Infrared singularities of QCD amplitudes with a massive parton

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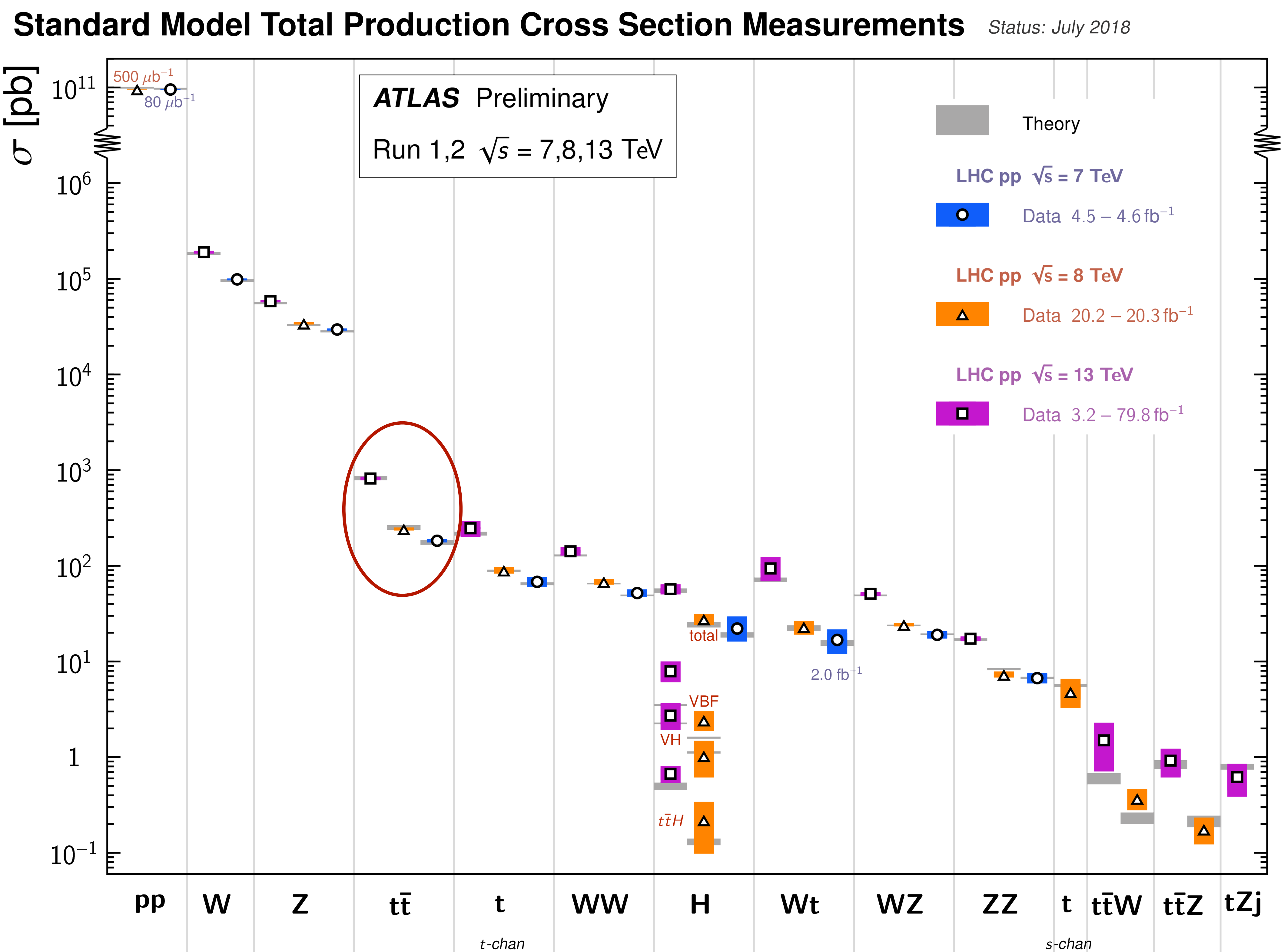
"10 Years MITP" Anniversary Event, May 8th, 2023

Outline

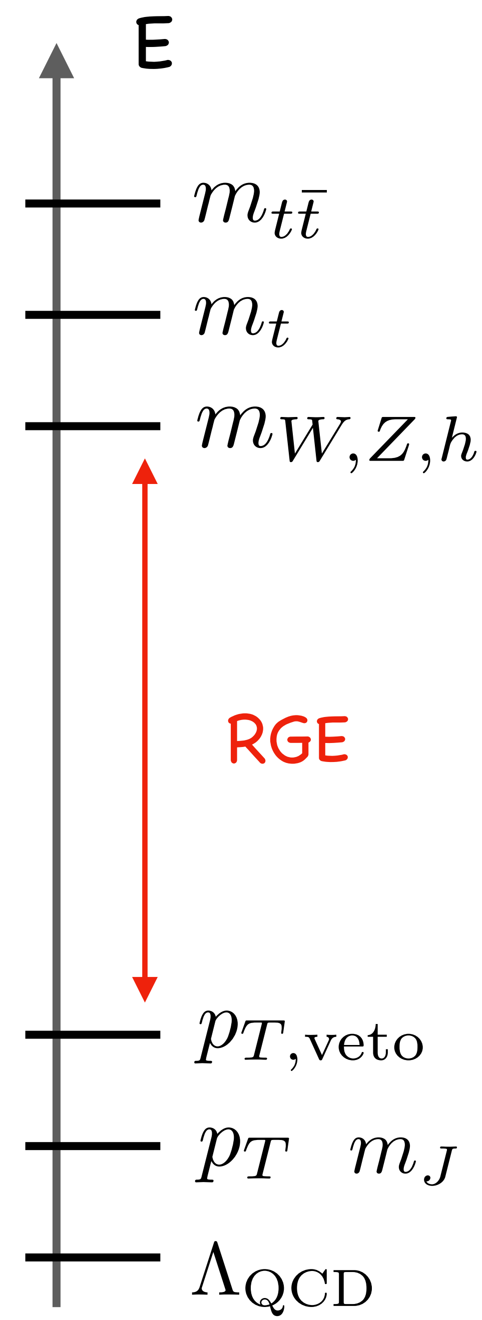
- Motivation
- Constraints to anomalous dimensions
    - Soft-collinear factorization - Kinematics
    - Non-Abelian Exponentiation Theorem - Color
    - Two-particle collinear limits
    - Small-mass limits
- Calculation of the Tripole Correlation
- Results

Measurements at the LHC



Precision Collider Observables

- Multi scales are involved in measurements
  - Fixed-order results are invalid due to large logarithms of scale ratios
  - Large logs need to be resummed to all orders in α_s
  - Renormalization-group evolutions are governed by anomalous dimensions



Effective field theory is powerful for scale separation and factorization

$$\sigma \sim H \otimes B_{a/N} \otimes B_{b/N} \otimes S \otimes \prod_i J$$

IR poles   UV poles in low energy matrix elements

In soft-collinear effective theory (SCET), IR poles of hard coefficients are in one-to-one correspondence to the UV poles of low-energy matrix elements.

