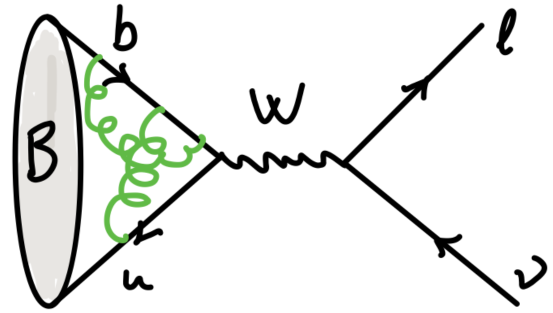
QED corrections to $B \to \mu\bar{\nu}$

Claudia Cornella (JGU Mainz)

based on 2212.14430 and ongoing work with M. Neubert and M.König

Motivation



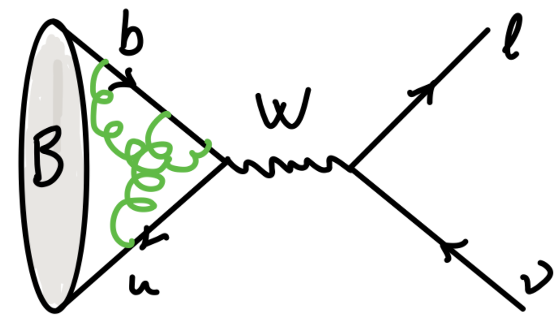
$$\Gamma \sim m_\ell^2 \, f_{B_u}^2 \, |V_{ub}|^2$$

Why $B \to \ell\nu$?

- direct determination of $|V_{ub}|$, largely unaffected by hadronic uncertainties
- chirality suppressed → powerful probe of (pseudo)scalar NP.
- testing Lepton Flavor Universality in charged currents

Motivation

Why $B \to \ell\nu$?



$$\Gamma \sim m_\ell^2 \, f_{B_u}^2 \, |V_{ub}|^2$$

- direct determination of $|V_{ub}|$, largely unaffected by hadronic uncertainties
- chirality suppressed → powerful probe of (pseudo)scalar NP.
- testing Lepton Flavor Universality in charged currents

Why QED corrections?

- Belle II will measure the τ, μ channels with $5-7\,\%$ uncertainty [Belle II Physics Book]
- QCD matrix element is know with $\mathcal{O}(<1\%)$ accuracy:

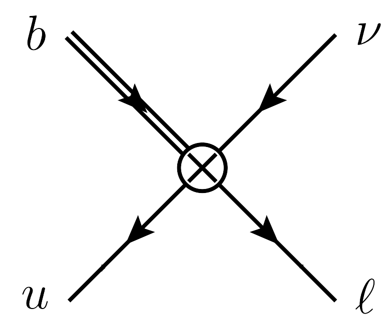
$$\langle 0 | \bar{u}\gamma^\mu\gamma_5 b | B_q(p) | \rangle = i f_{B_u} p^\mu \qquad f_{B_u} = 189.4 \pm 1.4 \text{ MeV}$$ [FNAL/MILC 1712.09262]

- QED corrections can be of similar magnitude or even larger, due to presence of large logarithms $\alpha \ln(m_b/m_\ell)$ and $\alpha \ln(m_\ell/E_s)$

QED corrections to exclusive B decays

- QED effects are well under control for $\mu > m_b$ as well as for $\mu \ll \Lambda_{\rm QCD}$:
  - all short distance ($\mu > m_b$) QED effects can be included in the **weak effective Lagrangian**

$$\mathcal{L}_{\rm eff} = -\frac{4G_F}{\sqrt{2}} K_{\rm EW}(\mu) V_{ub} \, (\bar{u}\gamma^\mu P_L b)(\bar{\ell}\gamma_\mu P_L \nu_\ell)$$

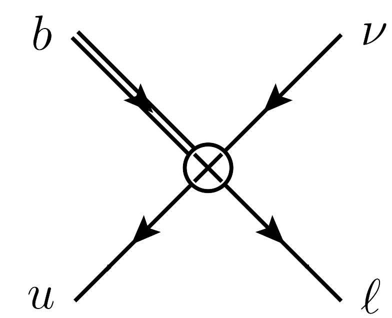


  - photons with energy much smaller than $\Lambda_{\rm QCD}$ cannot resolve the hadron structure and can be computed treating the B as **point-like.**

QED corrections to exclusive B decays

- QED effects are well under control for $\mu > m_b$ as well as for $\mu \ll \Lambda_{\text{QCD}}$:
  - all short distance ($\mu > m_b$) QED effects can be included in the **weak effective Lagrangian**

$$\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} (\bar{u}\gamma^\mu P_L b)(\bar{\ell}\gamma_\mu P_L \nu_\ell)$$



  - photons with energy much smaller than Λ_{QCD} cannot resolve the hadron structure and can be computed treating the B as **point-like.**

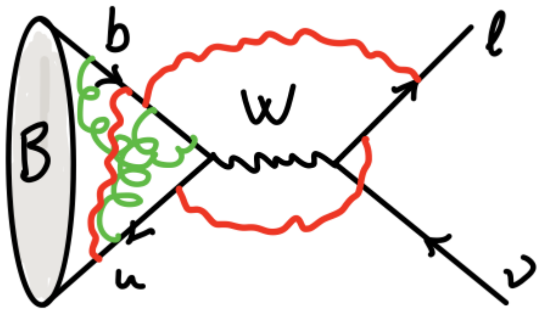
- Things are more complicated for $\Lambda_{\text{QCD}} < \mu < m_b$: very active research topic.

  QED factorization theorems available only for a few processes:

  - $B_s \to \mu^+\mu^-$ [Beneke, Bobeth, Szafron, 1708.09152,1908.07011]
  - $B \to \pi K, B \to D\pi$ [Beneke, Böer et al 2008.10615,2107.03819]
  - $B_s \to \mu^+\mu^-\gamma$ [Beneke, Bobeth, Wang 2008.12494]

QED corrections to exclusive B decays

Main challenges in formulating a factorization theorem:

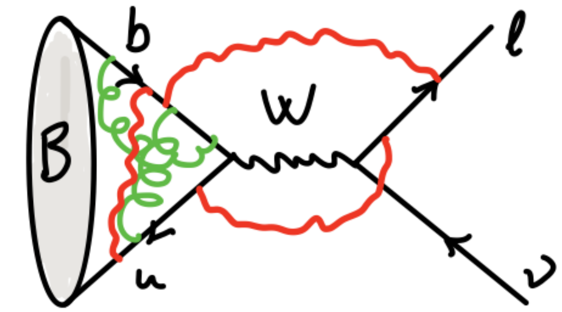


- unlike in QCD, **external states** can be **charged** in QED

  ⇒ “universal” hadronic quantities become process-dependent, e.g. decay constants are not constants anymore

QED corrections to exclusive B decays

Main challenges in formulating a factorization theorem:



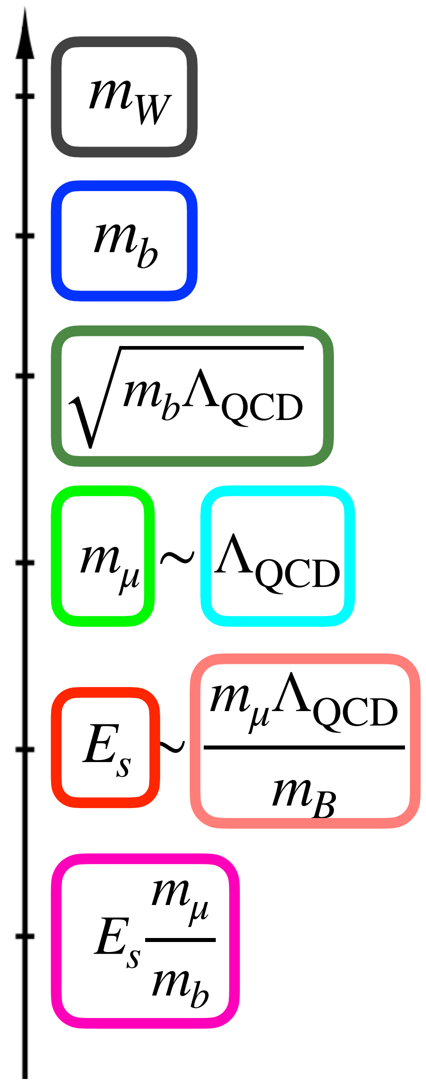
- unlike in QCD, **external states** can be **charged** in QED
  $\Rightarrow$ "universal" hadronic quantities become process-dependent, e.g. decay constants are not constants anymore

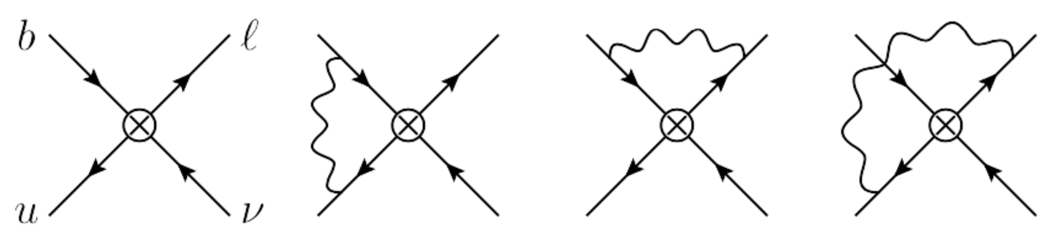
- beyond leading power convolutions have **endpoint divergences**.
  - cannot be dealt with using standard renormalization techniques and require appropriate subtractions
  - relevant here: the chiral suppression makes $B \to \mu\bar{\nu}$ is a genuine next-to-leading power process!

Scales

In the presence of QED corrections,

$B \to \mu\bar{\nu}$ is sensitive to eight different energy scales:



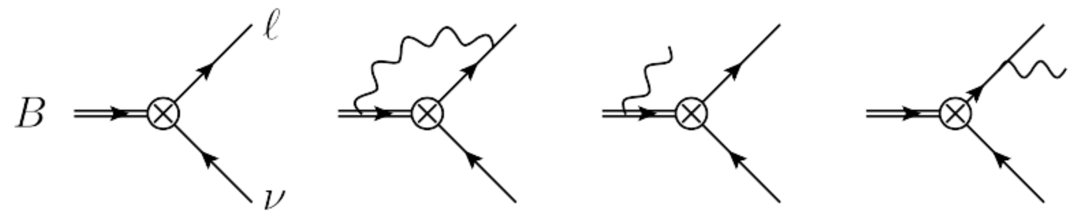


Fock-state description of B meson: $|\bar{u}b\rangle + |\bar{u}gb\rangle + \ldots$

[see: Beneke, Bobeth, Szafron 2019]

---

description of B meson as a point-like pseudo-scalar boson



[see: Isidori, Nabeebaccus, Zwicky 2020; Zwicky 2021;
isee: Dai, Kim, Leibovich 2021]

Strategy

We want to disentangle all these scales by

- identifying the **appropriate EFT** description
- deriving a **factorization theorem** to describe the multi-scale problem in terms of convolutions of single-scale objects

Strategy

We want to disentangle all these scales by

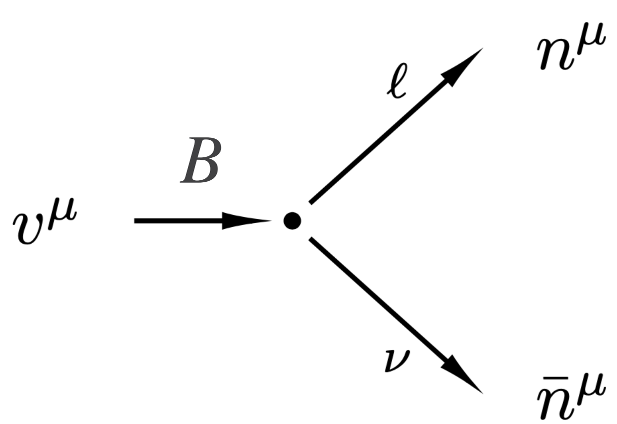
- identifying the **appropriate EFT** description
- deriving a **factorization theorem** to describe the multi-scale problem in terms of convolutions of single-scale objects

