Pushing the Limits of Theoretical Physics

MITP 10 Years Celebration

The quest for understanding long-distance singularities of scattering amplitudes

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The quest for understanding long-distance singularities of scattering amplitudes

Motivation:

Determine long-distance singularities beyond what is accessible in fixed-order calculations

Catani (1998); Mert-Aybat, Dixon & Sterman (2006); Becher & Neubert; EG & Magnea (2009); …

✓ Essential check of future amplitude computations.

✓ Cancellation of singularities in cross sections.

✓ Resummation of large logarithms.

✓ **Understand** the physical and mathematical principles underlining the structure of gauge-theory amplitudes

IR singularities are **universal** wrt the underlying hard process.

IR singularities are largely **theory-independent**.

Exponentiation: access to all-order properties.

Relation between general kinematics and **special limits** (soft, collinear, Regge,..)

Bootstrap!

Stepping **beyond the planar limit**.

IR Singularities using Wilson lines

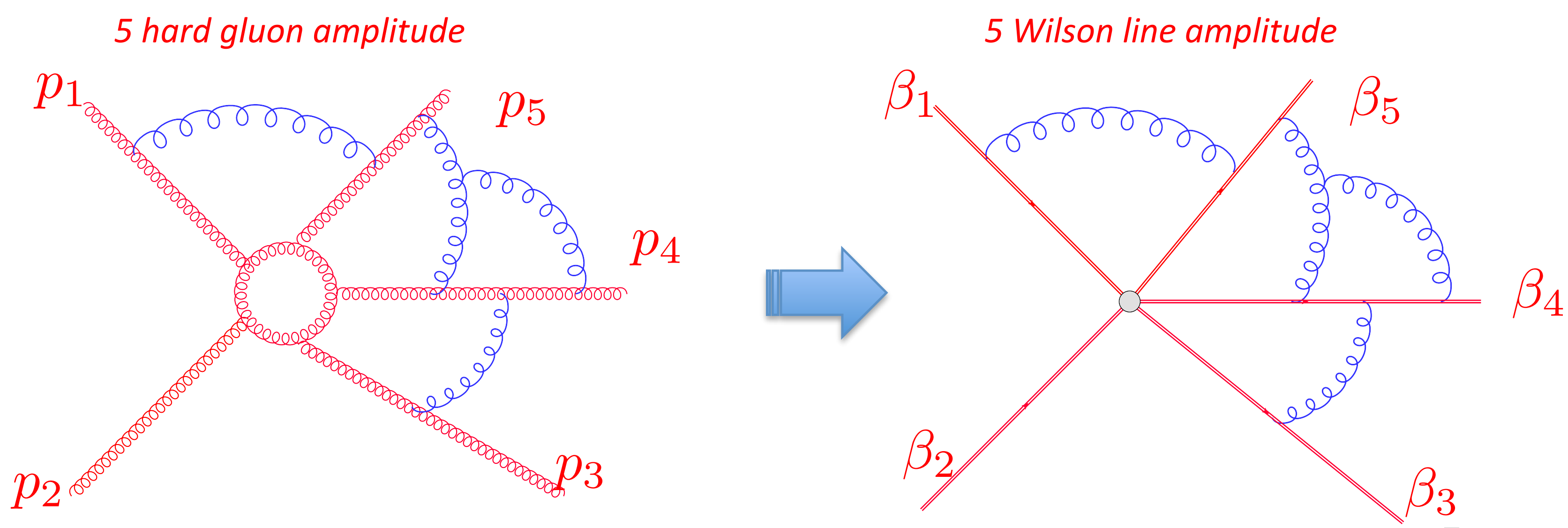
Factorization at fixed angles:

all kinematic invariants are simultaneously taken large $\quad p_i \cdot p_j = Q^2 \beta_i \cdot \beta_j \gg \Lambda^2$

Soft singularities **factorise** to all orders:

$$\mathcal{M}_J(p_i, \epsilon_{\rm IR}) = \sum_K \mathcal{S}_{JK}\left(\gamma_{ij}, \epsilon_{\rm IR}\right) H_K(p_i)$$

IR can be computed from Wilson lines — process independent!



$\mathcal{S} = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$ product of Wilson lines: $\quad \phi_{\beta_l} \equiv \mathcal{P}\exp\left[i g_s \int_0^\infty dt \beta_l \cdot A(t\beta_l)\right]$

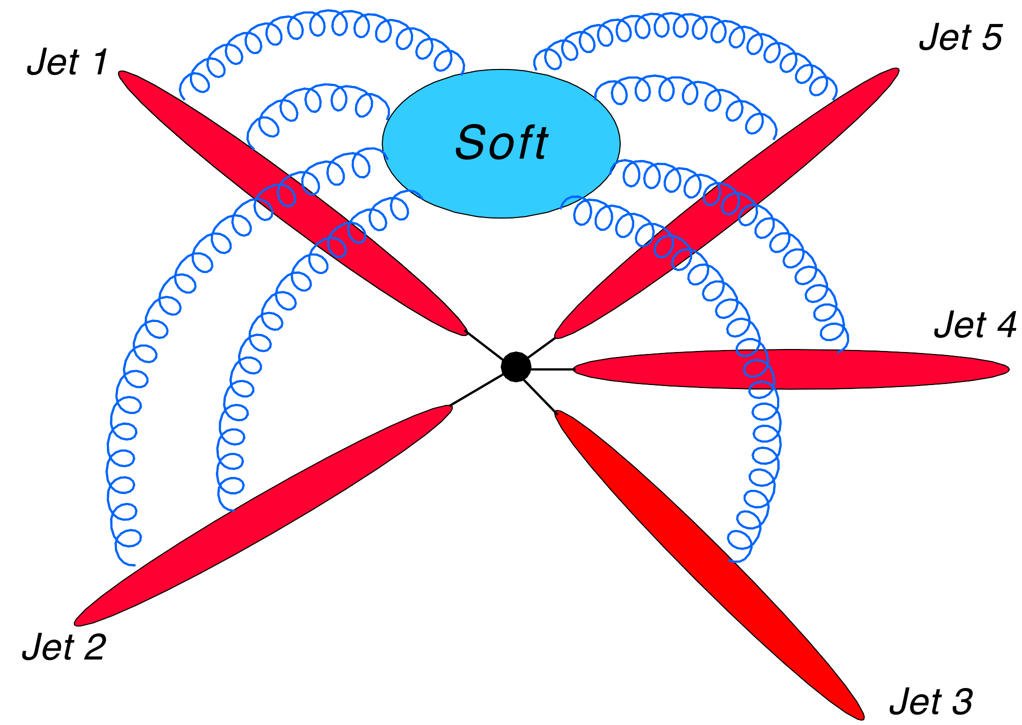
Due to **rescaling symmetry** it only depends on angles: $\quad \gamma_{ij} = \frac{2\beta_i \cdot \beta_j}{\sqrt{\beta_i^2 \beta_j^2}}$

$\mathcal{S}$ is **multiplicatively renormalizable** [Brandt, Neri & Sato]. Anomalous dimension: Γ

UV-IR connection, just as in the *cusp anom. dim.* [Korchemsky & Radyushkin (1986)]

IR Factorization of amplitudes with massless legs

Fixed angle scattering with **massless partons** $p_i^2 = 0$



$$s_{ij} \equiv 2p_i \cdot p_j = 2\beta_i \cdot \beta_j Q^2 \gg \Lambda^2$$

IR singularities can be factorised — all originate in soft and collinear regions of loop momenta

Soft (matrix in colour flow space)

Jets (colour singlet)

$$\mathcal{M}_N\left(p_i/\mu, \epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \epsilon\right) H_L\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}\right) \prod_{i=1}^{n} \frac{J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)}$$

The soft function: **lightlike** Wilson lines $\mathcal{S} = \langle \phi_{\beta_1} \otimes \phi_{\beta_2} \otimes \ldots \phi_{\beta_n} \rangle$

IR singularities in amplitudes with massless legs

Exponentiation:

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}\left(\lambda,\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

Rescaling symmetry of Wilson-line velocities & **soft/jet factorisation** imply**:**

$$\sum_{j\neq i}\frac{d\mathbf{\Gamma}}{d\log(-s_{ij})} = \Gamma_i^{\mathrm{cusp}}$$

Becher & Neubert, EG & Magnea (2009)

The Dipole Formula (full result to 2 loops): inhomogeneous solution = linear in the log

$$\mathbf{\Gamma}_{\mathrm{Dip.}}(\lambda,\alpha_s) = \frac{1}{4}\widehat{\gamma}_K(\alpha_s)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}(\alpha_s)$$

Lightlike Cusp anomalous dimension

Catani (1998); Dixon, Mert-Aybat & Sterman (2006)
Becher & Neubert, EG & Magnea (2009)

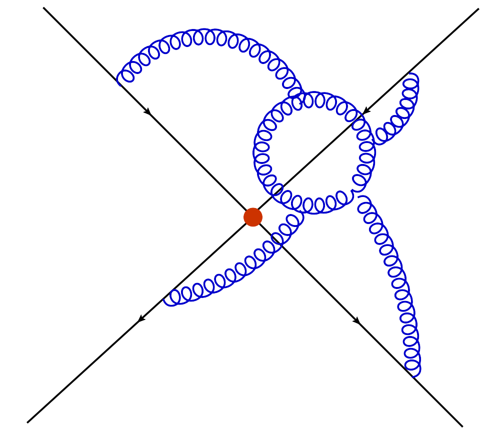
$$\Gamma_i^{\mathrm{cusp}}(\alpha_s) = \frac{1}{2}\widehat{\gamma}_K(\alpha_s)C_i + \sum_R g_R(\alpha_s)\frac{d_{RR_i}}{N_{R_i}} + \cdots$$

Corrections to the Dipole Formula

$$\sum_{j\neq i} \frac{d\boldsymbol{\Gamma}}{d\log(-s_{ij})} = \Gamma_i^{\rm cusp}$$ allows two types of **corrections to the dipole formula**:

Becher & Neubert, EG & Magnea (2009)

1. Corrections governed by higher Casimir contributions to the cusp anomalous dimension — **starting at 4 loops**:



$$\Gamma_i^{\rm cusp}(\alpha_s) = \frac{1}{2}\widehat{\gamma}_K(\alpha_s) C_i + \sum_R g_R(\alpha_s)\frac{d_{RR_i}}{N_{R_i}} + \cdots$$

