COLOURFUL DIPOLES FROM THE LHC TO THE CELESTIAL SPHERE

Lorenzo Magnea

Università di Torino - INFN Torino

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Outline

- Infrared factorisation of scattering amplitudes
- The celestial sphere
- Infrared factorisation on the celestial sphere
- A colourful conformal field theory
- Many open questions

INFRARED VISIONS

A. Sen, A.H. Mueller, J. Collins, G. Sterman, J. Botts, LM, S. Catani, L. Dixon, E. Gardi, T. Becher, M. Neubert, I. Feige, M. Schwartz, O. Erdogan, Y. Ma, …

The factorised amplitude

Infrared divergences in fixed-angle multi-particle scattering amplitudes factorise

$$\mathcal{A}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \mathcal{F}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

The infrared factor is a colour operator determined by a finite anomalous dimension matrix

$$\mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{P}\exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2, \epsilon)\right)\right] ,$$

All infrared poles arise from the scale integration, through the d-dimensional running coupling

$$\lambda\frac{\partial\alpha_s}{\partial\lambda} \equiv \beta(\alpha_s, \epsilon) = -2\epsilon\alpha_s - \frac{\alpha_s^2}{2\pi}\sum_{k=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^k b_k \, .$$

For massless theories, the all-order structure of the anomalous dimension in known, up to corrections due to higher-order Casimir operators of the gauge algebra

$$\Gamma_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2)\right) + \Delta_n\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) ,$$

$$\rho_{ijkl} = \frac{p_i\cdot p_j\, p_k\cdot p_l}{p_i\cdot p_l\, p_j\cdot p_k} = \frac{s_{ij}s_{kl}}{s_{il}s_{jk}} \, .$$

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Color basis notation

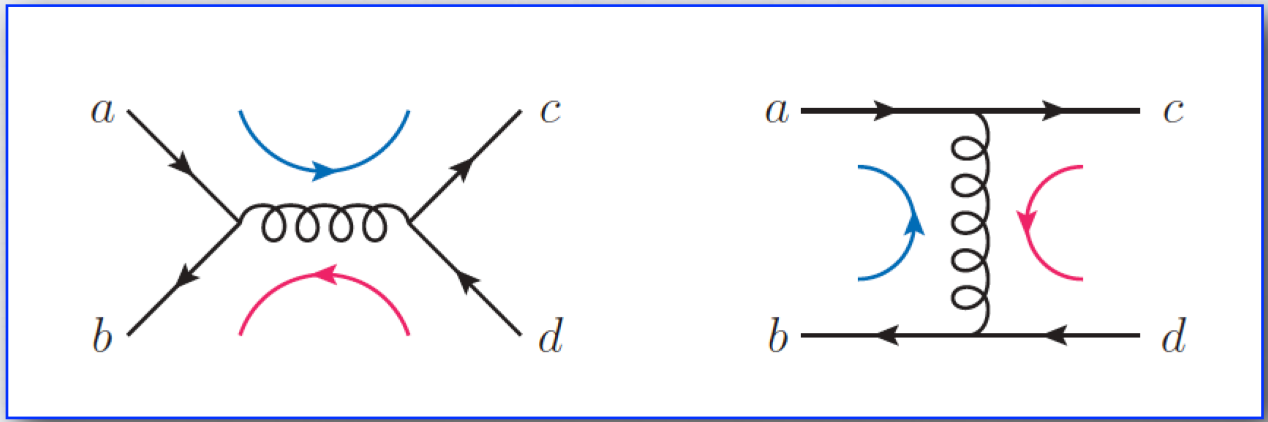
N. Kidonakis, G. Oderda, G. Sterman; M. Sjödahl; Y. Dokshitzer, G. Marchesini; M. Beneke, P. Falgari, C. Schwinn …

The amplitude can be expressed in a process-dependent orthonormal basis of colour tensors

$$\mathcal{A}_n^{a_1 \ldots a_n}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \sum_L \mathcal{A}_n^L\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) c_L^{a_1 \ldots a_n} \, .$$

$$\sum_{\{a_i\}} c_L^{a_1 \ldots a_n} \left(c_M^{a_1 \ldots a_n}\right)^* = \delta_{LM} \, .$$

A simple example is quark-antiquark scattering, where colour space is two-dimensional



Tree-level diagrams and leading color flows for quark-antiquark scattering

The amplitude is a vector in colour space, to all perturbative orders

$$\mathcal{A}_{abcd} = \mathcal{A}_1 \, c_{abcd}^{(1)} + \mathcal{A}_2 \, c_{abcd}^{(2)} \, , \qquad c_{abcd}^{(1)} = \delta_{ac}\delta_{bd} \, , \quad c_{abcd}^{(2)} = \delta_{ab}\delta_{cd} \, .$$

The exchange of a virtual gluon will shuffle the colour components, even if the gluon is soft

$$\text{QED}: \quad \mathcal{A}_{\text{div}} = \mathcal{Z} \, \mathcal{A}_{\text{Born}} \, ; \qquad \text{QCD}: \quad \left[\mathcal{A}_{\text{div}}\right]_J = \left[\mathcal{Z}\right]_{JK} \left[\mathcal{A}_{\text{Born}}\right]_K \, .$$

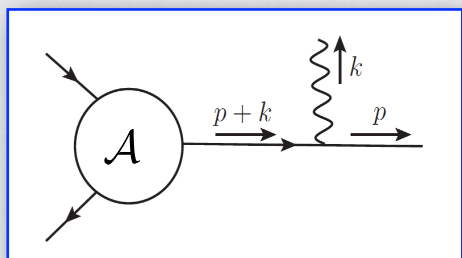
A. Bassetto, M. Ciafaloni, G. Marchesini; S. Catani, M. Seymour.

Color operator notation

A powerful basis-independent notation uses colour operators `inserting' soft gluons

$$\mathcal{A}_{n+1}^{a\, b_1 \ldots b_n}\Big|_{\text{soft}} \propto \sum_{i=1}^{n} \left[\mathbf{T}_i^a\right]^{b_i}_{\ c_i} \mathcal{A}_n^{b_1 \ldots c_i \ldots b_n} ,$$

Soft gluon operators are generators of the algebra in the representation of the emitter



$$g\mu^\epsilon\, \overline{u}_{s_i}(p_i)\, \gamma_\alpha \frac{\not p_i + \not k}{2 p_i \cdot k} \left(T^c\right)_{c_i d_i} \widehat{\mathcal{A}}_{s_1 \ldots s_n}^{c_1 \ldots d_i \ldots c_n}\left(\{p_j\}, k\right) \epsilon_\lambda^{*\,\alpha}(k) ,$$

At leading power in k :

$$g\mu^\epsilon \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} \left(T^c\right)_{c_i d_i} \left(\mathcal{A}_n\right)_{s_1 \ldots s_n}^{c_1 \ldots d_i \ldots c_n}\left(\{p_j\}\right) \equiv g\mu^\epsilon \frac{\beta_i \cdot \epsilon_\lambda^*(k)}{\beta_i \cdot k} \mathbf{T}_i \, \mathcal{A}_n\left(\{p_j\}\right) .$$

For different emitters :

$$\mathbf{T}_i\Big|_{\text{q, out}} \to T^a_{cd} , \qquad \mathbf{T}_i\Big|_{\bar{\text{q}}, \text{out}} \to -T^a_{dc} , \qquad \mathbf{T}_i\Big|_{\text{g, out}} \to -\mathrm{i} f^a_{\ cd} ,$$

Colour operators obey identities inherited by the algebra and dictated by gauge invariance

$$\left[\mathbf{T}_i^a, \mathbf{T}_i^b\right] = \mathrm{i} f^{ab}_{\ \ c}\, \mathbf{T}_i^c , \qquad \mathbf{T}_i \cdot \mathbf{T}_i \equiv \mathbf{T}_i^a \mathbf{T}_i^b \, \delta_{ab} = C_i^{(2)} , \qquad \sum_{i=1}^{n} \mathbf{T}_i = 0 ,$$

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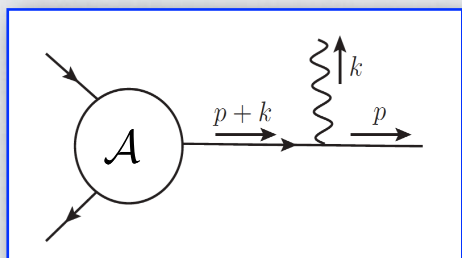

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when acting on the amplitude

E. Gardi, LM; T. Becher, M. Neubert.

The dipole formula

Let's take a closer look at the structure of the infrared anomalous dimension matrix.