In the following, we will

- describe the **EFT construction** across all scales
- discuss the **factorization & refactorization** of the **“virtual” amplitude**
- sketch the **low-energy theory** describing real emissions

Power counting and relevant modes

- In the B-meson rest frame, the muon and the neutrino are back to back.
  $\Rightarrow$ identify the **muon** direction of the "**collinear**" one.



$$p_B^\mu = m_B v^\mu\,, \qquad v^\mu = (1{,}0{,}0{,}0)$$

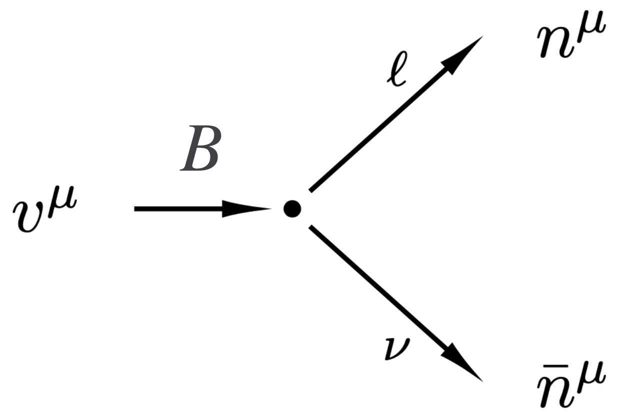
$$p_\mu^\mu = \frac{m_B}{2}\left(1 + \frac{m_\ell^2}{m_B^2}, 0, 0, +1 - \frac{m_\ell^2}{m_B^2}\right) \approx \frac{m_B}{2}(1, 0, 0, +1) = \frac{m_B}{2} n^\mu$$

$$p_\nu^\mu = \frac{m_B}{2}\left(1 - \frac{m_\ell^2}{m_B^2}, 0, 0, -1 + \frac{m_\ell^2}{m_B^2}\right) \approx \frac{m_B}{2}(1, 0, 0, -1) = \frac{m_B}{2} \bar{n}^\mu$$

$$\lambda_\ell = \frac{m_\ell}{m_b} \ll 1$$

Power counting and relevant modes

- In the B-meson rest frame, the muon and the neutrino are back to back.
  $\Rightarrow$ identify the **muon** direction of the "**collinear**" one.



$$p_B^\mu = m_B v^\mu\,, \qquad v^\mu = (1{,}0{,}0{,}0)$$

$$p_\mu^\mu = \frac{m_B}{2}\left(1+\frac{m_\ell^2}{m_B^2},0,0,+1-\frac{m_\ell^2}{m_B^2}\right) \approx \frac{m_B}{2}(1,0,0,\,+1) = \frac{m_B}{2}n^\mu$$

$$p_\nu^\mu = \frac{m_B}{2}\left(1-\frac{m_\ell^2}{m_B^2},0,0,\,-1+\frac{m_\ell^2}{m_B^2}\right) \approx \frac{m_B}{2}(1,0,0,\,-1) = \frac{m_B}{2}\bar{n}^\mu$$

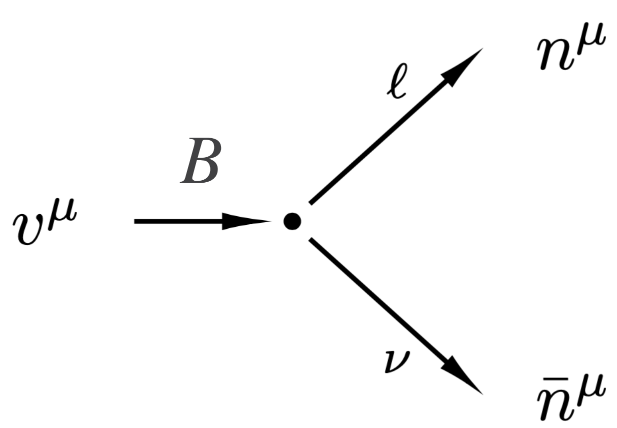
$$\lambda_\ell = \frac{m_\ell}{m_b} \ll 1$$

- Initial state quark are bound in the meson, with residual momenta of $\mathcal{O}(\Lambda_{\rm QCD})$

$$p_b = m_b v + k_b\,, \quad p_q = k_q \qquad k_b, k_q \sim m_b(\lambda,\lambda,\lambda) \qquad \lambda = \frac{\Lambda_{\rm QCD}}{m_b}$$

Power counting and relevant modes

- In the B-meson rest frame, the muon and the neutrino are back to back.
  $\Rightarrow$ identify the **muon** direction of the "**collinear**" one.



$$p_B^\mu = m_B v^\mu\,, \qquad v^\mu = (1,0,0,0)$$

$$p_\mu^\mu = \frac{m_B}{2}\left(1+\frac{m_\ell^2}{m_B^2},0,0,+1-\frac{m_\ell^2}{m_B^2}\right) \approx \frac{m_B}{2}(1,0,0,\,+1) = \frac{m_B}{2}n^\mu$$

$$p_\nu^\mu = \frac{m_B}{2}\left(1-\frac{m_\ell^2}{m_B^2},0,0,\,-1+\frac{m_\ell^2}{m_B^2}\right) \approx \frac{m_B}{2}(1,0,0,\,-1) = \frac{m_B}{2}\bar{n}^\mu$$

$$\lambda_\ell = \frac{m_\ell}{m_b} \ll 1$$

- Initial state quark are bound in the meson, with residual momenta of $\mathcal{O}(\Lambda_{\rm QCD})$

$$p_b = m_b v + k_b\,, \quad p_q = k_q \qquad k_b, k_q \sim m_b(\lambda,\lambda,\lambda) \qquad \lambda = \frac{\Lambda_{\rm QCD}}{m_b}$$

- Exp. cut on real radiation introduces a third expansion parameter $\lambda_E = \dfrac{E_s}{m_b}$

Power counting and relevant modes

Expansion parameters:

$$\lambda = \frac{\Lambda_{\rm QCD}}{m_b}$$

$$\lambda_\ell = \frac{m_\mu}{m_b} \sim \lambda$$

$$\lambda_E = \frac{E_s}{m_b} \sim \lambda_\ell^2$$

Power counting and relevant modes

Expansion parameters:

$$\lambda = \frac{\Lambda_{\rm QCD}}{m_b}$$

$$\lambda_\ell = \frac{m_\mu}{m_b} \sim \lambda$$

$$\lambda_E = \frac{E_s}{m_b} \sim \lambda_\ell^2$$

- Relevant modes for **virtual QED corrections**:
  - hard $(1, 1, 1)$
  - hard-collinear $(\lambda, 1, \lambda^{\frac{1}{2}})$
  - soft $(\lambda, \lambda, \lambda)$
  - collinear $(\lambda_\ell^2, 1, \lambda_\ell)$
  - soft-collinear $\lambda(\lambda_\ell^2, 1, \lambda_\ell)$
- Relevant modes for **real QED corrections**:
  - ultra-soft $(\lambda_E, \lambda_E, \lambda_E)$
  - ultra-soft-collinear $\lambda_E(\lambda_\ell^2, 1, \lambda_\ell)$

$\mu \sim m_b$: from the Fermi Theory to HQET $\times$ SCET-1

$\mu \sim m_b$: from the Fermi Theory to HQET $\times$ SCET-1

- Below m_b, radiation is too soft to affect the b-quark momentum
  $\Rightarrow$ the right description is **HQET**

$\mu \sim m_b$: from the Fermi Theory to HQET $\times$ SCET-1

- Below m_b, radiation is too soft to affect the b-quark momentum
  $\Rightarrow$ the right description is **HQET**
- Remaining fields can have large momenta, but small invariant mass
  $\Rightarrow$ the right EFT is **SCET-1**

$\mu \sim m_b$: from the Fermi Theory to HQET × SCET-1

- Below m_b, radiation is too soft to affect the b-quark momentum
  ⇒ the right description is **HQET**

- Remaining fields can have large momenta, but small invariant mass
  ⇒ the right EFT is **SCET-1**

- In practice, the fields & power counting we need at this stage are:

$$\nu \to \chi^{(\nu)}_{\overline{hc}}$$
$$\ell \to \chi^{(\ell)}_{hc}$$
$$b \to h_v$$
$$q \to \chi^{(q)}_{hc},\ q_s$$

(+ photons, gluons)

$$\chi^{(\ell)}_{hc}, \chi^{(q)}_{hc}, \chi^{(\nu)}_{\overline{hc}} \sim \lambda^{1/2}$$
$$h_v,\ q_s \sim \lambda^{3/2}$$
$$\mathcal{A}^{\mu}_{hc\perp},\ \mathcal{G}^{\mu}_{hc\perp} \sim \lambda^{1/2}$$

photons

gluons

Construction of SCET-1 basis

Construction of SCET-1 basis

To keep in mind when building the basis:

Construction of SCET-1 basis

To keep in mind when building the basis:

- All operators stem from the Fermi Lagrangian
  $\Rightarrow$ massless fields (neutrino and spectator) have to be left-handed

Construction of SCET-1 basis

To keep in mind when building the basis:

- All operators stem from the Fermi Lagrangian
  $\Rightarrow$ massless fields (neutrino and spectator) have to be left-handed

- The leading contribution to the amplitude is of order λ^6
  $\Rightarrow$ need operators with SCET-2 power counting λ^6 and mass dim 6

Construction of SCET-1 basis

To keep in mind when building the basis:

- All operators stem from the Fermi Lagrangian
  $\Rightarrow$ massless fields (neutrino and spectator) have to be left-handed
- The leading contribution to the amplitude is of order λ^6
  $\Rightarrow$ need operators with SCET-2 power counting λ^6 and mass dim 6
- Only two irreducible Dirac structures can enter the leptonic current in SCET-1: $\bar{\chi}_{hc}^{(\ell)}\, \Gamma_\ell\, P_L\, \chi_{\overline{hc}}^{(\nu)}$ with $\Gamma_\ell = 1\,, \gamma_\mu^\perp$

Construction of SCET-1 basis

To keep in mind when building the basis:

- All operators stem from the Fermi Lagrangian
  $\Rightarrow$ massless fields (neutrino and spectator) have to be left-handed

- The leading contribution to the amplitude is of order λ^6
  $\Rightarrow$ need operators with SCET-2 power counting λ^6 and mass dim 6

- Only two irreducible Dirac structures can enter the leptonic current in SCET-1: $\bar{\chi}_{hc}^{(\ell)}\,\Gamma_\ell\,P_L\,\chi_{\overline{hc}}^{(\nu)}$ with $\Gamma_\ell = 1\,,\gamma_\mu^\perp$
  - for $\Gamma_\ell = 1$ the muon is right-handed $\Rightarrow$ the chirality flip gives a factor m_ℓ, which we include in the operator definition

Construction of SCET-1 basis

Given this, we have **five classes of 4-fermion operators**:

A. Operators with soft spectator
B. Operators with hard-collinear spectator
C. Operators with soft spectator + hard-collinear photon
D. Operators with hard-collinear spectator + hard-collinear photon
E. Operators hard-collinear spectator + two perp objects

Construction of SCET-1 basis

Given this, we have **five classes of 4-fermion operators**:

A. Operators with soft spectator
B. Operators with hard-collinear spectator
C. Operators with soft spectator + hard-collinear photon
D. Operators with hard-collinear spectator + hard-collinear photon
E. Operators hard-collinear spectator + two perp objects

Only these are needed at $\mathcal{O}(\alpha)$.