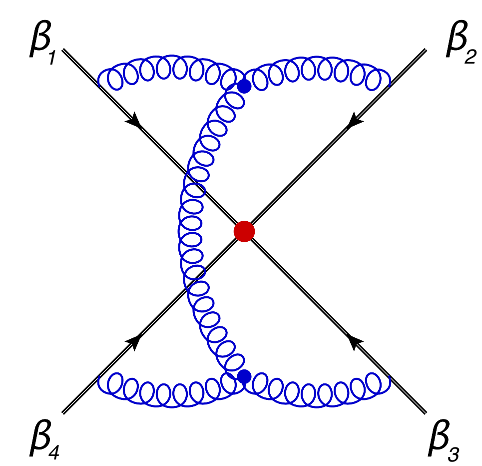
quartic Casimir $\frac{d_{RR_i}}{N_{R_i}} = \mathcal{D}^R_{iiii} = \frac{1}{4!}\sum_{\sigma\in\mathcal{S}_4}{\rm Tr}_R\left[T^{\sigma(a)}T^{\sigma(b)}T^{\sigma(c)}T^{\sigma(d)}\right]\mathbf{T}_i^a\mathbf{T}_i^b\mathbf{T}_i^c\mathbf{T}_i^d$

colour monster $\mathcal{D}^R_{ijkl} \equiv \frac{1}{4!}\sum_{\sigma\in\mathcal{S}_4}{\rm Tr}_R\left(T^{\sigma(a)}T^{\sigma(b)}T^{\sigma(c)}T^{\sigma(d)}\right)\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d$

2. Functions of **conformally-invariant cross ratios — starting at 3-loops**:

$$\mathbf{\Gamma} = \mathbf{\Gamma}_{\rm Dip.} + \mathbf{\Delta}(\rho_{ijkl})$$

$$\rho_{ijkl} = \frac{(p_i\cdot p_j)(p_k\cdot p_l)}{(p_i\cdot p_k)(p_j\cdot p_l)}$$



$\Delta_n^{(3)}$ was computed using Feynman diagrams in 2016

Ø. Almelid, C. Duhr, EG, Phys. Rev. Lett. **117**, 172002

The 3-loop correction to the soft anomalous dimension

Ø. Almelid, C. Duhr, EG
Phys. Rev. Lett. **117**, 172002

$$\Delta_n^{(3)}(z,\bar{z}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\le i<j<k<l\le n}\Big[\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\left(F\left(1-1/z\right)-F\left(1/z\right)\right) + \mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\left(F(1-z)-F(z)\right) + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\left(F\left(1/(1-z)\right)-F\left(1-1/(1-z)\right)\right)\Big] - \sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(\zeta_5+2\zeta_2\zeta_3\right)\Bigg\}$$

$$F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2\left(\mathcal{L}_{100}(z)+\mathcal{L}_{001}(z)\right)$$

$$\rho_{1234} = z\bar{z}$$
$$\rho_{1432} = (1-z)(1-\bar{z})$$

$\mathcal{L}_{10\ldots}(z)$ are the single-valued harmonic polylogarithms (SVHPLs) introduced by Francis Brown in 2009. They are single-valued in the region where $\bar{z} = z^*$.

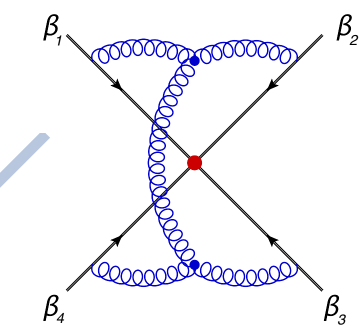
The result is very elegant. Can we re-derive it by boostrap?

Yes! Ø. Almelid, C. Duhr, EG, A. McLeod, C.D. White, JHEP 09 (2017) 073

Colour structure of the 3-loop soft anomalous dimension

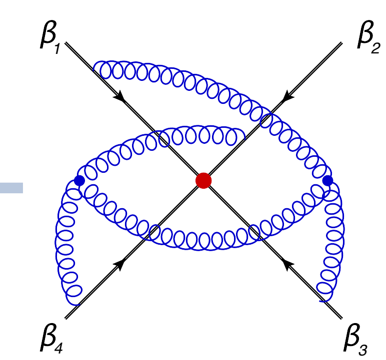
Non-Abelian exponentiation theorem [EG, Smillie, White (2013)] **implies that the Soft Anomalous Dimension has *fully connected* colour factors, such as** $f^{abe}f^{cde}\,\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d$

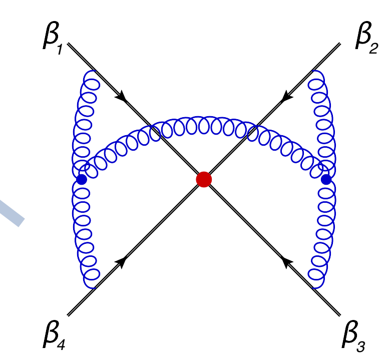
Applying **colour conservation** one finds that the answer can be expressed in terms of colour structures involving four generators correlating 3 and 4 lines.

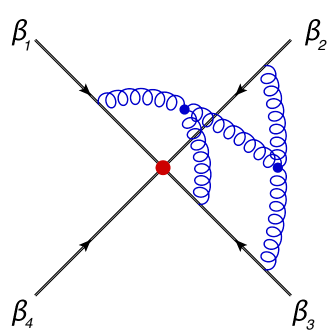


Bose symmetry then implies the structure:

$$
\begin{aligned}
\Delta_n^{(3)}(\{\rho_{ijkl}\}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\leq i<j<k<l\leq n}\Big[\; & \mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ikjl},\rho_{iljk}) \\
& + \mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijkl},\rho_{ilkj}) \\
& + \mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\,\mathcal{F}(\rho_{ijlk},\rho_{iklj})\Big] \\
& - C\sum_{i=1}^{n}\sum_{\substack{1\leq j<k\leq n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\Bigg\}
\end{aligned}
$$







Corrections to the light-like soft anomalous dimension through 4 loop

Using non-Abelian exponentiation and colour conservation [Becher & Neubert (2019)]

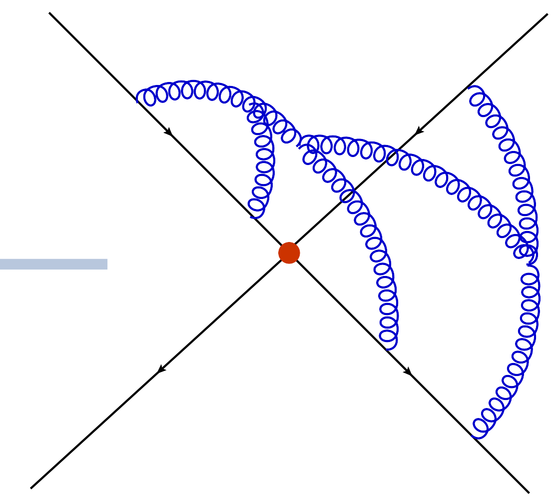
$$\Gamma_n\left(\{s_{ij}\},\lambda^2,\alpha_s\right) = -\frac{1}{4}\gamma_K(\alpha_s)\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_j\,\ln\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_i^n\gamma_i(\alpha_s)$$
$$+\frac{1}{2}f(\alpha_s)\sum_{(i,j,k)}f^{ade}f^{bce}\left(\mathbf{T}_i^a\mathbf{T}_i^b+\mathbf{T}_i^b\mathbf{T}_i^a\right)\mathbf{T}_j^c\mathbf{T}_k^d$$
$$+\sum_{(i,j,k,l)}f^{ade}f^{bce}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijlk},\rho_{iklj};\alpha_s)$$
$$-\frac{1}{2}\sum_R g_R(\alpha_s)\left[\sum_{(i,j)}\left(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\right)\ln\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\right]$$
$$+\sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\,\mathcal{G}_R(\rho_{ijlk},\rho_{iklj};\alpha_s)$$
$$+\sum_{(i,j,k,l)}\mathcal{T}_{ijkli}\,\mathcal{H}_1(\rho_{ijlk},\rho_{iklj};\alpha_s)$$
$$+\sum_{(i,j,k,l,m)}\mathcal{T}_{ijklm}\,\mathcal{H}_2(\rho_{ijkl},\rho_{ijmk},\rho_{ikmj},\rho_{jiml},\rho_{jlmi};\alpha_s)$$

But structures with an odd number of generators, $\mathcal{T}_{ijklm}=if^{adf}f^{bcg}f^{efg}\left(\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\mathbf{T}_m^e\right)_+$ are excluded based on symmetry under reversal of all lines (argument based on rapidity anomalous dimension, related to the soft one by conformal mapping)

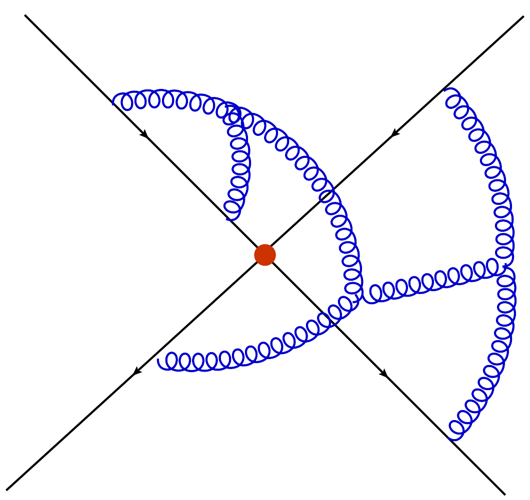
[Vladimirov (2017)]

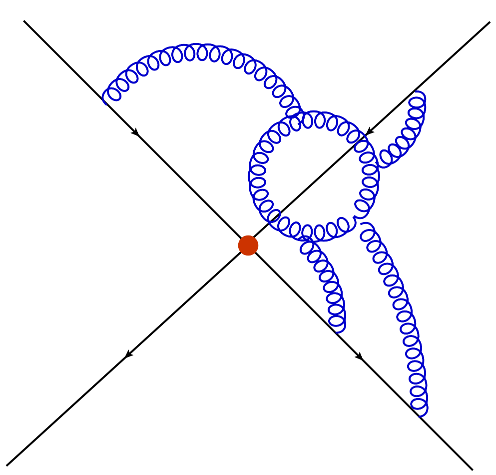
Corrections to the light-like soft anomalous dimension through 4 loop

