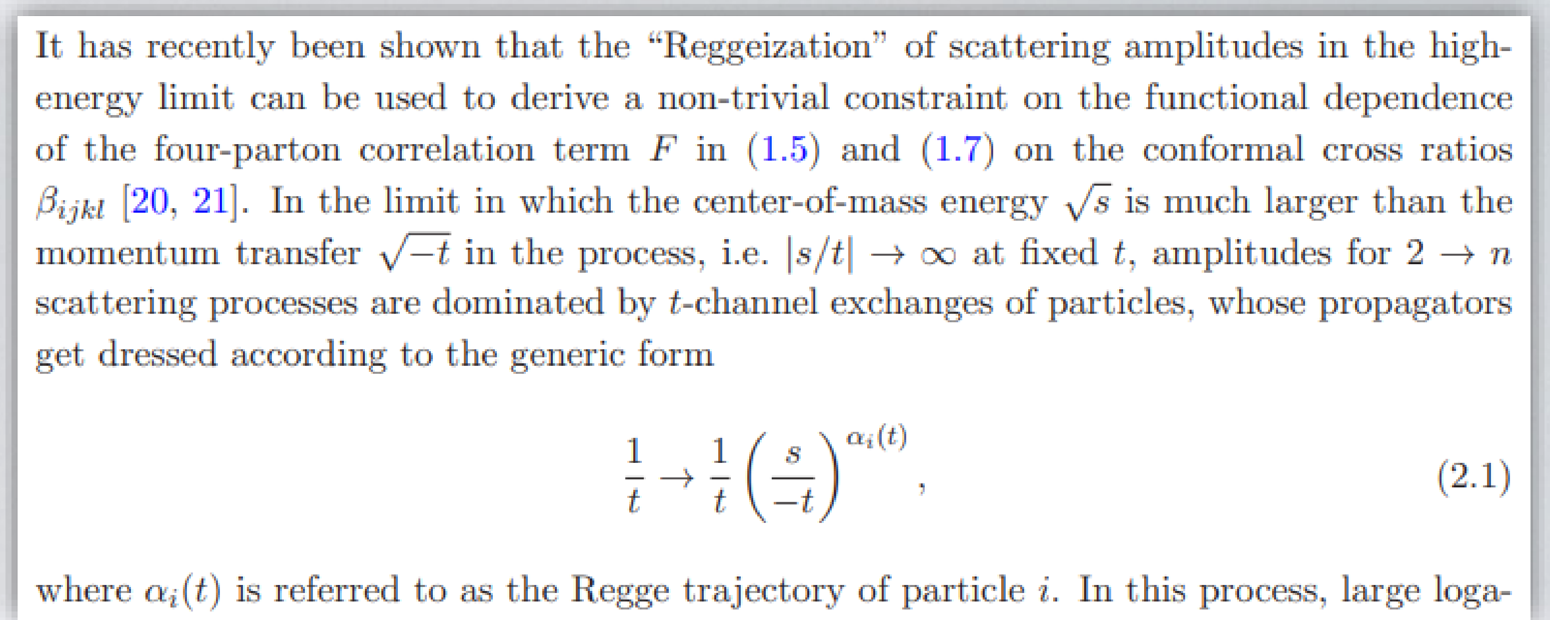
E-mail: `valentin-ahrens@gmx.de`, `neubertm@uni-mainz.de`,
`leonardov81@gmail.com`

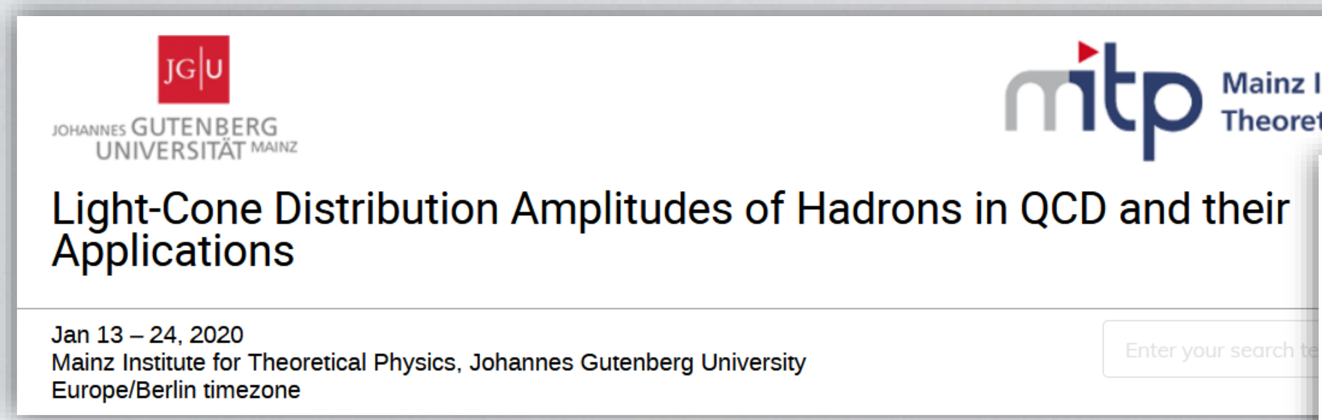


It has recently been shown that the "Reggeization" of scattering amplitudes in the high-energy limit can be used to derive a non-trivial constraint on the functional dependence of the four-parton correlation term F in (1.5) and (1.7) on the conformal cross ratios β_{ijkl} [20, 21]. In the limit in which the center-of-mass energy $\sqrt{s}$ is much larger than the momentum transfer $\sqrt{-t}$ in the process, i.e. $|s/t| \to \infty$ at fixed t, amplitudes for $2 \to n$ scattering processes are dominated by t-channel exchanges of particles, whose propagators get dressed according to the generic form

$$\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha_i(t)}, \qquad (2.1)$$

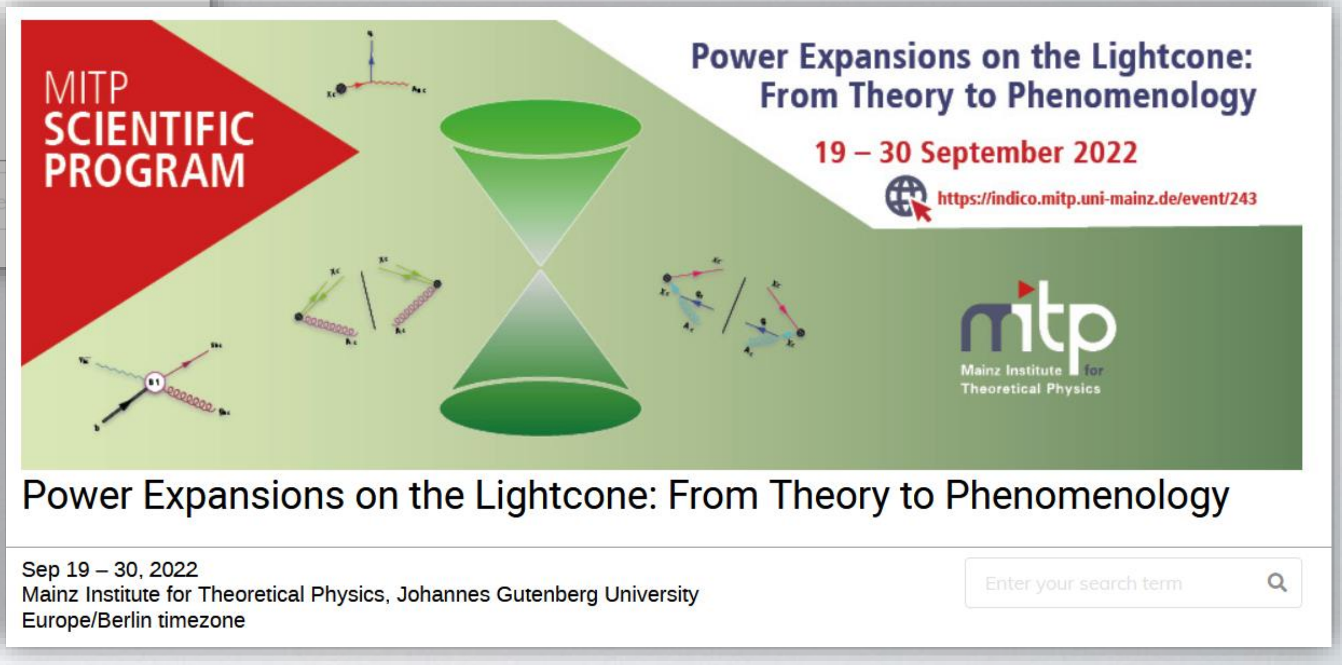
where $\alpha_i(t)$ is referred to as the Regge trajectory of particle i. In this process, large loga-

Over the years the MITP programs have been a great occasion to discuss these topics!



Light-Cone Distribution Amplitudes of Hadrons in QCD and their Applications

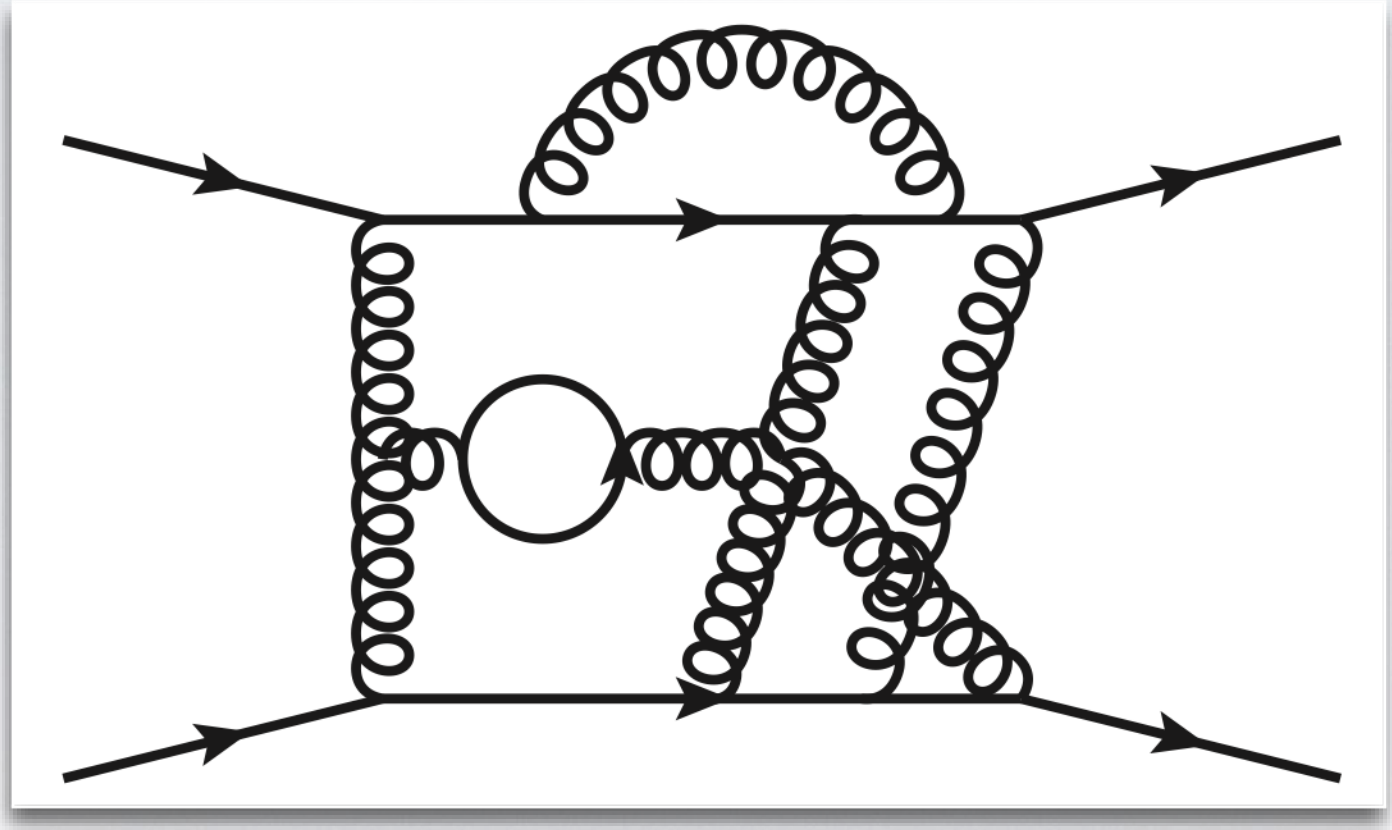
Jan 13 – 24, 2020
Mainz Institute for Theoretical Physics, Johannes Gutenberg University
Europe/Berlin timezone



