One-Loop Matching with Functional Methods

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"10 years MITP"
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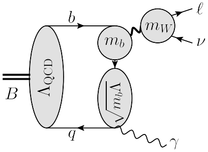
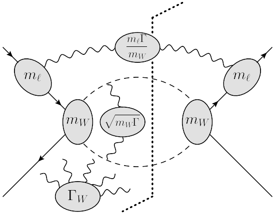
Effective Field Theories

Effective Field Theories (EFTs) are a **powerful tool** for processes with **multiple scales** present at the same time!

Upon closer inspection, any realistic process falls under this umbrella:



$B \to \ell\nu\gamma$



$e^+e^- \to W^+W^-$

A multitude of EFT constructions have been devised to deal with this:
SMEFT/LEFT, χPT, (b)HQET, (p)NRQFT, SCET...

Expand, factorize, renormalize, resum logs, ...

Match any **high-scale** New Physics to an agreed-upon parameterization*.

Low-energy phenomenology can be **extracted from reference results** derived using said parameterization!

Resummation of **logarithmic corrections** is an additional benefit.

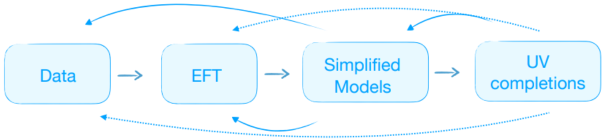
While technically **relatively simple**, complete one-loop matching to SMEFT(-like) EFTs does come with its own sources of nastiness!

Target basis needs to be known, can be tricky if **UV theory** has **multiple scales** itself or features **additional light degrees of freedom**.

Lots of book-keeping → **mistakes happen**.

EFTs in BSM Physics

On top of all that:



- Analyse data (and SM prediction)
- Find an EFT solution and investigate correlation with other observables.
- Implement the EFT in a simplified dynamical model.
- Find a reasonable UV completion.

These steps are complementary and not unidirectional...

Lots of work has been done in the last years!

[C. Cornella, MITP Workshop MODEL2019]

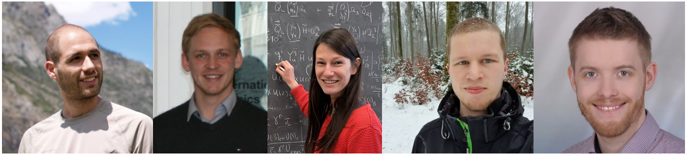
Once this needs to be done **multiple** times, work **adds up quickly**!

Functional One-Loop Matching - Why?

While giving up some **intuition from experience** in computing matching diagrammatically...

- Functional matching requires **no knowledge of a target basis**.
- Makes symmetry/multiplicity factors completely **trivial**.
- Leads to a very **algorithmic**, even brain-dead approach.
- Lends itself extremely well to **automation** *(→ see Javi's talk!)*

Brought to you by the MATCHETE collaboration:



Javier Fuentes-Martín, MK, Julie Pagès, Anders Eller Thomsen, Felix Wilsch

[Matchete (*2021*), JHEP 04 281], [Matchete (*2023*), JHEP 02 031], [Matchete (*2021*), arXiv:2212.04510]

Part I - Obtaining an Effective Lagrangian

Scenario: Two-scales, separated by a cutoff Λ. Want to compute scattering at energies $E < \Lambda$.

Split field content ϕ into **modes** $\omega > \Lambda$ ("hard") and $\omega < \Lambda$ ("soft"):

$$\phi \to \phi_H + \phi_S$$

Matrix elements **at low energies** derived from the generating functional:

$$Z[J_S] = \int \mathcal{D}\phi_S \mathcal{D}\phi_H \exp\left\{ iS(\phi_S, \phi_H) + i \int d^4x \, J_S(x)\phi_S(x) \right\}$$

From this, one defines the **Wilsonian effective action**:

$$\int \mathcal{D}\phi_H \exp\left\{ iS(\phi_S, \phi_H) \right\} \equiv \exp\left\{ iS_\Lambda(\phi_S) \right\}$$

$$\text{and also} \quad S_\Lambda = \int d^4x \, \mathcal{L}_{\text{eff}}(x)$$

Can compute S_Λ directly from this definition!