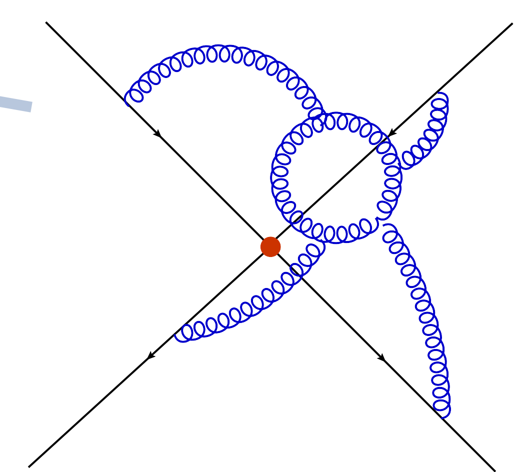
Using non-Abelian exponentiation, colour conservation and the absence of odd structures [Vladimirov (2017)] along with the relation with the lightlike cusp anomalous dimension [Becher & Neubert (2019)]



$$\begin{aligned}\mathbf{\Gamma}_n\left(\{s_{ij}\},\lambda^2,\alpha_s\right) =& -\frac{1}{4}\gamma_K(\alpha_s)\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_j\,\ln\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_{i}^{n}\gamma_i(\alpha_s)\\ &+\frac{1}{2}f(\alpha_s)\sum_{(i,j,k)}f^{ade}f^{bce}\left(\mathbf{T}_i^a\mathbf{T}_i^b+\mathbf{T}_i^b\mathbf{T}_i^a\right)\mathbf{T}_j^c\mathbf{T}_k^d\\ &+\sum_{(i,j,k,l)}f^{ade}f^{bce}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijlk},\rho_{iklj};\alpha_s)\\ &-\frac{1}{2}\sum_R g_R(\alpha_s)\left[\sum_{(i,j)}\left(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\right)\ln\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\right]\\ &+\sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\,\mathcal{G}_R(\rho_{ijlk},\rho_{iklj};\alpha_s)\end{aligned}$$



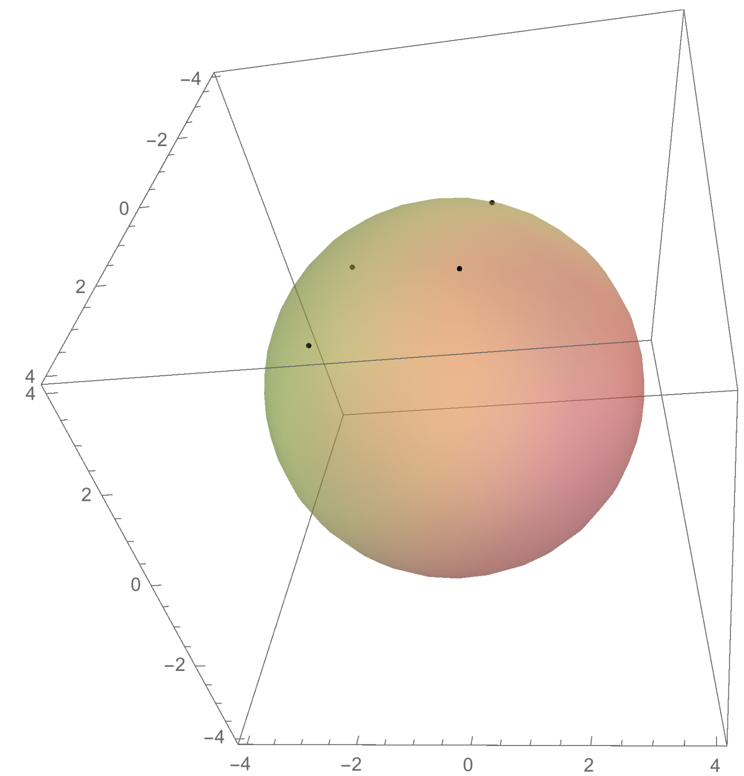




Direct computation is beyond what is currently possible.

The Space of Functions

n lightlike velocities β_i are described by
n points of the Riemann sphere



For a given set of 4 lines $\{i, j, k, l\}$ there are two independent cross ratios $\{\rho_{ijkl}, \rho_{ilkj}\}$

$$\rho_{ijkl} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left| \frac{(z_i - z_j)\,(z_k - z_l)}{(z_i - z_k)\,(z_j - z_l)} \right|^2$$

We can further use $\mathrm{SL}(2, \mathbb{C})$ invariance to fix three of the points:

$$z_i = z,\ z_j = 0,\ z_k = \infty,\ z_l = 1 \qquad \{\rho_{ijkl},\ \rho_{ilkj}\} = \{z\bar{z},\ (1-z)(1-\bar{z})\}$$

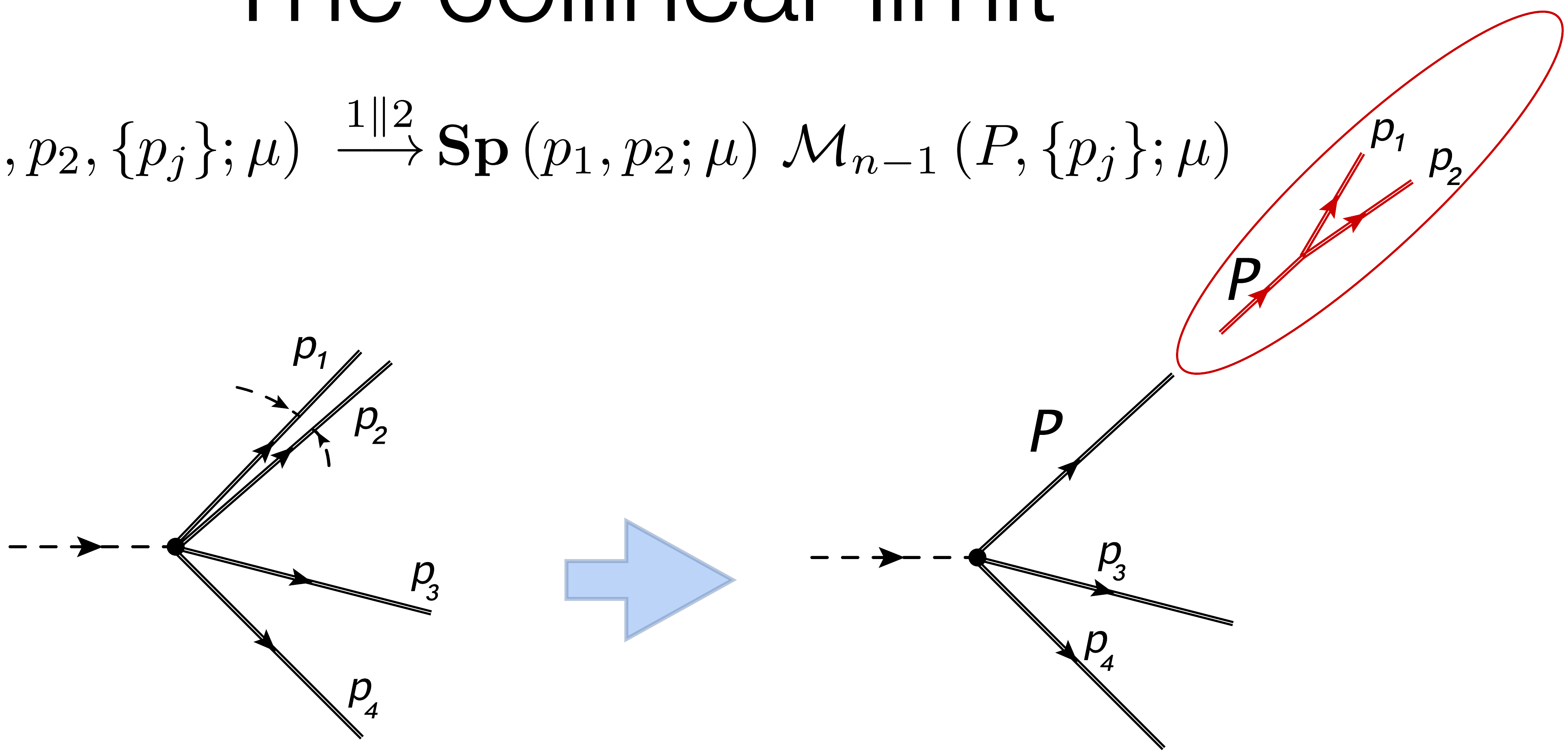
Iterated integrals on a Riemann sphere with n marked points are combinations of multiple polylogarithms (MPLs) with rational coefficients. F.C.S. Brown (2009)

- Absence of singularities in Euclidean kinematics implies: **Single-Valued** MPLs.
- If singularities **only** appear when points coincide, one obtains Single-Valued **HPLs,** i.e. Symbol alphabet: $\{z, \bar{z}, 1-z, 1-\bar{z}\}$
- At 3-loops we expect **pure functions** of **uniform weight** — a property of $\mathcal{N} = 4$ SYM

Ø. Almelid, C. Duhr, EG, A. McLeod, C.D. White (2017)

The collinear limit

$$\mathcal{M}_n\left(p_1, p_2, \{p_j\}; \mu\right) \xrightarrow{1\|2} \mathbf{Sp}\left(p_1, p_2; \mu\right)\, \mathcal{M}_{n-1}\left(P, \{p_j\}; \mu\right)$$



IR singularities of the splitting amplitude are those present in n-parton scattering (with 1&2) while not in (n-1)-parton scattering:

$$\Gamma_{\mathbf{Sp}} = \Gamma_n - \Gamma_{n-1}$$

Becher & Neubert (2009), …

The expectation (see e.g. [Catani, de Florian, Rodrigo 1112.4405, Feige & Schwartz 1403.6472]) is that the final-state splitting amplitude depends exclusively on the variables of the collinear pair.

This is *automatically realised* by the dipole formula for the singularities.