The dipole term :

$$\Gamma_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2)\right) = \frac{1}{2}\,\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \log\left(\frac{s_{ij}\, {\rm e}^{{\rm i}\pi\lambda_{ij}}}{\mu^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j + \sum_{i=1}^{n} \gamma_i\left(\alpha_s(\mu^2)\right) ,$$

The cusp anomalous dimension in the `Casimir scaling' limit:

$$\gamma_{K,r}(\alpha_s) = C_r^{(2)}\, \widehat{\gamma}_K(\alpha_s) \, ,$$

Corrections start at three loops, with quadrupoles:

$$F_{ijkl}\left(\{\rho\}\right) f_{abe} f_{cd}{}^{e}\, \mathbf{T}_i^a \mathbf{T}_j^a \mathbf{T}_k^c \mathbf{T}_l^d \, ,$$

Ø. Almelid, C. Duhr, E. Gardi; J. Henn, B. Mistlberger.

- The colour dipole is the natural structure arising at one loop from gluon exchange.
- The fact that it survives at two loops is a non-trivial consequence of symmetries.
- Field anomalous dimensions in color-uncorrelated terms govern collinear singularities.
- Unitarity phases contain crucial analytic information. For final-state pairs: $\lambda_{ij} = 1$.
- The cusp anomalous dimension plays a very special role: a universal infrared coupling.
- The structure emerges from the constraints of scale invariance in the soft limit.

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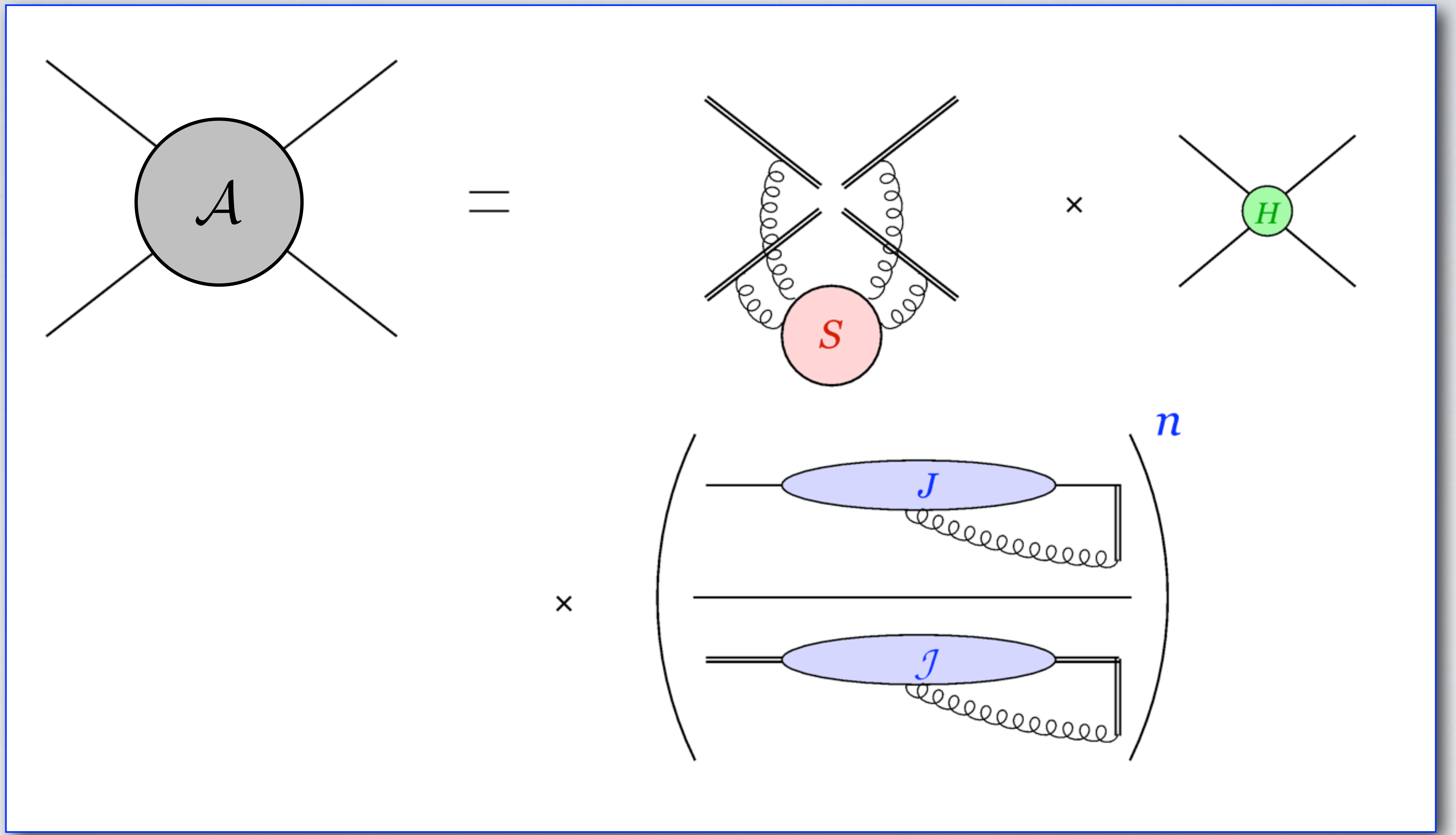
DEEP INFRARED VISIONS

Infrared factorisation: pictorial



A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes

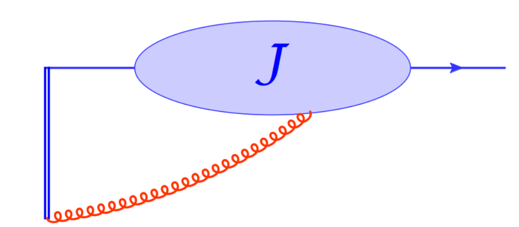
Operator Definitions

The precise functional form of this graphical factorisation is

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{E,\,i}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\mathcal{S}_n\left(\beta_i\cdot\beta_j\right)\;\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right)$$

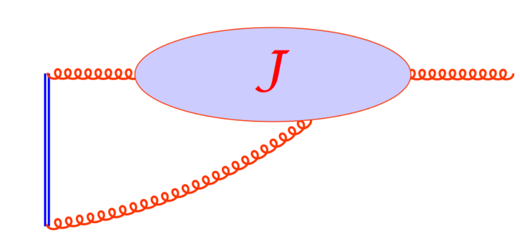
Here we introduced dimensionless four-velocities $\beta_i = p_i/Q$, and factorisation vectors n_i^μ, $n_i^2 \neq 0$ to define the jets in a gauge-invariant way. For outgoing quarks

$$\overline{u}_s(p)\,\mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p, s\,|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$



where Φ_n is the Wilson line operator along the direction n. For outgoing gluons

$$g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \equiv \langle k,\lambda|\,\Big[\Phi_n(\infty,0)\,\mathrm{i}D^\nu\,\Phi_n(0,\infty)\Big]\,|0\rangle\;,$$