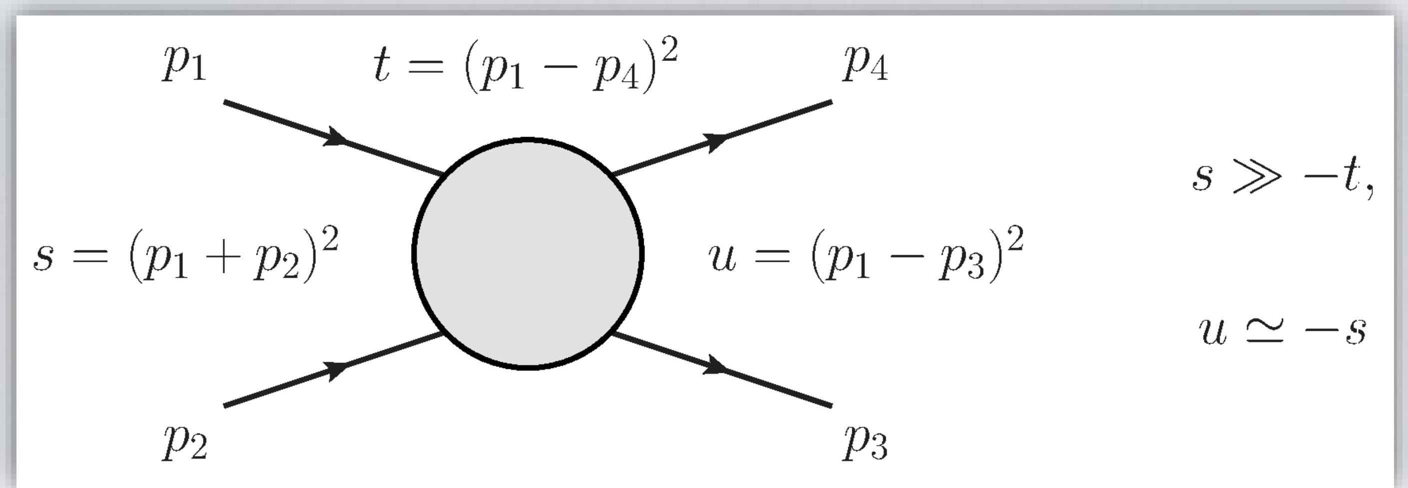
Power Expansions on the Lightcone: From Theory to Phenomenology

Sep 19 – 30, 2022
Mainz Institute for Theoretical Physics, Johannes Gutenberg University
Europe/Berlin timezone

FACTORISATION OF AMPLITUDES IN THE HIGH-ENERGY



TWO-PARTON SCATTERING AMPLITUDES



- Expansion in the strong coupling and in towers of (large) logarithms:

$$\mathcal{M}_{ij\to ij} = \mathcal{M}^{(0)} + \frac{\alpha_s}{\pi}\log\frac{s}{-t}\mathcal{M}^{(1,1)} + \frac{\alpha_s}{\pi}\mathcal{M}^{(1,0)}$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \log^2\frac{s}{-t}\mathcal{M}^{(2,2)} + \left(\frac{\alpha_s}{\pi}\right)^2 \log\frac{s}{-t}\mathcal{M}^{(2,1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{M}^{(2,0)} + \ldots$$

LL NLL NNLL

- Goal: develop a theory to calculate systematically the tower of logarithms at any order in the strong coupling expansion.

HIGH-ENERGY LIMIT

- Very interesting theoretical problem:
  - toy model for full amplitude, yet

    → retain rich dynamic in the 2D transverse plane,

    → non-trivial function spaces;
  - Understand the high-energy QCD asymptotic in terms of Regge poles and cuts;
  - predict amplitudes and other observables in overlapping limits:

    → soft limit, infrared divergences.

→ See talk by E. Gardi

MRK in N=4 SYM:
Dixon, Pennington, Duhr, 2012;
Del Duca, Dixon, Pennington, Duhr, 2013;
Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2019

- Relevant for phenomenology at the LHC and future colliders:
  - perturbative phenomenology of forward scattering, e.g.

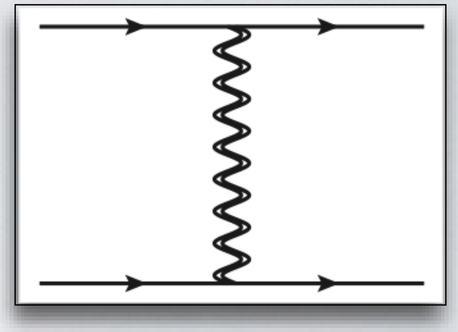
    → Deep inelastic scattering/saturation (small x = Regge, large Q^2 = perturbative),

    → Mueller-Navelet: pp → X+2jets, forward and backward.

See e.g. ***Andersen, Smillie, 2011; Andersen, Medley Smillie, 2016; Andersen, Hapola, Maier, Smillie, 2017; …***

TWO-PARTON SCATTERING AMPLITUDES

- LL tower: one-Reggeon exchange in the t-channel
(Regge pole in the complex angular momentum plane)



$$\frac{1}{t} \to \frac{1}{t}\left(\frac{s}{-t}\right)^{\frac{\alpha_s C_A}{\pi}\frac{r_\Gamma}{\epsilon}}$$

Regge, Gribov ~ 1960;
Lipatov; Fadin,Kuraev,Lipatov 1976

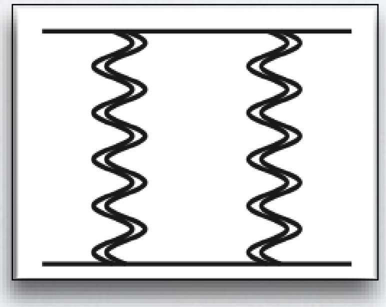
- LL amplitude

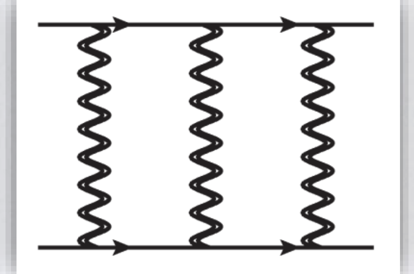
$$\mathcal{M}^{\mathrm{LL}} = e^{\frac{\alpha_s C_A L}{\pi}\frac{r_\Gamma}{\epsilon}}\mathcal{M}^{(0)}, \qquad r_\Gamma = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- Real amplitude at NLL: described by BFKL:

Fadin,Kuraev,Lipatov 1975-77; Balitsky,Lipatov 1978

- Beyond real NLL: compound states of multiple-Reggeon exchanges.





SCATTERING IN THE HIGH-ENERGY LIMIT

- Multiple Reggeon exchange contribution in scattering amplitudes elusive, until recently.
- First evidence of violation of Regge-pole factorization in

  Del Duca, Glover 2001;

- Interplay with the infrared factorization theorem investigated in

  Del Duca, Duhr, Gardi, Magnea, White 2011; Del Duca, Falcioni, Magnea, LV, 2013, 2014;

- High-energy scattering via Wilson lines:

  Korchemskaya, Korchemsky, 1994,1996; Balitsky 1995; Babansky, Balitsky 2002;

- Two-parton scattering from rapidity evolution of Wilson lines

  Caron-Huot, 2013; Caron-Huot, Gardi, LV, 2017; Caron-Huot, Gardi, Reichel, LV, 2017, 2020; Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021,2022.

  → This talk

- SCET-based formulation in

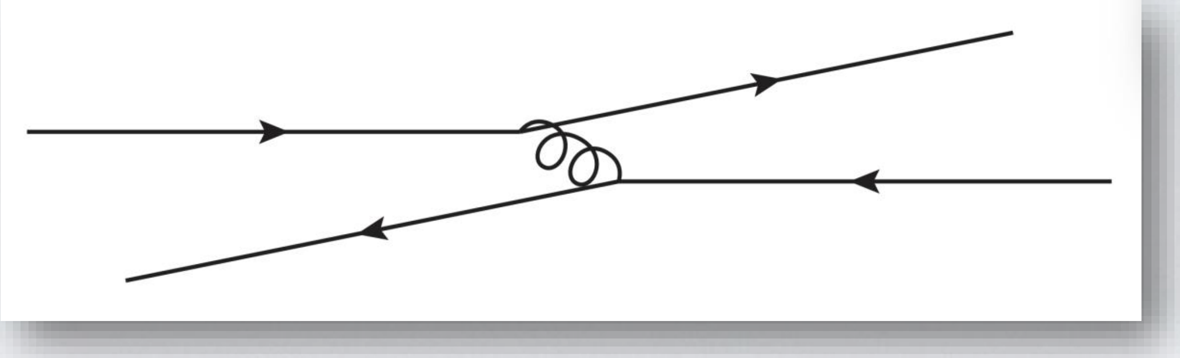
  Rothstein, Stewart 2016; Ridgway, Moult, Stewart, 2019, 2020.

- Calculation of multiple Reggeon exchanges within QCD also obtained in

  Fadin, Lipatov 2017; Fadin 2019, 2020.

FROM BALITSKY-JIMWLK TO AMPLITUDES

- The physical picture: high-energy limit = forward scattering:



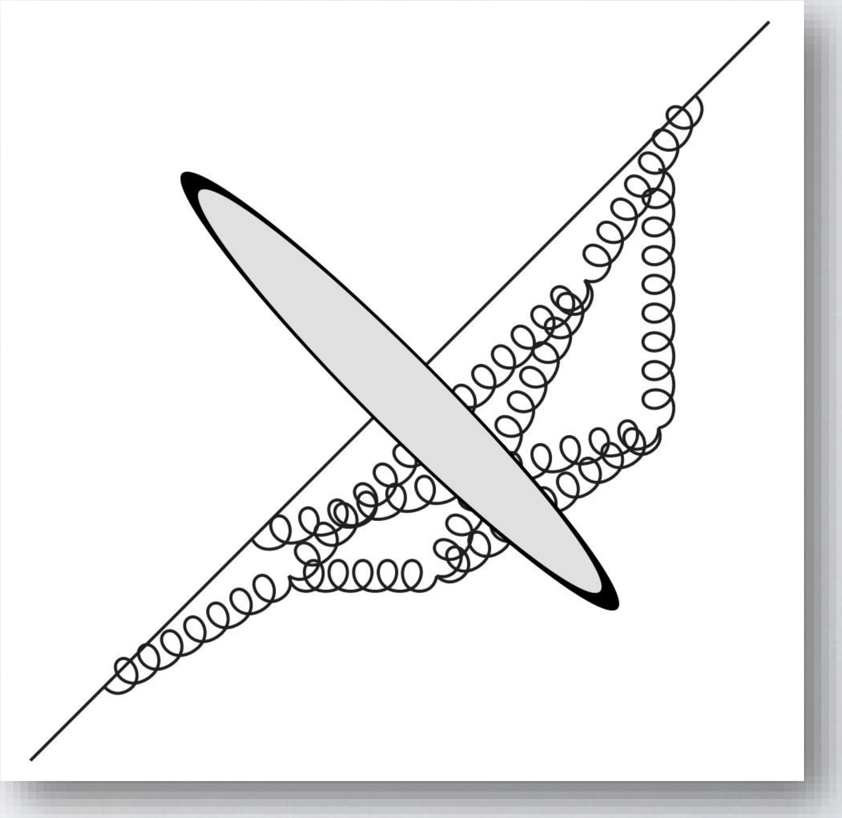
Korchemskaya,
Korchemsky, 1994, 1996;
Babansky, Balitsky, 2002;
Caron-Huot, 2013

- To leading power, the fast projectile and target described in terms of Wilson lines:

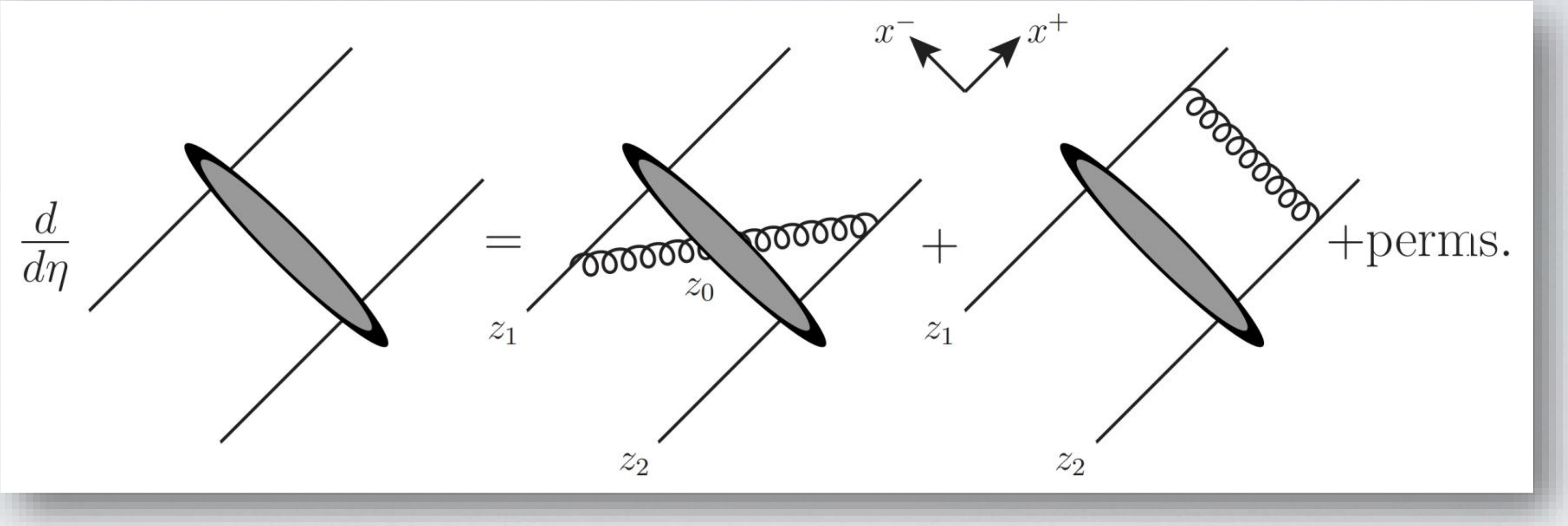
$$U(z_\perp) = \mathcal{P}\exp\left[ig_s\int_{-\infty}^{+\infty} A_+^a(x^+, x^-{=}0, z_\perp)dx^+ T^a\right].$$

- Upon evolution in energy (rapidity), emitted radiation gives additional Wilson lines!