Collinear limit constraints at 3 and 4 loops

Requiring that splitting amplitude is independent of the rest of the process implies:

$$\lim_{\rho_{12ij}\to 0} \mathcal{F}_R(\rho_{12ij},1;\alpha_s) = \frac{f(\alpha_s)}{2}$$

$$\lim_{\rho_{12ij}\to 0} \mathcal{G}_R(\rho_{12ij},1;\alpha_s) = -\frac{g_R(\alpha_s)}{12}\ln\rho_{12ij}$$

$$\begin{aligned}\mathbf{\Gamma}_n\left(\{s_{ij}\},\lambda^2,\alpha_s\right) =& -\frac{1}{4}\gamma_K(\alpha_s)\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_j\,\ln\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_i^n\gamma_i(\alpha_s)\\ &+\frac{1}{2}f(\alpha_s)\sum_{(i,j,k)}f^{ade}f^{bce}\left(\mathbf{T}_i^a\mathbf{T}_i^b+\mathbf{T}_i^b\mathbf{T}_i^a\right)\mathbf{T}_j^c\mathbf{T}_k^d\\ &+\sum_{(i,j,k,l)}f^{ade}f^{bce}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\rho_{ijlk},\rho_{iklj};\alpha_s)\\ &-\frac{1}{2}\sum_R g_R(\alpha_s)\left[\sum_{(i,j)}\left(\mathcal{D}^R_{iijj}+2\mathcal{D}^R_{iiij}\right)\ln\left(\frac{-s_{ij}}{\lambda^2}\right)+\sum_{(i,j,k)}\mathcal{D}^R_{ijkk}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\right]\\ &+\sum_R\sum_{(i,j,k,l)}\mathcal{D}^R_{ijkl}\,\mathcal{G}_R(\rho_{ijlk},\rho_{iklj};\alpha_s)\end{aligned}$$

Becher & Neubert (2020) *JHEP* 01 (2020) 025

Using the known four-loop result for the *lightlike cusp anomalous dimension*:

$$\lim_{\rho_{12ij}\to 0} \mathcal{G}_R(\rho_{12ij},1;\alpha_s) = -\frac{g_R(\alpha_s)}{12}\ln\rho_{12ij} = \left(\frac{\zeta_3^2}{8}+\frac{31\zeta_6}{96}\right)\ln\rho_{12ij}$$

Boels, Huber and Yang, 1705.03444
Moch et al., 1707.08315
Grozin, Henn, Stahlhofen, 1708.01221
Henn, Korchemsky and Mistlberger 1911.10174

This weight 6 coefficient cannot be obtained from a pure SVHPL ansatz for $\mathcal{G}_R(\rho_{ijlk},\rho_{iklj};\alpha_s)$ — SVMPLs with the additional letter $z-\bar{z}$ are essential!

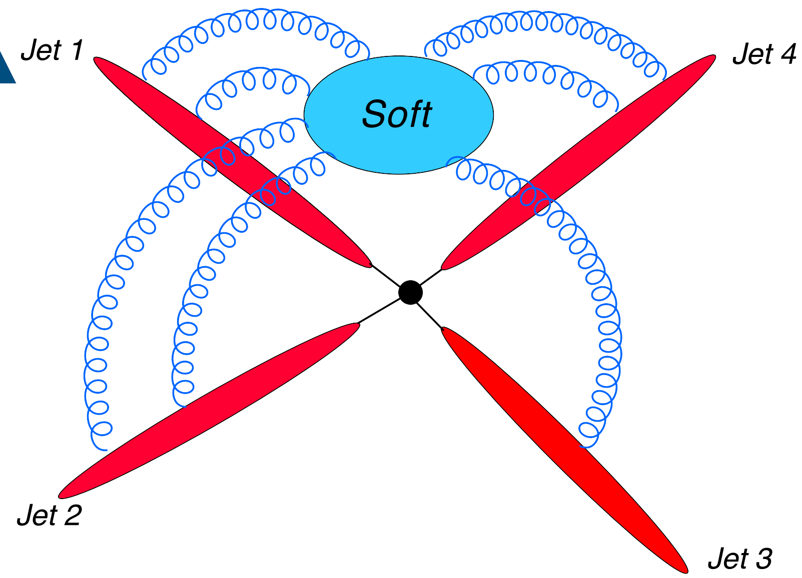
Minimal alphabet: $\{z,\bar{z},1-z,1-\bar{z},z-\bar{z}\}$

Duhr, EG, Maher & McLeod (to appear)

The Regge limit overlap with infrared singularities

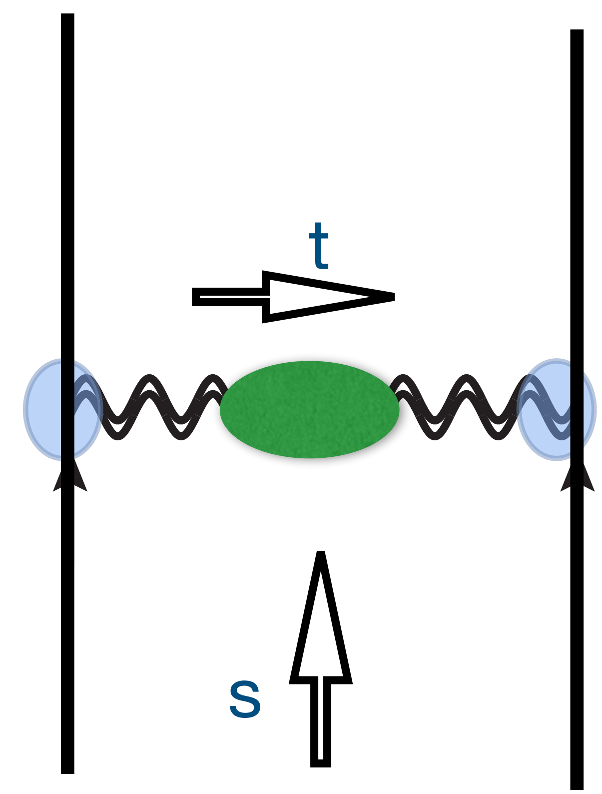
Exponentiation of infrared (soft) singularities $1/\epsilon$
in fixed-angle amplitude using factorisation into
Soft-Collinear-Hard subprocesses

IR



High-energy limit
of the soft
anomalous dimension

= Soft limit of BFKL



Regge limit

High-Energy limit:
BFKL resummation of log(s/t)
Regge factorization

The soft anomalous dimension in the high-energy limit — three loops

$$\mathbf{\Gamma}_{ij\to ij}\left(\alpha_s, L, \frac{-t}{\lambda^2}\right) = \Gamma_i\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \Gamma_j\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \frac{1}{2}\gamma_K(\alpha_s)\left[L\mathbf{T}_{\mathbf{t}}^{\mathbf{2}} + i\pi\mathbf{T}_{\mathbf{s-u}}^{\mathbf{2}}\right] + \sum_{\ell=3}^{\infty}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}\mathbf{\Delta}^{(\ell)}(L)$$

$$\mathbf{\Delta}^{(3)} = 0L^2 + i\pi\Big[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\Big]\frac{11}{4}\zeta_3 L + O(L^0)$$

$$L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) = \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}$$

Absence of $\alpha_s^3 L^k$ for $k \geq 1$ in the *real part* and for $k \geq 2$ in the *imaginary part*, is a non-trivial prediction from rapidity evolution, which underpins the structure of corrections to the dipole formula.

Caron-Huot, EG, Vernazza JHEP 06 (2017) 016

The **only** term in the *real part* of the soft anom. dim. linear in the high-energy logarithm is the cusp anomalous dimension, **generalising** the Korchemsky & Korchemskaya relation between the gluon Regge trajectory and cusp to 3 loops.

The soft anomalous dimension in the high-energy limit — four loops

Falcioni, EG, Maher, Milloy, Vernazza (2021)

$$\mathbf{\Gamma}_{ij\to ij}\left(\alpha_s, L, \frac{-t}{\lambda^2}\right) = \Gamma_i\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \Gamma_j\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \frac{1}{2}\gamma_K(\alpha_s)\left[L\mathbf{T_t^2} + i\pi\mathbf{T_{s-u}^2}\right] + \sum_{\ell=3}^{\infty}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}\mathbf{\Delta}^{(\ell)}(L)$$

$$L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) = \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}$$

$$\mathbf{\Delta}^{(4)}(L) = -L^3 i\pi\frac{\zeta_3}{24}\left[\mathbf{T}_t^2,[\mathbf{T}_t^2,\mathbf{T}_{s-u}^2]\right]\mathbf{T}_t^2 + L^2\mathbf{\Delta}^{(-,4,2)} + L^2\zeta_2\zeta_3\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24} - \frac{1}{4}\mathbf{T}_t^2[(\mathbf{T}_{s-u}^2)^2,\mathbf{T}_t^2] + \frac{3}{4}[\mathbf{T}_{s-u}^2,\mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2\right) + \mathcal{O}(L)$$

All Regge-limit constraints at four loops:

Signature even				Signature odd			
	L^3	L^2	L^1 (conj.)		L^3	L^2	L^1
$\mathcal{F}_A^{(+,4)}$	0	$-\frac{C_A}{8}\zeta_2\zeta_3$	0	$\mathcal{F}_A^{(-,4)}$	$i\pi\frac{C_A}{24}\zeta_3$	?	?
$\mathcal{F}_F^{(+,4)}$	0	0	0	$\mathcal{F}_F^{(-,4)}$	0	?	?
$\mathcal{G}_A^{(+,4)}$	0	$\frac{1}{2}\zeta_2\zeta_3$	$\frac{1}{6}g_A^{(4)}$				
$\mathcal{G}_F^{(+,4)}$	0	0	$\frac{1}{6}g_F^{(4)}$				
$\mathcal{H}_1^{(+,4)}$	0	0	0	$\mathcal{H}_1^{(-,4)}$	0	?	?
				$\tilde{\mathcal{H}}_1^{(-,4)}$	0	?	?