Want to compute the path integral perturbatively, split

$$\phi = \underbrace{\varphi}_{\text{classical field, satisfies eom}} + \underbrace{\eta}_{\text{quantum fluctuation}}$$

Expand in the quantum field η:

$$\mathcal{L} = \underbrace{\mathcal{L}(\varphi)}_{\text{tree-level}} + \underbrace{\eta_i \cdot \left[\frac{\delta \mathcal{L}}{\delta \phi_i}\right](\varphi)}_{\text{eom} \to 0} + \frac{1}{2}\bar{\eta}_i \eta_j \underbrace{\left[\frac{\delta^2 \mathcal{L}}{\delta \bar{\phi}_i \delta \phi_j}\right]}_{\text{fluctuation operator } \mathcal{Q}_{ij}(\varphi)}(\varphi) + \mathcal{O}(\eta^3)$$

(indices on fields here denote hard or soft)

The path-integral of the one-loop piece is **Gaussian**:

$$\exp\left\{iS^{(1)}_{\text{eff}}\right\} = \int \mathcal{D}\eta \exp\left\{\frac{1}{2}\bar{\eta}\mathcal{Q}\eta\right\}$$

From this, one finds the **effective action** given by

$$S_{\text{eff}}^{(1)} = \frac{i}{2}\text{STr}\big(\log \mathcal{Q}\big) = \pm\frac{i}{2}\int \frac{d^d k}{(2\pi)^d}\,\langle k|\text{tr}\log \mathcal{Q}|k\rangle$$

This **supertrace** generalizes the usual trace to operators with fermionic and bosonic fields with appropriate signs.

[Cohen et al (*2020*), 2011.02484]

One now **splits** the fluctuation operator into:

$$\mathcal{Q}_{ij} = \delta_{ij}\Delta_i^{-1} - X_{ij} = \Delta_i^{-1}\left(\delta_{ij} - \Delta_i X_{ij}\right)$$

$$\Delta_i^{-1} = \begin{cases} -(D^2 + M_i^2) & \text{scalar} \\ i\not{D} - M_i & \text{fermion} \\ g^{\mu\nu}(D^2 + M_i^2) & \text{vector} \end{cases} \qquad X_{ij} = \text{(interactions)}$$

and expands the log in the interactions:

$$S_{\text{eff}}^{(1)} = \frac{i}{2}\text{STr}(\log \Delta^{-1}) - \frac{i}{2}\sum_{n=1}^{\infty}\text{STr}[(\Delta X)^n]$$

This is just the usual **one-loop effective action** S_{eff}. How to get to the **Wilsonian** effective action S_Λ from here?

Remember, in S_Λ only the integral over **hard modes** was carried out!
$\rightarrow$ Need to "select" only those pieces of the integrals

$$\int \frac{d^d k}{(2\pi)^d} \langle k | \mathrm{tr} \log \mathcal{Q} | k \rangle$$

An **intuitive** way to realize this, would be to evaluate the integral with a lower momentum **cutoff** in place.

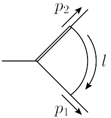
In **dimensional regularization** however, we can simply expand the **integrands** around this limit, by power-counting the loop momentum k like the heavy masses M.

[Beneke, Smirnov (*1997*), hep-ph/9711391]
[Jantzen (*2011*), 1111.2589]

Each propagator now has virtuality $k^2 \sim M^2$.