$$\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$$



FROM BALITSKY-JIMWLK TO AMPLITUDES



- This is expressed by the (nonlinear!) Balitsky-JIMWLK evolution equation:

$$\frac{d}{d\eta}UU \sim g_s^2 \int d^2z_0\, K(z_0, z_1, z_2)\, \Big[U(z_0)UU - UU\Big].$$

***NLL:* Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016; *(some) NNLL:* Caron-Huot, Gardi, Vernazza, 2017.**

- Shock = Lorentz-contracted target;
- 45° lines = fast projectile partons;
- Each parton crossing the shock gets a Wilson line
- Evolution in rapidity resums the high-energy log: $\eta = L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}.$

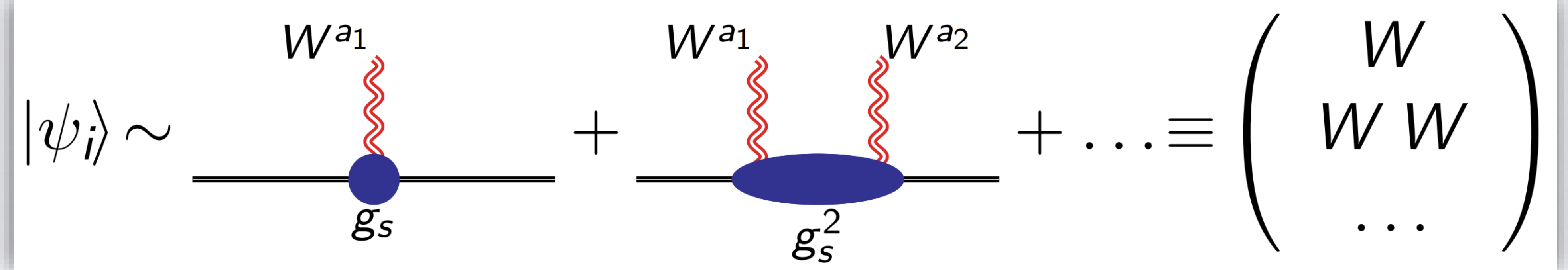
FROM BALITSKY-JIMWLK TO AMPLITUDES

- The Balitsky-JIMWLK equation is non-linear: leads to the phenomenon of saturation.
- For scattering amplitudes, we can consider the dilute regime: expand Wilson lines around unity in an effective degree of freedom dubbed as "Reggeon":

$$U^{\eta}(z_\perp) = \mathcal{P}\exp\left[ig_s\,\mathbf{T}^a\int_{-\infty}^{+\infty}dx^+\,A_+^a(x^+,x^-=0,z_\perp)\right] \equiv e^{ig_s\,\mathbf{T}^a\,W^a(z_\perp)}\,.$$

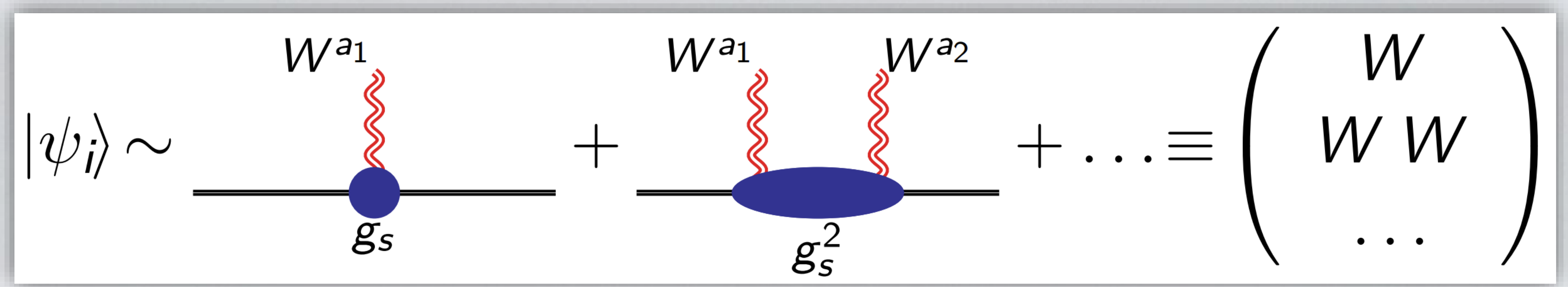
Caron-Huot, 2013

  - T^a group generator in the parton representation
  - $\eta = L$ (implicit) cutoff
- Scattering states (target and projectile) are expanded in Reggeon fields W^a:



FROM BALITSKY-JIMWLK TO AMPLITUDES

- Scattering states (target and projectile) are expanded in Reggeon fields W^a:



- Evolution in rapidity resums the high-energy log:

$$\frac{d}{dL}|\psi_i\rangle = -H|\psi_i\rangle.$$

H = Balitsky-JIMWLK Hamiltonian

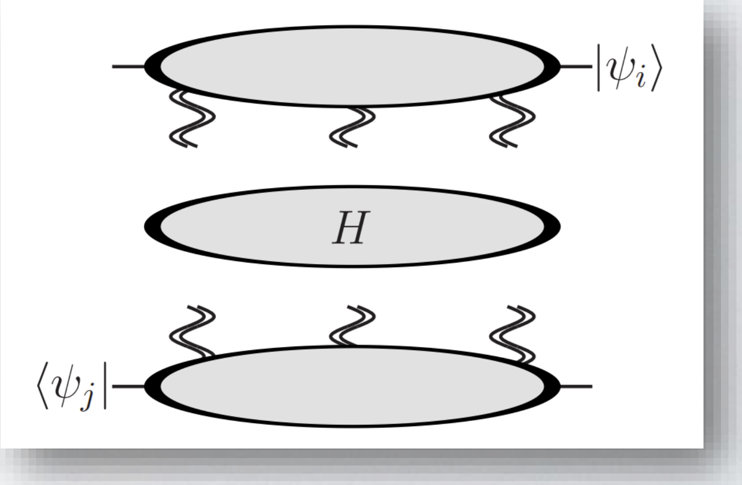
NLL:* *Balitsky Chirilli, 2013; Kovner, Lublinsky, Mulian, 2013, 2014, 2016; (some) NNLL: Caron-Huot, Gardi, Vernazza, 2017.

- Scattering amplitude: expectation value of Wilson lines evolved to equal rapidity:

$$\frac{i}{2s}\frac{1}{Z_i Z_j}\mathcal{M}_{ij\to ij} = \langle\psi_j|e^{-LH}|\psi_i\rangle.$$

(Z_i = collinear poles)

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017



FROM BALITSKY-JIMWLK TO AMPLITUDES

- Structure of the leading-order Balitsky-JIMWLK equation:

$$H\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix} = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & 0 & H_{5\to1} & \dots \\ 0 & H_{2\to2} & 0 & H_{4\to2} & 0 & \dots \\ H_{1\to3} & 0 & H_{3\to3} & 0 & H_{5\to3} & \dots \\ 0 & H_{2\to4} & 0 & H_{4\to4} & 0 & \dots \\ H_{1\to5} & 0 & H_{3\to5} & 0 & H_{5\to5} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}$$

$$\sim \begin{pmatrix} g_s^2 & 0 & g_s^4 & 0 & g_s^6 & \dots \\ 0 & g_s^2 & 0 & g_s^4 & 0 & \dots \\ g_s^4 & 0 & g_s^2 & 0 & g_s^4 & \dots \\ 0 & g_s^4 & 0 & g_s^2 & 0 & \dots \\ g_s^6 & 0 & g_s^4 & 0 & g_s^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} W \\ (W)^2 \\ (W)^3 \\ (W)^4 \\ (W)^5 \\ \dots \end{pmatrix}.$$

LO BFKL kernel

From LO B-JIMWLK

Terms in NNLO B-JIMWLK predicted by symmetry $H = H^T$

Caron-Huot, 2013, Caron-Huot, Gardi, LV, 2017

- At NLL we need $m \to m$ transition only → the LO BFKL kernel.
- At NNLL we need the $m \to m+2$ transition from the LO B-JIMWLK kernel.
- Define the reduced amplitude: subtract single-Reggeon exchange:

$$\frac{i}{2s}\hat{\mathcal{M}}_{ij\to ij} = \langle\psi_j|e^{-(H-H_{1\to1})L}|\psi_i\rangle \equiv \langle\psi_j|e^{-\hat{H}L}|\psi_i\rangle.$$

TWO PARTON SCATTERING AMPLITUDES

- Organizing principle: exploit symmetry under $s \leftrightarrow u$ exchange:
  → the amplitude decomposes into even (+) and odd (-) components under $s \leftrightarrow u$:

$$\mathcal{M}^{(\pm)}(s,t) = \tfrac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big).$$

- Expand the amplitude in terms of the signature-even combination of logarithms:

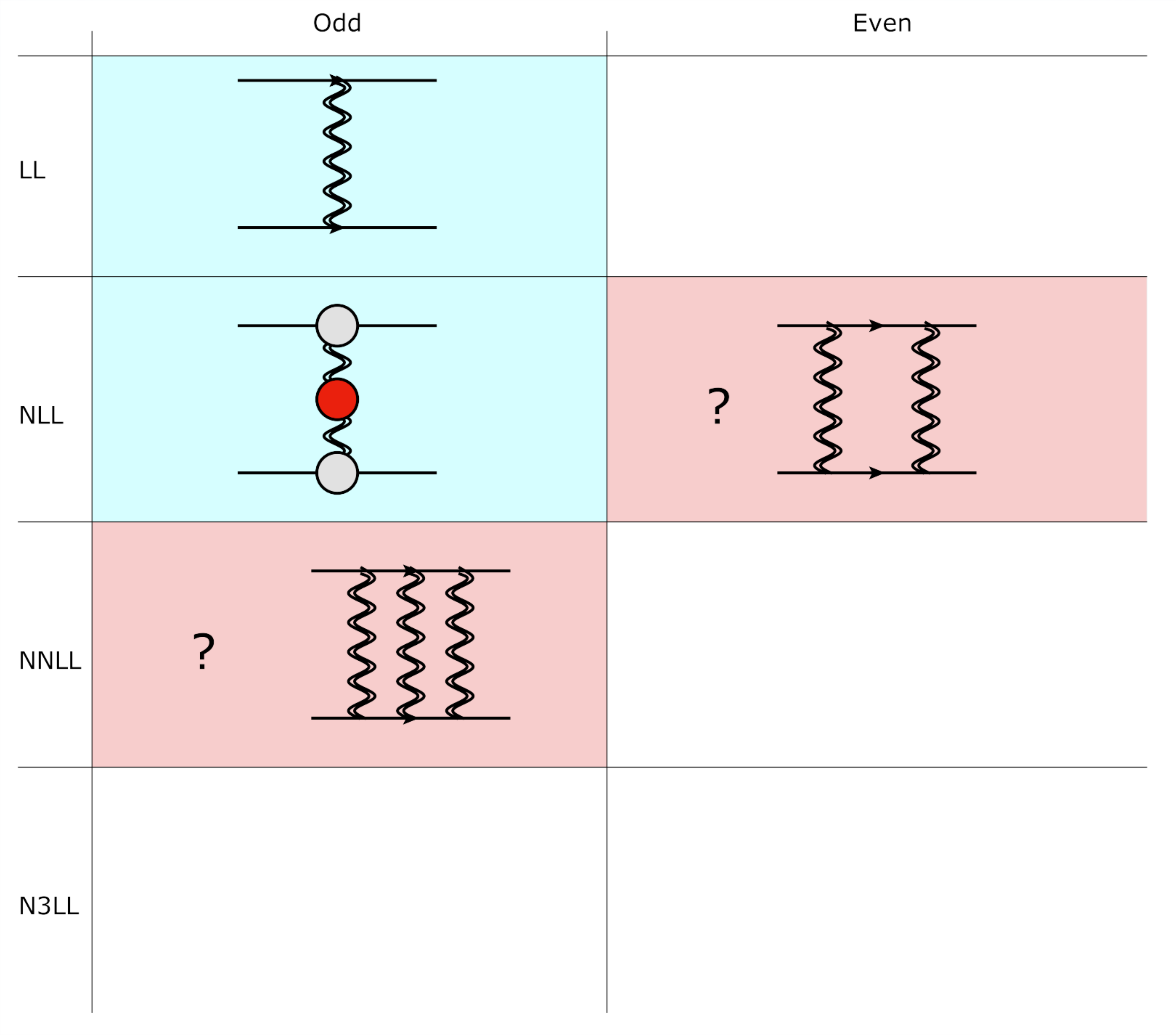
$$L \equiv \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} = \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) .$$

  → $M^{(+)}$ imaginary with even number of Reggeons
  → $M^{(-)}$ real with odd number of Reggeons

- Goals:
  - Calculate multiple Reggeon exchanges to high-order in perturbation theory
  - Understand the high-energy asymptotics of partonic amplitudes
  - Investigate implications for IR divergences
  - Do multiple Reggeon exchange exponentiate?