Soft-Collinear Factorization

For n-jet amplitudes:

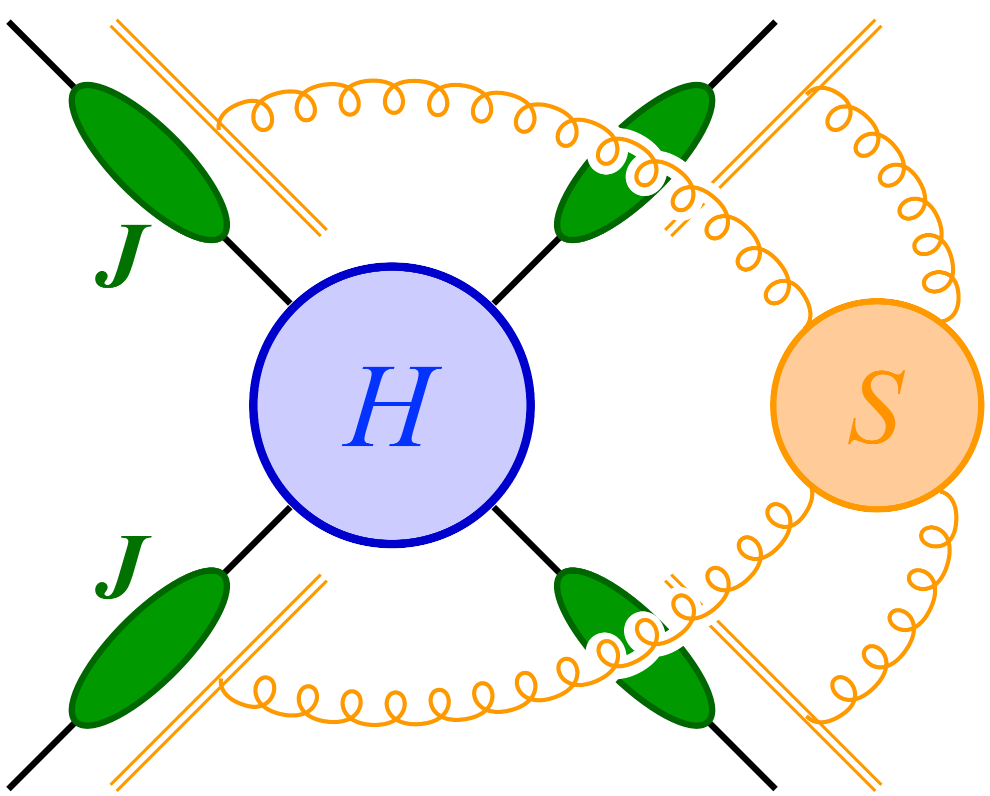
Matrix in color space

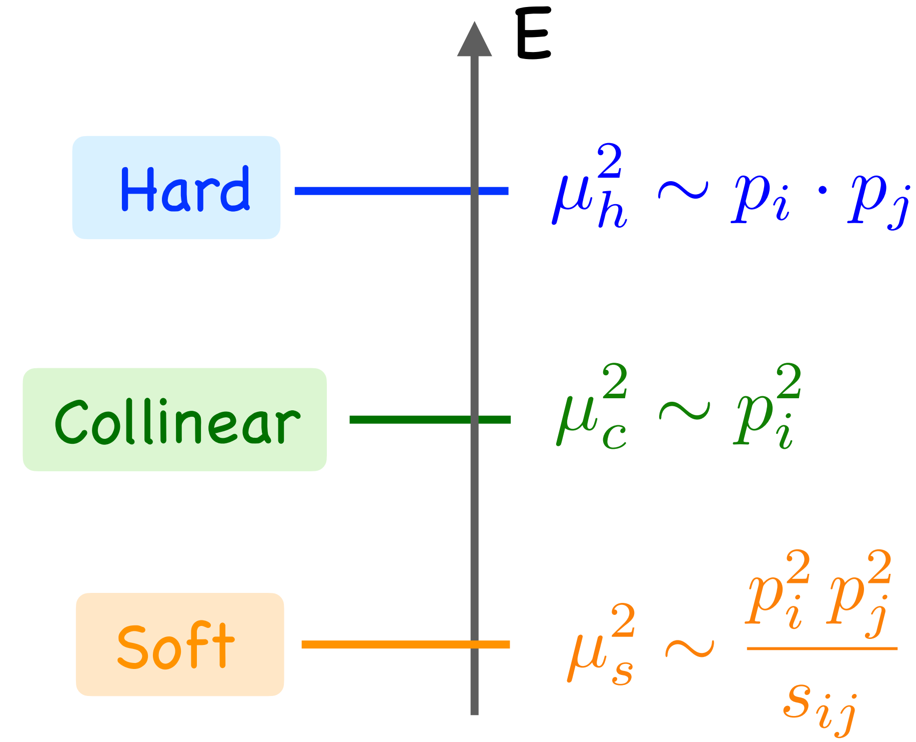
$$\underline{\text{Off-Shell Green's function}} = \boldsymbol{S}(\{\underline{\beta}\}, \epsilon) \prod_i J(L_i^2, \epsilon) |\mathcal{M}(\{\underline{s}\}, \epsilon)\rangle$$

UV renormalized
free of IR poles

$$\beta_{ij} = \ln \frac{(-s_{ij})\, \mu^2}{(-p_i^2)(-p_j^2)} \qquad \ln \frac{\mu^2}{-p_i^2} \qquad s_{ij} = \pm 2 p_i \cdot p_j$$

cusp angles





Soft-Collinear Factorization

Renormalization in $\overline{\text{MS}}$ scheme   Becher, Neubert, '09   Also see Catani, '98

$$|\mathcal{M}(\{\underline{s}\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{\underline{s}\},\mu)|\mathcal{M}(\epsilon,\{\underline{s}\})\rangle$$

Renormalization-group equation (RGE) gives

anomalous dimension

$$\boldsymbol{Z}(\epsilon,\{\underline{p}\},\mu) = \mathbf{P}\exp\left[\int_\mu^\infty \frac{d\mu'}{\mu'}\boldsymbol{\Gamma}(\{\underline{p}\},\mu')\right]$$