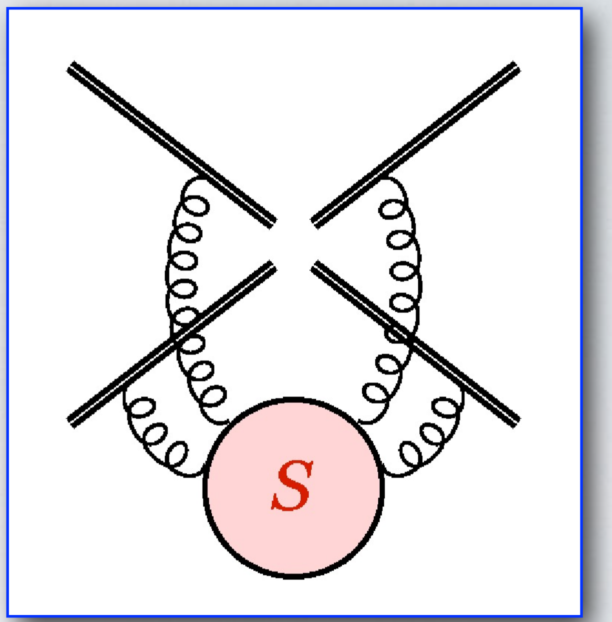


Becher, Bell 2010

Wilson line correlators

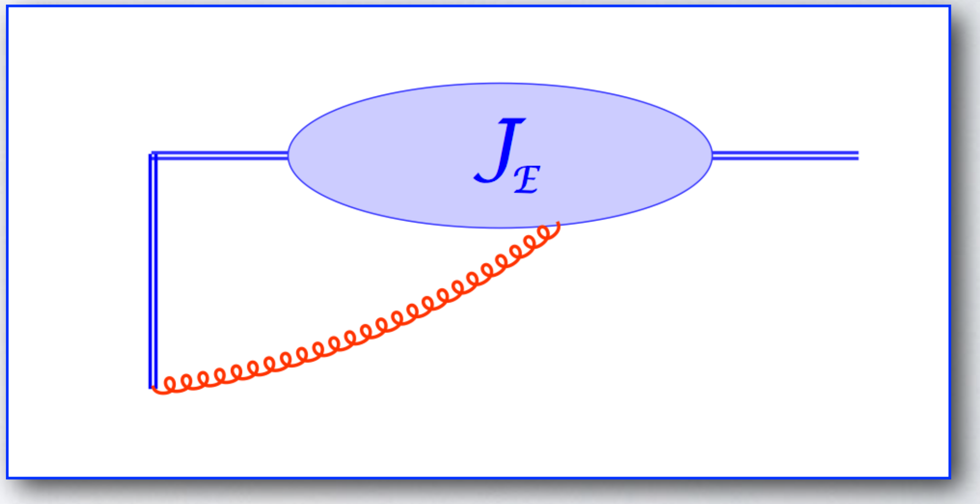
The soft function S is a color operator, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$\mathcal{S}_n\left(\beta_i \cdot \beta_j\right) = \langle 0 | \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0) \, | 0 \rangle$$



The soft jet function $J_{\mathcal{E}}$ contains soft-collinear poles: it is defined by replacing the field in the ordinary jet J with a Wilson line in the appropriate color representation.

$$\mathcal{J}_{\rm E}\left(\frac{(\beta \cdot n)^2}{n^2}\right) = \langle 0 | \, \Phi_\beta(\infty, 0) \, \Phi_n(0, \infty) \, | 0 \rangle$$



Wilson-line matrix elements exponentiate non-trivially and have tightly constrained functional dependence on their arguments. They are known to three loops.

On functional dependences

Straight semi-infinite Wilson lines are scale-invariant

$$\Phi_\beta(\infty,0) \equiv P\exp\left[ig\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right].$$

Correlators involving light-like Wilson lines break scale invariance due to collinear poles: a quantum `anomaly' proportional to the cusp anomalous dimension.

The anomaly must cancel in combination that are free from collinear poles

$$\widehat{\mathcal{S}}_{LK}(\rho_{ij},\alpha_s(\mu^2),\epsilon) \equiv \frac{\mathcal{S}_{LK}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right)}{\prod_{i=1}^n \mathcal{J}_{E,i}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu^2),\epsilon\right)}.$$

The reduced function depends only on scale-invariant combinations

$$\rho_{ij} \equiv \frac{\left(\beta_i\cdot\beta_j\right)^2\, n_i^2\, n_j^2}{\left(\beta_i\cdot n_i\right)^2\left(\beta_j\cdot n_j\right)^2}.$$

At the level of anomalous dimensions the cancellation is particularly striking

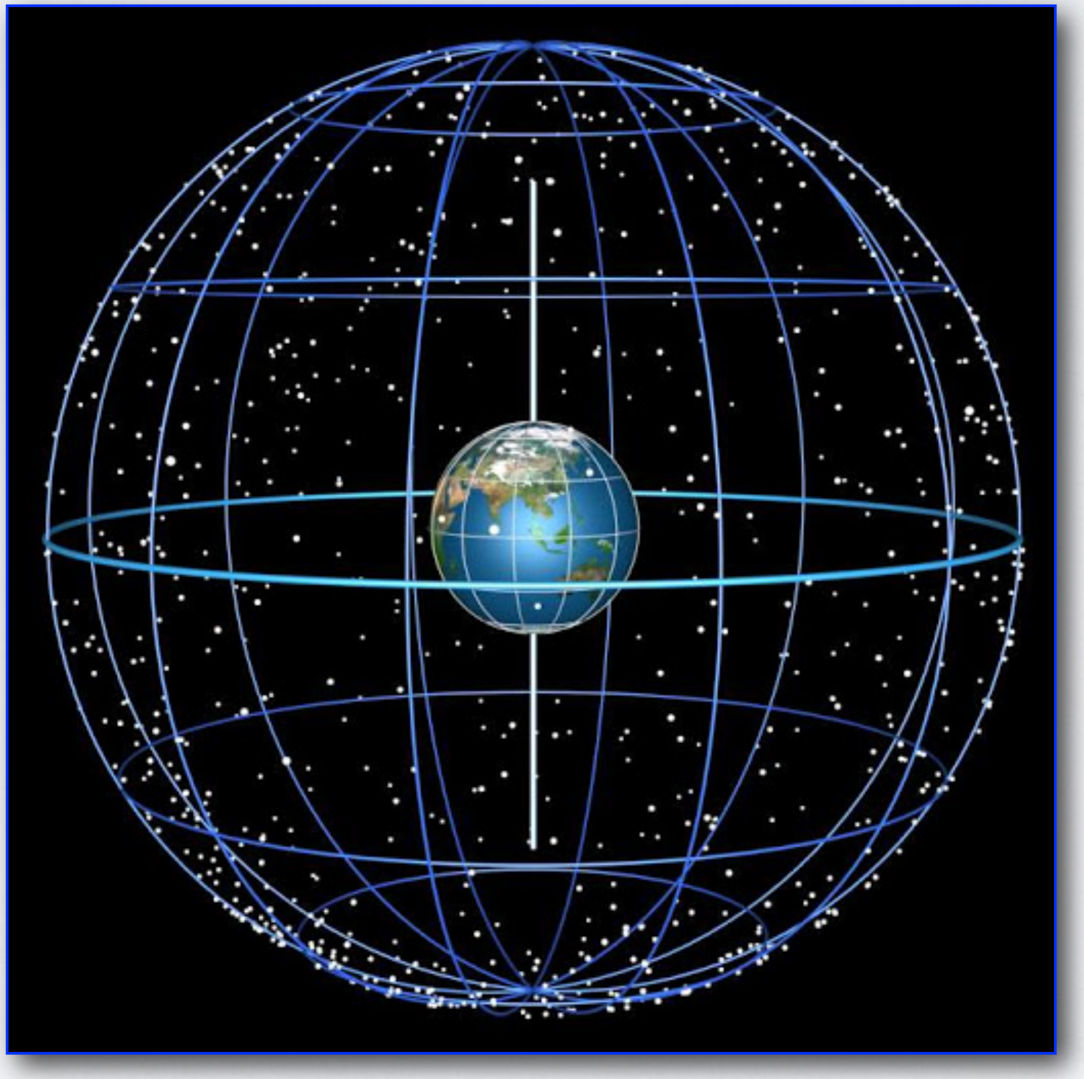
$$\Gamma_{KL}^{(\widehat{\mathcal{S}})}\left(\rho_{ij},\alpha_s(\mu^2)\right) = \Gamma_{KL}^{(\mathcal{S})}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right) - \delta_{KL}\sum_{i=1}^n \gamma_{\mathcal{J}_E}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2},\alpha_s(\mu^2),\epsilon\right),$$