Unitarity cut of the 3-loop web

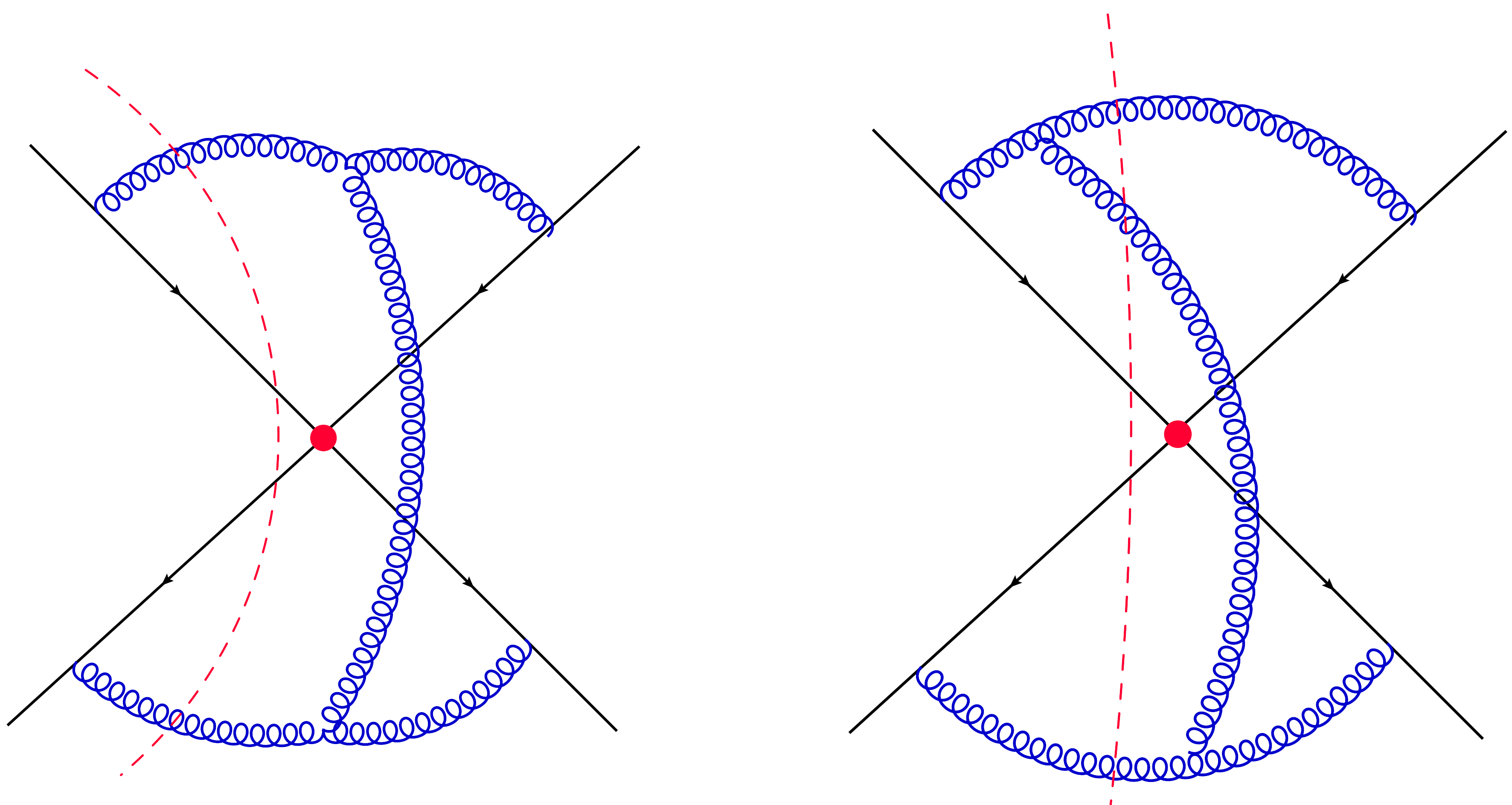
The analytic structure of webs can be explored by unitarity cuts using the relation between discontinuities and cuts: [Cutkosky (1960), ’t Hooft & Veltman (1974)]

$$\mathrm{Disc}_s F = (-1) \sum_{\{\mathrm{cuts}\}} \mathrm{Cut}_s F$$

Cutting a Wilson line, along with all the gluons emitted from it, yields a vanishing cut.

[Andries Waelkens, PhD (2017)]

Examples of vanishing cuts of the 3-loop web:

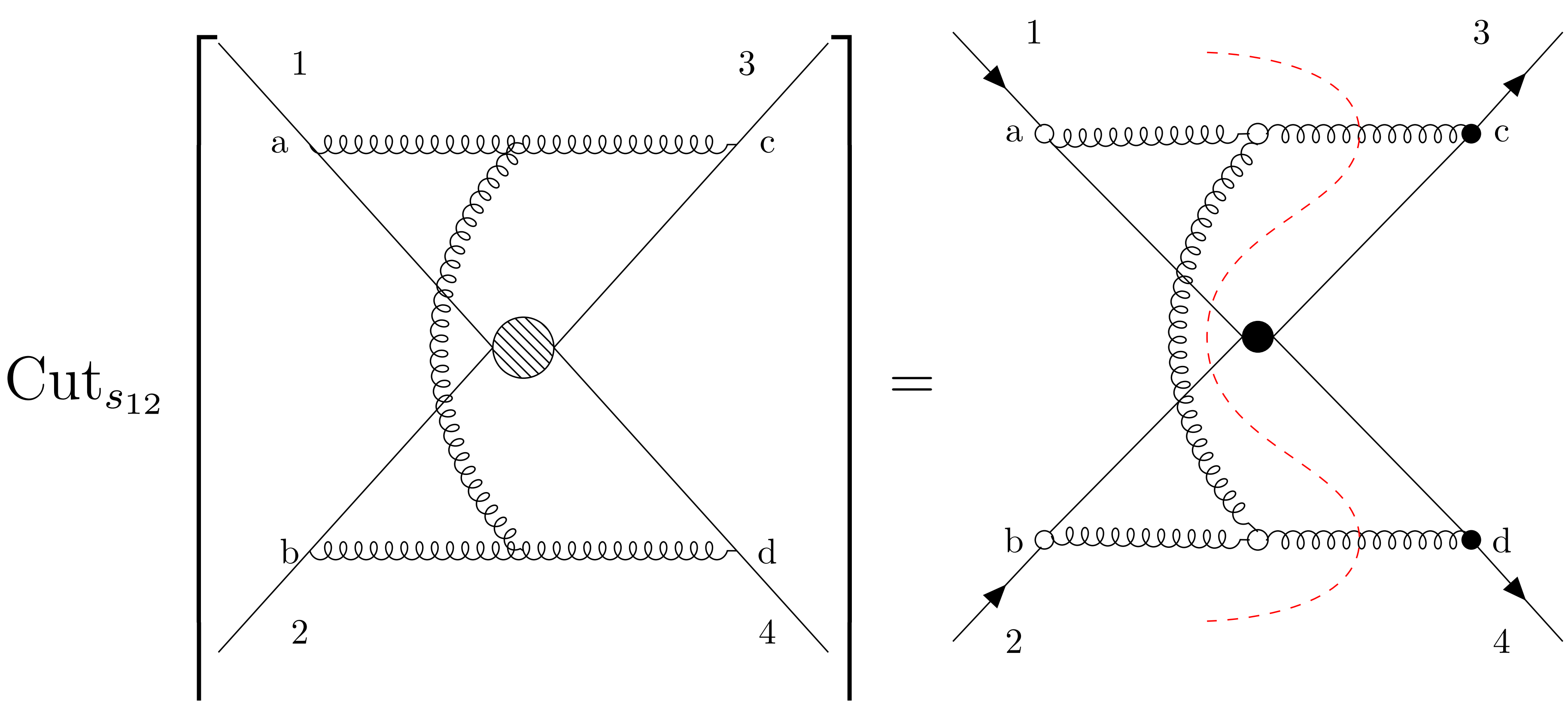


Unitarity cut of the 3-loop web

The analytic structure of webs can be explored by unitarity cuts using the relation between discontinuities and cuts: [Cutkosky (1960), ’t Hooft & Veltman (1974)]

$$\mathrm{Disc}_s F = (-1) \sum_{\{\mathrm{cuts}\}} \mathrm{Cut}_s F$$

There is just **a single non-vanishing cut** on the (12) channel:



Iterated unitarity cuts

The relation between **discontinuity** and **unitarity cuts** was generalised to <u>iterated</u> discontinuities and <u>iterated</u> unitary cuts in [Abreu, Britto, Duhr, EG (2014)] *by **excluding** all crossed cuts.*

$$\text{Disc}_{s_1,\ldots,s_k} F = (-1)^k \sum_{\{\text{cuts}\}} \text{Cut}_{s_1,\ldots,s_k} F$$

<u>Example</u>:
only the following 6 diagrams contribute to the double discontinuity (crossed cuts are not included in the sum).