RG invariance implies:

$$\boldsymbol{\Gamma}(\{\underline{s}\},\mu) = \boldsymbol{\Gamma}_s(\{\underline{\beta}\},\mu) + \sum_{i=1}^{n}\Gamma_c^i(L_i,\mu)\mathbf{1}$$

free of collinear scale p_i^2

$$\Gamma_c^i = -\Gamma_{\text{cusp}}^i L_i + \gamma_c^i$$

Becher, Hill, Lange, Neubert, '03

which leads to

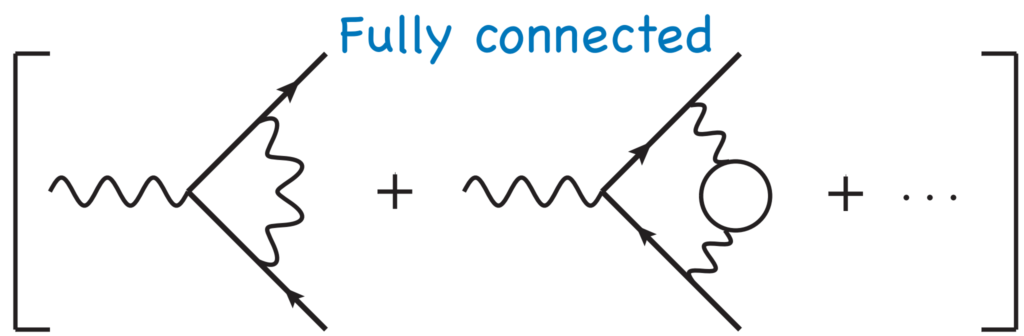
$$\frac{\partial\boldsymbol{\Gamma}_s}{\partial L_i} = -\frac{\partial\Gamma_c^i}{\partial L_i}\mathbf{1}$$

Non-Abelian Exponentiation Theorem

- UV poles of soft matrix elements can be written as exponentials of simpler quantities, which only receive contribution from color connected **webs**

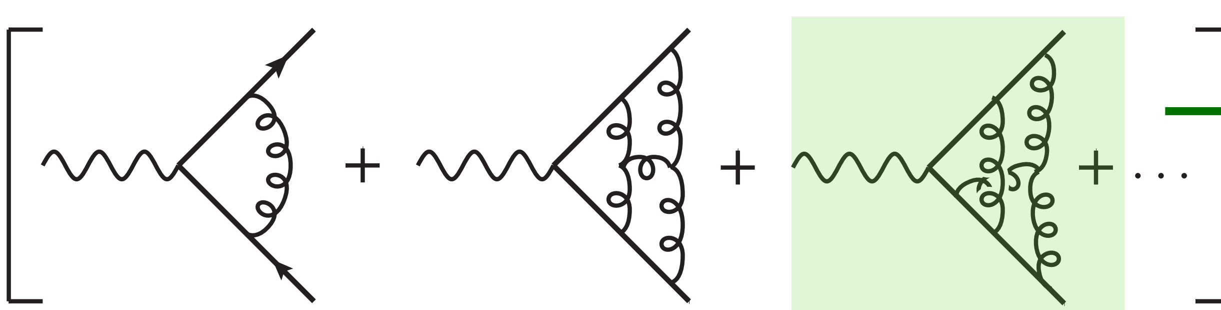
Abelian:

$$\mathcal{A} = \mathcal{A}_0 \exp$$

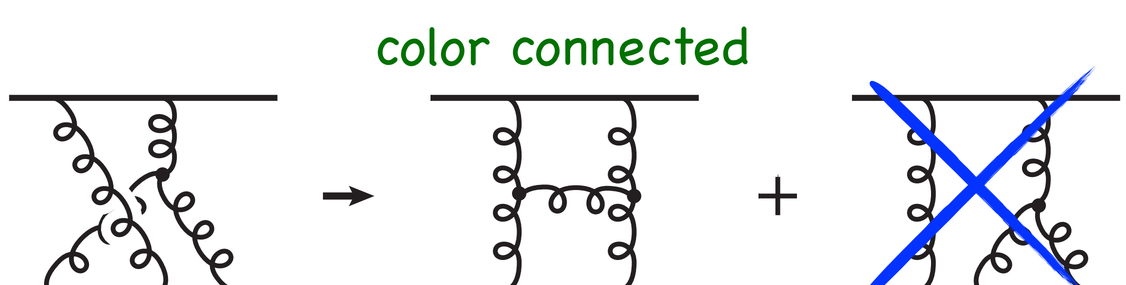


non-Abelian: Gatheral, '83 Frenkel & Taylor, '84

$$\mathcal{A} = \mathcal{A}_0 \exp$$



maximally non-abelian part of color



not contribute to exponent

Non-Abelian Exponentiation Theorem

- Generalize to multi-leg scattering amplitudes — Gardi et al. '09-'14

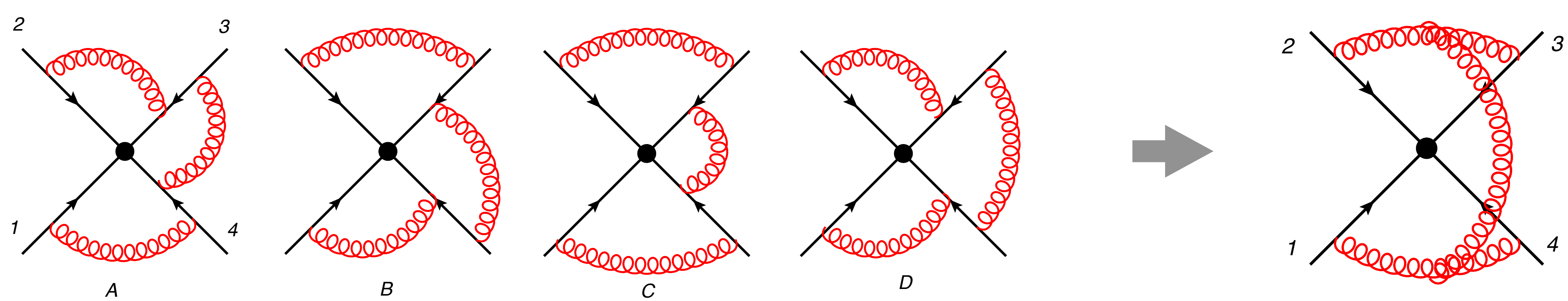
kinematic → $F(D)$; conventional color → $C(D')$

$$\boldsymbol{S} = \exp\left[\sum_{W}\sum_{D,D'} F(D)\, R^{(W)}_{DD'}\, C(D')\right]$$

$R^{(W)}_{DD'}$: project to maximally non-abelian part

- An example of **web** : $W_{(1,1,2,2)}$

Figure from Gardi, Smillie, White, 1304.7040



$$R^{(1,1,2,2)} = \frac{1}{6}\begin{bmatrix} 2 & 2 & -2 & -2 \\ 2 & 2 & -2 & -2 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

obtained by replica trick

Gardi, Laenen, Stavenga, White '10

$$\sum_{D,D'} F(D) R^{(W)}_{DD'} C(D') = \frac{1}{6}\left(-2F_A - 2F_B + F_C + F_D\right)$$

$$\times\, f^{abd} f^{bce} \boldsymbol{T}_1^c \boldsymbol{T}_2^a \boldsymbol{T}_3^d \boldsymbol{T}_4^e$$