- Singular terms in Γ_S must be diagonal.
- Finite diagonal terms in Γ_S must form ρ_{ij}'s.
- Off-diagonal terms in Γ_S must be finite, and must depend only on cross-ratios ρ_{ijkl}

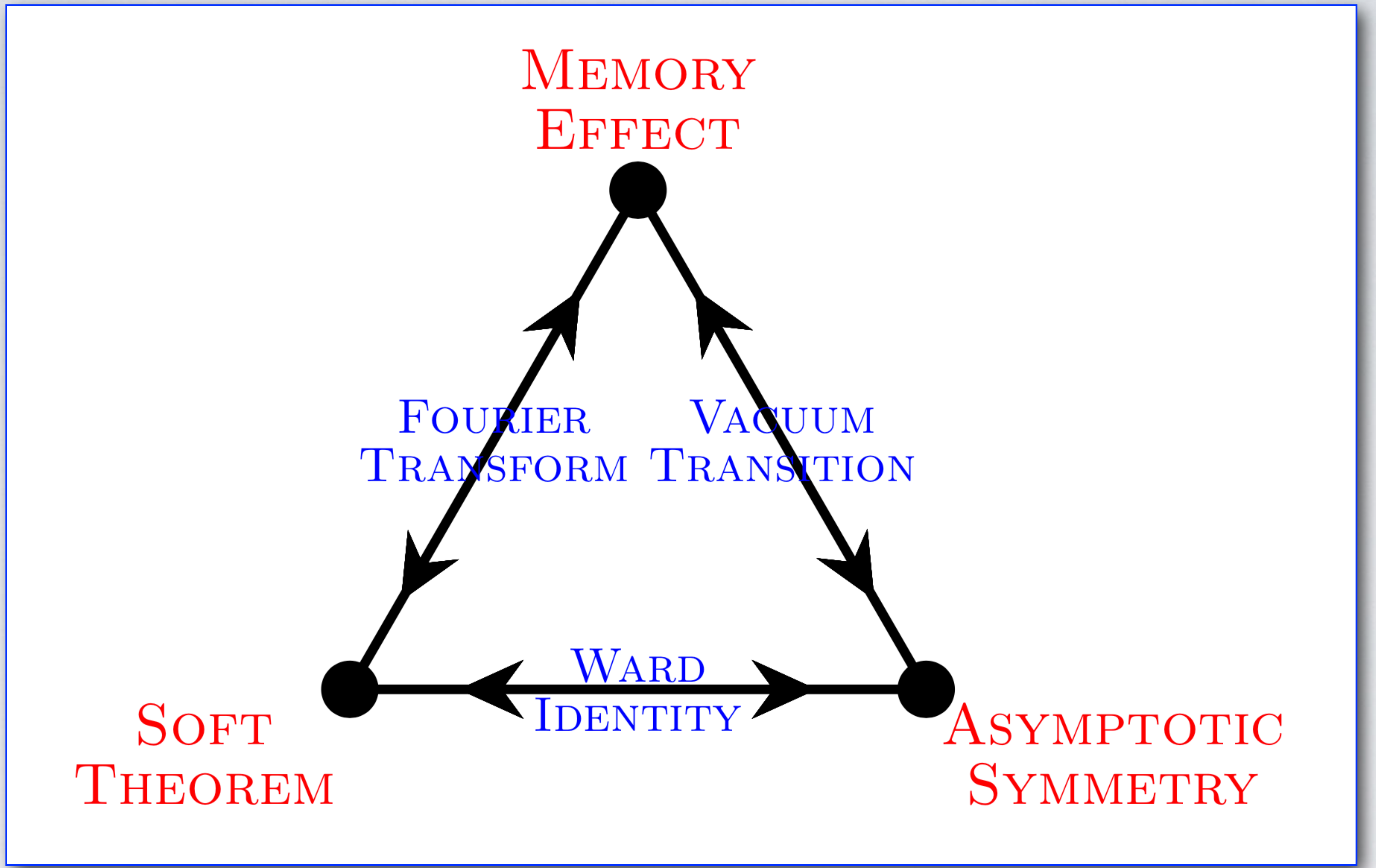
$$\sum_{i=1}^n\sum_{j\neq i}\frac{\partial}{\partial\ln\rho_{ij}}\Gamma_{KL}^{(\widehat{\mathcal{S}})}\left(\rho_{ij},\alpha_s\right) = \frac{1}{4}\gamma_K(\alpha_s)\,\delta_{KL}.$$

An exact equation for the soft anomalous dimension

THE CELESTIAL SPHERE



The Strominger Triangle



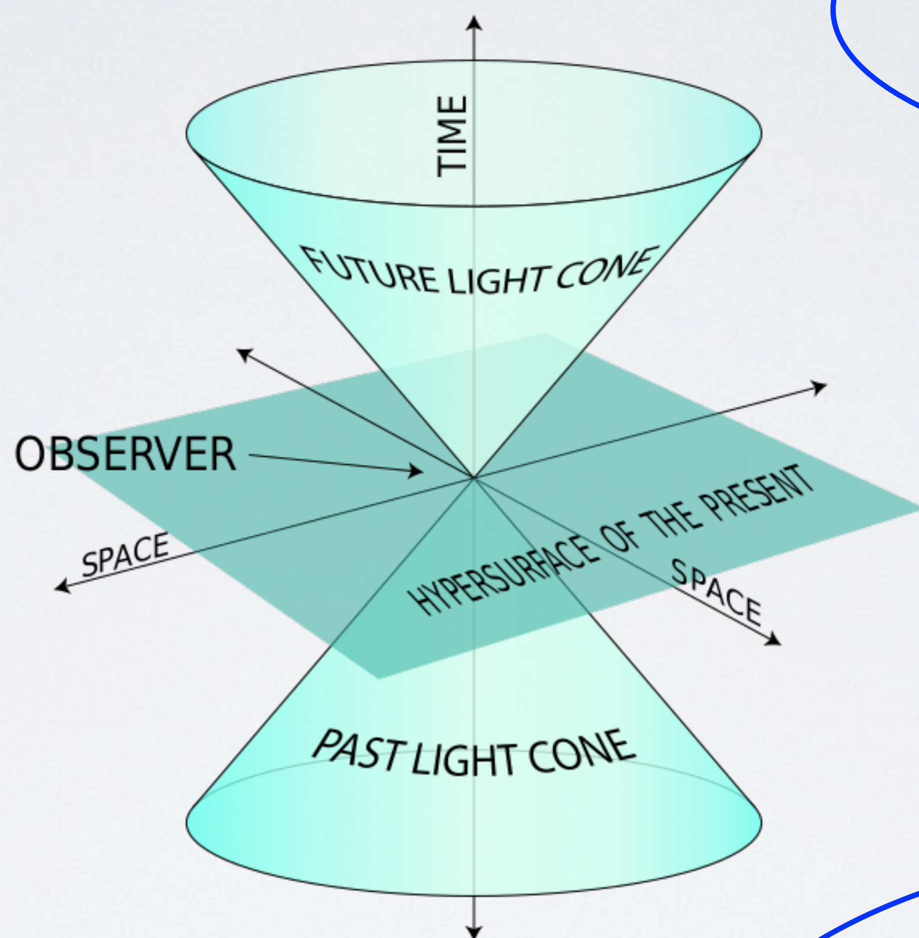
- A new viewpoint on infrared/long-distance phenomena in quantum field theory.
- A lesson from gravity: do not trivialise the behaviour and symmetries `at infinity'.
- Does this idea lead to new calculational techniques for non-abelian theories?

