Conformal symmetry and integrability in effective theories?

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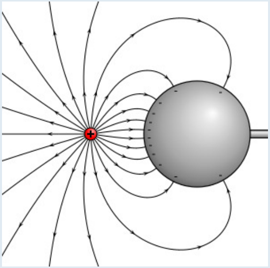
MITP, May 2023

L.I. Magnus (1831):
Inversion transformation

$$x^\mu \to \frac{x^\mu}{x^2}$$



- Small distances ↔ large distances ??
- Asymptotic freedom ↔ confinement ??

Collinear subgroup: SL(2) algebra

$SL(2, R)$ algebra

$$S_- = -\frac{d}{dz}$$
$$S_+ = z^2\frac{d}{dz} + 2j\,z$$
$$S_0 = z\frac{d}{dz} + j$$

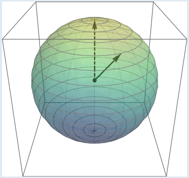
$SL(2)$ commutation relations

$$[S_+, S_-] = 2S_0 \qquad [S_0, S_\pm] = \pm S_+$$

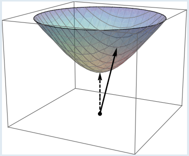
Conformal spin

$$j = \frac{1}{2}(l+s)$$

Non-compact spin



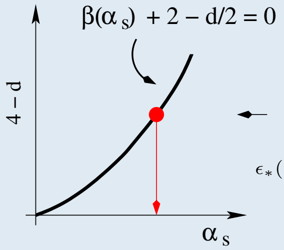
$SO(3)$



$SL(2,\mathbb{R}) \sim O(2,1)$

QCD?

QCD is not a conformal theory, but



conformal theory

$$\epsilon_*(a_s) = -\beta_0 a_s + \ldots\,, \qquad a_s = \frac{\alpha_s}{4\pi}$$

$$\mathcal{A}_{\rm QCD} = \mathcal{A}^{\rm conf}_{\rm QCD} + \frac{\beta(g)}{g}\Delta\mathcal{Q}$$

"Conformal QCD": QCD in $d-2\epsilon$ at Wilson-Fischer critical point $\beta(\alpha_S)=0$

V.B., A. Manashov, Eur.Phys.J.C 73 (2013) 2544

- No corrections $\sim\beta(\alpha_s)$ for the counterterms
  $\Rightarrow$ exact conformal symmetry of the QCD RG equations in MS-schemes

Selected applications: (1) mixing with total derivatives

Renormalization of off-forward matrix elements

$$\langle p'|\mathcal{O}_{Nk}|p\rangle = \langle p'|\bar{q}(0)\gamma_{\mu_1} \overleftarrow{D}_{\mu_2} \ldots \overleftarrow{D}_{\mu_k} \overrightarrow{D}_{\mu_{k+1}} \ldots \overrightarrow{D}_{\mu_N} q(0)|p\rangle$$

Mixing matrix with operators containing total derivatives with anomalous dimensions on the diagonal

$$\begin{pmatrix} \gamma_1 & 0 & 0 & 0 \\ * & \gamma_2 & 0 & 0 \\ * & * & \gamma_3 & 0 \\ * & * & * & \gamma_4 \end{pmatrix}$$

need off-diagonal elements

[evolution kernels in x-space are functions of two variables]

- In conformal theories, operators with different scaling dimensions are orthogonal

$$\langle O_1(x_1)\, O_2(x_2)\rangle = \frac{\text{const}}{|x_1 - x_2|^{2\Delta_1}}\, \delta_{\Delta_1 \Delta_2}$$

- Three-loop results for flavor-singlet operators for $N \leq 8$

VB, K.Chetyrkin, A.Manashov, PLB 834 (2022) 137409

- Gauge-invariant two-point functions
- Simple algorithmic implementation
- Straightforward to extend to four loops