Color connected

Construct Soft Anomalous Dimensions

- Kinematic dependences
  - Cusp angles - depend on collinear scales

$$\beta_{ij} = L_i + L_j - \ln \frac{\mu^2}{-s_{ij}} \qquad \beta_{Ij} = L_j - \ln \frac{m_I \mu}{-s_{Ij}}$$

$$\beta_{IJ} = \cosh^{-1}\left(\frac{-s_{IJ}}{2m_I m_J}\right) \qquad s_{ij} = \pm 2 p_i \cdot p_j \qquad L_i = \frac{\mu^2}{-p_i^2}$$

  - Conformal cross ratios - independent on collinear scales

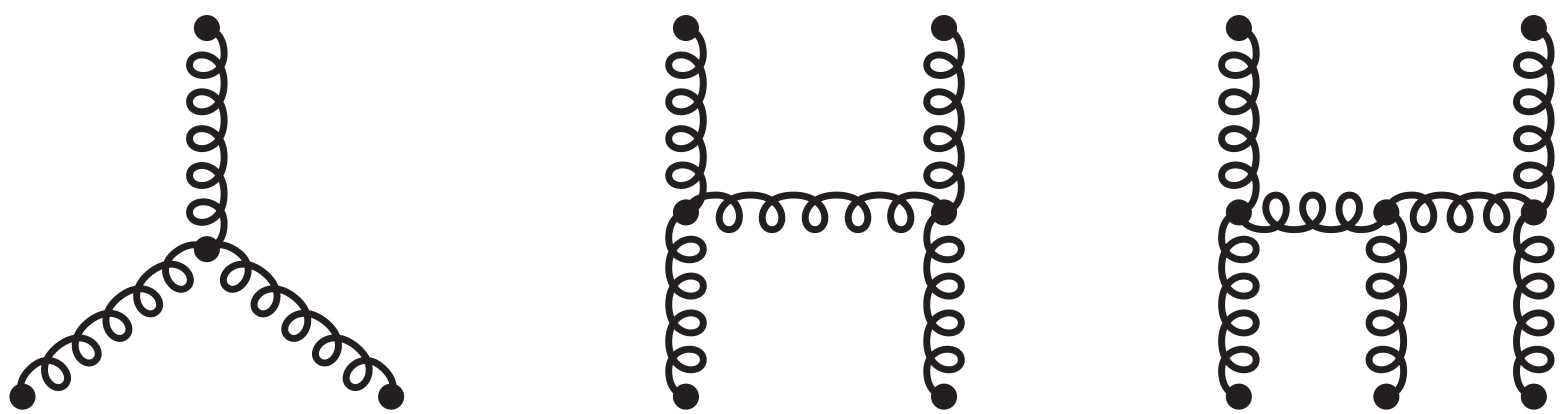
Purely massless: $$\beta_{ijkl} = \ln \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$ Becher, Neubert '09; Gardi, Magnea '09

One massive: $$r_{ijI} \equiv \frac{v_I^2\,(n_i \cdot n_j)}{2\,(v_I \cdot n_i)(v_I \cdot n_j)}$$ ZLL, Schalch '22

Construct Soft Anomalous Dimensions

- Color
  - Only color connected

$\boldsymbol{T}_i^a \boldsymbol{T}_j^a$ $\quad$ $if^{abc}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c$ $\quad$ $f^{abe}f^{cde}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d$



  - Symmetrization of external legs

apply $[\boldsymbol{T}_i^a, \boldsymbol{T}_i^b] = if^{abc}\boldsymbol{T}_i^c$

$$\boldsymbol{T}_i^a \boldsymbol{T}_j^a \qquad \mathcal{T}_{ijk} \equiv if^{abc}\left(\boldsymbol{T}_i^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c\right)_+ \qquad \mathcal{T}_{ijkl} \equiv f^{ade}f^{bce}\left(\boldsymbol{T}_i^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c\boldsymbol{T}_l^d\right)_+$$

color dipole

Sum over all the permutations of i,j,k,...

Becher, Neubert '09

Construct Soft Anomalous Dimensions

- Only linearly depends on cusp angles due to RG invariance

$$\frac{\partial \mathbf{\Gamma}_s}{\partial L_i} = -\frac{\partial \Gamma_c}{\partial L_i}\mathbf{1} = \Gamma^i_{\text{cusp}}\mathbf{1}$$

No terms like $\beta_{ij}^2, \ \beta_{ij}^3, \ \cdots$

Except for conformal cross ratios

- Symmetry properties

$$\mathcal{T}_{ijkl} = \mathcal{T}_{jilk} = -\mathcal{T}_{ikjl} = -\mathcal{T}_{ljki} = \mathcal{T}_{klij}$$

- Jacobi identity

$$\mathcal{T}_{iklj} + \mathcal{T}_{iljk} + \mathcal{T}_{ijkl} = 0$$

also for β_{ijkl}

only two of twenty-four are linearly independent

Construct Soft Anomalous Dimensions

- When kinematic functions hit color structures   Becher, Neubert, '09
  - Tripole correlations $if^{abc}\boldsymbol{T}_i^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c$ and $if^{abc}\boldsymbol{T}_I^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c$ vanish

    impossible to construct an anti-symmetric kinematic functions

    Explain only dipole structures for two-loop massless amplitudes !!!

  - $if^{abc}\boldsymbol{T}_I^a\boldsymbol{T}_J^b\boldsymbol{T}_k^c\, F\left(\beta_{IJ}, \ln\frac{v_I\cdot p_k}{v_J\cdot p_k}\right)$ and $if^{abc}\boldsymbol{T}_I^a\boldsymbol{T}_J^b\boldsymbol{T}_K^c\, F\left(\beta_{IJ},\beta_{IK},\beta_{JK}\right)$
    are allowed starting at two-loop order   Ferroglia, Neubert, Pecjak, Yang, '09

  - Kinematic functions correspond to $\mathcal{T}_{ijkl}$ and $\mathcal{T}_{ijkI}$ are odd functions