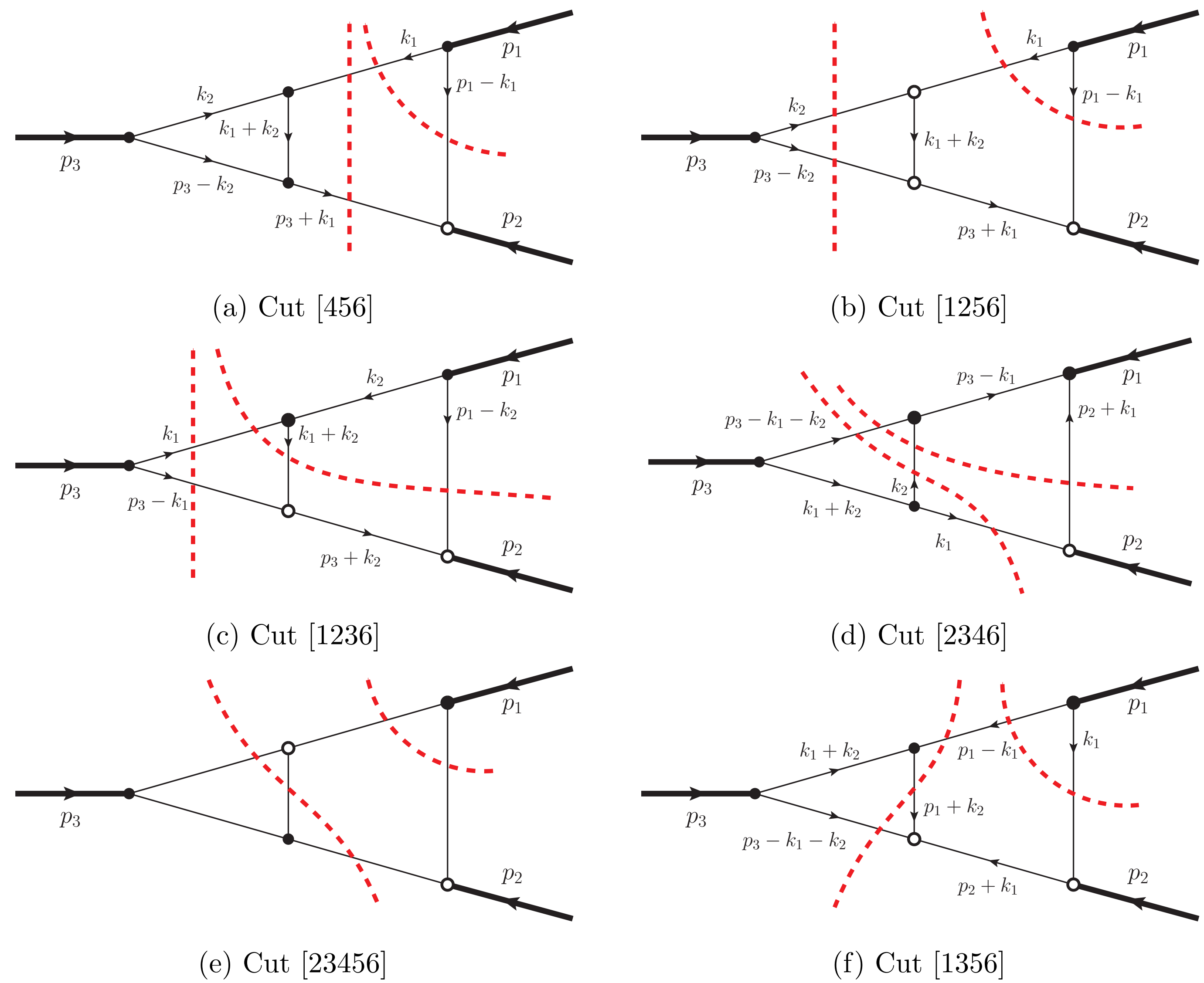
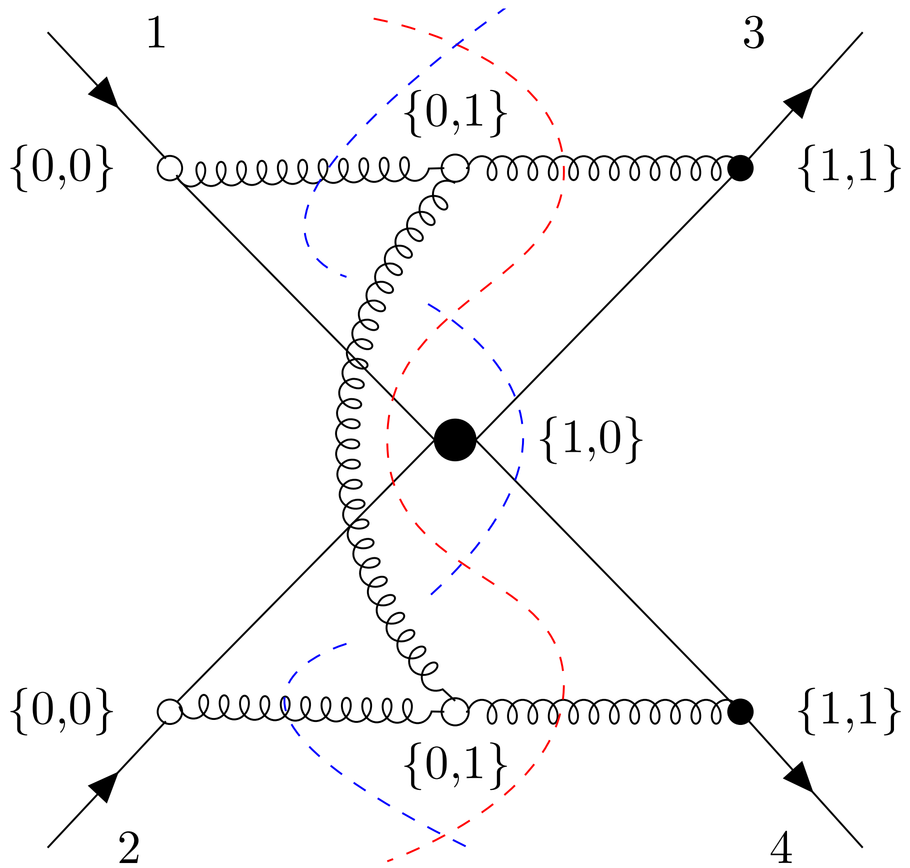


Figure 11: Cut diagrams contributing to the $\text{Cut}_{p_1^2} \circ \text{Cut}_{p_3^2}$ sequence of unitarity cuts.

Sequential unitarity cuts of the 3-loop web

<u>We observe:</u> There are no compatible (non-crossed) non-vanishing unitarity cuts on the (12) and (34) channels.

The red (12) and blue (34) channels cuts (the only non-vanishing cuts on the respective channels) are incompatible. Hence the iterated cut vanishes.



$$\mathrm{Disc}_{s_{12},s_{34}}\left[w_{(13)(24)}\right] = \mathrm{Cut}_{s_{12},s_{34}}\left[w_{(13)(24)}\right] = 0$$

Indeed, **using the computed expression** for the web:

$$\mathrm{Disc}_{s_{12},s_{34}}\left[w_{(13)(24)}\right] = 0$$

[Niamh Maher, PhD (2022)]

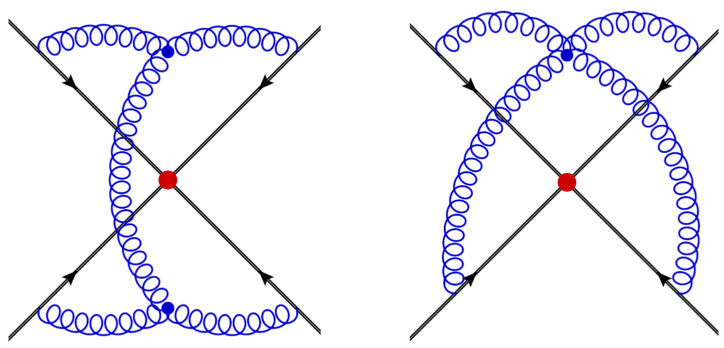
Note that this double discontinuity vanishes for **generic** (non-lightlike) Wilson lines.

Vanishing iterated discontinuities as constraints

$$\mathrm{Disc}_{s_{12},s_{34}}[w_{(13)(24)}] = \mathrm{Cut}_{s_{12},s_{34}}[w_{(13)(24)}] = 0$$

For generic Wilson lines:

Vanishing double discontinuity of **all** 3-loop webs correlating all 6 angles!



For (nearly) lightlike Wilson lines:

For the anomalous dimension, the vanishing of the double discontinuity above translates into an *adjacency condition* for the first two entries in the symbol: **any repeated entries are forbidden** (in fact, this applies throughout the symbol!)

$$z \otimes z$$

$$\bar{z} \otimes \bar{z}$$

$$1 - z \otimes 1 - z$$

$$1 - \bar{z} \otimes 1 - \bar{z}$$

[Niamh Maher, PhD (2022)]

Conclusions

The soft anomalous dimension of massless scattering:

- ✓ Colour structure is dictated by non-Abelian exponentiation: the anomalous dimension features only fully-connected colour structures.
- ✓ Kinematic dependence is constrained by factorisation & rescaling symmetry
- ✓ Space of functions: SVHPLs at 3 loops, SVMPLs at 4 loops
- ✓ Analytic structure is highly constrained at 3 loops by (non-Steinmann) vanishing double discontinuities (applies also in the non-lightlike case!).
- ✓ At 3 loops, constraints from collinear limits and the Regge limit allow to bootstrap the general-kinematics result.
- ✓ At 4 loops, despite recent progress, available constraints are not sufficient to fully determine the soft anomalous dimension.

IR Singularities in QCD using Wilson lines

IR singularities in general kinematics— state of the art:

- **Massless** particles scattering:

  3 loops for any number of legs [Almelid, Duhr & EG (2015)]

  reproduced by *bootstrap* [Almelid, Duhr, EG, McLeod & White (2017)]

  Constraints at **4 loops**

  [Vladimirov (2017), Becher & Neubert (2020); Falcioni et al. (2021)]

- **Massive+massless** particles scattering

  2 loops for any number of massive legs

  [Ferroglia, Neubert, Pecjak & Li Lin Yang (2009)]

  3 loops for 2 massive legs (angle-dependent cusp)

  [Grozin, Henn, Korchemsky & Marquard (2015)]

  4 loops for 2 massive legs (angle-dependent cusp) at small angles

  [Grozin, Lee & Pikelner (2022)]

  Partial calculation **3 loops for one massive, others massless**

  [Liu & Schalch (2022)]

IR singularities in amplitudes with massless legs

Exponentiation:

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda,\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

The Dipole Formula:

$$\Gamma_{\mathrm{Dip.}}(\lambda,\alpha_s) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s\right)$$

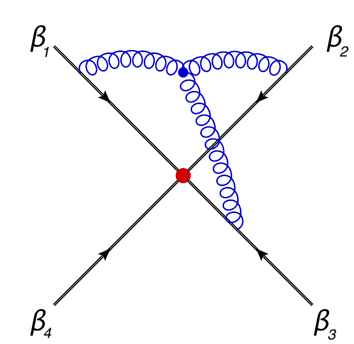
Lightlike Cusp anomalous dimension

Catani (1998)
Dixon, Mert-Aybat and Sterman (2006)
Becher & Neubert, EG & Magnea (2009)

Rescaling symmetry of Wilson-line velocities & **soft/jet factorisation** imply:

A. The anomalous dimension Γ to two loops is a ***dipole sum.***

(tripoles, $f_{abc}\,\mathbf{T}_1^a\mathbf{T}_2^b\mathbf{T}_3^c$, would be incompatible with rescaling symmetry.)

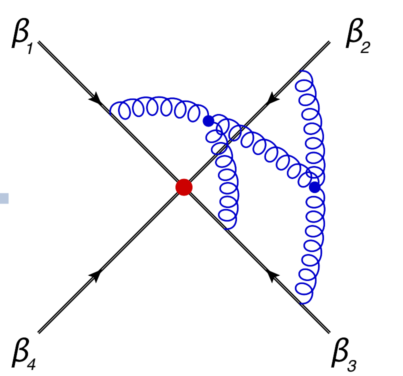


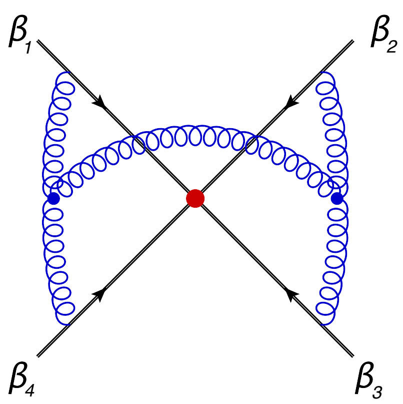
B. Strong constraints on higher-order corrections.