Construction of SCET-1 basis

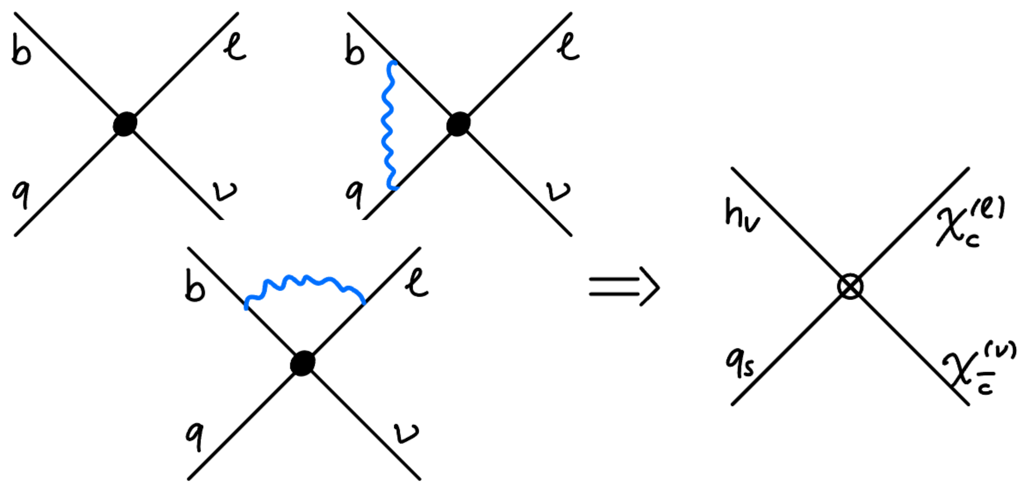
A. Operators with soft spectator

power counting

Construction of SCET-1 basis

A. Operators with soft spectator

- Obtained from hard matching:



power counting

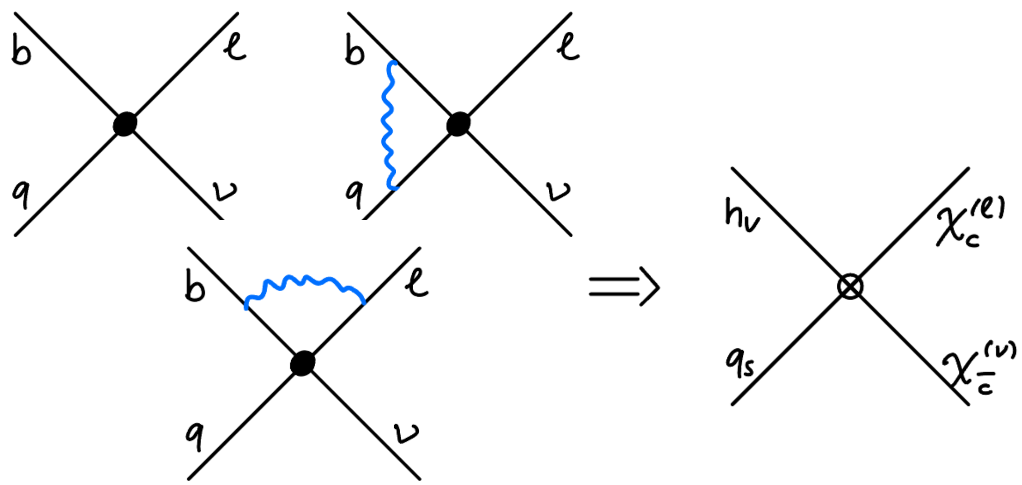
$$\mathcal{O}_{A,1}^{(5)} = m_\ell \left(\bar q_s \frac{\bar{\not n}}{2} P_L h_v \right) \left(\bar\chi_{hc}^{(\ell)} \frac{1}{i\bar n \cdot \overleftarrow{\partial}} P_L \chi_{\overline{hc}}^{(\nu)} \right)$$

$$\mathcal{O}_{A,2}^{(5)} = m_\ell \left(\bar q_s \frac{\not n}{2} P_L h_v \right) \left(\bar\chi_{hc}^{(\ell)} \frac{1}{i n \cdot \overrightarrow{\partial}} P_L \chi_{\overline{hc}}^{(\nu)} \right)$$

Construction of SCET-1 basis

A. Operators with soft spectator

- Obtained from hard matching:



power counting

$$\mathcal{O}_{A,1}^{(5)} = m_\ell \left(\bar{q}_s \frac{\bar{\not{n}}}{2} P_L h_v \right) \left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} P_L \chi_{\overline{hc}}^{(\nu)} \right)$$

$$\mathcal{O}_{A,2}^{(5)} = m_\ell \left(\bar{q}_s \frac{\not{n}}{2} P_L h_v \right) \left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{in\cdot\overrightarrow{\partial}} P_L \chi_{\overline{hc}}^{(\nu)} \right)$$

- Only the first contributes at tree level:

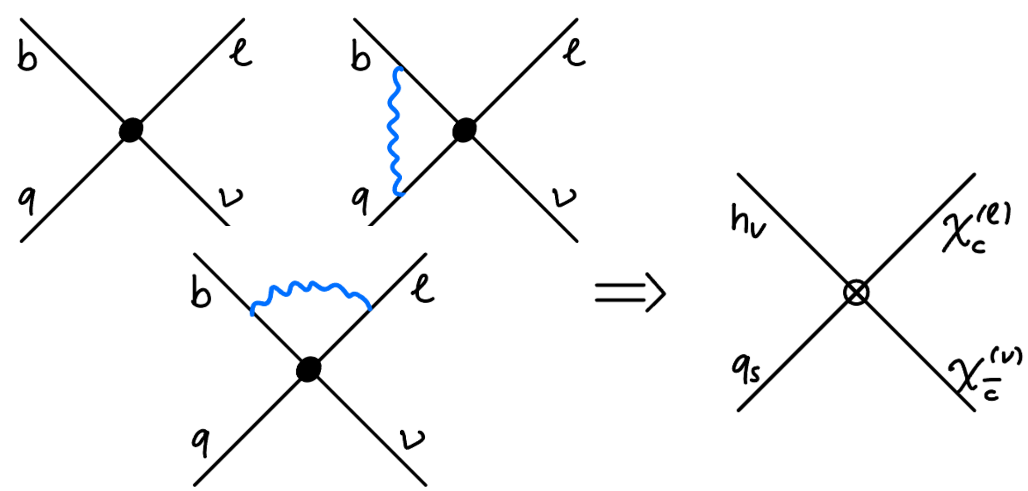
$$i\mathcal{A}_{\text{tree}} = -\frac{4G_F}{\sqrt{2}} V_{ub} \left[(\bar{v}_q \gamma_\perp^\mu P_L u_b)(\bar{u}_\ell \gamma_{\perp\mu} P_L v_\nu) + \frac{2m_\ell}{m_B} \left(\bar{v}_q \frac{\bar{\not{n}}}{2} P_L u_b \right) (\bar{u}_\ell P_L v_\nu) \right]$$

lepton EOM: $$\bar{u}_\ell \frac{\not{n}}{2} = \frac{m_\ell}{m_B} \bar{u}_\ell - \frac{2m_\ell^2}{m_B^2} \bar{u}_\ell \frac{\bar{\not{n}}}{2}\,, \qquad \frac{\bar{\not{n}}}{2} v_\nu = 0$$

Construction of SCET-1 basis

A. Operators with soft spectator

- Obtained from hard matching:



power counting

$$\mathcal{O}_{A,1}^{(5)} = m_\ell \left(\bar{q}_s \frac{\bar{\not{n}}}{2} P_L h_v\right) \left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} P_L \chi_{\overline{hc}}^{(\nu)}\right)$$

$$\mathcal{O}_{A,2}^{(5)} = m_\ell \left(\bar{q}_s \frac{\not{n}}{2} P_L h_v\right) \left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{in\cdot\overrightarrow{\partial}} P_L \chi_{\overline{hc}}^{(\nu)}\right)$$

- Only the first contributes at tree level:

no projection on B meson

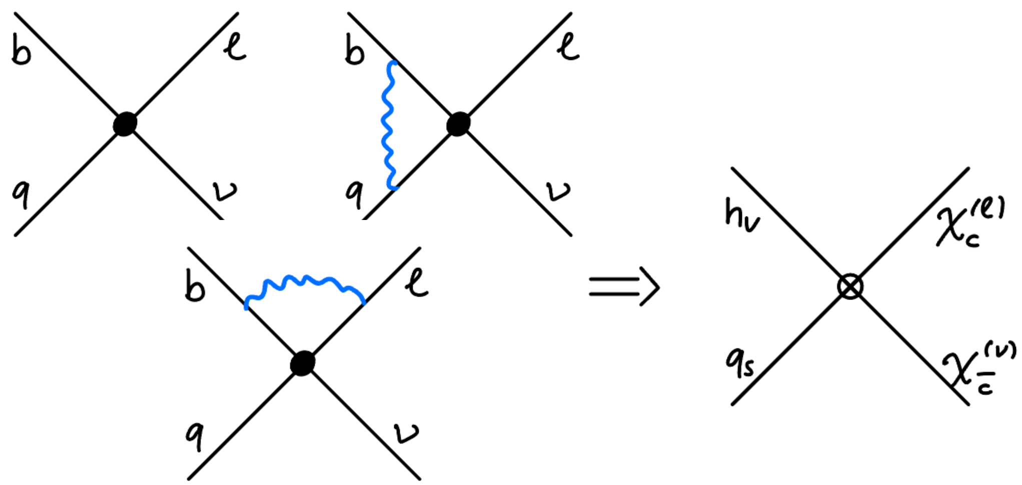
$$i\mathcal{A}_{\text{tree}} = -\frac{4G_F}{\sqrt{2}} V_{ub} \left[\cancel{(\bar{v}_q \gamma_\perp^\mu P_L u_b)(\bar{u}_\ell \gamma_{\perp\mu} P_L v_\nu)} + \frac{2m_\ell}{m_B}\left(\bar{v}_q \frac{\bar{\not{n}}}{2} P_L u_b\right)(\bar{u}_\ell P_L v_\nu)\right]$$

lepton EOM: $$\bar{u}_\ell \frac{\not{n}}{2} = \frac{m_\ell}{m_B}\bar{u}_\ell - \frac{2m_\ell^2}{m_B^2}\bar{u}_\ell \frac{\bar{\not{n}}}{2}, \qquad \frac{\bar{\not{n}}}{2} v_\nu = 0$$

Construction of SCET-1 basis

A. Operators with soft spectator

- Obtained from hard matching:



power counting

$$\boxed{\mathcal{O}^{(5)}_{A,1} = m_\ell \left(\bar q_s \frac{\not{\bar n}}{2} P_L h_v\right)\left(\bar\chi^{(\ell)}_{hc} \frac{1}{i\bar n\cdot\overleftarrow{\partial}} P_L \chi^{(\nu)}_{\overline{hc}}\right)}$$

$$\mathcal{O}^{(5)}_{A,2} = m_\ell \left(\bar q_s \frac{\not{\bar n}}{2} P_L h_v\right)\left(\bar\chi^{(\ell)}_{hc} \frac{1}{i\bar n\cdot\overrightarrow{\partial}} P_L \chi^{(\nu)}_{\overline{hc}}\right)$$

- Only the first contributes at tree level:

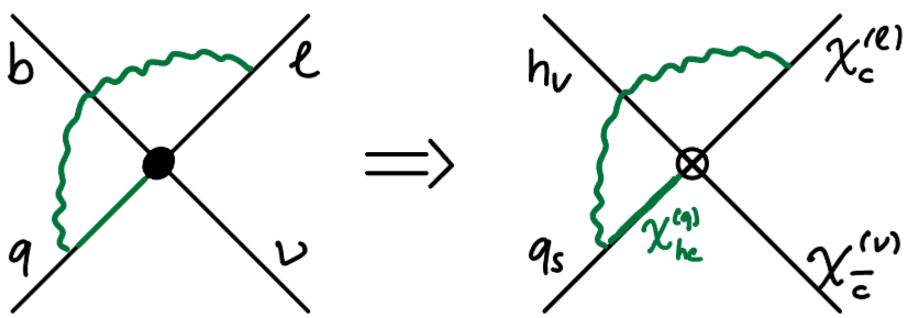
no projection on B meson

$$i\mathcal{A}_{\text{tree}} = -\frac{4G_F}{\sqrt{2}} V_{ub}\left[\cancel{(\bar v_q \gamma^\mu_\perp P_L u_b)(\bar u_\ell \gamma_{\perp\mu} P_L v_\nu)} + \boxed{\frac{2m_\ell}{m_B}\left(\bar v_q \frac{\not{\bar n}}{2} P_L u_b\right)(\bar u_\ell P_L v_\nu)}\right]$$

lepton EOM: $\bar u_\ell \frac{\not n}{2} = \frac{m_\ell}{m_B}\bar u_\ell - \frac{2m_\ell^2}{m_B^2}\bar u_\ell \frac{\not{\bar n}}{2}\,, \qquad \frac{\not{\bar n}}{2} v_\nu = 0$

Construction of SCET-1 basis

B. Operators with hard-collinear spectator



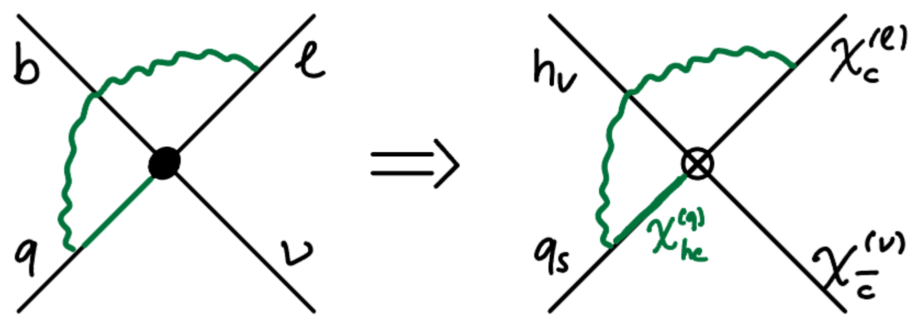
$$\mathcal{O}_{B,1}^{(7/2)} = \left(\bar{\chi}_{hc}^{(q)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} i\overleftarrow{\not{D}}_{\perp s} \frac{\not{\bar{n}}}{2} \gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_{hc}^{(\ell)} \gamma_{\perp\mu} P_L \chi_{\bar{C}}^{(\nu)}\right),$$

$$\mathcal{O}_{B,2}^{(7/2)} = \left(\bar{\chi}_{hc}^{(q)} \frac{\not{\bar{n}}}{2} P_L h_v\right)\left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} i\overleftarrow{\not{D}}_{\perp s} P_L \chi_{\overline{hc}}^{(\nu)}\right),$$

$$\mathcal{O}_{B}^{(4)} = m_\ell \left(\bar{\chi}_{hc}^{(q)} \frac{\not{\bar{n}}}{2} P_L h_v\right)\left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} P_L \chi_{\overline{hc}}^{\nu}\right).$$

Construction of SCET-1 basis

B. Operators with hard-collinear spectator

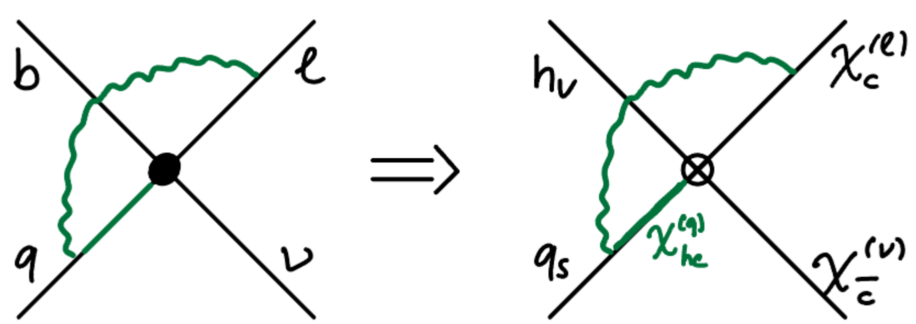


$$
\begin{aligned}
\mathcal{O}_{B,1}^{(7/2)} &= \left(\bar{\chi}_{hc}^{(q)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} i\overleftarrow{\not{D}}_{\perp s} \frac{\not{\bar{n}}}{2}\gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_{hc}^{(\ell)}\gamma_{\perp\mu} P_L \chi_{\bar{C}}^{(\nu)}\right), \\
\mathcal{O}_{B,2}^{(7/2)} &= \left(\bar{\chi}_{hc}^{(q)} \frac{\not{\bar{n}}}{2} P_L h_v\right)\left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} i\overleftarrow{\not{D}}_{\perp s} P_L \chi_{\overline{hc}}^{(\nu)}\right), \\
\mathcal{O}_{B}^{(4)} &= m_\ell \left(\bar{\chi}_{hc}^{(q)} \frac{\not{\bar{n}}}{2} P_L h_v\right)\left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} P_L \chi_{\overline{hc}}^{\nu}\right).
\end{aligned}
$$

- Their one-loop matrix elements reproduce the hard-collinear loops in the Fermi theory

Construction of SCET-1 basis

B. Operators with hard-collinear spectator



$$\mathcal{O}_{B,1}^{(7/2)} = \left(\bar{\chi}_{hc}^{(q)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} i\overleftarrow{\not{D}}_{\perp s} \frac{\not{\bar{n}}}{2} \gamma_\perp^\mu P_L h_v\right) \left(\bar{\chi}_{hc}^{(\ell)} \gamma_{\perp\mu} P_L \chi_{\bar{C}}^{(\nu)}\right),$$

$$\mathcal{O}_{B,2}^{(7/2)} = \left(\bar{\chi}_{hc}^{(q)} \frac{\not{\bar{n}}}{2} P_L h_v\right) \left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} i\overleftarrow{\not{D}}_{\perp s} P_L \chi_{\overline{hc}}^{(\nu)}\right),$$

$$\mathcal{O}_{B}^{(4)} = m_\ell \left(\bar{\chi}_{hc}^{(q)} \frac{\not{\bar{n}}}{2} P_L h_v\right) \left(\bar{\chi}_{hc}^{(\ell)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}} P_L \chi_{\overline{hc}}^{\nu}\right).$$

- Their one-loop matrix elements reproduce the hard-collinear loops in the Fermi theory
- B operators are power-enhanced with respect to A ones, but need one insertion of the (power-suppressed) soft-collinear interactions
  $\Rightarrow$ they contribute at the same order

Construction of SCET-1 basis

C. Operators with with soft spectator + hard-collinear photon

Construction of SCET-1 basis

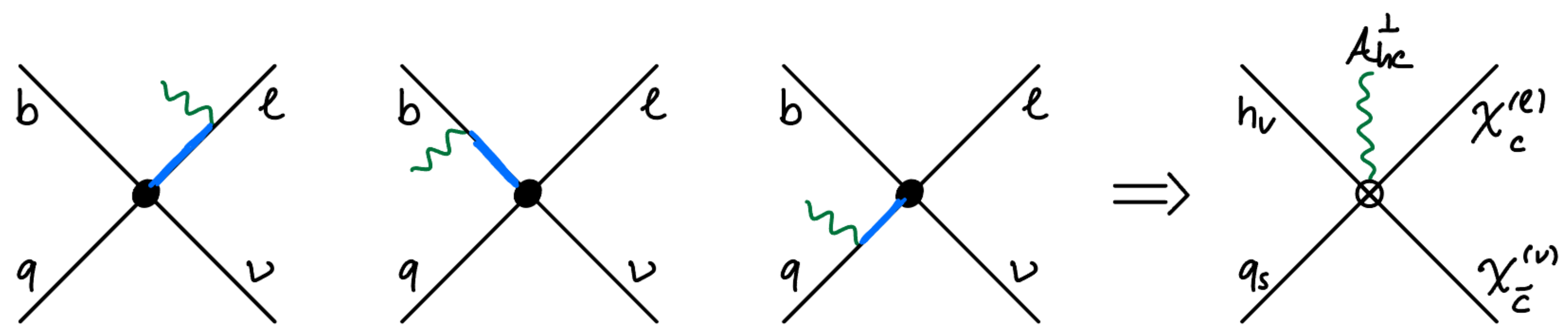
C. Operators with with soft spectator + hard-collinear photon

- Arise from hard-collinear emission from muon, b or spectator quark

Construction of SCET-1 basis

C. Operators with with soft spectator + hard-collinear photon

- Arise from hard-collinear emission from muon, b or spectator quark



$$\mathcal{O}_{C,1}^{(5)} = \frac{1}{in \cdot \partial_{hc}} \left(\bar{q}_s \frac{\not{\bar{n}}}{2} \gamma_\perp^\mu \mathcal{A}_{hc}^{(b)\perp} P_L h_v \right) \left(\bar{\chi}_{hc}^{(\ell)} \gamma_\mu^\perp P_L \chi_{\overline{hc}}^{(\nu)} \right)$$

$$\mathcal{O}_{C,2}^{(5)} = \frac{1}{i\bar{n} \cdot \partial_{hc}} \left(\bar{q}_s \frac{\not{n}}{2} P_L h_v \right) \left(\bar{\chi}_{hc}^{(\ell)} \mathcal{A}_{hc,\perp}^{(\ell)} P_L \chi_{\overline{hc}}^{(\nu)} \right) ,$$

$$\mathcal{O}_{C,3}^{(5)} = \left(\bar{q}_s \mathcal{A}_{hc,\perp}^{(u)} \frac{1}{i\bar{n} \cdot \overleftarrow{\partial}_{hc}} \frac{\not{n}}{2} \gamma_\perp^\mu P_L h_v \right) \left(\bar{\chi}_{hc}^{(\ell)} \gamma_\mu^\perp P_L \chi_{\overline{hc}}^{(\nu)} \right)$$

- Moving to SCET-2, these reproduce the collinear loops of the Fermi theory

$\mu \sim \sqrt{m_b \Lambda}$: from SCET-1 to SCET-2

$\mu \sim \sqrt{m_b \Lambda}$: from SCET-1 to SCET-2

- At $\mu \sim \sqrt{m_b \Lambda}$, integrate out hard-collinear modes and lower virtuality
  $\Rightarrow$ now collinear and soft modes live at the same scale: **SCET-2**

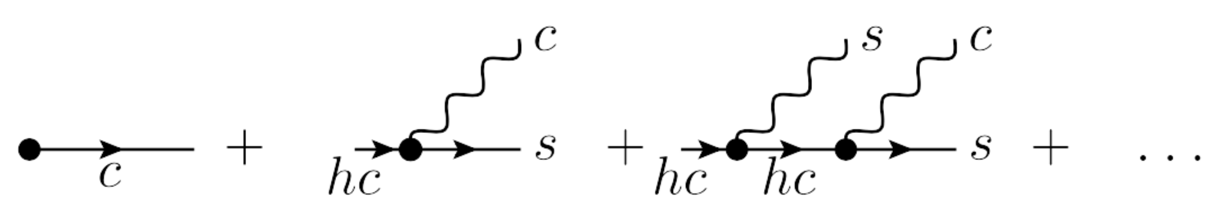
$\mu \sim \sqrt{m_b \Lambda}$: from SCET-1 to SCET-2

- At $\mu \sim \sqrt{m_b \Lambda}$, integrate out hard-collinear modes and lower virtuality
  $\Rightarrow$ now collinear and soft modes live at the same scale: **SCET-2**

$$p_c \sim (1, \lambda^2, \lambda)\,, \qquad p_s \sim (\lambda, \lambda, \lambda)\,, \qquad p_c^2 \sim p_s^2 \sim \mathcal{O}\left(\lambda^2\right)$$

- When integrating out hard-collinear modes, intermediate propagators introduce **non-local operators**:

$$\psi_{hc} \quad \rightarrow \quad \psi_c \quad + \quad \psi_c \cdot \psi_s \quad + \quad \psi_c \cdot \psi_s^2 + \ldots$$



$$\frac{1}{n \cdot \partial} q_s, \quad \left(\frac{1}{n \cdot \partial} \mathcal{A}^{\mu}_{\perp s}\right) \left(\frac{1}{n \cdot \partial} q_s\right), \quad \ldots$$

- $\Rightarrow$ now contain more fields, but are of the same order!