TWO PARTON SCATTERING AMPLITUDES

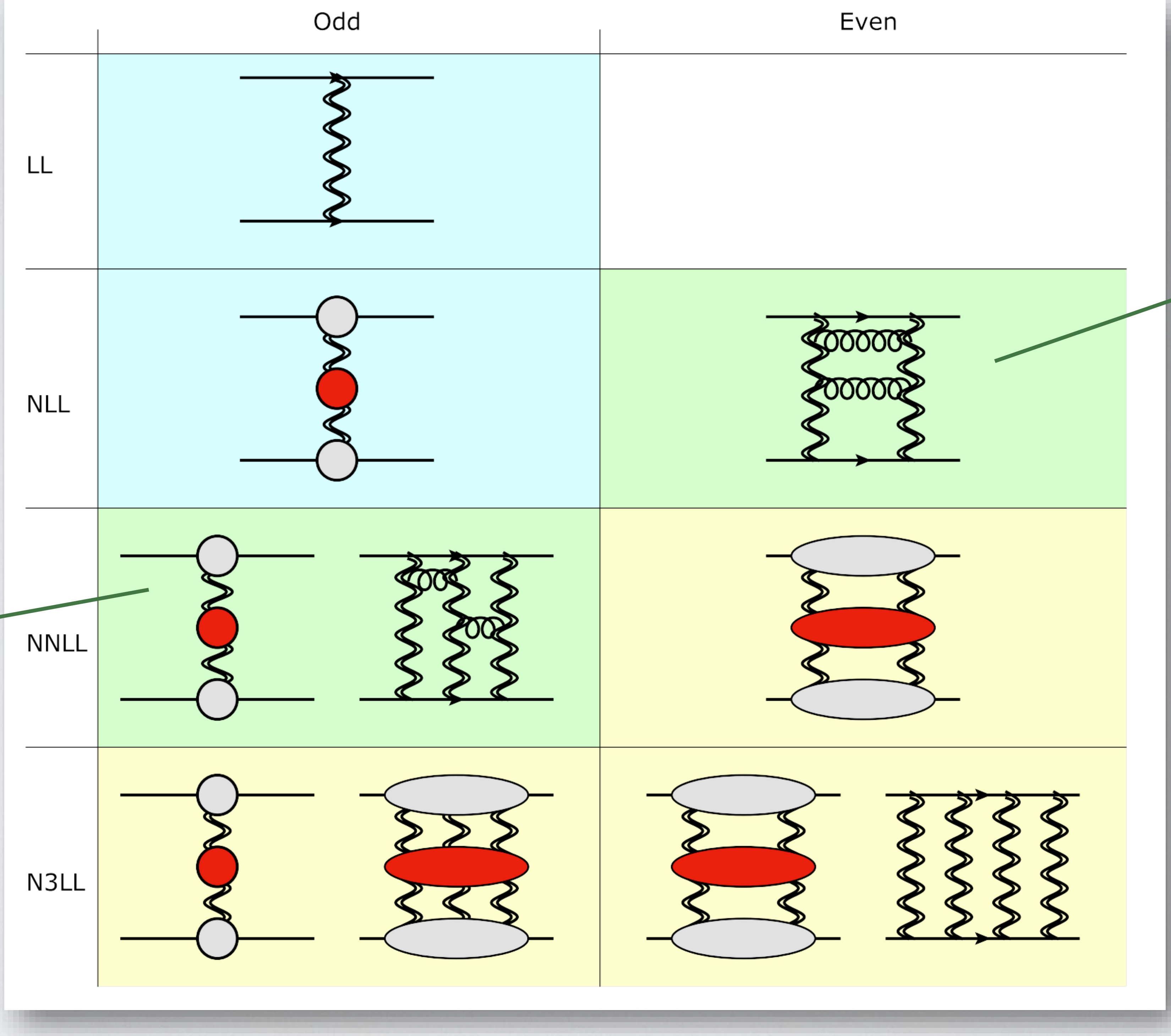
- Status pre ~ 2014:



	Odd	Even
LL		
NLL		?
NNLL	?	
N3LL		

TWO PARTON SCATTERING AMPLITUDES

- Developed a framework for the calculation of amplitudes in the high-energy limit;
- Systematic relation between logarithmic accuracy and number of Reggeons.



Analysed to 2 loops in Del Duca, Falcioni, Magnea, Vernazza 2014;

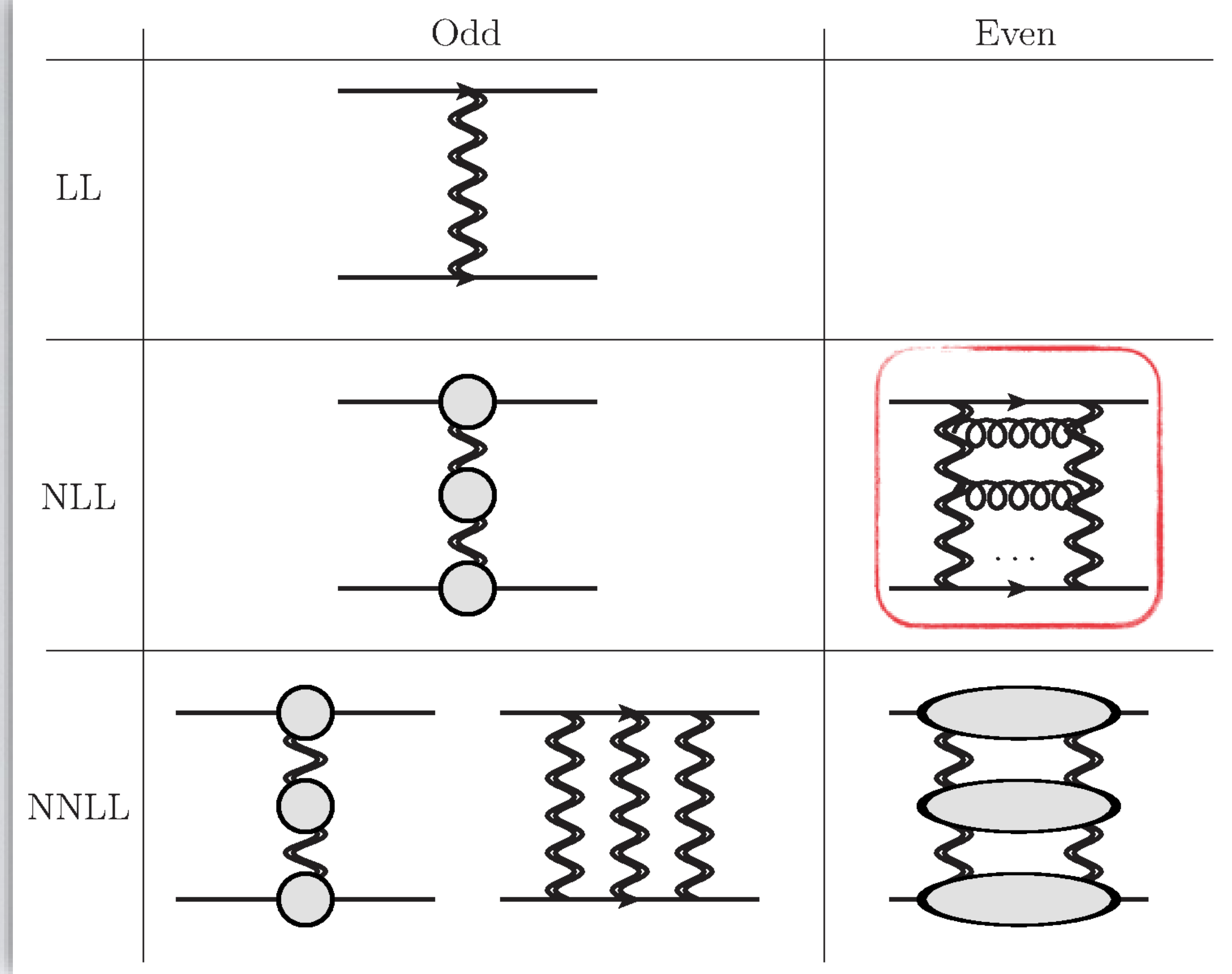
Calculated to 3 loops in Caron-Huot, Gardi, LV, 2017;

Calculated to 4 loops in Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021.

IR rivergences calculated to all orders in Caron-Huot, Gardi, Reichel, LV, 2017;

Finite terms calculated to 13 loops in Caron-Huot, Gardi, Reichel, LV, 2020.

THE TWO-REGGEON CUT



THE TWO-REGGEON CUT

- The amplitude takes the form of an iterated integral over the BFKL kernel:

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = \boxed{-i\pi} \frac{(B_0)^\ell}{(\ell-1)!} \int [{\rm D}k] \, \frac{p^2}{k^2(k-p)^2} \, \Omega^{(\ell-1)}(p,k) \, {\bf T}^2_{s-u} \, \mathcal{M}^{(0)}, \qquad B_0 = e^{\epsilon\gamma_{\rm E}} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}.$$

- One rung = apply once the BFKL kernel on the "target averaged wave function":

$$\Omega^{(\ell-1)}(p,k) = \hat{H} \, \Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - {\bf T}_t^2) \, \hat{H}_{\rm i} + (C_A - {\bf T}_t^2) \, \hat{H}_{\rm m}$$

- "Integration" part:

Caron-Huot, Gardi, Reichel, LV, 2017

$$\hat{H}_{\rm i} \, \Psi(p,k) = \int [{\rm D}k'] \, f(p,k,k') \Big[\Psi(p,k') - \Psi(p,k) \Big],$$

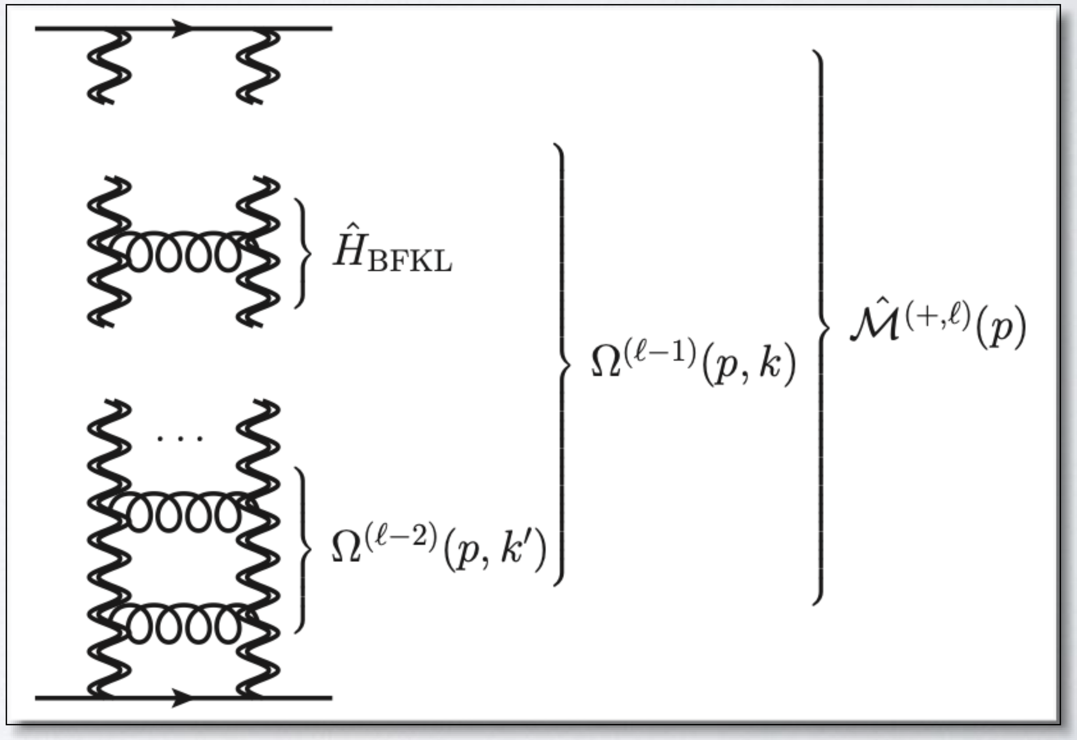
$$f(p,k',k) = \frac{k'^2}{k^2(k-k')^2} + \frac{(p-k')^2}{(p-k)^2(k-k')^2} - \frac{p^2}{k^2(p-k)^2}.$$

- "Multiplication" part:

$$\hat{H}_{\rm m} \Psi(p,k) = \frac{1}{2\epsilon} \left[2 - \left(\frac{p^2}{k^2} \right)^\epsilon - \left(\frac{p^2}{(p-k)^2} \right)^\epsilon \right] \Psi(p,k).$$

- Initial condition

$$\Omega^{(0)}(p,k) = 1.$$



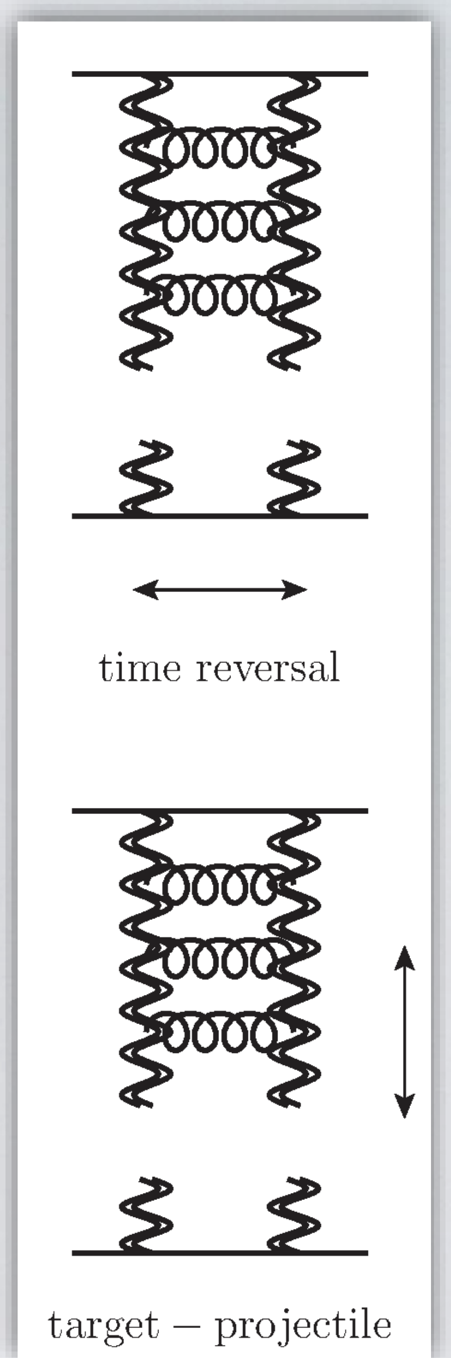
THE TWO-REGGEON CUT

- Exact solution in the adjoint channel: $\Omega = 1$.
- For generic color representations in d dimension eigenfunctions are not known:
  - → Iterative solution.
- General features:
  - → target-projectile, time reversal and crossing symmetry;
  - → outermost rungs are easy (multiplication);
  - → first non-trivial integration at 4-loops:

Caron-Huot, 2013

$$\hat{\mathcal{M}}^{(+,4)}_{\rm NLL} = i\pi \frac{(B_0)^4}{4!}\Big\{(C_A - \mathbf{T}_t^2)^3\Big(\frac{1}{(2\epsilon)^4} + \frac{175\zeta_5}{2}\epsilon + \mathcal{O}(\epsilon^2)\Big) + C_A(C_A-\mathbf{T}_t^2)^2\Big(-\frac{\zeta_3}{8\epsilon} - \frac{3}{16}\zeta_4 - \frac{167\zeta_5}{8}\epsilon + \mathcal{O}(\epsilon^2)\Big)\Big\}\mathbf{T}^2_{s-u}\, M^{(0)}.$$

- Integration in $d=2-2\epsilon$ involves Appell functions starting at 4 loops.
  - → How to predict higher orders?