Many directions

Electromagnetic, colour and gravitational memory effects

Asymptotically flat spacetimes and holography

Full conformal symmetry on the celestial sphere



Black hole soft hair and the information paradox

Soft, next-to soft, next-to-next-to soft

$$\mathcal{A}(\Delta_j, z_j) = \left(\prod_{i=1}^{n} \int_0^\infty \frac{d\omega_i}{\omega_i} \omega_i^{\Delta_i} \right) \mathbf{A}(\omega_j, z_j).$$

Celestial amplitudes

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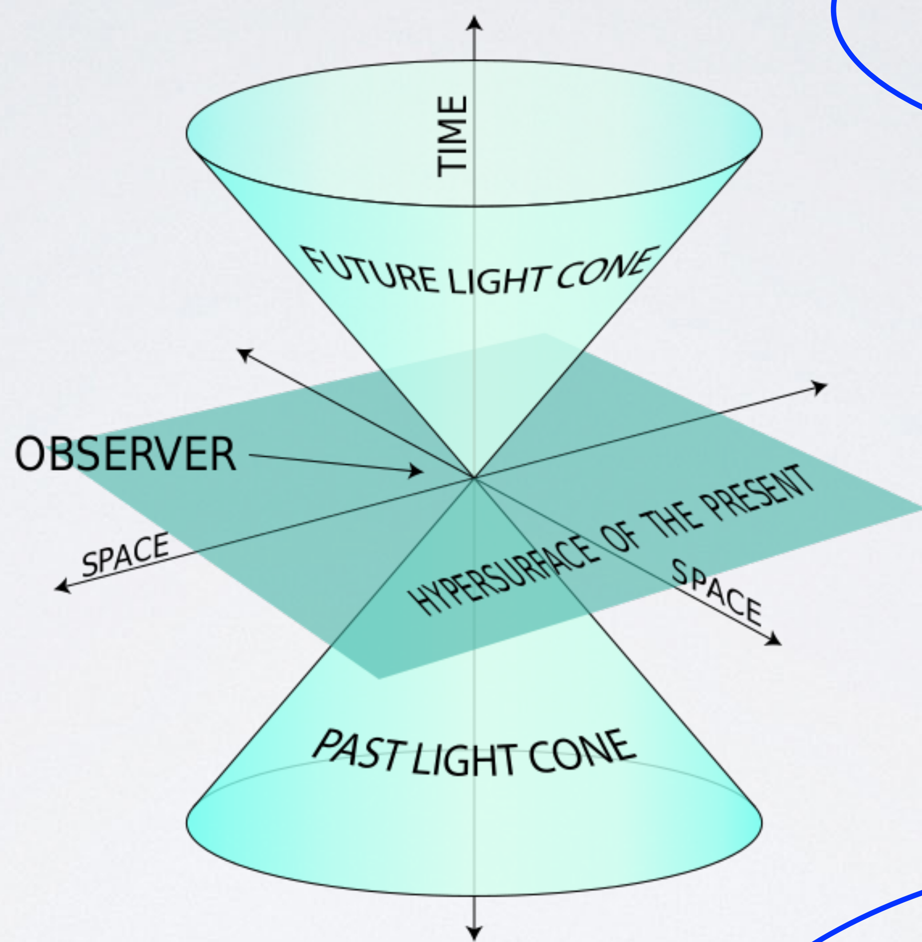

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Celestial amplitudes

See also: H.A. González, F. Rojas, 2104.12979

INFRARED VISIONS
ON THE CELESTIAL SPHERE

On dipole correlations

Let us begin by disentangling collinear poles (which are colour-singlets) from soft poles (which are colour-correlated). We replace the running scale λ with the fixed scale μ in the logarithmic term, and perform the colour sum using colour conservation.

$$\Gamma_n^{\text{dipole}}\left(\frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda,\epsilon)\right) = \frac{1}{2}\widehat{\gamma}_K\left(\alpha_s(\lambda,\epsilon)\right)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\left(\frac{-s_{ij}+i\eta}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j$$
$$-\sum_{i=1}^{n}\gamma_i\left(\alpha_s(\lambda,\epsilon)\right) - \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s(\lambda,\epsilon)\right)\ln\left(\frac{\mu^2}{\lambda^2}\right)\sum_{i=1}^{n}C_i^{(2)}$$
$$\equiv \Gamma_n^{\text{corr.}}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\lambda,\epsilon)\right) + \Gamma_n^{\text{singl.}}\left(\frac{\mu^2}{\lambda^2}, \alpha_s(\lambda,\epsilon)\right),$$

At one loop, integrating the colour-correlated term yields single soft poles, while the singlet term yields single collinear and double soft-collinear poles

$$\alpha_s\left(\lambda,\epsilon\right) = \alpha_s\left(\mu\right)\left(\frac{\lambda^2}{\mu^2}\right)^{-\epsilon},$$

$$\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\alpha_s(\lambda,\epsilon) = -\frac{1}{\epsilon}\,\alpha_s(\mu)\,, \quad \int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\ln\left(\frac{\lambda^2}{\mu^2}\right)\alpha_s(\lambda,\epsilon) = -\frac{1}{\epsilon^2}\,\alpha_s(\mu)\,, \quad (\epsilon<0)\,.$$

At h loops, multiple poles (up to order h+1) are generated by the β function. For conformal gauge theories the logarithm of the infrared factor has only single and double poles.