(Not) overcoming the chiral suppression

$$\left\langle 0 \left| \left(\frac{1}{n \cdot \partial} \overline{q}_s \right) \ldots h_v \right| B \right\rangle \sim \frac{1}{\lambda_B} \sim \mathcal{O}\left(\Lambda_{\mathrm{QCD}}^{-1} \right)$$

(Not) overcoming the chiral suppression

- Inverse derivative operators can probe the meson structure, and possibly overcome the chiral suppression

$$\left\langle 0 \left| \left(\frac{1}{n \cdot \partial} \overline{q}_s \right) \ldots h_v \right| B \right\rangle \sim \frac{1}{\lambda_B} \sim \mathcal{O}\left(\Lambda_{\rm QCD}^{-1} \right)$$

(Not) overcoming the chiral suppression

- Inverse derivative operators can probe the meson structure, and possibly overcome the chiral suppression

$$\left\langle 0 \left| \left(\frac{1}{n \cdot \partial} \bar{q}_s \right) \ldots h_v \right| B \right\rangle \sim \frac{1}{\lambda_B} \sim \mathcal{O}\left(\Lambda_{\mathrm{QCD}}^{-1} \right)$$

- Happens for $B_s \to \mu^+ \mu^-$, but not for $B \to \mu \bar{\nu}_\mu$ [Beneke, Bobeth, Szafron 2017, 2019]

(Not) overcoming the chiral suppression

- Inverse derivative operators can probe the meson structure, and possibly overcome the chiral suppression

$$\left\langle 0 \left| \left(\frac{1}{n\cdot\partial}\bar{q}_s\right) \dots h_v \right| B \right\rangle \sim \frac{1}{\lambda_B} \sim \mathcal{O}\left(\Lambda_{\rm QCD}^{-1}\right)$$

- Happens for $B_s \to \mu^+\mu^-$, but not for $B \to \mu\bar{\nu}_\mu$ [Beneke, Bobeth, Szafron 2017, 2019]
- For left-handed currents, these contributions come with evanescent Dirac structures:

(Not) overcoming the chiral suppression

- Inverse derivative operators can probe the meson structure, and possibly overcome the chiral suppression

$$\left\langle 0 \left| \left(\frac{1}{n \cdot \partial} \bar{q}_s \right) \ldots h_v \right| B \right\rangle \sim \frac{1}{\lambda_B} \sim \mathcal{O}\left(\Lambda_{\rm QCD}^{-1}\right)$$

- Happens for $B_s \to \mu^+\mu^-$, but not for $B \to \mu\bar{\nu}_\mu$ [Beneke, Bobeth, Szafron 2017, 2019]

- For left-handed currents, these contributions come with evanescent Dirac structures:

$$\left(\bar{v}\frac{\not{n}}{2}\gamma_\perp^\mu\gamma_\perp^\nu P_L u\right)_h \left(\bar{u}\gamma_\mu^\perp\gamma_\nu^\perp \left[\frac{v - a\gamma_5}{2}\right] v\right)_\ell = 2(v-a)\left(\bar{v}\frac{\not{n}}{2}P_L u\right)_h \left(\bar{u}P_R v\right)_\ell + \mathcal{O}(\epsilon)$$

$\Rightarrow$ structure-dependent contributions to $B \to \mu\bar{\nu}_\mu$ carry the same suppression as the tree level result!

SCET-2 basis

$$\mathcal{Q}_{A,1} = \frac{m_\ell}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s\frac{\not{\bar{n}}}{2}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{A,2} = \frac{m_\ell}{in\cdot\partial_c}\left(\bar{q}_s\frac{\not{n}}{2}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,1} = \left(\bar{q}_s\frac{1}{in\cdot\overleftarrow{\partial}_s}\mathcal{A}_c^{(u)\perp}\frac{i\overleftarrow{\not{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c}\frac{\not{n}\not{\bar{n}}}{4}\gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\gamma_\mu^\perp P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,2} = \left(\bar{q}_s\frac{1}{in\cdot\overleftarrow{\partial}_s}\mathcal{A}_c^{(u)\perp}\frac{\not{n}\not{\bar{n}}}{4}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\frac{i\overleftarrow{\not{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,3} = \left(\bar{q}_s\frac{1}{in\cdot\overleftarrow{\partial}_s}\mathcal{A}_c^{(u)\perp}\frac{\not{n}\not{\bar{n}}}{4}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\frac{m_\ell}{i\bar{n}\cdot\overleftarrow{\partial}_c}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,4} = \frac{1}{in\cdot\partial_c}\left(\bar{q}_s\frac{\not{n}}{2}\gamma_\perp^\mu\gamma_\perp^\nu P_L h_v\right)(\bar{\chi}_c^{(\ell)}\gamma_\mu^\perp\mathcal{A}_{c\nu}^{(b)\perp}P_L\chi_{\bar{c}}^{(\nu)})$$

$$\mathcal{Q}_{B,5} = \frac{1}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s\frac{\not{\bar{n}}}{2}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\mathcal{A}_c^{(\ell)\perp}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,6} = \left(\bar{q}_s\mathcal{A}_{c,\perp}^{(u)}\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}_c}\frac{\not{\bar{n}}}{2}\gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\gamma_\mu^\perp P_L\chi_{\bar{c}}^{(\nu)}\right)$$

SCET-2 basis

$$\mathcal{Q}_{A,1} = \frac{m_\ell}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s \frac{\slashed{\bar{n}}}{2} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{A,2} = \frac{m_\ell}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s \frac{\slashed{\bar{n}}}{2} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

descend directly from A-type operators in SCET-1

$$\mathcal{Q}_{B,1} = \left(\bar{q}_s \frac{1}{i n\cdot\overleftarrow{\partial}_s} \mathcal{A}_c^{(u)\perp} \frac{i\overleftarrow{\slashed{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c} \frac{\slashed{n}\slashed{\bar{n}}}{4} \gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \gamma_\mu^\perp P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,2} = \left(\bar{q}_s \frac{1}{i n\cdot\overleftarrow{\partial}_s} \mathcal{A}_c^{(u)\perp} \frac{\slashed{n}\slashed{\bar{n}}}{4} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \frac{i\overleftarrow{\slashed{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,3} = \left(\bar{q}_s \frac{1}{i n\cdot\overleftarrow{\partial}_s} \mathcal{A}_c^{(u)\perp} \frac{\slashed{n}\slashed{\bar{n}}}{4} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \frac{m_\ell}{i\bar{n}\cdot\overleftarrow{\partial}_c} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,4} = \frac{1}{i n\cdot\partial_c}\left(\bar{q}_s \frac{\slashed{n}}{2} \gamma_\perp^\mu \gamma_\perp^\nu P_L h_v\right)(\bar{\chi}_c^{(\ell)} \gamma_\mu^\perp \mathcal{A}_{c\nu}^{(b)\perp} P_L \chi_{\bar{c}}^{(\nu)})$$

$$\mathcal{Q}_{B,5} = \frac{1}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s \frac{\slashed{\bar{n}}}{2} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \mathcal{A}_c^{(\ell)\perp} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,6} = \left(\bar{q}_s \mathcal{A}_{c,\perp}^{(u)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}_c} \frac{\slashed{\bar{n}}}{2} \gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \gamma_\mu^\perp P_L \chi_{\bar{c}}^{(\nu)}\right)$$

SCET-2 basis

$$\mathcal{Q}_{A,1} = \frac{m_\ell}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s \frac{\bar{\not{n}}}{2} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{A,2} = \frac{m_\ell}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s \frac{\not{n}}{2} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

descend directly from A-type operators in SCET-1

$$\mathcal{Q}_{B,1} = \left(\bar{q}_s \frac{1}{in\cdot\overleftarrow{\partial}_s} \mathcal{A}_c^{(u)\perp} \frac{i\overleftarrow{\not{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c} \frac{\not{n}\bar{\not{n}}}{4} \gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \gamma_\mu^\perp P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,2} = \left(\bar{q}_s \frac{1}{in\cdot\overleftarrow{\partial}_s} \mathcal{A}_c^{(u)\perp} \frac{\not{n}\bar{\not{n}}}{4} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \frac{i\overleftarrow{\not{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,3} = \left(\bar{q}_s \frac{1}{in\cdot\overleftarrow{\partial}_s} \mathcal{A}_c^{(u)\perp} \frac{\not{n}\bar{\not{n}}}{4} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \frac{m_\ell}{i\bar{n}\cdot\overleftarrow{\partial}_c} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,4} = \frac{1}{in\cdot\partial_c}\left(\bar{q}_s \frac{\not{n}}{2} \gamma_\perp^\mu \gamma_\perp^\nu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \gamma_\mu^\perp \mathcal{A}_{c\nu}^{(b)\perp} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,5} = \frac{1}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s \frac{\bar{\not{n}}}{2} P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \mathcal{A}_c^{(\ell)\perp} P_L \chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,6} = \left(\bar{q}_s \mathcal{A}_{c,\perp}^{(u)} \frac{1}{i\bar{n}\cdot\overleftarrow{\partial}_c} \frac{\bar{\not{n}}}{2} \gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)} \gamma_\mu^\perp P_L \chi_{\bar{c}}^{(\nu)}\right)$$

descend directly from C-type operators in SCET-1

SCET-2 basis

$$\mathcal{Q}_{A,1} = \frac{m_\ell}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s\frac{\slashed{\bar{n}}}{2}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{A,2} = \frac{m_\ell}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s\frac{\slashed{n}}{2}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