Selected applications: (2) Kinematic power corrections in off-forward reactions

schematically

$$\mathsf{T}\{j(x)j(0)\} = \sum_N \Big\{ A_N^{\mu_1\cdots\mu_N} \underbrace{\mathcal{O}^N_{\mu_1\cdots\mu_N}}_{\text{twist-2 operators}} + B_N^{\mu_1\cdots\mu_N} \underbrace{\partial^\mu \mathcal{O}^N_{\mu,\mu_1\cdots\mu_N}}_{\text{descendants of twist 2}}$$

$$+ C_N^{\mu_1\cdots\mu_N} \underbrace{\partial^2 \mathcal{O}^N_{\mu_1\cdots\mu_N}}_{\text{descendants}} + D_N^{\mu_1\cdots\mu_N} \underbrace{\partial^\mu\partial^\nu \mathcal{O}^N_{\mu,\nu,\mu_1\cdots\mu_N}}_{\text{descendants}} + \ldots \Big\} + \ldots$$

$$\equiv \sum_N \mathbb{C}_N^{\mu_1\cdots\mu_N}(x,\partial)\, \mathcal{O}^N_{\mu_1\cdots\mu_N} + \text{quark-gluon operators}$$

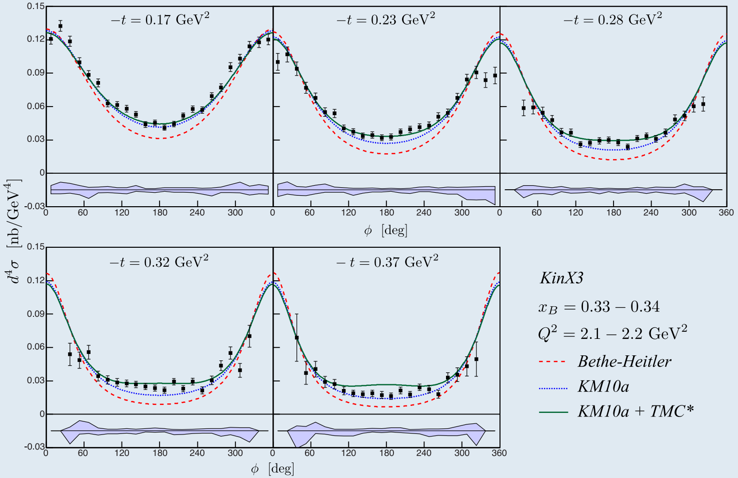
- In conformal field theories, the CFs of descendants are related to the CFs of twist-2 operators by symmetry and do not need to be calculated directly

$$A_N^{\mu_1\cdots\mu_N} \overset{O(4,2)}{\mapsto} \mathbb{C}_N^{\mu_1\cdots\mu_N}(x,\partial)$$

VB, Yao Ji, A. Manashov, JHEP 03 (2021) 051; JHEP 01 (2023) 078

Kinematic power corrections in DVCS

M. Defurne *et al.* [Hall A Collaboration] arXiv:1504.05453



GPD model: KM10a (Kumericki, Mueller, Nucl. Phys. B **841** (2010) 1)

Selected applications: (3) Lange-Neubert evolution equation

B-meson light-cone distribution amplitude

$$\langle 0|\bar{q}(zn)\not{n}\gamma_5 h_v(0)|B(v)\rangle = iF_B(\mu)\int_0^\infty d\omega\, e^{-i\omega z}\phi_+(\omega,\mu)$$

Lange-Neubert evolution equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s C_F}{\pi}H_{LN}\right)\phi_+(\omega,\mu) = 0\,,$$

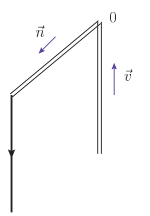
$$[H_{LN}f](\omega) = \frac{\alpha_s C_F}{\pi}\left\{-\int_0^\infty d\omega' \left[\frac{\omega}{\omega'}\frac{\theta(\omega'-\omega)}{\omega'-\omega} + \frac{\theta(\omega-\omega')}{\omega-\omega'}\right]_+ + \left[\ln\frac{\mu}{\omega} - \frac{5}{4}\right]f(\omega)\right\} + \mathcal{O}(\alpha_s^2)$$