Construct Soft Anomalous Dimensions

- Non-dipole building blocks up to three loops (only one massive parton)

$$\mathcal{T}_{iijj}\beta_{ij}\,,\quad \mathcal{T}_{iijj}\,,\quad \mathcal{T}_{jjII}\beta_{Ij}\,,\quad \mathcal{T}_{jjII}\,,\quad \mathcal{T}_{iijk}\beta_{ij}\,,\quad \mathcal{T}_{iijk}\beta_{jk}\,,\quad \mathcal{T}_{iijk}\,,\quad \mathcal{T}_{ijII}\beta_{ij}\,,$$

$$\mathcal{T}_{ijII}\beta_{Ii}\,,\quad \mathcal{T}_{iijI}\beta_{ij}\,,\quad \mathcal{T}_{iijI}\beta_{Ii}\,,\quad \mathcal{T}_{iijI}\beta_{Ij}\,,\quad \mathcal{T}_{ijkl}\beta_{ij}\,,\quad \mathcal{T}_{ijkI}\beta_{ij}\,,\quad \mathcal{T}_{ijkI}\beta_{Ij}\,,$$

$$\mathcal{T}_{ijII}\bar{F}^{[A]}_{\mathrm{h2}}(r_{ijI})\,,\quad \mathcal{T}_{iijI}\bar{F}^{[B]}_{\mathrm{h2}}(r_{ijI})\,,\quad \mathcal{T}_{ijkI}\bar{F}_{\mathrm{h3}}(r_{ijI},r_{ikI},r_{jkI})\,,\quad \mathcal{T}_{ijkl}\,\bar{F}_{4}(\beta_{ijkl},\beta_{ijkl}-2\beta_{ilkj})$$

not linear independent, so vanish

Symmetry property of $\mathcal{T}_{ijkl}$ has been taken into consideration

- Color conservation: $\sum_i \boldsymbol{T}_i + \sum_I \boldsymbol{T}_I = 0$ in the color-space formalism

massless (i), massive (I)

Catani, Seymour, '96

leads to

$$\sum_{(i,j)} \boldsymbol{T}_i\cdot\boldsymbol{T}_j + \sum_{I,j}\boldsymbol{T}_I\cdot\boldsymbol{T}_j = -\sum_j T_j^2 = -\sum_j C_{R_j}$$

$i \neq j$ ← (i,j); C_{R_j}: Casimir invariants

$$\mathcal{T}_{ijII} = \frac{1}{2}\left(\mathcal{T}_{jjiI}+\mathcal{T}_{iijI}\right) - \frac{1}{2}\sum_{k\neq i,j}\left(\mathcal{T}_{ijkI}+\mathcal{T}_{jikI}\right) - \frac{1}{2}\sum_{J\neq I}\left(\mathcal{T}_{ijIJ}+\mathcal{T}_{jiIJ}\right)$$

and a few more similar relations, which help to reduce the building blocks.

Construct Soft Anomalous Dimensions

- The detailed derivation is given by

$$\bar{\mathbf{\Gamma}}_s^{(3)} = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2} (\beta_{ij} \bar{f}_1 + \bar{h}_1) + \sum_{I,j} \boldsymbol{T}_I \cdot \boldsymbol{T}_j \left[\left(\bar{f}_1 + \frac{C_A^2}{8} \bar{f}_2 \right) \beta_{Ij} + \bar{h}_2 \right] + \sum_i \bar{c}_i + \sum_I \bar{c}_I$$
$$+ \sum_{(i,j)} \mathcal{T}_{iijj} \left(\beta_{ij}\ \bar{f}_3 + \bar{h}_3 \right) + \sum_{I,j} \mathcal{T}_{IIjj} \left(\beta_{Ij}\ \bar{f}_4 + \bar{h}_4 \right) + \sum_{(i,j,k)} \mathcal{T}_{iijk} \left(\beta_{jk}\ \bar{f}_5 + \bar{h}_5 \right) + \sum_I \sum_{(i,j)} \mathcal{T}_{iijI}\ \beta_{Ij}\ \bar{f}_6$$
$$+ \sum_I \sum_{(i,j)} \mathcal{T}_{ijII} \left[\beta_{ij}\ \bar{f}_7 + \bar{F}_{\text{h2}}(r_{ijI}) \right] + \sum_{(i,j,k,l)} \mathcal{T}_{ijkl} \left[\beta_{ijkl}\ \bar{f}_8 + \bar{F}_4(\beta_{ijkI}, \beta_{ijkl} - 2\beta_{ilkj}) \right]$$
$$+ \sum_I \sum_{(i,j,k)} \mathcal{T}_{ijkI} \bar{F}_{\text{h3}}(r_{ijI}, r_{ikI}, r_{jkI}) + \cdots .$$

Using constraint from soft-collinear factorization

$$\frac{\partial \bar{\mathbf{\Gamma}}_s^{(3)}}{\partial L_i} = C_{R_i} \left(\frac{C_A^2}{4} \bar{f}_5 - \bar{f}_1 \right) + \frac{C_A^2}{8} \sum_I \boldsymbol{T}_I \cdot \boldsymbol{T}_i (\bar{f}_2 + 2\bar{f}_5 - 2\bar{f}_7) + 2 \sum_{j \neq i} \mathcal{T}_{iijj} (\bar{f}_3 - \bar{f}_5) + \sum_I \mathcal{T}_{iiII} (\bar{f}_4 - 2\bar{f}_7)$$
$$+ \sum_I \sum_{j \neq i} \mathcal{T}_{jjiI} (\bar{f}_6 - 2\bar{f}_5) + \cdots$$

We have

$$\bar{f}_4 = 2\bar{f}_7 = \bar{f}_2 + 2\bar{f}_3 \,, \qquad \bar{f}_5 = \bar{f}_3 \,, \qquad \bar{f}_6 = 2\bar{f}_3$$

General Structure of Anomalous Dimension

- Using non-Abelian exponentiation and constraint from soft-collinear factorization

$$\frac{\partial \mathbf{\Gamma}_s}{\partial L_i} = -\frac{\partial \Gamma_c}{\partial L_i}\mathbf{1} = \Gamma^i_{\text{cusp}}\mathbf{1}$$

we derive the general formula of anomalous dimensions

$$\mathbf{\Gamma}(\{\underline{p}\},\{\underline{m}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\gamma_{\text{cusp}}(\alpha_s)\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\,\mathbf{1}$$

Becher, Neubert, 0901.0722, 0903.1126
Gardi, Magnea, 0901.1091

$$+\sum_{I,j}\boldsymbol{T}_I\cdot\boldsymbol{T}_j\,\gamma_{\text{cusp}}(\alpha_s)\ln\frac{m_I\mu}{-s_{Ij}} - \sum_{(I,J)}\frac{\boldsymbol{T}_I\cdot\boldsymbol{T}_J}{2}\gamma_{\text{cusp}}(\beta_{IJ},\alpha_s) + \sum_I \gamma^I(\alpha_s)\,\mathbf{1}$$

Becher, Neubert 0904.1021

$$+ f(\alpha_s)\sum_{(i,j,k)}\mathcal{T}_{iijk} + \sum_{(i,j,k,l)}\mathcal{T}_{ijkl}\,F_4(\beta_{ijkl},\beta_{ijkl}-2\beta_{ilkj},\alpha_s)$$