The Method of Regions - Example

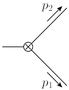
Take this simple scalar loop graph with $p_1^2 \sim p_2^2 \sim (p_1 \cdot p_2) \ll M^2$:



$$\sim \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \frac{1}{(l-p_1)^2} \frac{1}{(l+p_2)^2 - M^2}$$

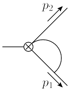
$l^2 \sim M^2$

$l^2 \sim (p_i \cdot p_j)$



$$\sim \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \frac{1}{l^2} \frac{1}{l^2 - M^2}$$

"hard region"



$$\sim \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \frac{1}{(l-p_1)^2} \frac{1}{(-M^2)}$$

"soft region"

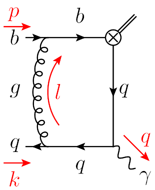
The hard region corresponds to the one-loop matching correction to an operator ϕ^3, the soft region to a matrix element of an operator ϕ^4

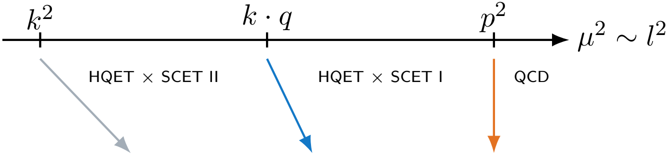
$$\left\langle T\{\phi^4, i\int d^4x \mathcal{L}(x)\}\right\rangle$$

The Method of Regions

This of course generalizes to any **more complicated** multiscale problem, too.

Each region comes with a **characteristic virtuality**, representing a **matching step**:





$$i\mathcal{A} = \mathcal{S}_i(\omega, \mu_\mathrm{s}) \otimes \mathcal{J}_{ij}(\omega, \nu, \mu_\mathrm{hc}) \otimes \mathcal{H}_j(\nu, \mu_\mathrm{h})$$

⇒ Region analysis an important tool, both for computing **ingredients** of **factorization formulas** as well as **checking** your EFT constructions!

Further features:

- **Power-corrections:** The method of regions allows to extract matching coefficients to **any order** in power-counting!
- **Overcounting?** Said that "hard region": $l^2 \sim M^2$, but we still integrate over **all** $d^d l$.
  $\rightarrow$ Not in **dimensional regularization** and as long as we identify $\epsilon_{\mathrm{IR}} = \epsilon_{\mathrm{UV}}$. Then the overlaps are all scaleless integrals.
- **Matching:** Along those lines, graphs with purely soft modes are automatically zero and thus the method automatically "selects" the correct amplitudes and directly yields matching coefficients.

From this it follows that evaluating the **effective action** in the **hard region**, directly yields the **Wilsonian effective action**!

[Fuentes et al (*2016*), 1607.02142]

$$S_{\mathrm{eff}}^{(1)}\Big|_{\mathrm{hard}} = \pm \frac{i}{2} \int_h \frac{d^d k}{(2\pi)^d} \langle k | \mathrm{tr} \log \mathcal{Q} | k \rangle = S_{\Lambda}^{(1)} \, .$$

Hard-Region Supertraces

We can now evaluate the supetraces in the hard region:

$$S_\Lambda^{(1)} = \frac{i}{2}\mathrm{STr}_h(\log\Delta^{-1}) - \frac{i}{2}\sum_{n=1}^{\infty}\mathrm{STr}_h[(\Delta X)^n]$$

The fluctuation operator and thus Δ and X depend on the classical fields φ and derivatives.

$\Rightarrow S_\Lambda$ is of the form $i\int d^4x \sum_i C_i(\mu)\cdot\mathcal{O}_i(\varphi, iD_\mu)$

with loop functions $C_i(\mu)$ and composite operators of **classical** fields and covariant derivatives.

This is **almost** an effective Lagrangian!

The expression is not manifestly **gauge-invariant** and contains **redundant operators**.

The supertraces contain **open covariant derivatives**:

$$\int \frac{d^d k}{(2\pi)^d} \mathrm{tr}\left(\mathcal{Q}(\varphi, iD_\mu)\right) \cdot \mathbf{1} \supset \ldots D_\mu \mathbf{1}$$

that **cannot** be set to zero, as is clear from e.g $F_{\mu\nu} = i[D_\mu, D_\nu]/g$.