- To all orders in perturbation theory

V.B., Yao Ji and A.Manashov, PRD **100** *(2019) 014023*

$$H_{LN}(a_s) = \Gamma_{\rm cusp}(a_s)\ln(iK(a_s)\mu e^{2\gamma_E}) + \text{const}$$

— $K(a_s) = v^\mu K_\mu(\epsilon_*(a_s), a_s)$ is the generator of special conformal trafos in $d-2\epsilon_*$



- Geometry is invariant under $v^\mu K_\mu$

$$(*) \quad [K, H_{LN}] = 0$$

- Dilatation invariance broken by the cusp

$$(**) \quad [D, H_{LN}] = \Gamma_{\text{cusp}}$$

- $(*) \mapsto H_{LN}$ is a function of K
- This function can be found from $(**) \mapsto$ answer

Solution at one loop

$\mathcal{H}_{LN}$ and K share the same eigenfunctions:

$$iKQ_s = sQ_s$$

- At one-loop order:

G.Bell, T.Feldmann, Yu-Ming Wang, M.Yip, JHEP 11 (2013) 191
V.B., A.Manashov, Phys.Lett.B 731 (2014) 316

in position space

$$K \;=\; z^2\partial_z + 2z \;\Rightarrow\; Q_s(z) = -\frac{1}{z^2}\exp\left\{\frac{is}{z}\right\}$$

in momentum space

$$\mathcal{K} = i\Big[\omega\partial_\omega^2 + 2\partial_\omega\Big] \;\Rightarrow\; Q_s(\omega) = \frac{1}{\sqrt{\omega s}}J_1(2\sqrt{\omega s})$$

eigenvalues ≡ anomalous dimensions

$$\mathcal{H}_{LN}Q_s(z) = \Big[\ln(i\mu K) - \psi(1) - \frac{5}{4}\Big]Q_s(z) = \Big[\ln(\mu s) - \psi(1) - \frac{5}{4}\Big]Q_s(z)$$

- At two-loops:

$$K = K^{(0)} + a_s K^{(1)} \qquad \Leftarrow \qquad \text{conformal Ward identities}$$

V.B., Yao Ji and A.Manashov, PRD **100** (2019) 014023

Complete integrability in effective theories?

- Integrability of one-loop RG equations in QCD

Lipatov '94
Faddeev, Korchemsky '95
Braun, Derkachov, Manashov '98

- Integrability of $N = 4$ SUSY YM

Minahan, Zarembo '03
Beisert, Staudacher '03

RG equations have $SL(2)$ symmetry $\quad [S_+, S_-] = 2S_0. \quad [S_0, S_\pm] = \pm S_+$

$$S_- = -\frac{d}{dz} \qquad S_+ = z^2\frac{d}{dz} + 2j\,z \qquad S_0 = z\frac{d}{dz} + j$$

two-particle RG kernels can be written in terms of the Casimir operator $\quad S^2 = J(J-1)$

Bukhvostov, Frolov, Lipatov, Kuraev, '85

e.g. quark-quark with opposite helicities

$$\mathcal{H}_{12} = 4\,[\psi(J_{12}) - \psi(2)] - 2/[J_{12}(J_{12}-1)] + 1$$

RG equations for multiparticle operators have a pair-wise structure

$$\mathcal{H}_{123} = \mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{31}$$

- (noncompact) spin chains, open (qFq) or closed (qqq), QISM

- Integrability in HQET

Braun, Derkachov, Manashov '14
Braun, Ji, Manashov '17-'18

- $SL(2)$ symmetry broken to

$$[S_+, \mathcal{H}_{qh}] = 0\,, \qquad [S_0, \mathcal{H}_{qh}] = 1$$

- Covariant representation for the heavy-light "Hamiltonian" (evolution kernel)