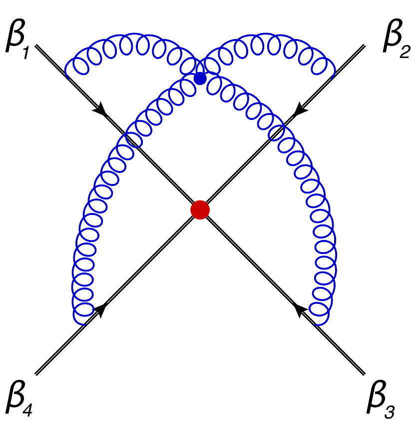
Corrections to the light-like soft anomalous dimension through 3 loop

Using non-Abelian exponentiation and colour conservation

$$\mathbf{\Gamma}_n\left(\{s_{ij}\},\lambda,\alpha_s\right) = -\frac{1}{4}\gamma_K(\alpha_s)\sum_{(i,j)}\mathbf{T}_i\cdot\mathbf{T}_j\,\ln\left(\frac{-s_{ij}}{\lambda}\right)+\sum_{i}^{n}\gamma_i(\alpha_s)$$
$$+\frac{1}{2}f(\alpha_s)\sum_{(i,j,k)}f^{ade}f^{bce}\left(\mathbf{T}_i^a\mathbf{T}_i^b+\mathbf{T}_i^b\mathbf{T}_i^a\right)\mathbf{T}_j^c\mathbf{T}_k^d$$
$$+\sum_{(i,j,k,l)}f^{ade}f^{bce}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,\mathcal{F}(\beta_{ijlk},\beta_{iklj};\alpha_s)$$







Diagrammatic origin of the 3-loop anomalous dimension near the lightlike limit

3-loop webs involving 4 Wilson lines

Single connected subgraph

Each web depends on all six angles - **can form conformally-invariant cross ratios** (cicrs)

Two connected subgraphs

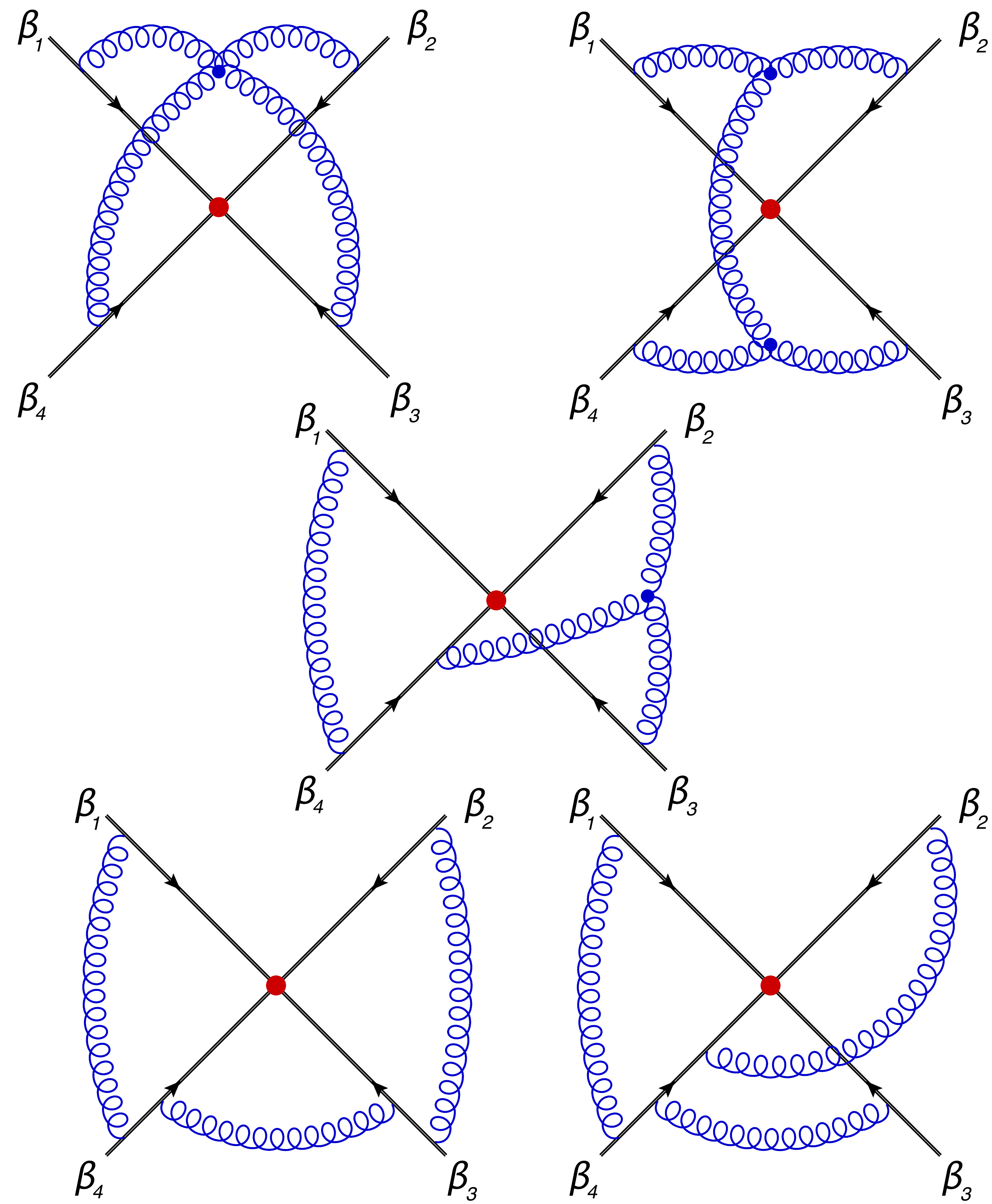
Depends on $\gamma_{14}, \gamma_{23}, \gamma_{24}, \gamma_{34}$ only.

Cannot form cicrs - yields products of logs for near lightlike kinematics

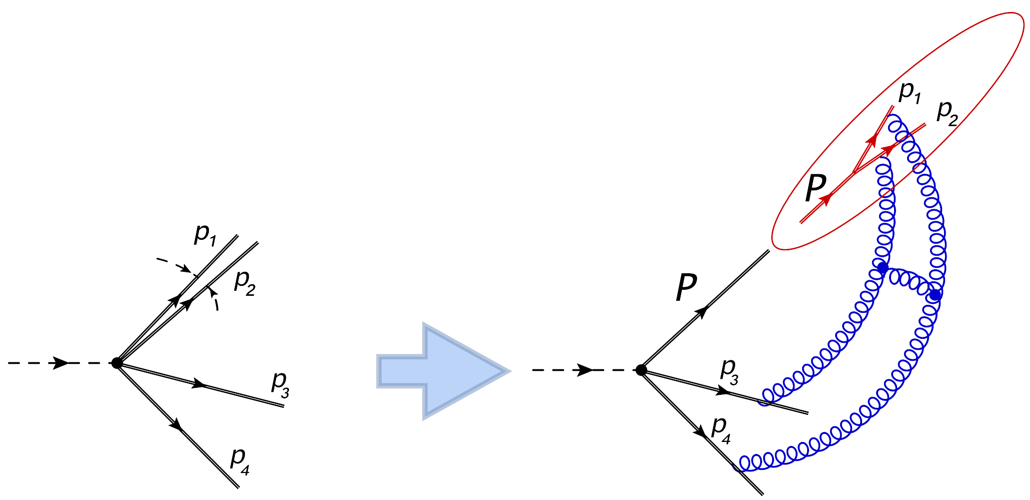
Three connected subgraphs (multiple gluon exchanges)

Depends on 3 angles only!

Cannot form cicrs - yields products of logs for near lightlike kinematics



Collinear limit constraints at 3 loops



The 3-loop splitting amplitude

$$\Delta^{(3)}_{\mathbf{Sp}} = (\Delta^{(3)}_n - \Delta^{(3)}_{n-1})\Big|_{1\|2} = -24\left(\frac{\alpha_s}{4\pi}\right)^3 (\zeta_5 + 2\zeta_2\zeta_3)\left[f^{abe}f^{cde}\left\{\mathbf{T}^a_1, \mathbf{T}^c_1\right\}\left\{\mathbf{T}^b_2, \mathbf{T}^d_2\right\} + \frac{1}{2}C_A^2\mathbf{T}_1\cdot\mathbf{T}_2\right]$$

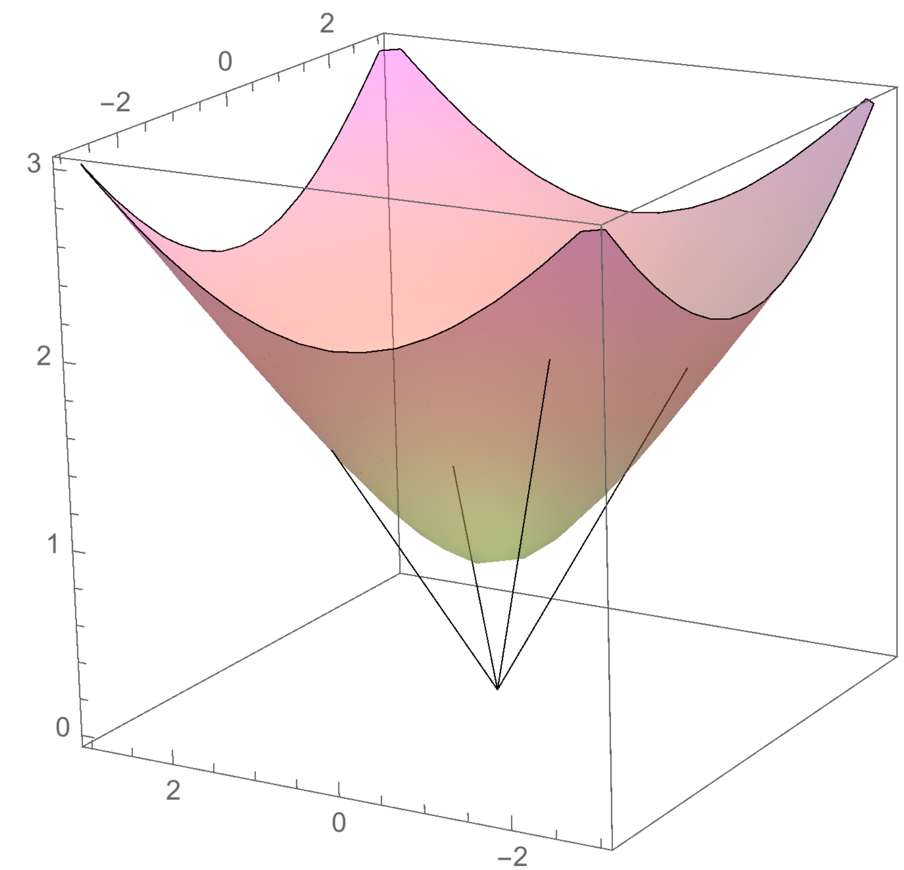
can be computed from $\Delta^{(3)}_{\mathbf{Sp}} = (\Delta^{(3)}_3 - \Delta^{(3)}_2)\Big|_{1\|2} = \Delta^{(3)}_3\Big|_{1\|2}$