THE TWO-REGGEON CUT

- Observations:

  1) The wavefunction *Ω(n)(p,k)* is finite as $\epsilon \to 0$:

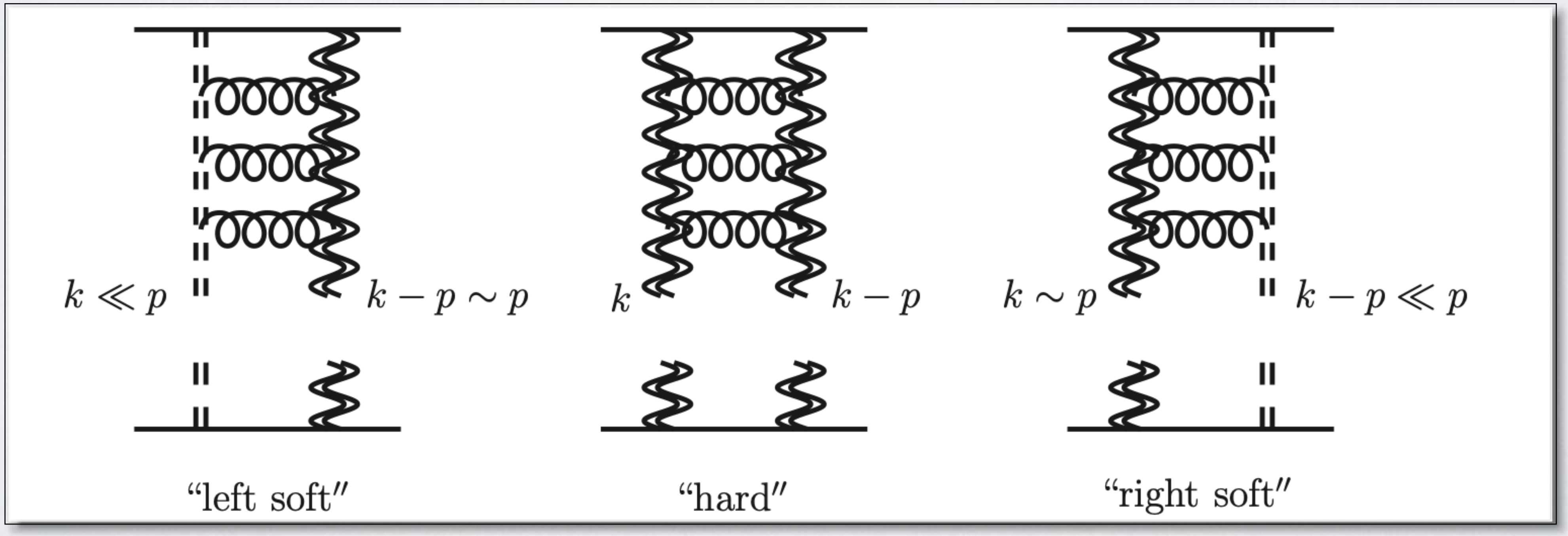
     → poles can only appear from the last integration.

  2) Evolution closes in the soft limit:

$$\int_{k\to 0} \Omega^{(\ell)}(p,k).$$

→ IR divergences occur only when a full rail goes soft!

→ Compute evolution in the (left) soft region and multiply by two.



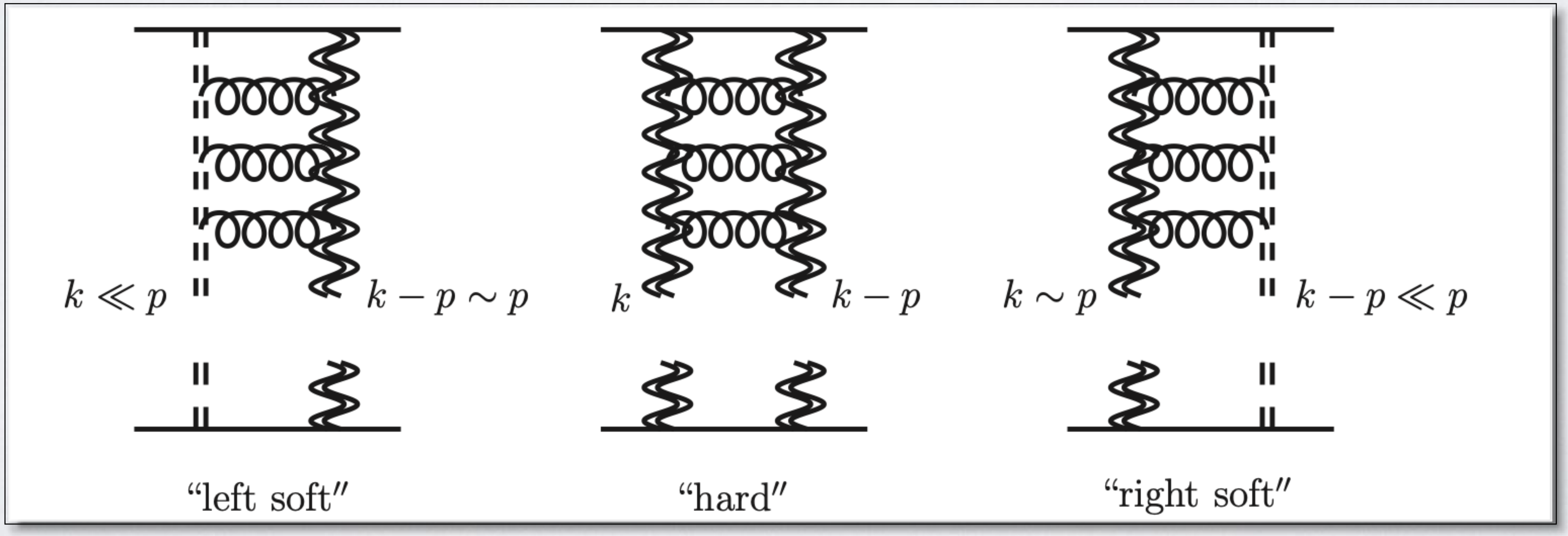
THE TWO-REGGEON CUT

- What about the finite part?
  - → Claim: the $\epsilon \to 0$ limit determined from evolution with $\epsilon = 0$.

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}\left(\frac{s}{-t}\right) = -i\pi\left\{\underbrace{\int [Dk]\,\frac{p^2}{k^2(p-k)^2}\Omega_s(p,k)\;\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\substack{\text{compute using soft limit}\\ \text{of wavefunction in } D \text{ dimensions}}} + \underbrace{\lim_{\epsilon\to 0}\int [Dk]\,\frac{p^2}{k^2(p-k)^2}\,\Omega_h^{(2d)}(p,k)\;\mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\text{compute in } D=2}\right\}.$$

Caron-Huot, Gardi, Reichel, LV, 2020

Recall: the wavefunction is finite, singularities are generated upon the last integration for $k \to 0$.



TWO-REGGEON CUT: SOFT APPROXIMATION

- The soft function is polynomial in $(p^2/k^2)^{\epsilon}$:

Γ functions

$$\hat{H}_{\mathrm{i}}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\boxed{\frac{B_n(\epsilon)}{B_0(\epsilon)}}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon},$$

$$\hat{H}_{\mathrm{m}}\left(\frac{p^2}{k^2}\right)^{n\epsilon} = \frac{1}{2\epsilon}\left[\left(\frac{p^2}{k^2}\right)^{n\epsilon} - \left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}\right].$$

Caron-Huot, Gardi, Reichel, LV, 2017

- Easy to compute to all orders: to $O(\epsilon^{-1})$ the amplitude reduces to a geometric series!

$$\boxed{\hat{\mathcal{M}}^{(+,\ell)}_{\mathrm{NLL}}|_s = i\pi\,\frac{1}{(2\epsilon)^{\ell}}\,\frac{B_0^{\ell}(\epsilon)}{\ell!}\,\left(1 - R(\epsilon)\frac{C_A}{C_A - \mathbf{T}_t^2}\right)^{-1}(C_A - \mathbf{T}_t^2)^{\ell-1}\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(0)} + \mathcal{O}(\epsilon^0),}$$

where

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2 + 10\zeta_6\right)\epsilon^6 + \mathcal{O}(\epsilon^7).$$

TWO-REGGEON CUT: D=2

- Translate the action of the BFKL kernel into a set of differential equations:

$$z\frac{d}{dz}\left[\hat{H}_{\rm 2d,i}\Psi(z,\bar{z})\right] = \hat{H}_{\rm 2d,i}\left[z\frac{d}{dz}\Psi(z,\bar{z})\right].$$

- The full algorithm requires to take care of contact terms,

$$\partial_z\partial_{\bar{z}}\,\log(z\bar{z}) = \pi\,\delta^2(z),$$

  and to considering the action of (1-z)d/dz as well.

Brown, 2004, 2013, Schnetz, 2013

Dixon, Pennington, Duhr, 2012; Del Duca, Dixon, Pennington, Duhr, 2013; Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek 2019, …

- The 2D wavefunction is expressed as function of SVHPLs, e.g.

$$\Omega^{(1)}_{\rm 2d} = \frac{1}{2}C_2\left(\mathcal{L}_0 + 2\mathcal{L}_1\right)$$

$$\Omega^{(2)}_{\rm 2d} = \frac{1}{2}C_2^2\left(\mathcal{L}_{0,0} + 2\mathcal{L}_{0,1} + 2\mathcal{L}_{1,0} + 4\mathcal{L}_{1,1}\right) + \frac{1}{4}C_1C_2\left(-\mathcal{L}_{0,1} - \mathcal{L}_{1,0} - 2\mathcal{L}_{1,1}\right).$$

  where $C_1=2C_A-T_t^2$, $C_2 = C_A-T_t^2$ and, e.g.,

$$\mathcal{L}_{0,1}(z,\bar{z}) = H_0(z)H_1(\bar{z}) + H_{0,1}(z) + H_{1,0}(\bar{z}).$$

TWO-REGGEON CUT: D=2

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}\left(\frac{s}{-t}\right) = -i\pi\Bigg\{ \underbrace{\int [Dk]\, \frac{p^2}{k^2(p-k)^2}\Omega_s(p,k)\; \mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\substack{\text{compute using soft limit} \\ \text{of wavefunction in } D \text{ dimensions}}} + \underbrace{\lim_{\epsilon\to 0}\int [Dk]\, \frac{p^2}{k^2(p-k)^2}\,\Omega^{(2\rm d)}_h(p,k)\; \mathbf{T}^2_{s-u}\mathcal{M}^{(\rm tree)}_{ij\to ij}}_{\text{compute in } D=2}\Bigg\}.$$

- Two methods to perform the last integration and sum consistently soft and hard region.

$$\hat{\mathcal{M}}^{(1)}|_{\epsilon^0} = 0, \qquad\qquad \hat{\mathcal{M}}^{(2)}|_{\epsilon^0} = 0,$$

$$\hat{\mathcal{M}}^{(3)}|_{\epsilon^0} = -i\pi\frac{(B_0)^3}{2!}\left[C_2^2\left(-\frac{11}{4}\zeta_3\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\hat{\mathcal{M}}^{(4)}|_{\epsilon^0} = -i\pi\frac{(B_0)^4}{3!}\left[C_1C_2^2\left(-\frac{3}{16}\zeta_4\right) + C_2^3\left(\frac{3}{16}\zeta_4\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