Braun, Manashov, '14

$$\mathcal{H}_{qh} = \ln(i\mu S_+^{qh}) - \frac{5}{4} + \gamma_E$$

- Complete integrability for qqh and qFh systems with selected quantum numbers

Braun, Ji, Manashov, JHEP 06 (2018) 017

- Three-particle Hamiltonian commutes with the "C"-entry of the transfer matrix
- Explicit analytic solutions for the "wave functions" using Derkachov, Korchemsky, Manashov, '03

Similarity transformation

recall

$$S_- = -\frac{d}{dz} \qquad S_+ = z^2\frac{d}{dz} + 2j\,z \qquad S_0 = z\frac{d}{dz} + j$$

- Heavy quarks only move a little, $z \sim 1/m$, suggests to rescale

$$S_-^{(h)} \mapsto \lambda S_-^{(h)}\,, \qquad S_+^{(h)} \mapsto \lambda^{-1} S_+^{(h)}\,, \qquad S_0^{(h)} \mapsto S_0^{(h)}$$

and consider the limit $\lambda \sim m \to \infty$

Factorization of heavy and light degrees of freedom: Hamiltonian

Heavy-light generators

$$S_+^{(qh)} \mapsto S_+^{(q)} + \lambda^{-1} S_+^{(h)} = S_+^{(q)} + \mathcal{O}(\lambda^{-1})$$
$$S_-^{(qh)} \mapsto S_-^{(q)} + \lambda S_-^{(h)} = \lambda S_-^{(h)} + \mathcal{O}(1)$$
$$S_0^{(qh)} = \mathcal{O}(1)$$

Two-particle Casimir operator

$$S_{qh}^2 = S_+^{(qh)} S_-^{(qh)} + S_0^{(qh)} (S_0^{(qh)} - 1) \mapsto \lambda S_+^{(q)} S_-^{(h)} + \mathcal{O}(1)$$

Hence $J_{(qh)} \sim \sqrt{\lambda}$ and the RG kernel simplifies to

$$\mathcal{H}_{qh} = 2\Big[\psi(J_{qh}) - \psi(1)\Big] + \mathcal{O}(1/S_{qh}^2) \mapsto \ln\Big(\lambda S_+^{(q)} S_-^{(h)}\Big) + \mathcal{O}(\lambda^{-1})$$

Since the "heavy" and "light" generators act on different spaces

$$\mathcal{H}_{qh} \mapsto \ln\Big(i\mu S_+^{(q)}\Big) + \ln\Big(-i\mu^{-1}\lambda S_-^{(h)}\Big)$$

where μ is an arbitrary parameter with dimension of mass. In HQET only light degrees of freedom remain

$$\mathcal{H}_{qh}^{\mathrm{HQET}} = \ln\Big(i\mu S_+^{(q)}\Big) + \text{ const}$$

Factorization of heavy and light degrees of freedom: Wave functions

- For qq, eigenfunctions of the Hamiltonian are usually sought on the space of polynomials that can be mapped to local composite operators:

$$\Psi_{n,k}(z_1,z_2) = (S_+^{(qq)})^k (z_1-z_2)^n\,, \qquad S_{qq}^2 \Psi_{n,k} = (n+2)(n+1)\Psi_{n,k}\,, \qquad k\geq 0$$

- For qh one needs eigenfunctions analytic in the lower half of the complex plane:

$$\Psi_n^{(\eta)}(z_q,z_h) = e^{-\frac{i}{\eta}S_+^{(qh)}}(z_q-z_h)^n \sim \frac{(z_q-z_h)^n}{(z_q-i\eta)^{n+2}(z_h-i\eta)^{n+2}}\,, \qquad \mathrm{Re}(\eta)>0$$