Celestial dipoles

Crucially, we now parametrise the light-cone momenta in celestial coordinates

$$p_i^\mu = \omega_i \left\{ 1 + z_i \bar{z}_i ,\, z_i + \bar{z}_i ,\, -\mathrm{i}\big(z_i - \bar{z}_i\big) ,\, 1 - z_i \bar{z}_i \right\} ,$$

where the energy ω_i and the sphere coordinates z_i have simple transformation properties under the Lorentz group acting as SL(2, C):

$$\omega' = |cz + d|^2 \omega , \qquad z' = \frac{az + b}{cz + d} ,$$

Mandelstam invariants are distances on the sphere

$$s_{ij} = 2 p_i \cdot p_j = 4 \omega_i \omega_j \big| z_i - z_j \big|^2 ,$$

which unpacks the logarithms

$$\log\left(-s_{ij} + \mathrm{i}\eta\right) = \log\left(\big|z_i - z_j\big|^2\right) + \log \omega_i + \log \omega_j + 2 \log 2 + \mathrm{i}\pi ,$$

Energies give new singlet terms

$$\Gamma_n^{\mathrm{dipole}}\left(\frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda, \epsilon)\right) \equiv \widehat{\Gamma}_n^{\mathrm{corr.}}\left(z_{ij}, \alpha_s(\lambda, \epsilon)\right) + \widehat{\Gamma}_n^{\mathrm{singl.}}\left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon)\right) ,$$

which take the form

$$\widehat{\Gamma}_n^{\mathrm{singl.}}\left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon)\right) = - \sum_{i=1}^{n} \gamma_i\big(\alpha_s(\lambda, \epsilon)\big) - \frac{1}{4} \widehat{\gamma}_K\big(\alpha_s(\lambda, \epsilon)\big) \sum_{i=1}^{n} \ln\left(\frac{-4\omega_i^2 + \mathrm{i}\eta}{\lambda^2}\right) C_i^{(2)} ,$$

Celestial dipoles

The colour-correlated term, responsible for all soft poles, is remarkably simple

$$\widehat{\Gamma}_n^{\rm corr.}\Big(z_{ij},\alpha_s(\lambda,\epsilon)\Big) \, = \, \frac{1}{2}\,\widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Big(|z_{ij}|^2\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\,.$$

Scale and coupling dependence are completely factored from colour and kinematics, and equal for all dipoles. The scale integral can this be performed in full generality, yielding

$$\begin{aligned}\mathcal{Z}_n^{\rm corr.}\Big(z_{ij},\alpha_s(\mu),\epsilon\Big) \, &\equiv \, \exp\left[\int_0^\mu\frac{d\lambda}{\lambda}\,\widehat{\Gamma}_n^{\rm corr.}\Big(z_{ij},\alpha_s(\lambda,\epsilon)\Big)\right] \\ &= \, \exp\left[-\,K\big(\alpha_s(\mu),\epsilon\big)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Big(|z_{ij}|^2\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\right],\end{aligned}$$

The scale factor K is well-known in QCD from form-factor calculations, and gives the perturbative Regge trajectory in the high-energy limit of four-point amplitudes. It is

G. Korchemsky, I.A. Korchemskaya; V. Del Duca, C. Duhr, E. Gardi, LM, C. White; G. Falcioni, L. Vernazza, …

$$K\big(\alpha_s(\mu),\epsilon\big) \, = \, -\frac{1}{2}\int_0^\mu\frac{d\lambda}{\lambda}\,\widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big)\,.$$

The function K can be computed order by order in terms of the cusp and the β function

$$\begin{aligned}K(\alpha_s,\epsilon) \, = \, &\frac{\alpha_s}{\pi}\frac{\widehat{\gamma}_K^{(1)}}{4\epsilon} + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\widehat{\gamma}_K^{(2)}}{8\epsilon} + \frac{b_0\widehat{\gamma}_K^{(1)}}{32\epsilon^2}\right) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\widehat{\gamma}_K^{(3)}}{12\epsilon} + \frac{b_0\widehat{\gamma}_K^{(2)} + b_1\widehat{\gamma}_K^{(1)}}{48\epsilon^2} + \frac{b_0^2\,\widehat{\gamma}_K^{(1)}}{192\epsilon^3}\right) + \mathcal{O}\big(\alpha_s^4\big)\,,\end{aligned}$$

$\beta \to 0$

$$K(\alpha_s,\epsilon) \, = \, \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n\frac{\widehat{\gamma}_K^{(n)}}{4n\epsilon}\,,$$

COLOUR
ON THE CELESTIAL SPHERE

Hints of a celestial theory

The colour-correlated term in the anomalous dimension matrix is strongly reminiscent of conformal field theory results. One needs only go so far as Joe Polchinski's book to find

2.3 The expectation value of a product of exponential operators on the plane is

$$\left\langle \prod_{i=1}^{n} :e^{ik_i \cdot X(z_i,\bar{z}_i)}: \right\rangle = iC^X (2\pi)^D \delta^D(\textstyle\sum_{i=1}^{n} k_i) \prod_{\substack{i,j=1 \\ i<j}}^{n} |z_{ij}|^{\alpha' k_i \cdot k_j} ,$$

with C^X a constant. This can be obtained as a limit of the expectation value (6.2.17) on the sphere, which we will obtain by several methods in chapter 6.

A correlator of vertex operators in a free-boson theory (such as the bosonic string) has the correct form, up to the substitution of momenta with colour matrices.

This was noticed by N. Kalyanapuram in 2011.11412, for the simple case of QED. He writes

Nande, Pate and Strominger, 1705.00608

$$\ln\left(\mathcal{A}^{soft}_{n,s=1}|_{vir}\right) = -\frac{1}{8\pi^2\epsilon}\sum_{i\neq j} e_i e_j \ln|z_i - z_j|^2 .$$

The result is formally reproduced by introducing vertex operators with electric charges

$$V_j(z_j,\bar{z}_j) =: e^{ie_j\varphi(z_i,\bar{z}_j)} : \quad \longrightarrow \quad \langle V_1(z_1,\bar{z}_1)\cdots V_n(z_n,\bar{z}_n)\rangle = A^{soft}_n|_{vir,s=1} .$$

Lie-algebra-valued free bosons

It is natural to mimic the bosonic string, considering free bosons spanning the gauge algebra.

$$S(\phi) \,=\, \frac{1}{2\pi}\int d^2z\,\partial_z\phi^a(z,\bar{z})\,\partial_{\bar{z}}\phi_a(z,\bar{z})\,,$$

The free bosons could be organised in a matrix field :
gauge generators at different points must then be taken to commute

$$\Phi_r(z,\bar{z}) \,\equiv\, \phi_a(z,\bar{z})\,T^a_{r,\,z}\,,$$

The well-known results for free bosons in d=2 can be directly transcribed.

The equations of motions are:

$$\partial_z\,\partial_{\bar{z}}\,\phi^a(z,\bar{z}) \,=\, 0\,,$$

implying that the derivatives of the fields are (anti)holomorphic

A normal-ordered product can be defined, obeying the classical equation of motion

$$:\phi^a(z,\bar{z})\,\phi^b(w,\overline{w}):\,=\,\phi^a(z,\bar{z})\,\phi^b(w,\overline{w}) + \frac{1}{2}\,\delta^{ab}\log|z-w|^2\,,$$

There is a traceless conserved energy-momentum tensor, and conserved Noether currents

$$T(z) \,=\, -\,:\partial_z\phi^a(z,\bar{z})\,\partial_z\phi_a(z,\bar{z}):\,,$$

$$j^a(z) \,=\, \partial_z\phi^a(z,\bar{z})\,,$$

Matrix vertex operators

Guided by the QED example, we can tentatively define a matrix-valued vertex operator

$$V(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_z \cdot \phi(z,\bar{z})} \,: \, = \, :\, \mathrm{e}^{\mathrm{i}\kappa \Phi(z,\bar{z})} \,:,$$

A `single-copy' of the string vertex operator!

In colour space, this is a matrix in the representation of $\mathbf{T}_z$, defined on the boundary sphere and acting on the bulk colour degrees of freedom. But is it a conformal primary field?

For conventional vertex operators (as for example for bosonic strings)

$$V_{\text{c.s.}}(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}k^\mu X_\mu(z,\bar{z})} \,: \quad \longrightarrow \quad h = \frac{1}{4}\, k^\mu k^\nu \eta_{\mu\nu} = \frac{k^2}{4},$$

The same calculation yields

$$V(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_z \cdot \phi(z,\bar{z})} \,: \quad \longrightarrow \quad h = \frac{\kappa^2}{4}\, \mathbf{T}_z \cdot \mathbf{T}_z = \frac{\kappa^2}{4}\, C_r^{(2)},$$

Crucially, this is a positive real number and not a matrix. For consistency, two-point functions must evaluate to a power of the distance given by the conformal weight $\Delta = h + \bar{h}$. Indeed

$$\big\langle V(z_1,\bar{z}_1) V(z_2,\bar{z}_2) \big\rangle \sim |z_{12}|^{-2\Delta},$$

by colour conservation $\mathbf{T}_1 + \mathbf{T}_2 = 0$

Note analogies with other constructions.