This leads to a constraint upon requiring
that the same may also be obtained from $(\Delta^{(3)}_4 - \Delta^{(3)}_3)\Big|_{1\|2}$

Kinematic variables

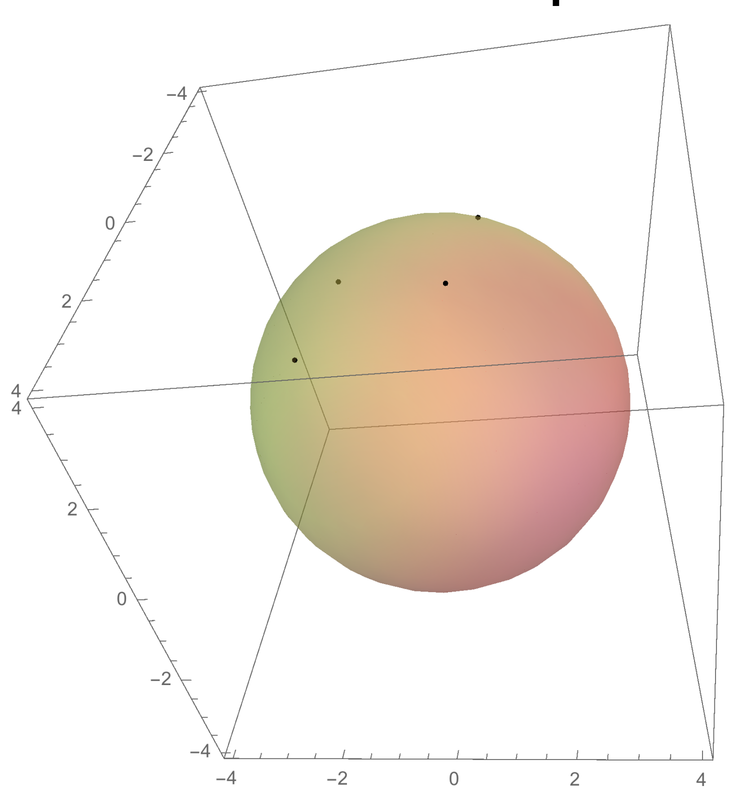
Ø. Almelid, C. Duhr, EG, A. McLeod, C.D. White, “Bootstrapping the QCD soft anomalous dimension” JHEP 09 (2017) 073

For $\beta_i^2 \neq 0$: using **rescaling symmetry** the velocities β_i map to a hyperbolic 3D space:



$$(\beta_i^0)^2 - (\beta_i^1)^2 - (\beta_i^2)^2 - (\beta_i^3)^2 = R^2, \quad \beta_i^0 > 0$$

The **lightlike limit** corresponds to the boundary of this space: β_i map to points on a Riemann sphere.



Parametrising: $$\beta_i = \left(1 + \frac{z_i \bar{z}_i}{4}, \frac{z_i + \bar{z}_i}{2}, \frac{z_i - \bar{z}_i}{2i}, 1 - \frac{z_i \bar{z}_i}{4}\right)$$

maps angles to distances: $2\beta_i \cdot \beta_j = |z_i - z_j|^2$

The **rescaling-invariant** kinematic variables are:

$$\rho_{ijkl} = \frac{(\beta_i \cdot \beta_j)\,(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left|\frac{(z_i - z_j)\,(z_k - z_l)}{(z_i - z_k)\,(z_j - z_l)}\right|^2$$

The quest for understanding long-distance singularities of scattering amplitudes

outline

✓ Factorisation & rescaling symmetry for lightlike Wilson lines - dipole formula

✓ soft anomalous dimension at 3-loop

✓ Colour structure: non-Abelian exponentiation; webs at 3 and 4 loops

✓ Space of functions and constraints from collinear and Regge limits
— the bootstrap program

✓ Analytic structure: vanishing cuts and discontinuities

The soft anomalous dimension in the high-energy limit — four loops

Falcioni, EG, Maher, Milloy, Vernazza (2021)

$$\mathbf{\Gamma}_{ij\to ij}\left(\alpha_s, L, \frac{-t}{\lambda^2}\right) = \Gamma_i\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \Gamma_j\left(\alpha_s, \frac{-t}{\lambda^2}\right) + \frac{1}{2}\gamma_K(\alpha_s)\left[L\mathbf{T}_{\mathbf{t}}^{\mathbf{2}} + i\pi\mathbf{T}_{\mathbf{s-u}}^{\mathbf{2}}\right] + \sum_{\ell=3}^{\infty}\left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^{\ell}\mathbf{\Delta}^{(\ell)}(L)$$

$$L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) = \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}$$

Explicit computation from *rapidity evolution equations*:

$$\mathbf{\Delta}^{(4)}(L) = -L^3 i\pi\frac{\zeta_3}{24}\left[\mathbf{T}_t^2, [\mathbf{T}_t^2, \mathbf{T}_{s-u}^2]\right]\mathbf{T}_t^2 + L^2\mathbf{\Delta}^{(-,4,2)} + L^2\zeta_2\zeta_3\left(\frac{d_{AA}}{N_A} - \frac{C_A^4}{24} - \frac{1}{4}\mathbf{T}_t^2[(\mathbf{T}_{s-u}^2)^2, \mathbf{T}_t^2] + \frac{3}{4}[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]\mathbf{T}_t^2\mathbf{T}_{s-u}^2\right) + \mathcal{O}(L)$$

Comparing terms linear in L in IR and Regge pole we further conjecture:

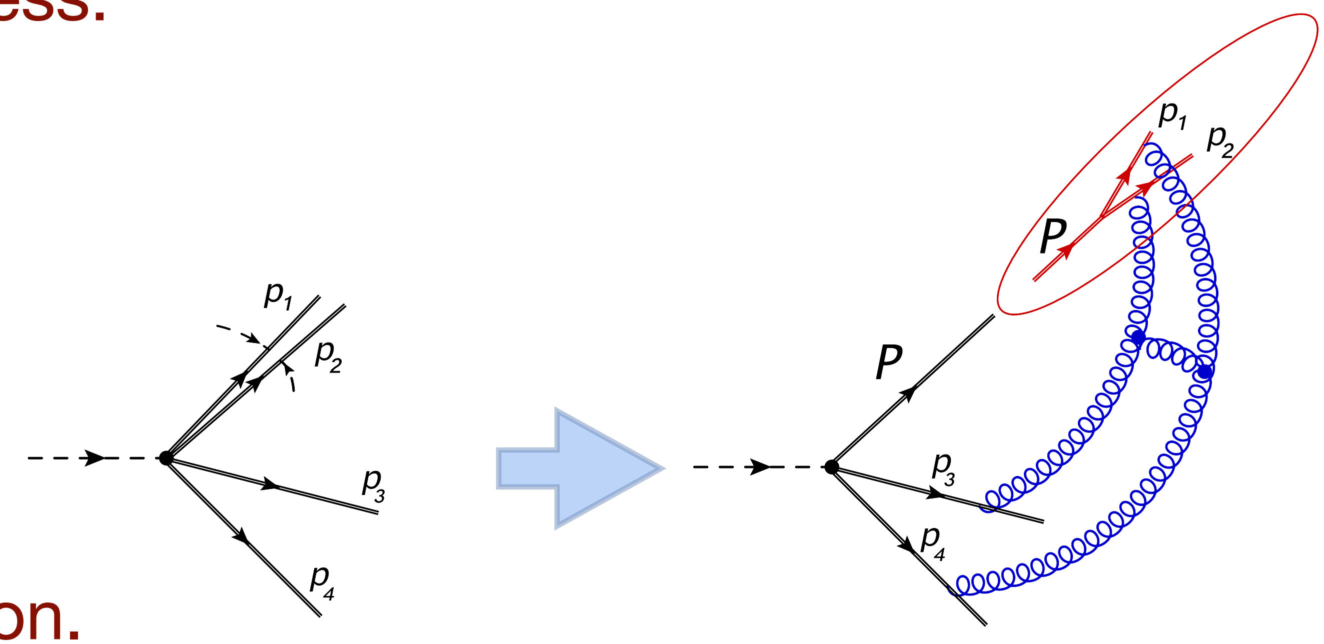
$$-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\mathbf{\Gamma}_{ij\to ij}^{(+)}\left(\alpha_s, L, \frac{-t}{\lambda^2}\right) \quad\longleftrightarrow\quad C_A\tilde{\alpha}_g(t)L$$

The collinear limit at 3 loops

Starting at 3 loops there are diagrams that could introduce correlation between collinear partons and the rest of the process:

$$\Gamma_{\mathbf{Sp}}(p_1, p_2; \mu) = \Gamma_{\mathbf{Sp}}^{\mathrm{dip.}}(p_1, p_2; \mu) + \Delta_{\mathbf{Sp}}$$

Requiring that the splitting amplitude singularities are independent of the rest of the process amounts to a constraint on the structure of the correction.



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can be computed from $\Delta_{\mathbf{Sp}}^{(3)} = \left(\Delta_3^{(3)} - \Delta_2^{(3)}\right)\Big|_{1\|2} = \Delta_3^{(3)}\Big|_{1\|2}$

Requiring that **the same** may also be obtained from $\left(\Delta_4^{(3)} - \Delta_3^{(3)}\right)\Big|_{1\|2}$ implies:

$$\lim_{\rho_{12ij}\to 0} \mathcal{F}_R(\rho_{12ij}, 1; \alpha_s) = \frac{f(\alpha_s)}{2}$$

[Becher & Neubert (2009); Dixon, EG & Magnea (2010); Almelid, Duhr, EG, McLeod & White (2017), Becher & Neubert (2020)]