- Rescaling $z_h \mapsto \lambda^{-1} z_h$ implies $z_h \ll z_q$ and also $n = \mathcal{O}(\lambda^{1/2})$
- Extracting the leading behavior at $\lambda\to\infty$ one breaks $SL(2)$ so that the states with different η are no longer degenerate and the system "chooses" a particular solution that satisfies the residual symmetry to the special conformal transformation

$$\Psi_n^{(\eta)}(z_q,z_h) \mapsto \Psi_{(s)}(z_q)\,, \qquad S_+^{(q)}\Psi_{(s)}(z_q) = is\,\Psi_{(s)}(z_q)$$

- From this condition $\eta = \mathcal{O}(\lambda^{-1/2})$ so that $s\equiv n\eta = \mathcal{O}(1)$ and

$$\Psi_{(s)}(z_q) = \frac{1}{(i\eta)^{n+2}}\frac{1}{z_q^2}e^{is/z_q}\Big(1+\mathcal{O}(\lambda^{-1/2})\Big),$$

reproducing the result in arXiv:1402.5822 up to a different normalization

Factorization of heavy and light degrees of freedom: Transfer matrix and conserved charges

- **closed spin chain for N light fields:**

Monodromy matrix

$$T_N(u) = L_1(u)L_2(u)\dots L_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}, \qquad L_k(u) = u + i \begin{pmatrix} S_0^{(k)} & S_-^{(k)} \\ S_+^{(k)} & -S_0^{(k)} \end{pmatrix}$$

For $SL(2)$-invariant systems the proper object to consider is the transfer matrix

$$t_N(u) = \mathrm{Tr}[T_N(u)] = A_N(u) + D_N(u)$$

$$[t_N(u), t_N(v)] = 0, \qquad [S_\alpha, t_N(u)] = 0 \qquad [\mathcal{H}_N, t_N(u)] = 0$$

- **closed spin chain for N light and one heavy field:**

$$T_{N+1}(u) = \left\{ u + i \begin{pmatrix} S_0^{(h)} & \lambda S_-^{(h)} \\ \lambda^{-1} S_+^{(h)} & -S_0^{(h)} \end{pmatrix} \right\} \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix},$$

and in the scaling limit

$$t_{N+1}(u) \underset{\lambda\to\infty}{\longmapsto} \lambda i S_-^{(h)} C_N(u) + \mathcal{O}(\lambda^0)$$

for the simplest case of three quarks

$$C_2(u) = u\mathbb{Q}_1 + \mathbb{Q}_2\,, \qquad \mathbb{Q}_1 = i(S_+^{(1)} + S_+^{(2)})\,, \qquad \mathbb{Q}_2 = S_0^{(1)}S_+^{(2)} - S_0^{(2)}S_+^{(1)}.$$

Braun, Derkachov, Manashov '14

- **open spin chain for N light and one heavy field:**

$$\mathbf{t}_{N+1}(u)\Big|_{z_0\to\frac{z_h}{\lambda}} \underset{\lambda\to\infty}{\longmapsto} \lambda 2i\Big(u-\frac{i}{2}\Big)S_-^{(h)}\mathbb{C}_N(u)+\mathcal{O}(\lambda^0).$$

For the simplest case $N=2$

$$\mathbb{C}_2(u)=2i\left(u+\frac{i}{2}\right)\left(u^2\mathbb{Q}_1+\mathbb{Q}_2\right),$$

where

$$\mathbb{Q}_1=S_+^{(1)}+S_+^{(2)},$$

$$\mathbb{Q}_2=[\omega_2^2-j_2(j_2-1)]S_+^{(1)}-S_+^{(1)}\Big(S_+^{(1)}S_-^{(2)}+S_0^{(1)}S_0^{(2)}\Big)-S_0^{(1)}\Big(S_0^{(2)}S_+^{(1)}-S_0^{(1)}S_+^{(2)}\Big).$$