To this end, one uses the **Covariant-Derivative Expansion (CDE)** and introduces the unitary operator U_D satisfying

$$U_D = \exp(iD \cdot \partial_k)\,, \qquad U_D^\dagger U_D = \mathbf{1}\,, \qquad U_D \cdot \mathbf{1} = \mathbf{1}\,.$$

Now insert $\mathcal{Q} \cdot \mathbf{1} = U_D^\dagger \mathcal{Q} U_D \cdot \mathbf{1}$ and commute U_D to the left.

The resulting expression will have all all **open derivatives in commutators** and thus be expressed through covariant derivatives and field-strength tensors.

[Gaillard (*1986*), Nucl. Phys. B 268]
[Chan (*1986*), Phys. Rev. Lett. 57]
[Cheyette (*1988*), Nucl. Phys. B. 297]
[Henning et al (*2018*), JHEP 01 123]

Advantages of the Functional Formalism

The functional matching procedure with CDE yields the **complete effective action**. Just extract the fluctuation operator from the Lagrangian, compute

$$S_\Lambda^{(1)} = \frac{i}{2}\mathrm{STr}_h(\log \Delta^{-1}) - \frac{i}{2}\sum_{n=1}^{\infty}\mathrm{STr}_h[(\Delta X)^n]$$

and truncate the sum in the last term.

No need to enumerate diagram **topologies**, compute **symmetry factors**.

The result is **manifestly gauge-invariant**, operators with $F_{\mu\nu}$ and iD_μ emerge in a **transparent** way.

Lends itself extremely well to **automation**.

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[Das Bakshi et al (*2018*), 1808.04403]
[Cohen et al (*2020*), 2012.07851]
[**Matchete** collaboration (*2020*), 2012.08506]
[Carmona et al (*2021*), 2112.10787]
→ stay tuned for Javi's talk!

The Matching Checklist

Straightforward algorithm:

- Define your **UV Lagrangian** $\mathcal{L}_{\rm UV}$ and identify heavy fields.
- Compute the **equations of motions** for the **heavy** fields and solve for them to **express** them **through soft** (classical) modes:

$$\frac{\delta \mathcal{L}_{\rm UV}}{\delta \phi_H} = 0 \qquad \Rightarrow \qquad \phi_H = \varphi_H(\varphi_S)$$

- Split $\phi = \varphi + \eta$ Compute the **fluctuation operator** with at least one of the η a hard mode:

$$\mathcal{Q}_{ij} = \frac{\delta^2 \mathcal{L}_{\rm UV}}{\delta \bar{\eta}_i \delta \eta_j}$$

- Compute the **hard supertraces**, with power-type traces **up to the desired order**, replace classical hard modes with their eom.
- Perform the **CDE expansion**.

Part II - Reduction to a Basis

Redundant Operators and Field Redefinitions

Our procedure generates interaction terms with **derivatives** in $\mathcal{L}_{\text{eff}}$, some containing objects like

$$D^2\phi\,,\qquad i\slashed{D}\psi\,,\qquad D_\mu F^{\mu\nu}\,,\qquad \ldots$$

Operators containing such expressions are deemed **redundant** and can be removed.

Use **IBP relations** to bring as many derivatives as possible into this form!

Caution: **Field redefinitions** and **equation-of-motion identities** are **not equivalent** in the presence of **power-corrections**!