Almelid, Duhr, Gardi 1507.00047 + McLeod, White 1706.10162

← Starting at 3L

$$+\sum_I\sum_{(i,j)}\mathcal{T}_{ijII}\,F_{\text{h2}}(r_{ijI},\alpha_s) + \sum_I\sum_{(i,j,k)}\mathcal{T}_{ijkI}\,F_{\text{h3}}(r_{ijI},r_{ikI},r_{jkI},\alpha_s)$$

ZLL, Schalch 2207.02864

+ [non-dipole contributions involving two or more massive partons]

Ferroglia, Neubert, Pecjak, Yang, '09 only up to two loops

$$+\,\mathcal{O}(\alpha_s^4)$$

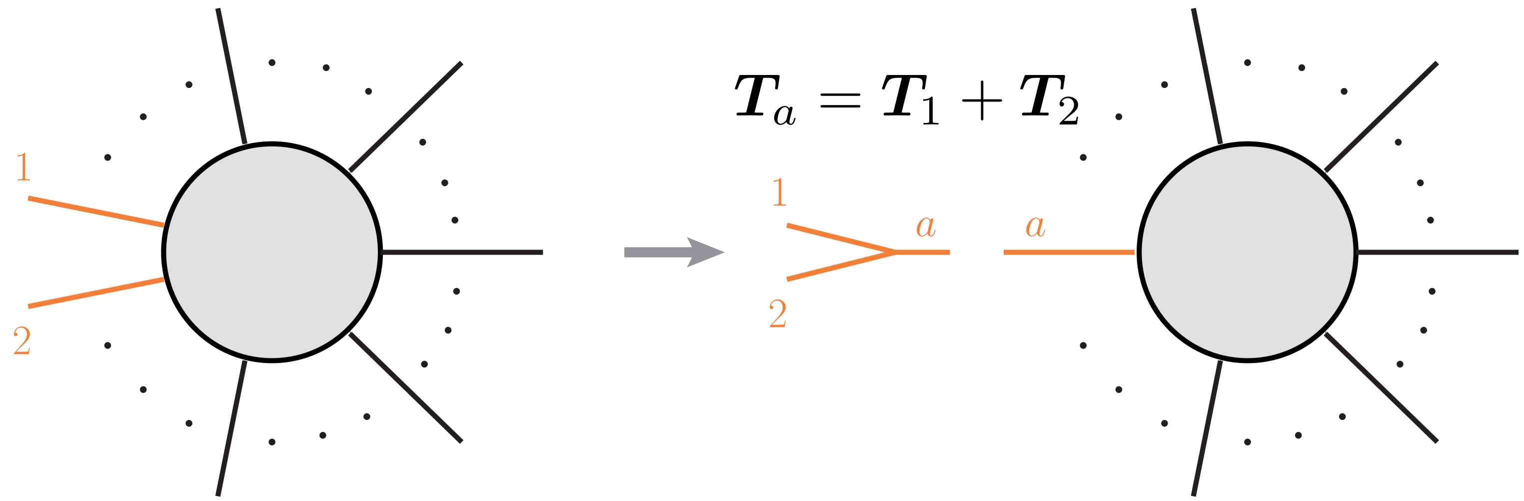
violation of Casimir scaling due to $d_R^{abcd}\boldsymbol{T}_i^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c\boldsymbol{T}_l^d$

Boels, Huber, Yang, '17

Two-Particle Collinear Limit

- When particle 1 and 2 become collinear

Berends, Giele, ’89 ,Bern, Chalmers, ’95
Kosower, ’99, Mangano, Parke, ’05



n-particle scattering Splitting function (n-1)-particle scattering

$$|\mathcal{M}(\{p_1, p_2, p_3, \ldots, p_n\}, \epsilon)\rangle \simeq \mathbf{Sp}(\{p_1, p_2\})|\mathcal{M}(\{p_a, p_3, \ldots, p_n\}, \epsilon)\rangle$$

leads to $\Gamma_{\mathrm{Sp}}(\{p_1, p_2\}, \mu) = \boldsymbol{\Gamma}\left(\{p_1, p_2, \ldots, p_n\}, \{\underline{m}\}, \mu\right) - \boldsymbol{\Gamma}\left(\{p_a, \ldots, p_n\}, \{\underline{m}\}, \mu\right)$

must be independent of color generators for particles other than 1 and 2, so

$$\lim_{\omega \to -\infty} F_4(\omega, \omega, \alpha_s) = \frac{f(\alpha_s)}{2}, \qquad F_{\mathrm{h2}}(0, \alpha_s) = 3f(\alpha_s), \qquad F_{\mathrm{h3}}(0, r, r, \alpha_s) = 2f(\alpha_s)$$

Almelid, Duhr, Gardi, ’15

ZLL, Schalch, ‘22

Two-Particle Collinear Limit

- The detailed derivation is given by

$$\boldsymbol{T}_a = \boldsymbol{T}_1 + \boldsymbol{T}_2$$

$$\begin{aligned}
\boldsymbol{\Gamma}_{\rm Sp}(\{p_1,p_2\},\mu) =& \boldsymbol{\Gamma}\left(\{p_1,p_2,\dots,p_n\},\{\underline{m}\},\mu\right) - \boldsymbol{\Gamma}\left(\{p_a,\dots,p_n\},\{\underline{m}\},\mu\right) \\
=& \gamma_{\rm cusp}(\alpha_s)\left[\boldsymbol{T}_1\cdot\boldsymbol{T}_2\left(\ln\frac{\mu^2}{-s_{12}} + \ln[z(1-z)]\right) + C_{R_1}\ln z + C_{R_2}\ln(1-z)\right] \\
&+ \left[\gamma^1(\alpha_s) + \gamma^2(\alpha_s) - \gamma^a(\alpha_s)\right]\boldsymbol{1} \\
&+ \left[f(\alpha_s) + 4F_4(\omega_{ij},\omega_{ij},\alpha_s)\right]\left(-\frac{C_A^2}{4}\boldsymbol{T}_1\cdot\boldsymbol{T}_2 - 2\mathcal{T}_{1122}\right) \\
&+ 4\sum_{i\neq 1,2}\mathcal{T}_{12ii}\left[f(\alpha_s) - 2F_4(\omega_{ij},\omega_{ij},\alpha_s)\right] \\
&+ 2\sum_{I}\mathcal{T}_{12II}\left[F_{\rm h2}(0,\alpha_s) - f(\alpha_s) - 4F_4(\omega_{ij},\omega_{ij},\alpha_s)\right] \\
&+ 2\sum_{I}\sum_{i\neq 1,2}\left(\mathcal{T}_{12iI} + \mathcal{T}_{21iI}\right)\left[F_{\rm h3}(0,r_{1iI},r_{1iI},\alpha_s) - 4F_4(\omega_{ij},\omega_{ij},\alpha_s)\right] \\
&+ \cdots
\end{aligned}$$