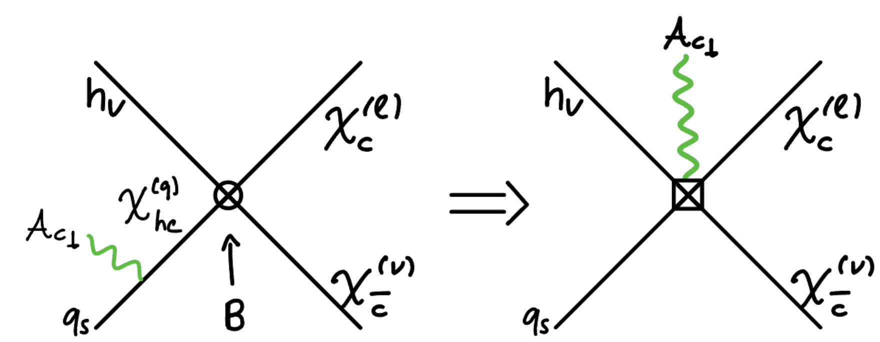
descend directly from A-type operators in SCET-1

$$\mathcal{Q}_{B,1} = \left(\bar{q}_s\frac{1}{in\cdot\overleftarrow{\partial}_s}\mathcal{A}_c^{(u)\perp}\frac{i\overleftarrow{\slashed{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c}\frac{\slashed{n}\slashed{\bar{n}}}{4}\gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\gamma_\mu^\perp P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,2} = \left(\bar{q}_s\frac{1}{in\cdot\overleftarrow{\partial}_s}\mathcal{A}_c^{(u)\perp}\frac{\slashed{n}\slashed{\bar{n}}}{4}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\frac{i\overleftarrow{\slashed{D}}_{\perp,s}}{i\bar{n}\cdot\overleftarrow{\partial}_c}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,3} = \left(\bar{q}_s\frac{1}{in\cdot\overleftarrow{\partial}_s}\mathcal{A}_c^{(u)\perp}\frac{\slashed{n}\slashed{\bar{n}}}{4}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\frac{m_\ell}{i\bar{n}\cdot\overleftarrow{\partial}_c}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

stem from matching B-type SCET-1 operators to SCET-2 at tree level



$$\mathcal{Q}_{B,4} = \frac{1}{in\cdot\partial_c}\left(\bar{q}_s\frac{\slashed{\bar{n}}}{2}\gamma_\perp^\mu\gamma_\perp^\nu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\gamma_\mu^\perp\mathcal{A}_{c\nu}^{(b)\perp}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,5} = \frac{1}{i\bar{n}\cdot\partial_c}\left(\bar{q}_s\frac{\slashed{\bar{n}}}{2}P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\mathcal{A}_c^{(\ell)\perp}P_L\chi_{\bar{c}}^{(\nu)}\right)$$

$$\mathcal{Q}_{B,6} = \left(\bar{q}_s\mathcal{A}_{c,\perp}^{(u)}\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}_c}\frac{\slashed{\bar{n}}}{2}\gamma_\perp^\mu P_L h_v\right)\left(\bar{\chi}_c^{(\ell)}\gamma_\mu^\perp P_L\chi_{\bar{c}}^{(\nu)}\right)$$

descend directly from C-type operators in SCET-1

Factorization formula (virtual corrections)

convolution

$$\mathscr{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = \sum_j H_j\ S_j\ K_j\ + \sum_i H_i \otimes J_j \otimes S_i \otimes K_i\,,$$

Factorization formula (virtual corrections)

convolution

$$\mathscr{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = \sum_j H_j\, S_j\, K_j + \sum_i H_i \otimes J_j \otimes S_i \otimes K_i\,,$$

- hard function: matching corrections at $\mu \sim m_b$

Factorization formula (virtual corrections)

convolution

$$\mathscr{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = \sum_j H_j\ S_j\ K_j\ + \sum_i H_i \otimes J_j \otimes\ S_i \otimes K_i\,,$$

- hard function: matching corrections at $\mu \sim m_b$
- hard-collinear function: matching corrections at $\mu \sim (m_b \Lambda_{\text{QCD}})^{1/2}$

Factorization formula (virtual corrections)

convolution

$$\mathscr{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = \sum_j H_j \; S_j \; K_j + \sum_i H_i \otimes J_j \otimes S_i \otimes K_i \, ,$$

- hard function: matching corrections at $\mu \sim m_b$
- hard-collinear function: matching corrections at $\mu \sim (m_b \Lambda_{\text{QCD}})^{1/2}$
- collinear function: leptonic matrix elements, $\mu \sim m_\mu$

Factorization formula (virtual corrections)

$$\mathscr{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = \sum_j H_j\, S_j\, K_j + \sum_i H_i \otimes J_j \overset{\text{convolution}}{\otimes} S_i \otimes K_i \,,$$

- hard function: matching corrections at $\mu \sim m_b$
- hard-collinear function: matching corrections at $\mu \sim (m_b \Lambda_{\text{QCD}})^{1/2}$
- collinear function: leptonic matrix elements, $\mu \sim m_\mu$
- soft (& soft-collinear) function: HQET B meson matrix elements

Factorization formula (virtual corrections)

convolution

$$\mathscr{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = \sum_j \underbrace{H_j\ S_j\ K_j}_{\text{SCET-1 operators with soft spectator (A-type)}} + \sum_i H_i \otimes J_j \otimes S_i \otimes K_i\,,$$

- hard function: matching corrections at $\mu \sim m_b$
- hard-collinear function: matching corrections at $\mu \sim (m_b \Lambda_{\text{QCD}})^{1/2}$
- collinear function: leptonic matrix elements, $\mu \sim m_\mu$
- soft (& soft-collinear) function: HQET B meson matrix elements

Factorization formula (virtual corrections)

$$\mathscr{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = \sum_j \underbrace{H_j\ S_j\ K_j}_{\text{SCET-1 operators with soft spectator (A-type)}} + \sum_i \underbrace{H_i \otimes J_j \overset{\text{convolution}}{\otimes} S_i \otimes K_i}_{\text{SCET-1 operators with hc spectator (B-type)}},$$

- hard function: matching corrections at $\mu \sim m_b$
- hard-collinear function: matching corrections at $\mu \sim (m_b\Lambda_{\text{QCD}})^{1/2}$
- collinear function: leptonic matrix elements, $\mu \sim m_\mu$
- soft (& soft-collinear) function: HQET B meson matrix elements

Factorization formula (virtual corrections)

- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$
$$\left[H_A(m_b) S_A + \int d\omega \int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x) S_B(\omega)\right] \qquad \omega = n \cdot p_u$$

Factorization formula (virtual corrections)

- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$
$$\Big[\underbrace{H_A(m_b) S_A}_{\substack{\text{SCET-1 operators with soft} \\ \text{spectator (A-type)}}} + \int d\omega \int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x) S_B(\omega) \Big] \qquad \omega = n\cdot p_u$$

Factorization formula (virtual corrections)

- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$

$$\Big[\underbrace{H_A(m_b) S_A}_{} + \int d\omega \int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x) S_B(\omega)\Big] \qquad \omega = n \cdot p_u$$

SCET-1 operators with soft spectator (A-type)

$$S_A = \langle O_A \rangle$$

$$O_A = \bar{n}_\mu \bar{u}_s \gamma^\mu P_L h_v S^\dagger_{v_\ell}$$

Factorization formula (virtual corrections)

- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$

$$\Bigg[\underbrace{H_A(m_b) S_A}_{\text{SCET-1 operators with soft spectator (A-type)}} + \int d\omega \underbrace{\int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x) S_B(\omega)}_{\text{SCET-1 operators with hc spectator (B-type)}}\Bigg] \qquad \omega = n\cdot p_u$$

$$S_A = \langle O_A \rangle$$

$$O_A = \bar{n}_\mu\, \bar{u}_s \gamma^\mu P_L h_v\, S^\dagger_{v_\ell}$$

Factorization formula (virtual corrections)

- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$

$$\Big[\underbrace{H_A(m_b) S_A}_{\text{SCET-1 operators with soft spectator (A-type)}} + \int d\omega \int_0^1 dx\, \underbrace{H_B(m_b,x)\, J_B(m_b\omega,x) S_B(\omega)}_{\text{SCET-1 operators with hc spectator (B-type)}}\Big]$$

$\omega = n \cdot p_u$

SCET-1 operators with soft spectator (A-type)

SCET-1 operators with hc spectator (B-type)

$$S_A = \langle O_A \rangle$$

$$O_A = \bar{n}_\mu\, \bar{u}_s \gamma^\mu P_L h_v\, S^\dagger_{v_\ell}$$

$$S_B = \langle O_B \rangle$$

$$O_B(\omega) = \bar{n}_\mu \int \frac{dt}{2\pi} e^{i\omega t} \bar{u}_s(tn)[tn,0]\gamma^\mu P_L h_v(0)\, S^\dagger_{v_\ell}(0)$$

Factorization formula (virtual corrections)

- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$
\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)
$$
$$
\left[H_A(m_b) S_A + \int d\omega \int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x) S_B(\omega) \right]
$$

$\omega = n \cdot p_u$

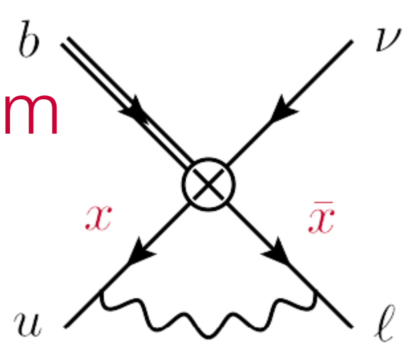
Factorization formula (virtual corrections)

- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub}\, \frac{m_\ell}{m_b}\, K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$
$$\left[H_A(m_b) S_A + \int d\omega \int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x) S_B(\omega) \right] \qquad \omega = n\cdot p_u$$

- Focus on second term:
  - Hard and jet function share a variable x = collinear momentum momentum fraction carried by the spectator
  - They scale as $H_B \sim x^{-\epsilon}, J_B \sim x^{-1-\epsilon}$

    $\Rightarrow$ $H_B \otimes J_B$ has an endpoint divergence in $x = 0$!



Factorization formula (virtual corrections)

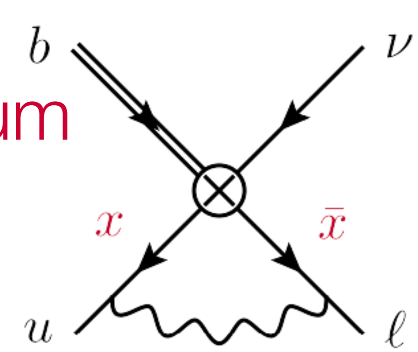
- Neglecting $\mathcal{O}(\alpha\alpha_s)$ corrections, two main contributions

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$
$$\left[H_A(m_b) S_A + \int d\omega \int_0^1 dx\, H_B(m_b, \boxed{x})\, J_B(m_b\omega, \boxed{x}) S_B(\omega) \right]$$

$\omega = n \cdot p_u$

- Focus on second term:
  - Hard and jet function share a variable x = collinear momentum momentum fraction carried by the spectator
  - They scale as $H_B \sim x^{-\epsilon}, J_B \sim x^{-1-\epsilon}$

    $\Rightarrow$ $H_B \otimes J_B$ has an endpoint divergence in $x = 0$!



- This cannot be removed with standard RG techniques, but is systematically treatable with **refactorization-based subtraction (RBS) scheme**

[Liu, MN 2019; Liu, Mecaj, MN, Wang 2020; Beneke et al. 2022; Liu, MN, Schnubel, Wang 2022]

Refactorization

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar\nu} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub}\, \frac{m_\ell}{m_b}\, K_A(m_\ell)\, \bar u(p_\ell) P_L v(p_\nu)$$
$$\left[H_A(m_b) S_A + \int d\omega \int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x) S_B(\omega) \right]$$

Refactorization

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$
$$\left[H_A(m_b) S_A + \int d\omega \boxed{\int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x)} S_B(\omega) \right]$$

- Start from the second term

Refactorization

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar\nu} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\,\bar u(p_\ell) P_L v(p_\nu)$$

$$\left[H_A(m_b) S_A + \int d\omega \boxed{\int_0^1 dx\, H_B(m_b,x)\, J_B(m_b\omega,x)} S_B(\omega)\right]$$

$$\boxed{\int_0^1 dx\, \left[H_B(m_b,x) J_B(m_b\omega,x) - \theta(\lambda - x)[[H_B(m_b,x)]][[J_B(m_b\omega,x)]]\right]}$$

$0 < \lambda < 1$ $\quad$ $[[f]] =$ singular part of f for $x \to 0$

- Start from the second term
- Remove the divergence from $H_B \otimes J_B$ with a plus subtraction

Refactorization

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub} \frac{m_\ell}{m_b} K_A(m_\ell)\, \bar{u}(p_\ell) P_L v(p_\nu)$$

$$\left[\boxed{H_A(m_b) S_A} + \int d\omega \boxed{\int_0^1 dx\, H_B(m_b, x)\, J_B(m_b\omega, x)}\, S_B(\omega)\right]$$

$$\boxed{H_A(m_b) S_A^{(\Lambda)}}$$

$$\boxed{\int_0^1 dx\, \left[H_B(m_b, x) J_B(m_b\omega, x) - \theta(\lambda - x)[[H_B(m_b, x)]][[J_B(m_b\omega, x)]]\right]}$$