Field Type	Redundant operators	Field redefinition
Real scalar φ	$\chi\mathcal{D}(\varphi)$	$\varphi \to \varphi + \chi$
Complex scalar ϕ	$\chi\mathcal{D}(\phi) + \mathcal{D}\left(\phi^\dagger\right)\Delta$	$\phi \to \phi + \frac{1}{2}(\chi^\dagger + \Delta)$
Majorana fermion η	$\chi\mathcal{D}(\eta) + \mathcal{D}\left(\eta^T\right)\Delta$	$\eta \to \eta + iC(\chi^T - \Delta)$
Dirac fermion ψ	$\chi\mathcal{D}(\psi) + \mathcal{D}\left(\bar\psi\right)\Delta$	$\psi \to \psi - \frac{i}{2}(\bar\chi + \Delta)$
Real vector field A	$\mathcal{D}(A_\mu)\,\chi^\mu$	$A_\mu \to A_\mu - \chi_\mu$
Complex vector field A	$\mathcal{D}(A_\mu)\,\chi^\mu + \mathcal{D}\left(A^\dagger_\mu\right)\Delta^\mu$	$A_\mu \to A_\mu - \frac{1}{2}(\chi^\dagger_\mu + \Delta_\mu)$

[**Matchete** collaboration (*2022*), arXiv:2212.04510]

Dirac Algebra and Evanescent Operators

Effective Lagrangians might contain **operators** that are **related** to each other **by Dirac algebra**: Fierz identities, Chisholm identities.

Allow us to **reduce** the operator **list further**, but rely on **four-dimensional Dirac algebra**!

These identities thus **do not hold** in **dimensional regularization** with $d = 4 - 2\epsilon$.

Difference between original and reduced operator is formally of $\mathcal{O}(\epsilon)$:

$$R_{\ell e} = (\bar{\ell} e)(\bar{e} \ell) \qquad \xrightarrow{d=4} \qquad -\frac{1}{2}(\bar{\ell}\gamma_\mu \ell)(\bar{e}\gamma^\mu e) = -\frac{1}{2} Q_{\ell e}$$

$$\Rightarrow E_{\ell e} = R_{\ell e} + \frac{1}{2} Q_{\ell e}$$

$E_{\ell e}$ does **not** generate **tree-level** matrix elements, things change when it is inserted into **loop graphs**.

Consider 2HDM, integrate out second Higgs doublet Φ:

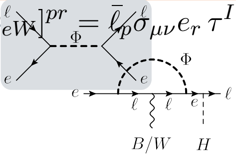
$$\mathcal{L} \supset \mathcal{L}_{\mathrm{SM}} + D_\mu \Phi^\dagger D^\mu \Phi - M_\Phi^2\, \Phi^\dagger \Phi - \left(y_{\Phi e}^{pr}\, \bar{\ell}_p \Phi e_r + \mathrm{h.c.} \right)$$

Generates, amongst many other operators:

$$[R_{\ell e}]^{prst} = (\bar{\ell}_p e_r)(\bar{e}_s \ell_t)$$

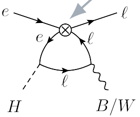
$$-\frac{1}{2}[Q_{\ell e}]^{ptsr} = -\frac{1}{2}(\bar{\ell}_p \gamma_\mu \ell_t)(\bar{e}_s \gamma^\mu e_r)$$

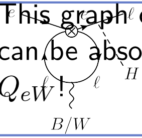
$$[Q_{eW}]^{pr} = \bar{\ell}_p \sigma_{\mu\nu} e_r\, \tau^I H W^{I\mu\nu}$$



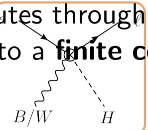
The operators $R_{\ell e}$ and $Q_{\ell e}$ generate **different** amplitudes for the **dipole transitions**!

$R_{\ell e} + \frac{1}{2} Q_{\ell e}$





This graph contributes through $\epsilon \cdot \left(1/\epsilon_{\mathrm{UV}}\right)$ and can be absorbed into a **finite counterterm** to Q_{eW}!