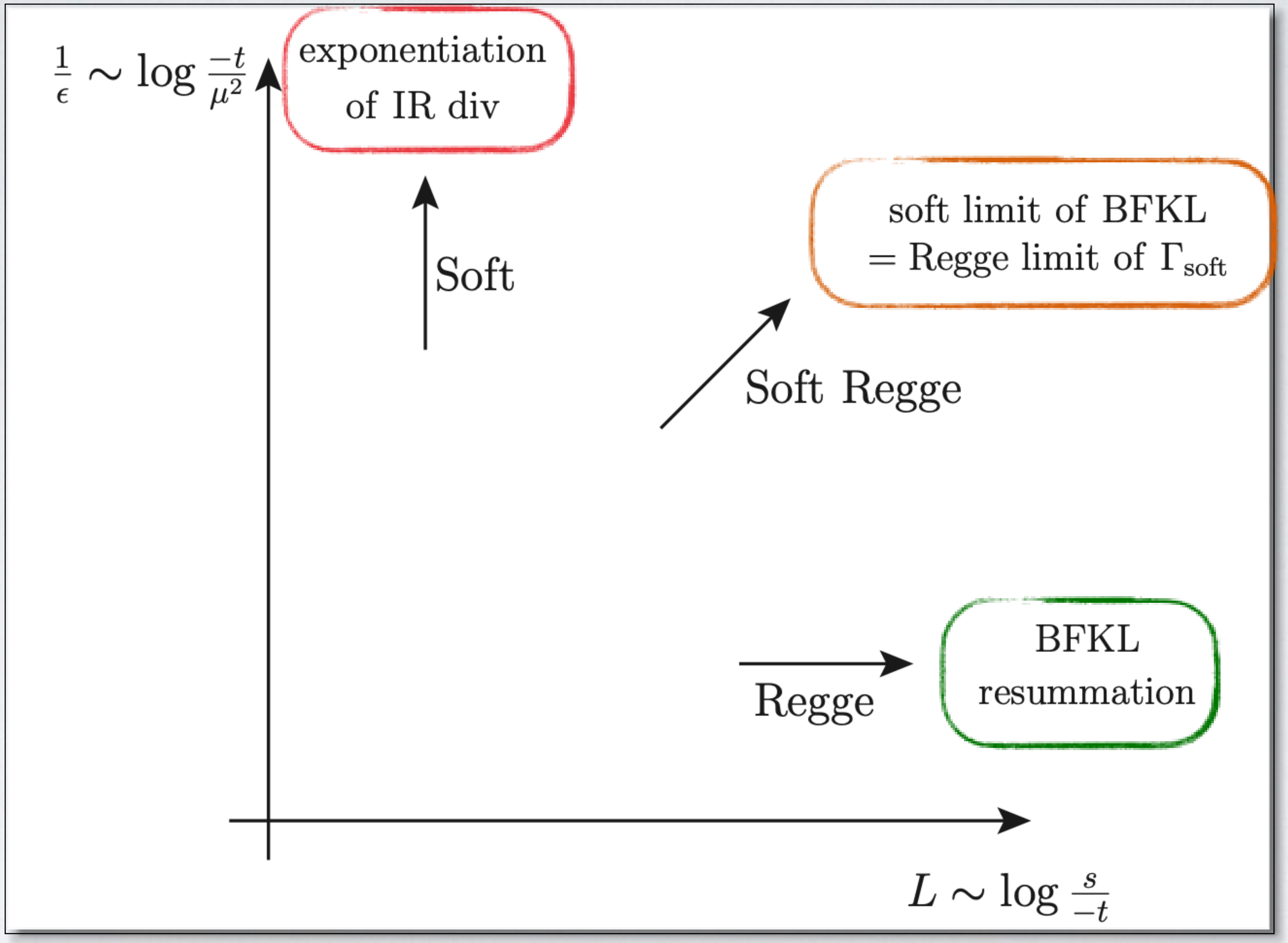
$$\hat{\mathcal{M}}^{(5)}|_{\epsilon^0} = -i\pi\frac{(B_0)^5}{4!}\left[C_2^4\left(-\frac{717}{16}\zeta_5\right) + C_1C_2^3\left(\frac{333}{16}\zeta_5\right) + C_1^2C_2^2\left(-\frac{5}{2}\zeta_5\right)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},$$

$$\begin{aligned}\hat{\mathcal{M}}^{(6)}|_{\epsilon^0} = -i\pi\frac{(B_0)^6}{5!}\bigg[&C_2^5\left(-\frac{2879}{32}\zeta_3^2 + \frac{5}{32}\zeta_6\right) + C_1C_2^4\left(\frac{2637}{32}\zeta_3^2 - \frac{5}{32}\zeta_6\right) \\ &+ C_1^2C_2^3\left(-\frac{399}{16}\zeta_3^2\right) + C_1^3C_2^2\left(\frac{39}{16}\zeta_3^2\right)\bigg]\mathbf{T}^2_{s-u}\mathcal{M}^{(0)},\end{aligned}$$

$$\ldots$$

Caron-Huot, Gardi, Reichel, LV, 2020

APPLICATION: INFRARED DIVERGENCES



- Use amplitudes calculated in the high-energy limit to extract the soft anomalous dimension in that limit;
- Bootstrap the result to constrain the structure of infrared divergences in general kinematic.

→ ***See talk by E. Gardi***

APPLICATION: NUMBER THEORY

- Applications: 2) number theory.

$$\hat{\mathcal{M}}_{\rm h}^{(11)} = \frac{i\pi}{8!}\Bigg\{C_2^2C_A^8\Bigg(-\frac{44253\,g_{533}}{5120}-\frac{652795\zeta_3^2\zeta_5}{2048}-\frac{818318 27\zeta_{11}}{327680}\Bigg)$$
$$+C_2^3C_A^7\Bigg(\frac{510873\,g_{533}}{5120}+\frac{10645591\zeta_3^2\zeta_5}{2048}+\frac{14761239427\zeta_{11}}{1966080}\Bigg)$$
$$+\ldots+C_2^8C_A^2\Bigg(-\frac{2158233\,g_{533}}{5120}-\frac{852453151\zeta_3^2\zeta_5}{2048}-\frac{1295244371839\zeta_{11}}{655360}\Bigg)$$
$$+C_2^9C_A\Bigg(\frac{6979863\,g_{533}}{5120}+\frac{2225183081\zeta_3^2\zeta_5}{2048}+\frac{741771390019\zeta_{11}}{655360}\Bigg)$$
$$+C_2^{10}\Bigg(\frac{1094181\,g_{533}}{2560}+\frac{2638860059\zeta_3^2\zeta_5}{1024}+\frac{4498262900131\zeta_{11}}{655360}\Bigg)\Bigg\}.$$

Brown, 2004, 2013; Schnetz, 2013

- Hard regions: only odd ζ_n, consistent with 2D wavefunction made of SVHPLs.
- Finite (hard) amplitude contains g_{533} at 11 loops:

Caron-Huot, Gardi, Reichel, LV, 2020

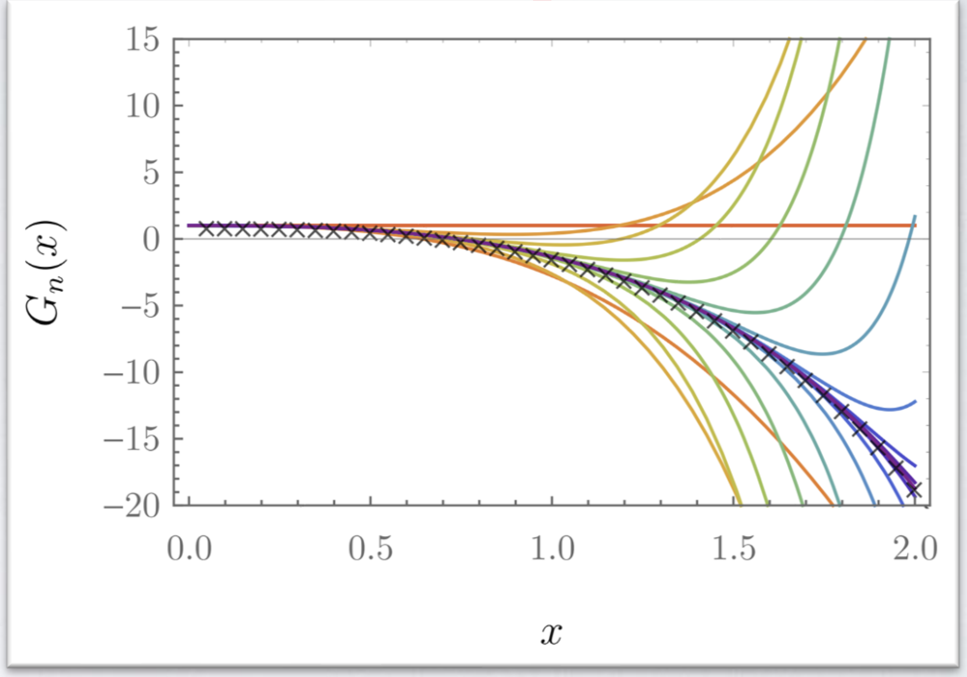
$$g_{5,3,3} = -\frac{4}{7}\zeta_2^3\zeta_5+\frac{6}{5}\zeta_2^2\zeta_7+45\zeta_2\zeta_9+\zeta_{5,3,3}\,.$$

→ no exponentiation in terms of **Γ** functions.

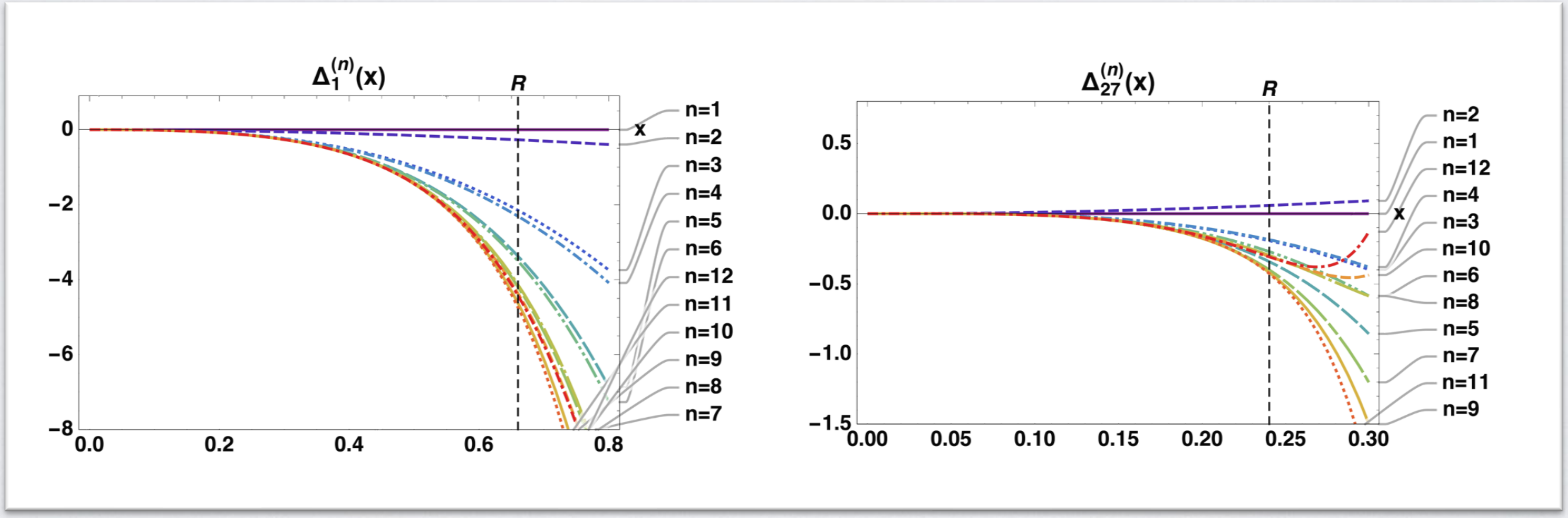
APPLICATION: NUMERICAL STUDIES

- Applications: 3) numerical studies.
- The soft anomalous dimension has an infinite radius of convergence: entire function, free of singularities for any finite $x=\alpha_s/\pi\, L$.