Braun, Ji, Manashov '17

Heavy-light eigenfunctions

- $C_N(u)$ and $B_N(u)$ are unitarily equivalent
- Eigenstates of B_N provide the basis for Sklyanin's SOV

Derkachov, Korchemsky, Manashov '03

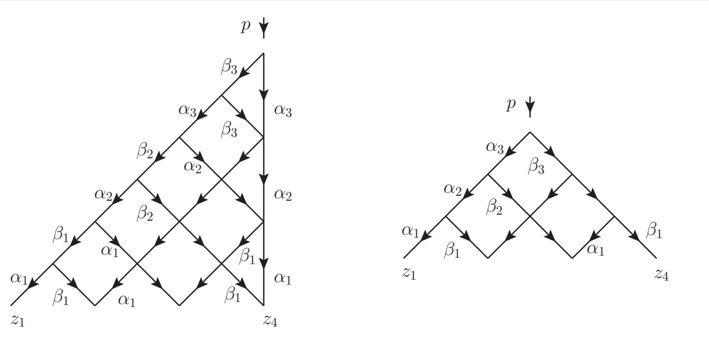


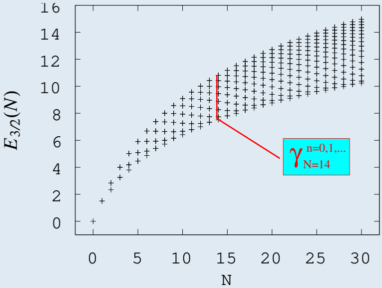
Figure: Diagrammatic representation for the eigenfunctions of the open (left) and closed (right) homogeneous spin chains for $N = 4$. The indices are $\alpha_k = s - ix_k$ and $\beta_k = s + ix_k$.

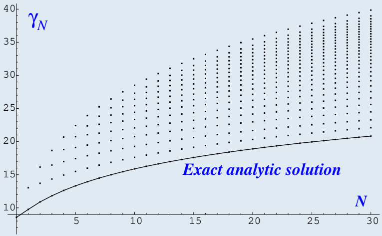
- Here: include impurities and allow for different spins

Heavy-light eigenfunctions vs. large-spin limit of light ones

- The spectrum of qFq (left) and qqq (right) QCD spin chains

Braun, Derkachov, Korchemsky, Manashov '99





- **Conjecture: For the lowest part of the spectrum in the large-spin limit**

$$\Psi^{(\eta\sim s/\sqrt{\lambda})}_{u_1,\ldots,u_{N+1}}(z_1,\ldots,z_N,z_0=z_h/\lambda) \overset{\lambda\to\infty}{\longmapsto} \Psi^{\text{heavy}}_{s,u_1,\ldots,u_{N-1}}(z_1,\ldots,z_N)$$

- Example:

$$\Psi_{n,q_3=0}(z_0,z_1,z_2) = \frac{z_{01}^{n+3}+z_{12}^{n+3}+z_{20}^{n+3}}{z_{01}z_{12}z_{20}} \overset{\lambda\to\infty}{\longmapsto} \frac{1}{z_1z_2z_{12}}\left(e^{is/z_1}-e^{is/z_2}\right)$$

B-meson LCDAs: Twist-three

[Braun:2015pha]

$$\phi_-(\omega,\mu) = \int_\omega^\infty \frac{d\omega'}{\omega'}\phi_+(\omega',\mu) + \int_0^\infty ds\, J_0(2\sqrt{\omega s})\, \eta_3^{(0)}(s,\mu)$$

$$\phi_3(\underline{\omega},\mu) = \int_0^\infty ds\Big[\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\,\underline{\omega}) + \frac{1}{2}\int_{-\infty}^\infty dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\,\underline{\omega})\Big],$$

where up to $1/N_c^2$ corrections

$$\eta_3(s,x,\mu) = L^{\gamma_3(x)/\beta_0}\, R(s;\mu,\mu_0)\,\eta_3(s,x,\mu_0)$$