Vertex operator construction of Kac-Moody algebras: → $U^\alpha(z) = z^{\alpha^2/2} :e^{i\alpha\cdot Q(z)}:.$ not the same

Reggeon fields for high-energy scattering: → $U(z) = e^{ig_s T^a W^a(z)}.$ closely related

(Caron-Huot 2013)

A conformal correlator

Our construction from the beginning targeted the n-point correlator

$$\mathcal{C}_n\Big(\{z_i\},\kappa\Big) \equiv \left\langle \prod_{i=1}^{n} V(z_i,\bar{z}_i) \right\rangle .$$

The calculation is a textbook exercise: it can be done with oscillators, after expanding the free fields in modes on the sphere, or computing the path integral (Polchinski). The result is

$$\mathcal{C}_n\Big(\{z_i\},\kappa\Big) = C(N_c)\,\exp\left[\frac{\kappa^2}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Big(|z_{ij}|^2\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\right],$$

$$\sum_{i=1}^{n}\mathbf{T}_i = 0\,,$$

reproducing the structure of the gauge theory infrared operator. Note that

- The correlator has support only on colour conserving configurations
- The field normalisation κ maps to the integral K, carrying scale and regulator dependence.
- In a path integral evaluation on a curved surface (say, a finite sphere with radius R) the correlator acquires a scale-dependent `Weyl' factor, which in this setting maps to an undetermined colour-singlet collinear contribution.

$$\mathcal{W}_n\Big(\{z_i\},\kappa\Big) = \exp\left[-\frac{1}{2}\sum_{i=1}^{n} C_i^{(2)}\,g(z_i,\bar{z}_i)\right],$$

A tree-level soft theorem

Real emission of a soft massless gauge boson from a fixed angle hard amplitude factorises in any non-abelian theory in the form

$$\langle c| \otimes \langle \lambda| \mathcal{A}_{g, f_1 \ldots f_n}(k, p_1, \ldots, p_n) \rangle_{\text{soft}} = \epsilon_\lambda(k) \cdot J^c(k) \, |\mathcal{A}_{f_1 \ldots f_n}(p_1, \ldots, p_n) \rangle \,,$$

The tree-level soft-gluon current has the classic eikonal form and is gauge-invariant

$$\mathbf{J}^\mu(k) = g \sum_{i=1}^n \mathbf{T}_i \, \frac{\beta_i^\mu}{\beta_i \cdot k} \,,$$

$$k \cdot \mathbf{J}^\mu(k) = g \sum_{i=1}^n \mathbf{T}_i = 0 \,,$$

The tree-level soft theorem is reproduced by the Ward identity for the Noether current associated with invariance under field translations in the Lie algebra. Using the conformal operator product expansion one finds

A. Strominger, T. He, P. Mitra, A. Nande, M. Pate, W. Fan, A. Fotopoulos, T.R. Taylor, …

$$\left\langle \partial_z \phi^a(z, \bar{z}) \prod_{i=1}^n V(z_i, \bar{z}_i) \right\rangle \simeq -\frac{\mathrm{i}}{2} \sum_{i=1}^n \frac{\mathbf{T}_i^a}{z - z_i} \, \mathcal{C}_n\Big(\{z_i\}, \kappa\Big) \,.$$

where the poles as $z \to z_i$ are collinear poles, since the celestial theory is energy-independent.

Celestial Sudakov parametrisation

In order to study collinear limits, it is useful to build a Sudakov parametrisation on the sphere

$$p_1^\mu = x p^\mu + p_\perp^\mu - \frac{p_\perp^2}{2xp\cdot n} n^\mu \,,$$
$$p_2^\mu = (1-x) p^\mu - p_\perp^\mu - \frac{p_\perp^2}{2(1-x)p\cdot n} n^\mu \,,$$

One can fix the light-like Sudakov vector n^μ, and then compute the collinear momentum p^μ

$$n^\mu = \frac{1}{2}\Big\{1,0,0,-1\Big\} \quad \to \quad n\cdot p_i = \omega_i \,, \quad n\cdot p = \omega_1 + \omega_2 \equiv \omega \,,$$

$$p^\mu = \omega \Big\{1 + z\bar{z},\, z+\bar{z},\, -\mathrm{i}(z-\bar{z}),\, 1 - z\bar{z}\Big\} \,,$$

The point z is on the line joining z_1 and z_2 :

$$z = x z_1 + (1-x) z_2, \qquad \omega_1 = x\omega, \qquad \omega_2 = (1-x)\omega.$$

Once the collinear direction is fixed, the transverse momentum vector can be computed

$$p_\perp^\mu = \omega x(1-x) \Big\{(1-2x)\big(z_1\bar{z}_2 + \bar{z}_1 z_2\big) + 2x z_1\bar{z}_1 - 2(1-x) z_2\bar{z}_2,$$
$$z_{12} + \bar{z}_{12},\, -\mathrm{i}\big(z_{12} - \bar{z}_{12}\big),\, -(1-2x)\big(z_1\bar{z}_2 + \bar{z}_1 z_2\big) - 2x z_1\bar{z}_1 + 2(1-x) z_2\bar{z}_2\Big\} \,.$$

It is antisymmetric, and satisfies

$$p_\perp^2 = -x(1-x) s_{12} = -4x(1-x)\omega_1\omega_2 \big|z_{12}\big|^2 \,,$$

Collinear limits

The operator product expansion governs the collinear limit on the sphere. One can transcribe the textbook result substituting colour operators for momenta.

$$: \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_1 \cdot \phi(z_1, \bar{z}_1)} : : \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_2 \cdot \phi(z_2, \bar{z}_2)} : \sim \left| z_{12} \right|^{\kappa^2 \mathbf{T}_1 \cdot \mathbf{T}_2} : \mathrm{e}^{\mathrm{i}\kappa \left(\mathbf{T}_1 + \mathbf{T}_2 \right) \cdot \phi(z, \bar{z})} : ,$$

Note that the exact collinear limit is outside the validity of the original factorisation. But it can be approached: on the gauge theory side, one defines a splitting anomalous dimension

$$\Gamma_{\mathrm{Sp.}}(p_1, p_2) \equiv \Gamma_n(p_1, p_2, \ldots, p_n) - \Gamma_{n-1}(p, p_3, \ldots, p_n) \Big|_{\mathbf{T}_p \to \mathbf{T}_1 + \mathbf{T}_2} .$$

(Becher-Neubert 2009)

The OPE encodes collinear factorisation: the n-point correlator reduces to (n-1)-points, with the \`merged' point carrying the sum of the colours of (only!) the two collinear particles.

The calculation of the splitting function is then the same as in the gauge theory, but requires reinstating the energy dependence, which is not encoded by the conformal correlator.

$$\Gamma_{\mathrm{Sp.}}(p_1, p_2) = \frac{1}{2} \widehat{\gamma}_K(\alpha_s) \Big[\ln \left(\frac{-s_{12} + \mathrm{i}\eta}{\mu^2} \right) \mathbf{T}_1 \cdot \mathbf{T}_2 - \ln x \; \mathbf{T}_1 \cdot (\mathbf{T}_1 + \mathbf{T}_2) - \ln(1 - x) \; \mathbf{T}_2 \cdot (\mathbf{T}_1 + \mathbf{T}_2) \Big] ,$$

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MANY QUESTIONS

Many Questions

- The choice of the gauge coupling.