Caron-Huot, Gardi, Reichel, LV, 2017, 2020

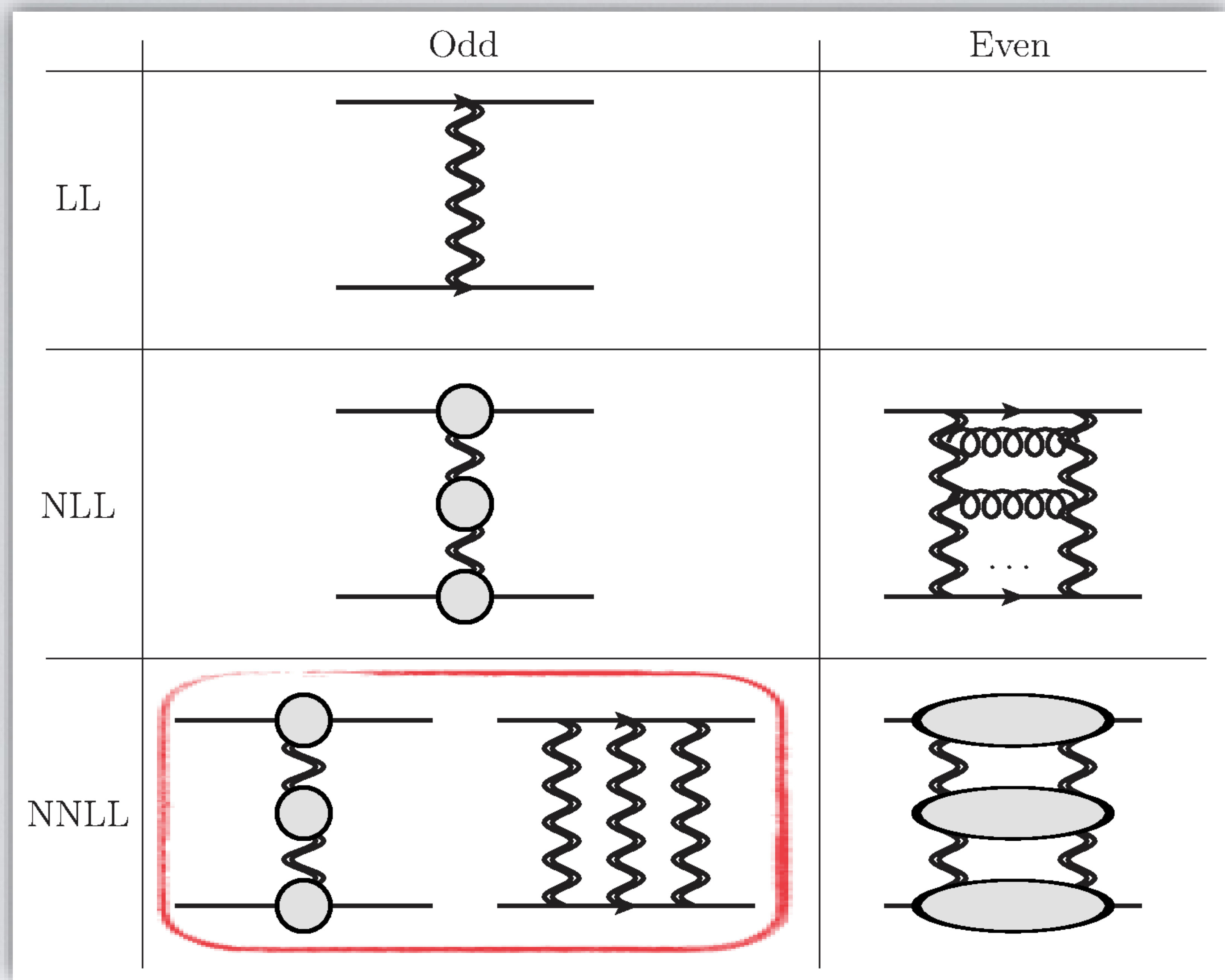


- The finite amplitude is an alternating series, whose coefficients grows geometrically:



- Finite radius of convergence in $\alpha_s/\pi\, L$ *that* stabilises to $|R| \simeq 0.66$ for singlet, $|R| \simeq 0.24$ for 27 representation, by means of a Padé approximant (pole at $-|R|$).

THE THREE-REGGEON CUT



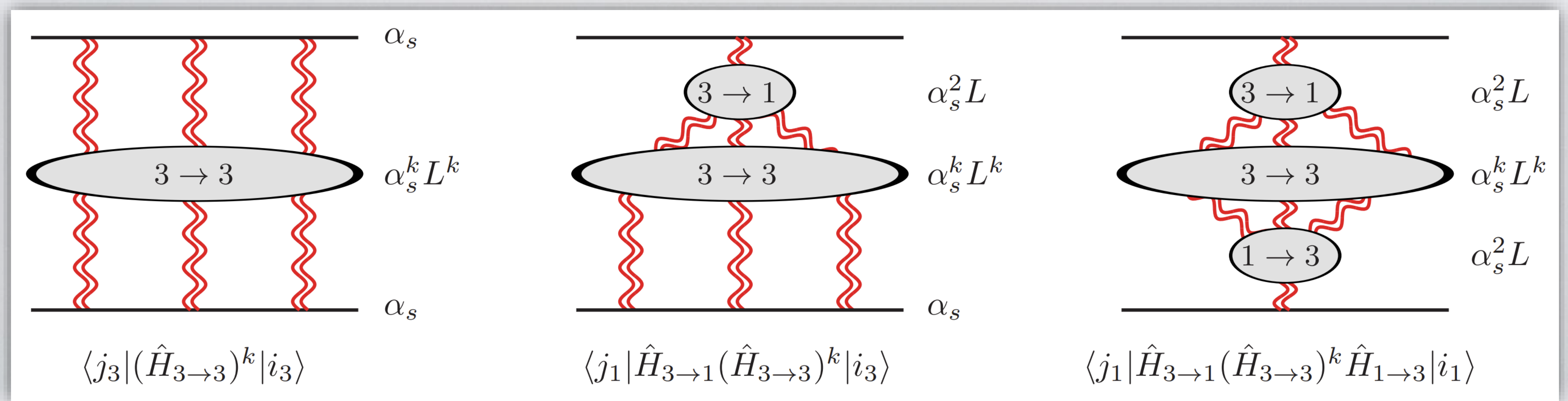
THE THREE-REGGEON CUT

Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021

- To all orders the amplitude takes the form

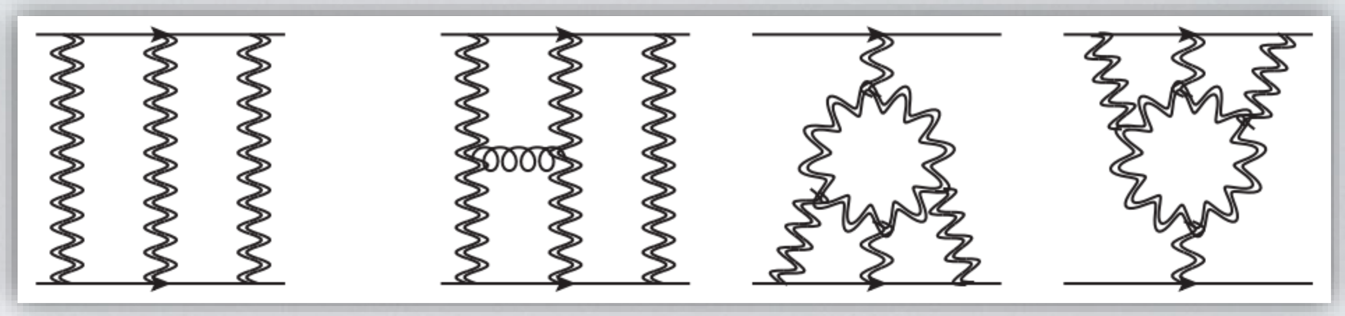
$$
\begin{aligned}
\frac{i}{2s}\hat{\mathcal{M}}^{(-),\mathrm{NNLL}}_{ij\to ij} = \left(\frac{\alpha_s}{\pi}\right)^2 \Bigg\{ r_\Gamma^2 \pi^2 \Bigg[&\sum_{k=0}^{\infty} \frac{(-X)^k}{k!} \langle j_3|\hat{H}^k_{3\to 3}|i_3\rangle \\
&+ \sum_{k=1}^{\infty} \frac{(-X)^k}{k!} \Big[\langle j_1|\hat{H}_{3\to 1}\hat{H}^{k-1}_{3\to 3}|i_3\rangle + \langle j_3|\hat{H}^{k-1}_{3\to 3}\hat{H}_{1\to 3}|i_1\rangle \Big] \\
&+ \sum_{k=2}^{\infty} \frac{(-X)^k}{k!} \langle j_1|\hat{H}_{3\to 1}\hat{H}^{k-2}_{3\to 3}\hat{H}_{1\to 3}|i_1\rangle \Bigg]^{\mathrm{LO}} + \langle j_1|i_1\rangle^{\mathrm{NNLO}} \Bigg\} .
\end{aligned}
$$

- In diagrams:

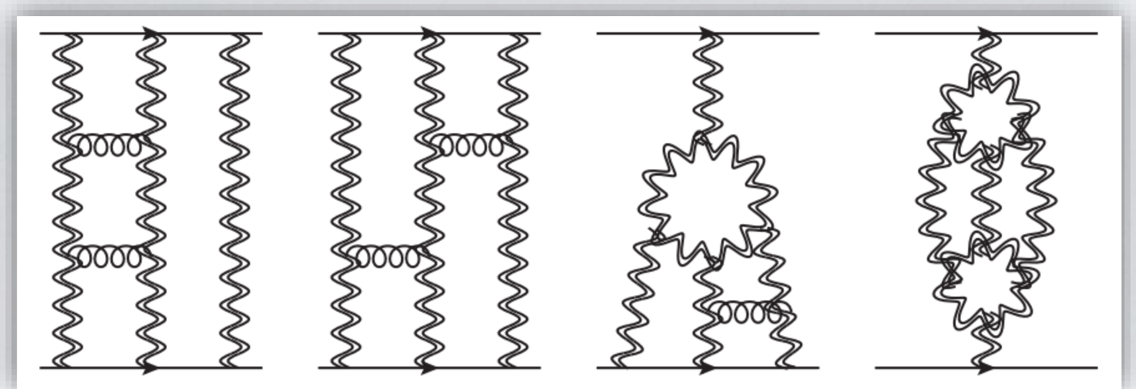


THE THREE-REGGEON CUT

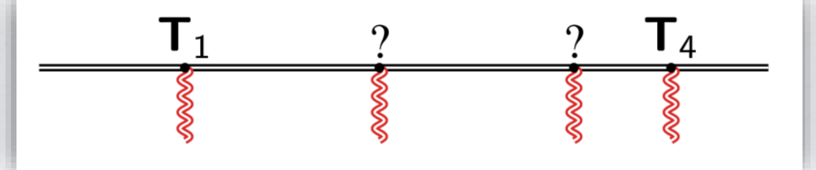
- Two and three loops:



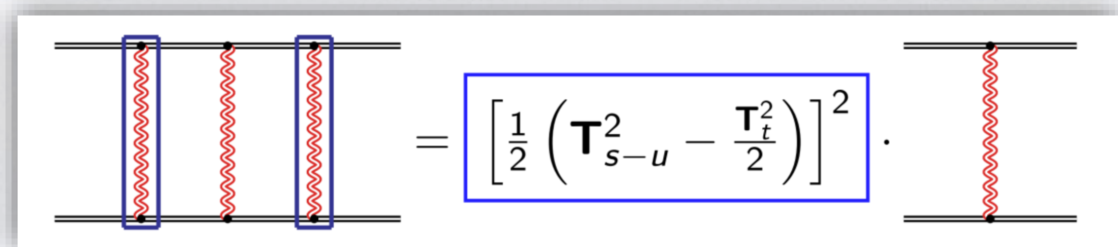
- Four loops:



- The integrals are relatively easy. It requires some work to express the color factors as operators acting on the tree level amplitude:
- Outmost generators clearly associated with external particles

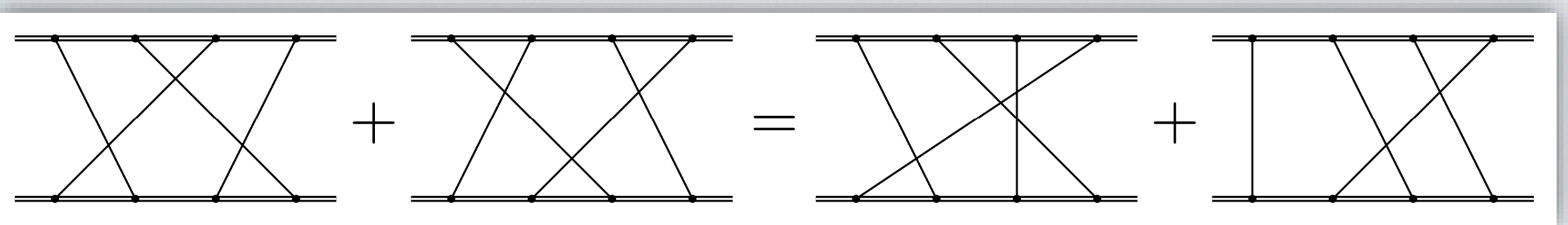


- At lowest order there is no ambiguity



THE THREE-REGGEON CUT

- Starting at three loops one has entangled contributions: needs identities such as



- After some work we obtain

$$\hat{\mathcal{M}}^{(-,4,2)} = \frac{\pi^2 r_\Gamma^4}{2}\left[\frac{1}{\epsilon^4}\mathbf{K}^{(4)} + \left(\frac{1}{\epsilon}\zeta_3 + \frac{3}{2}\zeta_4\right)\mathbf{K}^{(1)} + \mathcal{O}(\epsilon)\right]\hat{\mathcal{M}}_{\text{tree}},$$

with

$$\begin{aligned}\mathbf{K}^{(4)} = &\frac{1}{96}\Big[\mathbf{T}_{s-u}^2, \left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\Big]\mathbf{T}_t^2 + \frac{7}{576}\mathbf{T}_t^2\left[\left(\mathbf{T}_{s-u}^2\right)^2, \mathbf{T}_t^2\right] \\ &- \frac{1}{192}\left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\mathbf{T}_t^2\mathbf{T}_{s-u}^2 - \frac{5}{192}\mathbf{T}_{s-u}^2\left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\mathbf{T}_t^2,\end{aligned}$$

$$\begin{aligned}\mathbf{K}^{(1)} = &\frac{49}{48}\Big[\mathbf{T}_{s-u}^2, \left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\Big]\mathbf{T}_t^2 - \frac{47}{288}\mathbf{T}_t^2\left[\left(\mathbf{T}_{s-u}^2\right)^2, \mathbf{T}_t^2\right] \\ &+ \frac{101}{96}\left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\mathbf{T}_t^2\mathbf{T}_{s-u}^2 - \frac{49}{48}\mathbf{T}_{s-u}^2\left[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2\right]\mathbf{T}_t^2 + \frac{1}{24}\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\right).\end{aligned}$$

Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021

- The result holds in every gauge theory.
- Applications: extract infrared divergences and finite terms, etc. → ***See talk by E. Gardi***

APPLICATION: REGGE POLE AND CUT

- Before the development of QCD and perturbation theory, scattering amplitudes have been studied as an analytic function in the complex angular momentum plane.
- In this context, the amplitude is expected to be given in terms of Regge pole and cut:

$$A_{LL} \propto \underbrace{\frac{s^{\alpha_g(t)}}{t}}_{\text{“Regge pole”}} , \qquad A_{\mathrm{NLL}} \propto \underbrace{\int d\nu\, c(\nu)\, s^{E(\nu)}}_{\text{“Regge cut”}} .$$

Regge, Gribov ~ 1960; Lipatov; Fadin, Kuraev, Lipatov 1976.