$$\eta_3^{(0)}(s,\mu) = L^{N_c/\beta_0}\, R(s;\mu,\mu_0)\eta_3^{(0)}(s,\mu_0)$$

$$Y_3(s,x\,|\,\underline{\omega}) = -\int_0^1 du\,\sqrt{us\omega_1}\, J_1(2\sqrt{us\omega_1})\,\omega_2\, J_2(2\sqrt{\bar{u}s\omega_2})\, {}_2F_1\left(\begin{matrix}-\frac{1}{2}-ix,\, -\frac{1}{2}+ix \\ 2\end{matrix}\,\Big|\, -\frac{u}{\bar{u}}\right)$$

$$R(s;\mu,\mu_0) = L^{3C_F/(2\beta_0)}\exp\left[-\int_{\mu_0}^{\mu}\frac{d\tau}{\tau}\,\Gamma_{cusp}(\alpha_s(\tau))\,\ln(\tau s/s_0)\right], \qquad L = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

B-meson LCDAs: Twist-four

[Braun:2017liq]

$$\Phi_4(\underline{\omega}) = \frac{1}{2}\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;1}^{(+)}(s,x\,|\underline{\omega})\,,$$

$$(\Psi_4 + \widetilde{\Psi}_4)(\underline{\omega}) = -\int_0^\infty ds \int_{-\infty}^\infty dx\, \eta_4^{(+)}(s,x,\mu)\, Y_{4;2}^{(+)}(s,x\,|\underline{\omega})\,,$$

$$(\Psi_4 - \widetilde{\Psi}_4)(\underline{\omega}) = 2\int_0^\infty \frac{ds}{s}\left(-\frac{\partial}{\partial\omega_2}\right)\left\{\eta_3^{(0)}(s,\mu)\, Y_3^{(0)}(s\,|\underline{\omega}) + \frac{1}{2}\int_{-\infty}^{\infty} dx\, \eta_3(s,x,\mu)\, Y_3(s,x\,|\underline{\omega})\right\}$$

$$-\int_0^\infty ds \int_{-\infty}^\infty dx\, \varkappa_4^{(-)}(s,x,\mu)\, Z_{4;2}^{(-)}(s,x\,|\underline{\omega})\,,$$

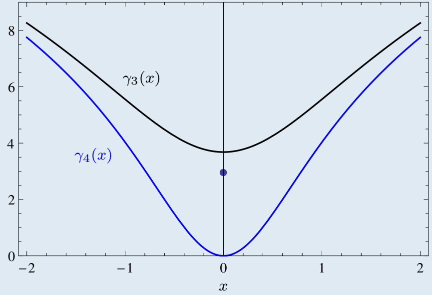
where up to $1/N_c^2$ corrections

$$\eta_4^{(+)}(s,x,\mu) = L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \eta_4^{(+)}(s,x,\mu_0)$$

$$\varkappa_4^{(-)}(s,x,\mu) = L^{\gamma_4(x)/\beta_0} R(s;\mu,\mu_0)\, \varkappa_4^{(-)}(s,x,\mu_0)$$

Anomalous dimensions

[Braun:2017liq]



$$\gamma_3(x) = N_c\Big[\psi\Big(3/2+ix\Big)+\psi\Big(3/2-ix\Big)+2\gamma_E\Big]$$

$$\gamma_3^{(0)} = \gamma_3(x=i/2) = N_c$$

$$\gamma_4(x) = N_c\Big[\psi\Big(ix\Big)+\psi\Big(-ix\Big)+2\gamma_E\Big]$$

Summary and Outlook

- Conformal symmetry of QCD at the Wilson-Fischer critical point offers an ample set of calculational tools for high-orders and higher-twists

- Similarity transformation for the generators of symmetry transformations may provide a method to study implications for effective theories, HQET as example