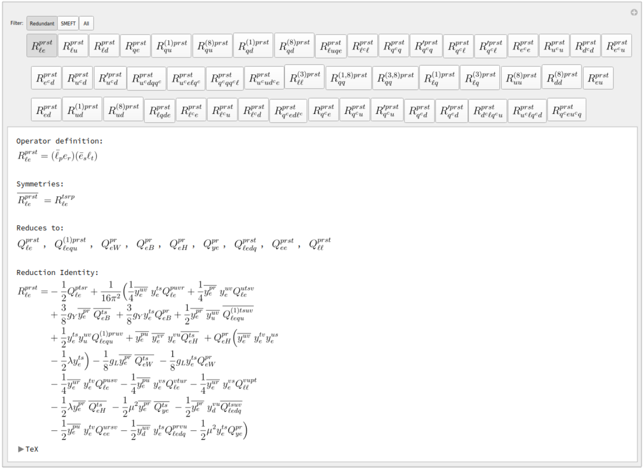
We can treat these contributions in a **similar** fashion **to the matching**:

- **Reduce** operators using $d = 4$ identities.
- Define **evanescent** operators as **difference** between original and reduced.
- From the resulting effective Lagrangian, evaluate the **one-loop effective action** with **one** evanescent insertion.
- We are only interested in **UV poles** $\rightarrow$ evaluate the **hard region** only, with appropriate **IR regulators** in place.
- **Absorb** these contributions into **matching coefficients** of **physical** operators, then **drop** evanescent operators!

Note: This procedure is **almost identical** to obtaining **counterterms** for the **physical** operators!

For all the details, and an **exhaustive** list of **one-loop** evanescent shifts for the **SMEFT** (even in an **interactive** form!) see our paper

[**Matchete** collaboration (*2023*), JHEP 02 031]


Filter: Redundant SMEFT All

$R_{\ell e}^{prst}$ $R_{\ell u}^{prst}$ $R_{\ell d}^{prst}$ R_{qe}^{prst} $R_{qu}^{(1)prst}$ $R_{qu}^{(8)prst}$ $R_{qd}^{(1)prst}$ $R_{qd}^{(8)prst}$ $R_{\ell uqe}^{prst}$ $R_{\ell^c\ell}^{prst}$ $R_{q^cq}^{prst}$ $R_{q^cq}^{\prime prst}$ $R_{q^c\ell}^{prst}$ $R_{q^c\ell}^{\prime prst}$ $R_{e^ce}^{prst}$ $R_{u^cu}^{prst}$ $R_{d^cd}^{prst}$ $R_{e^cu}^{prst}$

$R_{e^cd}^{prst}$ $R_{u^cd}^{prst}$ $R_{u^cd}^{\prime prst}$ $R_{u^cdqq^c}^{prst}$ $R_{u^ce\ell q^c}^{prst}$ $R_{q^cqq^c\ell}^{prst}$ $R_{u^cud^ce}^{prst}$ $R_{\ell\ell}^{(3)prst}$ $R_{qq}^{(1,8)prst}$ $R_{qq}^{(3,8)prst}$ $R_{\ell q}^{(1)prst}$ $R_{\ell q}^{(3)prst}$ $R_{uu}^{(8)prst}$ $R_{dd}^{(8)prst}$ R_{eu}^{prst}

R_{ed}^{prst} $R_{ud}^{(1)prst}$ $R_{ud}^{(8)prst}$ $R_{\ell qde}^{prst}$ $R_{\ell^ce}^{prst}$ $R_{\ell^cu}^{prst}$ $R_{\ell^cd}^{prst}$ $R_{q^ced\ell^c}^{prst}$ $R_{q^ce}^{prst}$ $R_{q^cu}^{prst}$ $R_{q^cu}^{\prime prst}$ $R_{q^cd}^{prst}$ $R_{q^cd}^{\prime prst}$ $R_{d^c\ell q^cu}^{prst}$ $R_{u^c\ell q^cd}^{prst}$ $R_{q^ceu^cq}^{prst}$