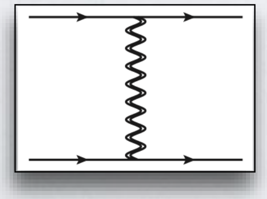
- While the Regge cut arises exclusively due to MR contributions to the amplitude, MR exchanges do contribute also to the Regge pole.

Eden, Landshoff, Olive, Polkinghorne, 1966; P. D. B. Collins, 2009

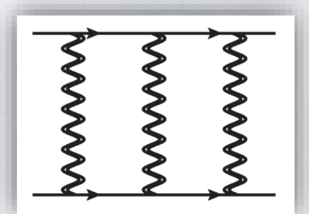
- This is evident in the large-Nc limit, where it is known that the amplitude only features a Regge pole, and yet, MR contributions are present.

Mandelstam 1963; P. D. B. Collins 2009

- It is also known that Regge cuts only arise due to nonplanar diagrams: the Regge cut should be identified as the nonplanar part of the MR contribution, while the Regge pole corresponds to SR plus the planar MR contributions:



$$\mathcal{M}^{(-)}_{ij\to ij} = \underbrace{\mathcal{M}^{(-)\,\mathrm{SR}}_{ij\to ij} + \mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{planar}}} + \underbrace{\mathcal{M}^{(-)\,\mathrm{MR}}_{ij\to ij}\Big|_{\mathrm{nonplanar}}}$$
$$= \mathcal{M}^{(-)\,\mathrm{pole}}_{ij\to ij} + \mathcal{M}^{(-)\,\mathrm{cut}}_{ij\to ij} .$$



APPLICATION: REGGE POLE AND CUT

- With this definition we are able to extract unambiguously the Regge trajectory at three loops, matching our calculation of the Regge-cut contribution with the recent calculations of two-parton scattering at three loops in QCD:

Caola, Chakraborty, Gambuti,
von Manteuffel, Tancredi, 2021

$$\mathcal{M}^{(-)}_{ij\to ij} = Z_i(t)\,\bar{D}_i(t)\,Z_j(t)\,\bar{D}_j(t)\left[\left(\frac{-s}{-t}\right)^{C_A\tilde{\alpha}_g(t)} + \left(\frac{-u}{-t}\right)^{C_A\tilde{\alpha}_g(t)}\right]\mathcal{M}^{\rm tree}_{ij\to ij} + \sum_{n=2}^{\infty}\frac{\alpha_s}{4\pi}L^{n-2}\mathcal{M}^{(-,n,n-2)\,\rm cut}_{ij\to ij},$$

with

$$\hat{\tilde{\alpha}}^{(3)}_g = K^{(3)} + C_A^2\left(\frac{297029}{93312} - \frac{799\zeta_2}{1296} - \frac{833\zeta_3}{216} - \frac{77\zeta_4}{192} + \frac{5}{24}\zeta_2\zeta_3 + \frac{\zeta_5}{4}\right) + C_A n_f\left(\frac{103\zeta_2}{1296} + \frac{139\zeta_3}{144} - \frac{5\zeta_4}{96} - \frac{31313}{46656}\right)$$
$$+ C_F n_f\left(\frac{19\zeta_3}{72} + \frac{\zeta_4}{8} - \frac{1711}{3456}\right) + n_f^2\left(\frac{29}{1458} - \frac{2\zeta_3}{27}\right) + \mathcal{O}(\epsilon),$$

$$K_{\rm cusp}(\alpha_s(\mu^2)) \equiv -\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma^{\rm cusp}_A(\alpha_s(\lambda^2))\,.$$

Gardi, Falcioni, Maher, Milloy, LV, 2021.

- The Regge-pole contribution is universal among all two-parton scattering processes, but theory dependent (i.e. different in N=4 SYM, QCD, etc);
- The Regge–cut contribution is different for each channel but depends only on the action of color operators in the gauge theory considered.

CONCLUSION

- Modern approach to high-energy scattering via Wilson lines:
  - → new theoretical control up to NNLL.
  - → 2 → 2 amplitudes obtained by iteration of the Balitsky-JIMWLK Hamiltonian.

- Imaginary part at NLL obtained to all orders in the strong coupling:
  - → Extracted the soft anomalous dimension to all orders;
  - → Numerical studies on the convergence of the perturbative expansion.

- Real part at NNLL obtained up to four loops:
  - → Extracted the corresponding term of the soft anomalous dimension;
  - → Real part of the 2 → 2 amplitude in QCD and N=4 SYM at four loops.
  - → Identified the Regge pole as the planar contribution of single- and multi-Reggeon exchange, and the Regge cut as the non-planar part of the multi-Reggeon exchange.

EXTRA SLIDES

TWO-REGGEON CUT: D=2

- Introduce complex variables

$$\frac{k}{p} = \frac{z}{z-1}, \qquad \frac{k'}{p} = \frac{w}{w-1}.$$

Caron-Huot, Gardi, Reichel, LV, 2020

- BFKL kernel in D=2:

$$\hat{H}_{\mathrm{2d}} = (2C_A - \mathbf{T}_t^2)\hat{H}_{\mathrm{2d,i}} + (C_A - \mathbf{T}_t^2)\hat{H}_{\mathrm{2d,m}}$$

- “Integration” part:

$$\hat{H}_{\mathrm{2d,i}} = \frac{1}{4\pi}\int d^2w\, K(w,\bar{w},z,\bar{z})\Big[\Psi(w,\bar{w}) - \Psi(z,\bar{z})\Big],$$

$$K(w,\bar{w},z,\bar{z}) = \frac{1}{\bar{w}(z-w)} + \frac{2}{(z-w)(\bar{z}-\bar{w})} + \frac{1}{w(\bar{z}-\bar{w})}.$$

- “Multiplication” part:

$$\hat{H}_{\mathrm{2d,m}} = \frac{1}{2}\log\left[\frac{z}{(1-z)^2}\frac{\bar{z}}{(1-\bar{z})^2}\right]\Psi(z,\bar{z}).$$

REGGE VS INFRARED FACTORISATION

- Applications: 1) test (and predict) the analytic structure of infrared divergences.
- The infrared divergences of amplitudes are controlled by a renormalization group equation:

$$\mathcal{M}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right),$$

where $\boldsymbol{Z}_n$ is given as a path-ordered exponential of the soft-anomalous dimension:

Becher, Neubert, 2009; Gardi, Magnea, 2009

$$\mathbf{Z}_n\left(\{p_i\},\mu,\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right)\right\},$$

- The soft anomalous dimension for scattering of massless partons is an operator in color space given by

$$\mathbf{\Gamma}_n\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) = \mathbf{\Gamma}_n^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) + \mathbf{\Delta}_n\left(\{\rho_{ijkl}\}\right).$$

- Given M_n as calculated in the high-energy limit, use IR factorisation to extract the soft anomalous dimension.

→ ***See talk by E. Gardi***

TWO-REGGEON CUT: IR SINGULARITIES

- Expand the soft anomalous dimension in the high-energy logarithm:

$$\mathbf{\Gamma}\left(\alpha_s(\lambda)\right) = \mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s(\lambda), L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s(\lambda), L\right) + \ldots \,.$$

- At LL gluon Reggeization fixes $\mathbf{\Gamma}_{\rm LL}$ from gluon trajectory:

$$\mathbf{\Gamma}_{\rm LL}\left(\alpha_s(\lambda)\right) \;=\; \frac{\alpha_s(\lambda)}{\pi}\frac{\gamma_K^{(1)}}{2} L\, \mathbf{T}_t^2 \;=\; \frac{\alpha_s(\lambda)}{\pi} L\, \mathbf{T}_t^2 \,.$$

- At NLL

$$\mathbf{\Gamma}_{\rm NLL} = \mathbf{\Gamma}_{\rm NLL}^{(+)} + \mathbf{\Gamma}_{\rm NLL}^{(-)},$$

- with

$$\mathbf{\Gamma}_{\rm NLL}^{(+)} = \frac{\alpha_s(\lambda)}{\pi} \sum_{i=1}^{2} \left(\frac{\gamma_K^{(1)}}{2} C_i \log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)} \right) + \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2 \frac{\gamma_K^{(2)}}{2} L\, \mathbf{T}_t^2,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi \frac{\alpha_s(\lambda)}{\pi} \mathbf{T}_{s-u}^2 \boxed{+ O(\alpha_s^4 L^3) .}$$

Del Duca, Duhr, Gardi, Magnea, White, 2011

TWO-REGGEON CUT: IR SINGULARITIES

- Derive an infrared-factorised representation of the reduced amplitude:

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = \exp\left\{ -\frac{\alpha_s(\mu)}{\pi}\frac{B_0(\epsilon)}{2\epsilon}\, L\,\mathbf{T}_t^2 \right\} \Bigg[\mathbf{Z}^{(-)}_{\rm NLL}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right) \mathcal{H}^{(-)}_{\rm LL}\left(\{p_i\},\mu,\alpha_s(\mu)\right) + \cancel{\mathbf{Z}^{(+)}_{\rm LL}\left(\frac{s}{t},\mu,\alpha_s(\mu)\right) \mathcal{H}^{(+)}_{\rm NLL}\left(\{p_i\},\mu,\alpha_s(\mu)\right)} \Bigg],$$

No poles

- By matching we get the soft anomalous dimension to all orders:

$$\boxed{\mathbf{\Gamma}^{(-,\ell)}_{\rm NLL} = \frac{i\pi}{(\ell-1)!}\left(1 - R\left(\frac{x}{2}(C_A-\mathbf{T}_t^2)\right)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\Bigg|_{x^{\ell-1}} \mathbf{T}^2_{s-u}\,,}$$

with

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(2\zeta_3^2 + 10\zeta_6\right)\epsilon^6 + \ldots .$$

Caron-Huot, Gardi, Reichel, LV, 2017

THE THREE-REGGEON CUT

Falcioni, Gardi, Milloy, LV, 2020; Falcioni, Gardi, Maher, Milloy, LV, 2021

- The NNLL amplitude at four loops reads

$$\hat{\mathcal{M}}^{(-,4,2)} = \frac{r_\Gamma^4 \pi^2}{144} \left[C_{\mathcal{M}}^{(-4)} \frac{1}{\epsilon^4} + C_{\mathcal{M}}^{(-1)} \frac{f_\epsilon}{\epsilon} + \mathcal{O}(\epsilon) \right] \mathcal{M}^{\text{Tree}}, \quad \text{with}$$

$$C_{\mathcal{M}}^{(-4)} = \frac{1}{2}\mathbf{C}_{33}^{(4,-4)} - \frac{C_A^4}{72} - \frac{1}{6}\frac{d_{AA}}{N_A} + \frac{1}{2}C_A(d_i + d_j), \quad C_{\mathcal{M}}^{(-1)} = \mathbf{C}_{33}^{(4,-1)} + \frac{101 C_A^4}{36} + \frac{110}{3}\frac{d_{AA}}{N_A} - 104 C_A(d_i + d_j)\,.$$

The result holds in every gauge theory.

- Applications: 1) extract infrared divergences. → ***See talk by E. Gardi***
- Applications: 2) finite terms:

$$\mathcal{H}^{(-,4,2)} = \left\{ \frac{C_A^2}{2} \left(\hat{\alpha}_g^{(2,0)} \right)^2 + \frac{3}{16}\zeta_4\zeta_2 C_{\boldsymbol{\Delta}}^{(4)} \right\} \mathcal{M}^{\text{tree}}, \quad \mathbf{C}_{\Delta}^{(4,2)} = \frac{1}{4}\mathbf{T}_t^2[\mathbf{T}_t^2, (\mathbf{T}_{s-u}^2)^2] + \frac{3}{4}[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2 + \frac{d_{AA}}{N_A} - \frac{C_A^4}{24}\,.$$

→ We calculate it in QCD and N=4 SYM

- QCD:

$$\mathcal{H}_{\text{QCD}}^{(-,4,2)} = \left\{ C_A^2 T_F^2 n_f^2 \frac{49}{1458} + C_A^3 T_F n_f \left(\frac{7\zeta_3}{216} - \frac{707}{2916} \right) + C_A^4 \left(\frac{\zeta_3^2}{128} - \frac{101\zeta_3}{864} + \frac{10201}{23328} \right) + \frac{3}{16}\zeta_4\zeta_2 C_{\boldsymbol{\Delta}}^{(4)} \right\} \mathcal{M}^{\text{tree}}.$$

- N=4 SYM (from QCD according to maximum trascendentality):

$$\mathcal{H}_{\mathcal{N}=4}^{(-,4,2)} = \left\{ \frac{C_A^4}{128}\zeta_3^2 + \frac{3}{16}\zeta_4\zeta_2 C_{\boldsymbol{\Delta}}^{(4)} \right\} \mathcal{M}^{\text{tree}}.$$

Matches the large Nc limit

New non-planar term, proportional to Δ(4,2)