$0 < \lambda < 1$ $\quad [[f]] =$ singular part of f for $x \to 0$

- Start from the second term
- Remove the divergence from $H_B \otimes J_B$ with a plus subtraction
- Add the subtraction term back, combining it with the other term in the factorization formula

Refactorization

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar\nu} = -\frac{4G_F}{\sqrt{2}} K_{\text{EW}}(\mu) V_{ub}\,\frac{m_\ell}{m_b}\,K_A(m_\ell)\,\bar u(p_\ell) P_L v(p_\nu)$$

$$\left[\boxed{H_A(m_b) S_A} + \int d\omega \boxed{\int_0^1 dx\, H_B(m_b,x)\, J_B(m_b\omega,x)} S_B(\omega)\right]$$

$\Lambda = \lambda\, m_b$

$$\boxed{H_A(m_b) S_A^{(\Lambda)}}$$

$$\boxed{\int_0^1 dx\, \left[H_B(m_b,x) J_B(m_b\omega,x) - \theta(\lambda - x)[[H_B(m_b,x)]][[J_B(m_b\omega,x)]]\right]}$$

$0 < \lambda < 1$

$[[f]] =$ singular part of f for $x \to 0$

- Start from the second term
- Remove the divergence from $H_B \otimes J_B$ with a plus subtraction
- Add the subtraction term back, combining it with the other term in the factorization formula

Decay constant

$$S_A^{(\Lambda)} = \langle 0 |\, O_A^{(\Lambda)} \,| B^-(v) \rangle = -\frac{i\sqrt{m_B}}{2}\, F(\mu,\Lambda)\ \langle 0 |\, S_{v_B}^{(B)} S_{v_\ell}^{(\ell)\dagger} \,| 0 \rangle$$

$$O_A^{(\Lambda)} = \bar{u}_s \not{\bar{n}} P_L h_{v_B}\, S_{v_\ell}^{(\ell)\dagger} \left[1 + Q_\ell Q_u \frac{\alpha}{2\pi} \frac{e^{\epsilon\gamma_E}\,\Gamma(\epsilon)}{\epsilon\,(1-\epsilon)} \int d\omega\, \phi_-(\omega) \left(\frac{\mu^2}{\omega\Lambda} \right)^{\epsilon} \right]$$

Decay constant

- The "new" soft function $S_A^{(\Lambda)}$ defines a renormalized **decay "constant"**:

$$S_A^{(\Lambda)} = \langle 0 |\, O_A^{(\Lambda)} \,| B^-(v) \rangle = -\frac{i\sqrt{m_B}}{2} F(\mu, \Lambda)\, \langle 0 |\, S_{v_B}^{(B)} S_{v_\ell}^{(\ell)\dagger} \,| 0 \rangle$$

$$O_A^{(\Lambda)} = \bar{u}_s \not{n} P_L h_{v_B} S_{v_\ell}^{(\ell)\dagger} \left[1 + Q_\ell Q_u \frac{\alpha}{2\pi} \frac{e^{\epsilon \gamma_E} \Gamma(\epsilon)}{\epsilon (1-\epsilon)} \int d\omega\, \phi_-(\omega) \left(\frac{\mu^2}{\omega \Lambda} \right)^{\epsilon} \right]$$

Decay constant

- The "new" soft function $S_A^{(\Lambda)}$ defines a renormalized **decay "constant"**:

$$S_A^{(\Lambda)} = \langle 0 |\, O_A^{(\Lambda)} \,| B^-(v)\rangle = -\frac{i\sqrt{m_B}}{2}\, F(\mu,\Lambda)\ \langle 0 |\, S_{v_B}^{(B)} S_{v_\ell}^{(\ell)\dagger} \,| 0\rangle$$

$$O_A^{(\Lambda)} = \bar{u}_s \not{\bar{n}} P_L h_{v_B} S_{v_\ell}^{(\ell)\dagger} \left[1 + Q_\ell Q_u \frac{\alpha}{2\pi} \frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)}{\epsilon(1-\epsilon)} \int d\omega\, \phi_-(\omega) \left(\frac{\mu^2}{\omega\Lambda}\right)^{\epsilon} \right]$$

- Evolution in μ and Λ well-defined and insensitive to IR regulators:

$$\frac{d\ln F}{d\ln\mu} = C_F \frac{3\alpha_s}{4\pi} - \frac{3\alpha}{4\pi}\left(Q_\ell^2 - Q_b^2 + \frac{2}{3} Q_\ell Q_u \ln\frac{\Lambda^2}{\mu^2} \right)$$

$$\frac{d\ln F}{d\ln\Lambda} = Q_\ell Q_u \frac{\alpha}{2\pi} \left[\int d\omega\, \phi_-(\omega) \ln\frac{\omega\Lambda}{\mu^2} - 1 + \ldots \right]$$

Decay constant

- The "new" soft function $S_A^{(\Lambda)}$ defines a renormalized **decay "constant"**:

$$S_A^{(\Lambda)} = \langle 0 |\, O_A^{(\Lambda)} \,| B^-(v) \rangle = -\frac{i\sqrt{m_B}}{2} F(\mu,\Lambda)\ \langle 0 |\, S_{v_B}^{(B)} S_{v_\ell}^{(\ell)\dagger} \,| 0 \rangle$$

$$O_A^{(\Lambda)} = \bar{u}_s \not{n} P_L h_{v_B} S_{v_\ell}^{(\ell)\dagger} \left[1 + Q_\ell Q_u \frac{\alpha}{2\pi} \frac{e^{\epsilon\gamma_E}\Gamma(\epsilon)}{\epsilon(1-\epsilon)} \int d\omega\, \phi_-(\omega) \left(\frac{\mu^2}{\omega\Lambda} \right)^{\epsilon} \right]$$

- Evolution in μ and Λ well-defined and insensitive to IR regulators:

$$\frac{d\ln F}{d\ln\mu} = C_F \frac{3\alpha_s}{4\pi} - \frac{3\alpha}{4\pi} \left(Q_\ell^2 - Q_b^2 + \frac{2}{3} Q_\ell Q_u \ln\frac{\Lambda^2}{\mu^2} \right)$$

$$\frac{d\ln F}{d\ln\Lambda} = Q_\ell Q_u \frac{\alpha}{2\pi} \left[\int d\omega\, \phi_-(\omega) \ln\frac{\omega\Lambda}{\mu^2} - 1 + \ldots \right]$$

- Related to lattice QCD result for the B decay constant:

$$\sqrt{m_B} f_B^{\mathrm{QCD}} = \left[1 - C_F \frac{\alpha_s(m_b)}{2\pi} \right] F(m_b, m_b) \Big|_{\alpha\to 0}$$

Virtual corrections to $B \to \mu\bar{\nu}$

Decay amplitude including virtual QED corrections at $\mathcal{O}(\alpha)$:

$$\mathcal{A}^{\text{virtual}}_{B\to\ell\bar{\nu}} = i\sqrt{2}G_F\,K_{\text{EW}}(\mu)\,V_{ub}\,\frac{m_\ell}{m_b}\,\sqrt{m_B}\,F(\mu,m_b,w)\,\bar{u}(p_\ell)P_L\,v(p_\nu)\Big[\mathcal{M}_{2p}(\mu)+\mathcal{M}_{3p}(\mu)\Big]$$

with:

$$\begin{aligned}\mathcal{M}_{2p}(\mu) &= 1+\frac{C_F\,\alpha_s}{4\pi}\left[\frac{3}{2}\ln\frac{m_b^2}{\mu^2}-2\right]\\ &\quad+\frac{\alpha}{4\pi}\left\{Q_b^2\left[\frac{3}{2}\ln\frac{m_b^2}{\mu^2}-2\right]-Q_\ell Q_b\left[\frac{1}{2}\ln^2\frac{m_b^2}{\mu^2}+2\ln\frac{m_b^2}{\mu^2}-3\ln\frac{m_\ell^2}{\mu^2}+1+\frac{5\pi^2}{12}\right]\right.\\ &\quad\left.+2Q_\ell Q_u\int_0^\infty d\omega\,\phi_-(\omega)\ln\frac{m_b\omega}{\mu^2}+Q_\ell^2\left[\frac{1}{\epsilon_{\text{IR}}}\left(\ln\frac{m_B^2}{m_\ell^2}-2\right)+\frac{1}{2}\ln^2\frac{m_\ell^2}{\mu^2}-\frac{1}{2}\ln\frac{m_\ell^2}{\mu^2}+2+\frac{5\pi^2}{12}\right]\right\}\\ \mathcal{M}_{3p}(\mu) &= \frac{\alpha}{\pi}\,Q_\ell Q_u\int_0^\infty d\omega\int_0^\infty d\omega_g\,\phi_{3g}(\omega,\omega_g)\left[\frac{1}{\omega_g}\ln\left(1+\frac{\omega_g}{\omega}\right)-\frac{1}{\omega+\omega_g}\right]\end{aligned}$$

$\Rightarrow$ significant hadronic uncertainties in $\mathcal{O}(\alpha)$ terms! [CC, König, Neubert 2022]

Virtual corrections to $B \to \mu\bar{\nu}$

Decay amplitude including virtual QED corrections at $\mathcal{O}(\alpha)$:

$$\mathcal{A}_{B\to\ell\bar{\nu}}^{\text{virtual}} = i\sqrt{2}G_F\,K_{\text{EW}}(\mu)\,V_{ub}\,\frac{m_\ell}{m_b}\,\sqrt{m_B}\,F(\mu,m_b,w)\;\bar{u}(p_\ell)P_L\,v(p_\nu)\Big[\mathcal{M}_{2p}(\mu)+\mathcal{M}_{3p}(\mu)\Big]$$

with:

$$\begin{aligned}\mathcal{M}_{2p}(\mu) &= 1+\frac{C_F\,\alpha_s}{4\pi}\left[\frac{3}{2}\ln\frac{m_b^2}{\mu^2}-2\right]\\ &\quad+\frac{\alpha}{4\pi}\Bigg\{Q_b^2\left[\frac{3}{2}\ln\frac{m_b^2}{\mu^2}-2\right]-Q_\ell Q_b\left[\frac{1}{2}\ln^2\frac{m_b^2}{\mu^2}+2\ln\frac{m_b^2}{\mu^2}-3\ln\frac{m_\ell^2}{\mu^2}+1+\frac{5\pi^2}{12}\right]\\ &\quad+2Q_\ell Q_u\int_0^\infty d\omega\,\phi_-(\omega)\ln\frac{m_b\omega}{\mu^2}+Q_\ell^2\left[\frac{1}{\epsilon_{\text{IR}}}\left(\ln\frac{m_B^2}{m_\ell^2}-2\right)+\frac{1}{2}\ln^2\frac{m_\ell^2}{\mu^2}-\frac{1}{2}\ln\frac{m_\ell^2}{\mu^2}+2+\frac{5\pi^2}{12}\right]\Bigg\}\\ \mathcal{M}_{3p}(\mu) &= \frac{\alpha}{\pi}\,Q_\ell Q_u\int_0^\infty d\omega\int_0^\infty d\omega_g\,\phi_{3g}(\omega,\omega_g)\left[\frac{1}{\omega_g}\ln\left(1+\frac{\omega_g}{\omega}\right)-\frac{1}{\omega+\omega_g}\right]\end{aligned}$$

IR divergence cancels against real soft photon emission

$\Rightarrow$ significant hadronic uncertainties in $\mathcal{O}(\alpha)$ terms!