  Our construction lends support to the idea the the cusp anomalous dimension should be taken as the definition of the strong coupling in the infrared.

  How far can one take this definition?

  S. Catani, B. Webber, G. Marchesini; A. Grozin et al.; A. Banfi et al.; O. Erdogan, G. Sterman; S. Catani, D. DeFlorian, M. Grazzini.

- Scale and regulator dependence.

  It is remarkable, and necessary, that infrared singularities be hidden in the matching condition between the gauge theory and the conformal theory.

  How can one make this correspondence more precise?

- Beyond the free theory.

  The celestial conformal theory certainly has corrections involving structure constants (as confirmed by the structure of Δ). The deformed theory is still conformal.

  What drives the deformation?

- Constraints from vast field theory data.

  Soft and collinear factorisation kernels are known to three loops, and in the massive case to two loops. In most cases their remarkable simplicity is only partly explained.

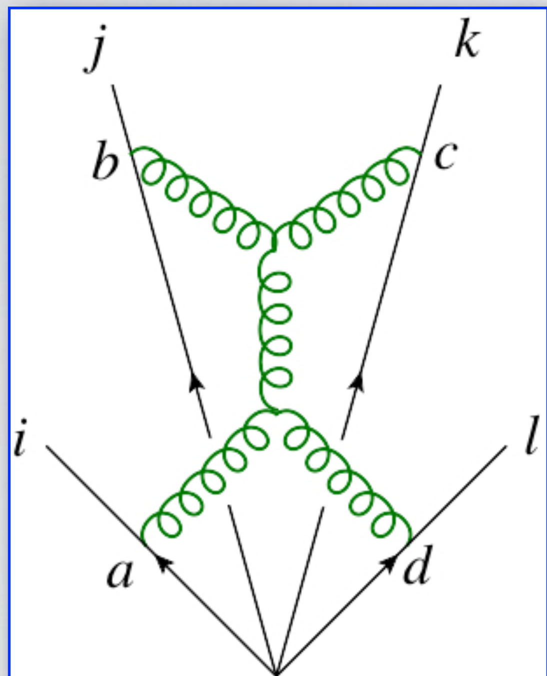
  How can we harness these data to constrain the celestial theory?

The exploration has just begun!

Non-trivial gauge data

- Quadrupole corrections two the correlator at three loops and beyond

Ø. Almelid, C. Duhr, E. Gardi (2015); with A. McLeod and C. White (2017)



$$\Delta_n^{(3)}(\rho_{ijkl}) = 16\, f_{abe} f_{cde} \Big\{ -C \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ j,k\neq i}} \big\{ \mathbf{T}_i^a, \mathbf{T}_i^d \big\}\, \mathbf{T}_j^b \mathbf{T}_k^c + \sum_{1\le i<j<k<l\le n} \Big[\mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d\, \mathcal{F}(\rho_{ikjl}, \rho_{iljk}) + \mathbf{T}_i^a \mathbf{T}_k^b \mathbf{T}_j^c \mathbf{T}_l^d\, \mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) + \mathbf{T}_i^a \mathbf{T}_l^b \mathbf{T}_j^c \mathbf{T}_k^d\, \mathcal{F}(\rho_{ijlk}, \rho_{iklj}) \Big] \Big\},$$

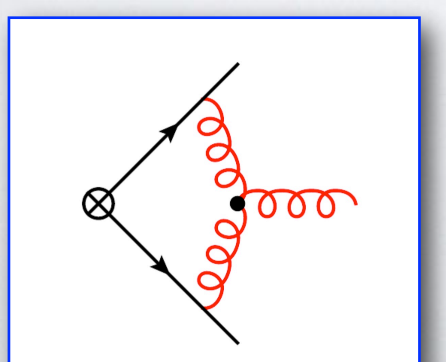
$$\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) = F(1 - z_{ijkl}) - F(z_{ijkl}),$$

$$z_{ijkl}\bar{z}_{ijkl} = \rho_{ijkl}, \qquad (1 - z_{ijkl})(1 - \bar{z}_{ijkl}) = \rho_{ilkj}$$

$$F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2 [\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)].$$

- Quantum corrections to the tree-level soft-gluon current

S. Catani, M. Grazzini (2000); L. Dixon, E. Herrmann, K. Yan, H-X Zhu (2020)



$$\mathbf{J}^{\mu}_{(1)}(k) = -\frac{\mathrm{i}}{16\pi^2} \left[\frac{C_A}{\epsilon^2} \mathbf{J}^{\mu}_{(0)}(k) + \frac{1}{\epsilon} \mathrm{i} f_{abc} \sum_{i\neq j} T_i^b T_j^c \left(\frac{\beta_i^\mu}{\beta_i \cdot k} - \frac{\beta_j^\mu}{\beta_j \cdot k} \right) \log \left(\frac{\mu^2 (-2\beta_i \cdot \beta_j)}{(-2\beta_i \cdot k)(-2\beta_j \cdot k)} \right) \right]$$

with L. Bianchi and M. Meineri

Towards an interacting theory?

- Interactions in the d=2 theory are constrained by gauge and euclidean invariance, whether the theory is conformal or not. With up to four fields one finds

$$\mathcal{L}\left(\phi^a\right) = \frac{1}{2}\,\partial_\mu\phi^a\partial^\mu\phi_a + \mathrm{i}\,\frac{\lambda_1}{6}\,\varepsilon^{\mu\nu}\,f^{abc}\,\phi_a\,\partial_\mu\phi_b\,\partial_\nu\phi_c - \frac{\lambda_2}{24}\,f^{abe}f_e{}^{cd}\,\phi_b\,\phi_d\,\partial_\mu\phi_a\,\partial^\mu\phi_c + \ldots$$

In fact, $\lambda_1 = \lambda_2 = 1$ yields the leading terms of the WZNW action, while $\lambda_1 = 0$, $\lambda_2 = 1$ yields the principal chiral model, which is not conformally invariant.

- Correlators in the WZNW model must obey the Knizhnik-Zamolodchikov equation, but this fails for the gauge-theory correlator, and cannot be compensated by quadrupoles.

$$\mathcal{Z}_n\left(z_i\right) \equiv \exp\left[\mathcal{E}_n\left(z_i\right)\right] \quad \to \quad \frac{\partial\mathcal{Z}_n}{\partial z_i} = K\,\mathbf{T}_i\cdot\sum_{k\neq i}\frac{\mathbf{T_k}}{z_k - z_i} + \frac{1}{2}\left[\mathcal{E}_n\left(z_i\right), \frac{\partial\mathcal{E}_n}{\partial z_i}\right] + \ldots$$

H. Nastase, F. Rojas, C. Rubio (2021)

Giving up full d=2 conformal invariance? A new colourful CFT? A deformed CFT? ….


merci
thank you
gracias
obrigado
danke
go raibh maith agat
спасибо
grazie
mochchakkeram
dakujem
bedankt
arigatô
ngiyabonga
terima kasih
sukriya
teşekkür ederim
dziękuję
chokrane
sagolun
tapadh
takk
謝謝
mauruuru
barka
tau
kia ora
ありがとう
grazzi
мерси
diolch
tanemirt
mersi
dhanyavad
shukriya
spas
dank je
welalin
gratias ago
misaotra
matondo
dhanyavadagalu
hvala
rahmet
kop khun krap
mercé
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nandri
didi madloba
mahalo
taiku
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kam sah hamnida
faafetai lava
paldies
kop khun
najis tuke
mamnun
trugarez
asante
Баярлалаа
vinaka
ευχαριστώ
kiitos
dankie
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