Almelid, Duhr, Gardi, '15

ZLL, Schalch, '22

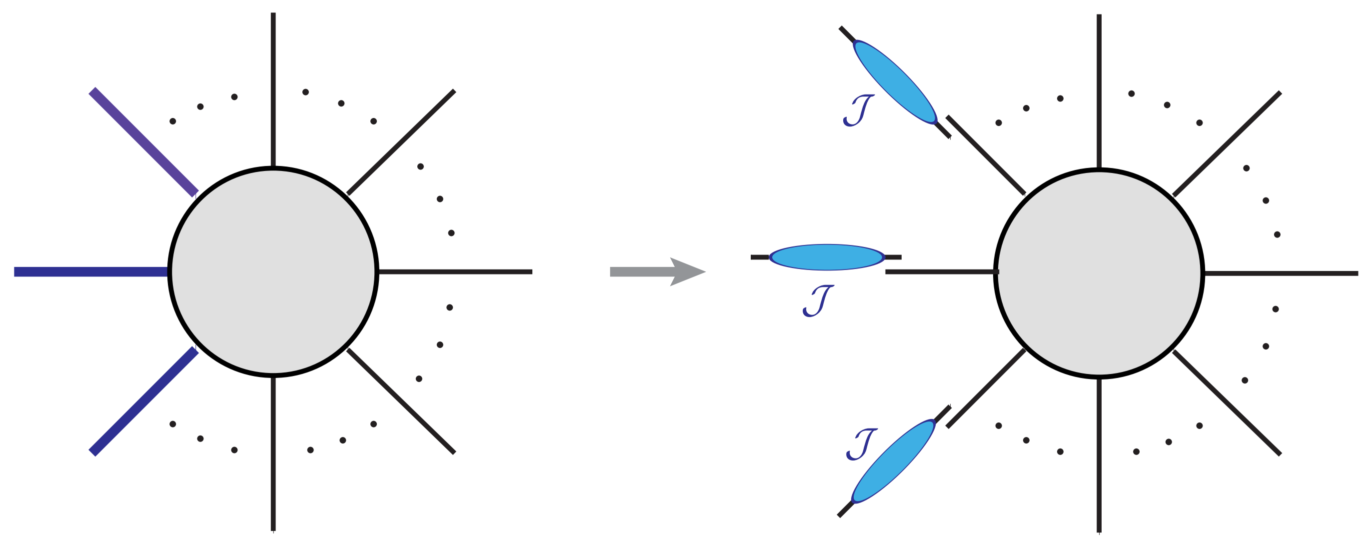
where we have used the relation:

$$\sum_{(i,j)}^{i,j\neq 1,2}\mathcal{T}_{12ij} = -\frac{C_A^2}{8}\boldsymbol{T}_1\cdot\boldsymbol{T}_2 - \mathcal{T}_{1122} - \sum_{i\neq 1,2}\mathcal{T}_{12ii} - \sum_{I}\mathcal{T}_{12II} - \sum_{I,i}^{i\neq 1,2}\left(\mathcal{T}_{12iI} + \mathcal{T}_{12Ii}\right) - \sum_{(I,J)}\mathcal{T}_{12IJ}$$

Small-Mass Limit

- When masses of external legs are much smaller than the hard scales

Mitov, Moch, '06
Becher, Melnikov, '07



collinear singularities regularized by masses

purely massless amplitude

$$\lim_{m\to 0} |\mathcal{M}(\{\underline{p}\},\{\underline{m}\},\epsilon)\rangle \simeq \prod_I \mathcal{J}_I(\{\underline{m}\},\epsilon) S(\{\underline{m},\epsilon\}) |\mathcal{M}(\{\underline{p}\},\{\underline{0}\},\epsilon)\rangle$$

Heavy quark loops

$$\mathbf{\Gamma}(\{\underline{p}\},\{\underline{m}\to 0\},\mu) - \mathbf{\Gamma}(\{\underline{p}\},\{\underline{0}\},\mu) = \sum_I \left[C_{R_I}\gamma_{\rm cusp}(\alpha_s)\ln\frac{\mu}{m_I} + \gamma^Q - \gamma^q \right]$$

There is no color exchange between different external legs, so

$$F_{\rm h2}(0,\alpha_s) = 3f(\alpha_s), \qquad \lim_{v_I^2\to 0} F_{\rm h3}(r_{ijI},r_{ikI},r_{jkI},\alpha_s) = 2f(\alpha_s) + 4F_4(\beta_{ijkI},\beta_{ijkI}-2\beta_{kjiI},\alpha_s)$$

ZLL, Schalch, '22

Small-Mass Limit

- In small-mass limit

$$\lim_{m\to 0}\beta_{IJ} = \lim_{m\to 0}\cosh^{-1}\left(\frac{-s_{IJ}}{2m_I m_J}\right) \simeq \ln\frac{\mu}{m_I} + \ln\frac{\mu}{m_J} - \ln\frac{\mu^2}{-s_{IJ}}$$

we have

$$\begin{aligned}\boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{m}\to 0\},\mu) - \boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{0}\},\mu) = &\sum_{I,i}\boldsymbol{T}_I\cdot\boldsymbol{T}_i\ln\frac{m_I}{\mu} + \sum_{(I,J)}\boldsymbol{T}_I\cdot\boldsymbol{T}_J\ln\frac{m_I}{\mu} + \sum_I\left(\gamma^Q - \gamma^q\right)\\ &+\sum_I\sum_{(i,j)}\Big[\mathcal{T}_{ijII}F_{\rm h2}(0,\alpha_s) - \left(\mathcal{T}_{ijII} + \mathcal{T}_{iiIj} + \mathcal{T}_{jjiI}\right)f(\alpha_s)\Big]\\ &+\sum_I\sum_{(i,j,k)}\mathcal{T}_{ijkI}\left[\lim_{v_I^2\to 0}F_{\rm h3}(r_{ijI},r_{ikI},r_{jkI},\alpha_s) - 4F_4(\beta_{ijkI},\beta_{ijkI}-2\beta_{kjiI},\alpha_s)\right]\end{aligned}$$

using

$$\mathcal{T}_{ijII} = \frac{1}{2}\left(\mathcal{T}_{jjiI} + \mathcal{T}_{iijI}\right) - \frac{1}{2}\sum_{k\neq i,j}\left(\mathcal{T}_{ijkI} + \mathcal{T}_{jikI}\right) - \frac{1}{2}\sum_{J\neq I}\left(\mathcal{T}_{ijIJ} + \mathcal{T}_{jiIJ}\right)$$

$$F_{\rm h2}(0,\alpha_s) = 3f(\alpha_s), \qquad \lim_{v_I^2\to 0}F_{\rm h3}(r_{ijI},r_{ikI},r_{jkI},\alpha_s) = 2f(\alpha_s) + 4F_4(\beta_{ijkI},\beta_{ijkI}-2\beta_{kjiI},\alpha_s)$$