Operator definition:

$$R_{\ell e}^{prst} = (\bar{\ell}_p e_r)(\bar{e}_s \ell_t)$$

Symmetries:

$$\overline{R_{\ell e}^{prst}} = R_{\ell e}^{tsrp}$$

Reduces to:

$$Q_{\ell e}^{prst}\,,\ Q_{\ell equ}^{(1)prst}\,,\ Q_{eW}^{pr}\,,\ Q_{eB}^{pr}\,,\ Q_{eH}^{pr}\,,\ Q_{ye}^{pr}\,,\ Q_{\ell edq}^{prst}\,,\ Q_{ee}^{prst}\,,\ Q_{\ell\ell}^{prst}$$

Reduction Identity:

$$\begin{aligned}
R_{\ell e}^{prst} = & -\frac{1}{2}Q_{\ell e}^{ptsr} + \frac{1}{16\pi^2}\Big(\frac{1}{4}\overline{y_e^{uv}}\, y_e^{ts} Q_{\ell e}^{puvr} + \frac{1}{4}\overline{y_e^{pr}}\, y_e^{uv} Q_{\ell e}^{utsv} \\
& + \frac{3}{8}g_Y \overline{y_e^{pr}}\, \overline{Q_{eB}^{ts}} + \frac{3}{8}g_Y y_e^{ts} Q_{eB}^{pr} + \frac{1}{2}\overline{y_e^{pr}}\, \overline{y_u^{uv}}\, \overline{Q_{\ell equ}^{(1)tsuv}} \\
& + \frac{1}{2}y_e^{ts} y_u^{uv} Q_{\ell equ}^{(1)pruv} + \overline{y_e^{pu}}\, \overline{y_e^{vr}}\, y_e^{vu}\overline{Q_{eH}^{ts}} + Q_{eH}^{pr}\Big(\overline{y_e^{uv}}\, y_e^{tv} y_e^{us} \\
& - \frac{1}{2}\lambda y_e^{ts}\Big) - \frac{1}{8}g_L \overline{y_e^{pr}}\, \overline{Q_{eW}^{ts}} - \frac{1}{8}g_L y_e^{ts} Q_{eW}^{pr} \\
& - \frac{1}{4}\overline{y_e^{ur}}\, y_e^{tv} Q_{\ell e}^{pusv} - \frac{1}{4}\overline{y_e^{pu}}\, y_e^{vs} Q_{\ell e}^{vtur} - \frac{1}{4}\overline{y_e^{ur}}\, y_e^{vs} Q_{\ell\ell}^{vupt} \\
& - \frac{1}{2}\lambda \overline{y_e^{pr}}\, \overline{Q_{eH}^{ts}} - \frac{1}{2}\mu^2 \overline{y_e^{pr}}\, \overline{Q_{ye}^{ts}} - \frac{1}{2}\overline{y_e^{pr}}\, y_d^{vu}\overline{Q_{\ell edq}^{tsuv}} \\
& - \frac{1}{2}\overline{y_e^{pu}}\, y_e^{tv} Q_{ee}^{ursv} - \frac{1}{2}\overline{y_d^{uv}}\, y_e^{ts} Q_{\ell edq}^{prvu} - \frac{1}{2}\mu^2 y_e^{ts} Q_{ye}^{pr}\Big)
\end{aligned}$$

▶ TeX

Conclusions

Conclusions

- One-loop matching is a **repetitive**, **mechanical**, but **crucial** task in BSM physics.
- **Functional matching** with the **CDE** and the **Method of Regions** are an economical way to compute effective Lagrangians.
- The **target basis** needs **not** to be known.
- Instead of constructing **Feynman graphs**, matching is a matter of evaluating a **simple** expression and applying **power-counting**.
- IBP relations, field redefinitions and treatment of evanescent operators are **straightforward**, yield the reduced **operator basis**.
- Evanescent treatment is just deriving **counterterms with extra steps** $\rightarrow$ computing **anomalous dimensions** very simple from here.
- Well-suited for **automation**! ($\rightarrow$ Javi's talk next!)

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Bonus slides

There are no bonus slides.