- This construction leads to new integrable models with reduced symmetry and “nonstandard” Hamiltonian
  In a few cases, such models can be identified as related to known effective theories

- Correspondence works for conserved charges/hamiltonians; more subtle for eigenfunctions

- Analogies exist with the semiclassical expansion of the solution to Baxter Equation

Belitsky:2006en,Beccaria:2007uj

Supplementary slides

QCD in $d\neq 4$ dimensions

- QCD in the $d = 4-2\epsilon$

$$S = \int d^dx \Big\{ \bar q \not{D} q + \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} - \bar c^a \partial_\mu (D^\mu c)^a + \frac{1}{2\xi}(\partial_\mu A^{a,\mu})^2 \Big\}$$

where $D_\mu = \partial_\mu - igM^\epsilon A^a_\mu T^a$, M is scale parameter

- The renormalized action

$$q \to Z_q q\,, \qquad A \to Z_A A\,, \qquad c \to Z_c c\,, \qquad g \to Z_g g\,, \qquad \xi \to Z_\xi \xi\,,$$

Here $Z_\xi = Z_A^2$ and all RG factors are defined using minimal subtraction

$$Z = 1 + \sum_{j=1}^{\infty} \epsilon^{-j} \sum_{k=j}^{\infty} z_{jk} \left(\frac{\alpha_s}{4\pi}\right)^k, \qquad \alpha_s = \frac{g^2}{4\pi}$$

notation: $a = \dfrac{\alpha_s}{4\pi}$

where z_{jk} are ϵ-independent constants

- Note that we do not send $\epsilon \to 0$ so that renormalized quantities explicitly depend on ϵ

- Critical coupling

$$\beta_g(g) = M\frac{dg}{dM} = 0 \ \Rightarrow a_*(\epsilon) = -\frac{\epsilon}{\beta_0} - \left(\frac{\epsilon}{\beta_0}\right)^2 \frac{\beta_1}{\beta_0} + O(\epsilon^3)$$

Conformal Ward identities

- Exploit invariance of the path integral under change of variables $\Phi \to \Phi + \delta_C \Phi, \qquad C = D, K, \ldots$

$$\langle [\mathcal{O}_1](x)[\mathcal{O}_2](y)\rangle = \int \mathcal{D}\Phi \, [\mathcal{O}_1](x)[\mathcal{O}_2](y)\, e^{S(\Phi)}$$

$$\boxed{\langle \delta_C[\mathcal{O}_1](x)[\mathcal{O}_2](y)\rangle + \langle [\mathcal{O}_1](x)\delta_C[\mathcal{O}_2](y)\rangle + \langle \delta_C S\, [\mathcal{O}_1](x)[\mathcal{O}_2](y)\rangle}$$

- Variation of the QCD action

$$\boxed{\delta_D S = \int d^d x\, \mathcal{N}(x)\,, \qquad \delta_K^\mu S = \int d^d x\, 2x^\mu \Big(\mathcal{N}(x) - (d-2)\partial^\rho \mathcal{B}_\rho(x)\Big)}$$

where

$$\mathcal{N}(x) = 2\epsilon \left(\frac{1}{4}F^2 + \frac{1}{2\xi}(\partial A)^2\right),$$

$$\mathcal{B}_\rho(x) = \bar{c} D^\rho c - \frac{1}{\xi} A^\rho (\partial A) = \delta_{\text{BRST}}(\bar{c}^a A^a_\mu)$$

results in quantum corrections (deformation) of the generators of symmetry transformations

- Dilatations: modification of scaling dimensions $\Delta_N \to \Delta_N + \gamma_N$
- Special conformal transformations:
  - — modification of scaling dimensions $\Delta_N \to \Delta_N + \gamma_N$
  - — extra terms that do not spoil conformal algebra ("conformal anomaly")
  - — extra BRST-exact and EOM contributions

VB, A.Manashov, S.Moch, M.Strohmaier, PLB 793 (2019) 78