[CC, König, Neubert 2022]

Virtual corrections to $B \to \mu\bar{\nu}$

Decay amplitude including virtual QED corrections at $\mathcal{O}(\alpha)$:

$$\mathcal{A}_{B\to\ell\bar{\nu}}^{\text{virtual}} = i\sqrt{2}G_F\, K_{\text{EW}}(\mu)\, V_{ub}\, \frac{m_\ell}{m_b}\sqrt{m_B}\, F(\mu, m_b, w)\, \bar{u}(p_\ell) P_L\, v(p_\nu) \Big[\mathcal{M}_{2p}(\mu) + \mathcal{M}_{3p}(\mu)\Big]$$

with:

large double logarithms

$$\begin{aligned}\mathcal{M}_{2p}(\mu) &= 1 + \frac{C_F\,\alpha_s}{4\pi}\left[\frac{3}{2}\ln\frac{m_b^2}{\mu^2} - 2\right] \\ &+ \frac{\alpha}{4\pi}\left\{Q_b^2\left[\frac{3}{2}\ln\frac{m_b^2}{\mu^2} - 2\right] - Q_\ell Q_b\left[\frac{1}{2}\ln^2\frac{m_b^2}{\mu^2} + 2\ln\frac{m_b^2}{\mu^2} - 3\ln\frac{m_\ell^2}{\mu^2} + 1 + \frac{5\pi^2}{12}\right]\right. \\ &\left. +2Q_\ell Q_u\int_0^\infty d\omega\, \phi_-(\omega)\ln\frac{m_b\omega}{\mu^2} + Q_\ell^2\left[\frac{1}{\epsilon_{\text{IR}}}\left(\ln\frac{m_B^2}{m_\ell^2} - 2\right) + \frac{1}{2}\ln^2\frac{m_\ell^2}{\mu^2} - \frac{1}{2}\ln\frac{m_\ell^2}{\mu^2} + 2 + \frac{5\pi^2}{12}\right]\right\}\end{aligned}$$

$$\mathcal{M}_{3p}(\mu) = \frac{\alpha}{\pi}\, Q_\ell Q_u \int_0^\infty d\omega \int_0^\infty d\omega_g\, \phi_{3g}(\omega,\omega_g)\left[\frac{1}{\omega_g}\ln\left(1 + \frac{\omega_g}{\omega}\right) - \frac{1}{\omega + \omega_g}\right]$$

$\Rightarrow$ significant hadronic uncertainties in $\mathcal{O}(\alpha)$ terms! [CC, König, Neubert 2022]

$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

$$\mathcal{L}_{\mathrm{y}} = y\, e^{-im_B(v\cdot x)} \varphi_B \left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)} \right) + \mathrm{h.c.}$$

$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

- Below $\mu \sim \Lambda_{\rm QCD}$ quarks hadronize: move to effective description with a **Yukawa theory**, with the meson treated as a **heavy scalar**

$$\mathcal{L}_{\rm y} = y\, e^{-im_B(v\cdot x)}\varphi_B \left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)}\right) + \text{h.c.}$$

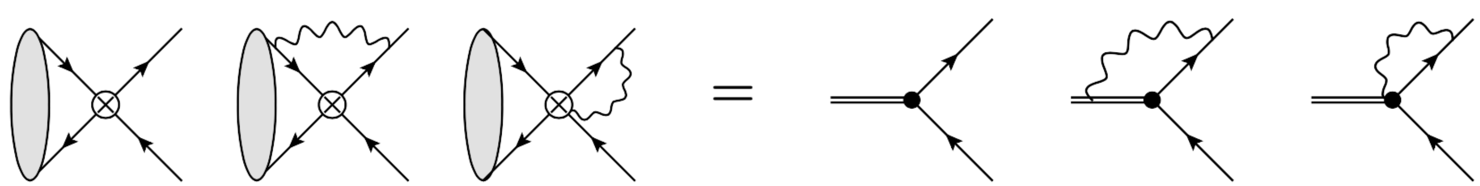
$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

- Below $\mu \sim \Lambda_{\rm QCD}$ quarks hadronize: move to effective description with a **Yukawa theory**, with the meson treated as a **heavy scalar**

$$\mathcal{L}_{\rm y} = y\, e^{-im_B(v\cdot x)}\varphi_B \left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)}\right) + \text{h.c.}$$

- Yukawa coupling is fixed by matching hadronic matrix elements between this and the previous description:

$$\langle \ell\, \nu \,|\, \mathcal{L}_{\rm SCET\,II \otimes HQET} \,|\, B \rangle \quad = \quad \langle \ell\, \nu \,|\, \mathcal{L}_{\rm SCET\,II \otimes HSET} \,|\, B \rangle$$



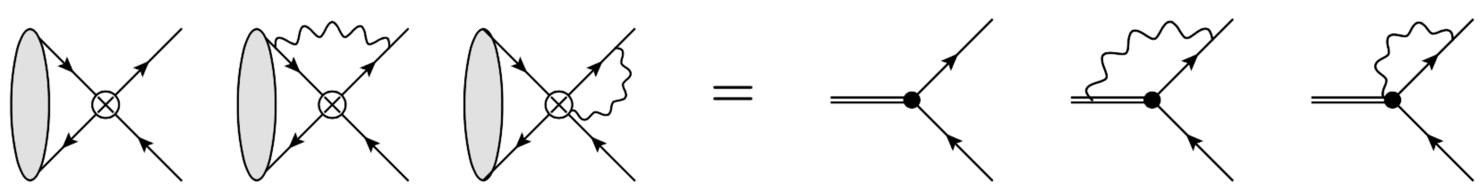
$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

- Below $\mu \sim \Lambda_{\rm QCD}$ quarks hadronize: move to effective description with a **Yukawa theory**, with the meson treated as a **heavy scalar**

$$\mathcal{L}_{\rm y} = y\, e^{-im_B(v\cdot x)} \varphi_B \left(\bar{\chi}_c^{(\ell)} P_L \chi_{\bar{c}}^{(\nu)}\right) + \text{h.c.}$$

- Yukawa coupling is fixed by matching hadronic matrix elements between this and the previous description:

$$\langle \ell\, \nu \,|\, \mathcal{L}_{\rm SCET\,II \otimes HQET} \,|\, B \rangle \quad = \quad \langle \ell\, \nu \,|\, \mathcal{L}_{\rm SCET\,II \otimes HSET} \,|\, B \rangle$$



- Since $\Lambda_{\rm QCD} \sim m_\mu$, we integrate out the **muon** in the same step and describe it as a **boosted heavy lepton** field: $\ell(x) = e^{-im_\ell v_\ell \cdot x} \chi_{v_\ell}(x)$

$\Rightarrow$ low-E theory is a heavy scalar effective theory $\otimes$ bHLET

$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

$$Y_v^{(\mathrm{s})}(x) = \mathcal{P}\exp\left\{ie\int_{-\infty}^{0} ds\, v\cdot A_s(x+sv)\right\}$$

$$Y_v^{(\mathrm{sc})}(x) = \mathcal{P}\exp\left\{ie\int_{-\infty}^{0} ds\, v\cdot A_{\mathrm{sc}}(x+sv)\right\}$$

$$W_s(\omega_s,\mu) = \left[\sum_{n_s=0}^{\infty}\prod_{i=1}^{n_s}\int d\Pi_i(q_i)\right]\left|\langle n_s\gamma_s(q_i)|\, Y_v^{(\mathrm{s})} Y_n^{(\mathrm{s})\dagger}\,|0\rangle\right|^2 \delta\left(\omega_s - q_0^{(\mathrm{s})}\right)\,,$$

$$W_{sc}(\omega_{sc},\mu) = \left[\sum_{n_{sc}=0}^{\infty}\prod_{j=1}^{n_s}\int d\Pi_j(q_j)\right]\left|\langle n_{sc}\gamma_{sc}(q_j)|\, Y_{\bar{n}}^{(\mathrm{sc})\dagger} Y_{v_l}^{(\mathrm{sc})}\,|0\rangle\right|^2 \delta\left(\omega_{sc} - q_0^{(\mathrm{sc})}\right)$$

$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

- It's a **theory of Wilson lines**: all interactions of the B and the muon with ultrasoft and ultrasoft-collinear photons can be moved into Wilson lines, and decoupled from them via field redefinitions:

$$Y_v^{(\mathrm{s})}(x) = \mathcal{P}\exp\left\{ie\int_{-\infty}^{0} ds\, v\cdot A_s(x+sv)\right\}$$
$$Y_v^{(\mathrm{sc})}(x) = \mathcal{P}\exp\left\{ie\int_{-\infty}^{0} ds\, v\cdot A_{\mathrm{sc}}(x+sv)\right\}$$

- Real corrections are matrix elements of these Wilson lines:

$$W_s(\omega_s,\mu) = \left[\sum_{n_s=0}^{\infty}\prod_{i=1}^{n_s}\int d\Pi_i(q_i)\right]\left|\langle n_s\gamma_s(q_i)|\, Y_v^{(\mathrm{s})}Y_n^{(\mathrm{s})\dagger}\,|0\rangle\right|^2\delta\left(\omega_s - q_0^{(\mathrm{s})}\right)\,,$$
$$W_{sc}(\omega_{sc},\mu) = \left[\sum_{n_{sc}=0}^{\infty}\prod_{j=1}^{n_s}\int d\Pi_j(q_j)\right]\left|\langle n_{sc}\gamma_{sc}(q_j)|\, Y_{\bar{n}}^{(\mathrm{sc})\dagger}Y_{v_l}^{(\mathrm{sc})}\,|0\rangle\right|^2\delta\left(\omega_{sc} - q_0^{(\mathrm{sc})}\right)$$

$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

- Convoluted with the measurement function involving the experimental cut, they yields the complete radiative function:

$$S(E_s,\mu) = \int_0^\infty d\omega_s \int_0^\infty d\omega_{sc} \boxed{\theta\left(\frac{E_s}{2} - \omega_s - \omega_{sc}\right)} W_s(\omega_s,\mu) W_{sc}(\omega_{sc},\mu)$$

- Integration and renormalisation of the bare functions can be carried out in Laplace space ⇒ resummation of ultra-soft and ultrasoft-collinear logs.

- **Full factorization formula**:

$\mu < \Lambda \sim m_\mu$: understanding the low-energy theory

- Convoluted with the measurement function involving the experimental cut, they yields the complete radiative function:

$$S(E_s, \mu) = \int_0^\infty d\omega_s \int_0^\infty d\omega_{sc} \boxed{\theta\left(\frac{E_s}{2} - \omega_s - \omega_{sc}\right)} W_s(\omega_s, \mu) W_{sc}(\omega_{sc}, \mu)$$

- Integration and renormalisation of the bare functions can be carried out in Laplace space ⇒ resummation of ultra-soft and ultrasoft-collinear logs.

- **Full factorization formula**:

$$\Gamma = \underbrace{|\mathcal{A}^{\text{virtual}}|^2}_{\text{non-radiative}} \otimes \underbrace{W_{us}(\mu) \otimes W_{usc}(\mu)}_{\text{radiative}}$$

Conclusions

- We derived a factorization formula for QED corrections to $B \to \mu\nu$, which **separates all scales** in the process and allows for the resummation of the associated logarithms.

- We used a multi-step matching procedure involving a combination of **HQET**, **SCET 1 & 2,** and **boosted HLET**. The derivation can be systematically extended to higher orders in the coupling constants and power counting.

- The exchange of virtual hard collinear photons gives rise to **structure-dependent** corrections. Unlike in $B_s \to \mu\mu$, these are not enhanced. Nevertheless, they are present and constitute a potentially important source of uncertainty in the prediction.

- Channels with **other lepton flavors** cannot be obtained simply by replacing the muon mass: they have different scale hierarchies, matching threshold, EFT constructions….