ZLL, Schalch, ‘22

State of Art of Anomalous Dimensions

- Four-loop γ_{cusp} and $\gamma^{q/g}$

Henn, Smirnov, Smirnov, Steinhauser, '16; + Lee, '16;
Lee, Smirnov, Smirnov, Steinhauser, '17,'19;
von Manteuffel, Schabinger, '19, + Panzer '20, + Agarwal '21;

massless form factors
γ_{cusp} and $\gamma^{q/g}$ are complete

Davies, Vogt, Ruijl, Ueda, Vermaseren, '16;
Moch, Ruijl, Ueda, Vermaseren, Vogt, '17, '18

splitting functions for γ_{cusp}

Grozin, '18; Henn, Peraro, Stahlhofen, Wasser, '19;
Henn, Korchemsky, Mistlberger, '19;

Wilson loop for γ_{cusp}

Brüser, Grozin, Henn, Stahlhofen, '19;
Brüser, Dlapa, Henn, Yan, '20

angle-dependent $\gamma_{\text{cusp}}(\beta)$
still incomplete at 4 loops

- Three-loop $\gamma_{\text{cusp}}(\beta)$ and γ^Q

Grozin, Henn, Korchemsky, Marquard, '14,'15
Brüser, ZLL, Stahlhofen, '19 only for γ^Q

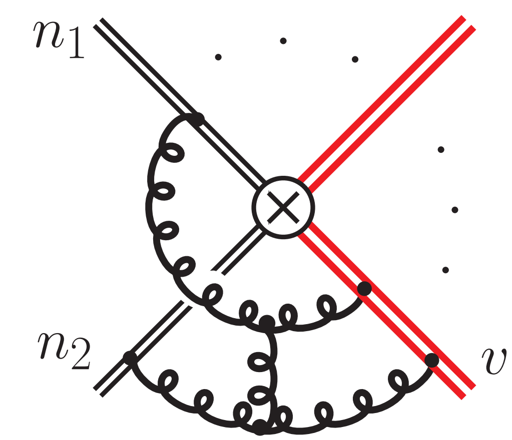
$\gamma_{\text{cusp}}(\beta)$ and γ^Q are complete at 3 loops

- Three-loop non-dipole terms

Almelid, Duhr, Gardi, '15; + McLeod, White, '17 f and F_4
ZLL, Stahlhofen, '20 only for f

Calculation of Tripole Correlation

- We calculate three-loop tripole correlation



$$F_{\mathrm{h2}}(r, \alpha_s) = \left(\frac{\alpha_s}{4\pi}\right)^3 \mathcal{F}_{\mathrm{h2}}(r) + \mathcal{O}(\alpha_s^4)$$

$$\text{with} \quad r = \frac{v^2(n_1 \cdot n_2)}{2(v \cdot n_1)(v \cdot n_2)}$$

- Regularization of IR poles
  - Configuration space with exponential regulators

Gardi, '13;
Almelid, Duhr, Gardi, '15

$$i g_s \int_0^\infty dt\ n_i \cdot A(t n_i) \to i g_s \int_0^\infty dt\ e^{-iwt\sqrt{n_i^2 - i0}} n_i \cdot A(t n_i)$$

Calculation in Feynman gauge and Mellin-Barnes representation

Perform asymptotic expansion $(n_i \cdot n_j)/\sqrt{n_i^2 n_j^2} \to \infty$

  - Momentum space with off-shellness

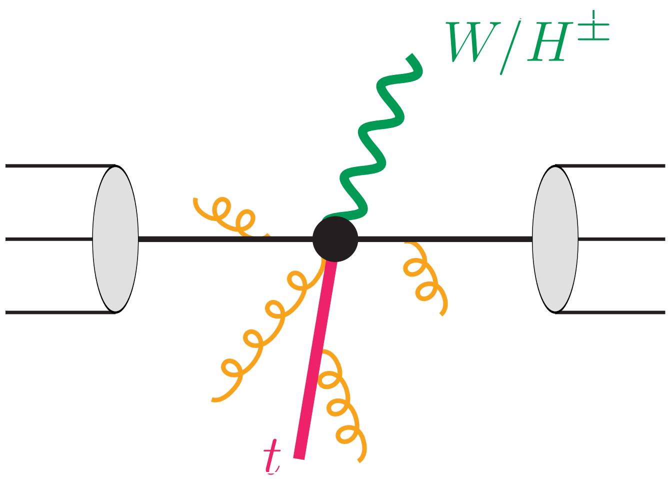
Ferroglia, Neubert, Pecjak, Yang, '09

$$\frac{1}{n \cdot k} \to \frac{1}{n \cdot k + \delta} \qquad \frac{1}{v \cdot k} \to \frac{1}{v \cdot k + \delta'}$$

Gauge invariance is not guaranteed !

IR Regulator

- In SCET, low-energy matrix elements are free of IR poles



Construct a soft function at cross section level

$$\boldsymbol{S}(\omega) = \langle 0 | \boldsymbol{Y}_{n_1}^\dagger \boldsymbol{Y}_{n_2}^\dagger \boldsymbol{Y}_v^\dagger \, \delta(\omega - v \cdot \hat{p}) \, \boldsymbol{Y}_v \, \boldsymbol{Y}_{n_1} \boldsymbol{Y}_{n_2} | 0 \rangle$$

Pick up momenta of all soft emissions

IR poles are regularized by the low-energy measurement

Soft function has applications in phenomenology, e.g.

$$\sigma(pp \to tW) \sim f_{a/N} \otimes f_{b/N} \otimes H_{ab \to tW} \otimes S$$

describe the soft-gluon effect near threshold in tW associated production

- Several advantages
  - ω is the only dimensionful variable, does NOT increase complexity of integrals
  - Gauge invariance is preserved, calculation can be in general covariant gauge
  - Calculation in momentum space, IBP and differential equation (DE) can be used

Phase Space to Loop Integration

- Soft function is defined at cross section level

$$\boldsymbol{S}(\omega) = \langle 0 | \boldsymbol{Y}_{n_1}^\dagger \boldsymbol{Y}_{n_2}^\dagger \boldsymbol{Y}_v^\dagger \, \delta(\omega - v \cdot \hat{p}) \, \boldsymbol{Y}_v \, \boldsymbol{Y}_{n_1} \boldsymbol{Y}_{n_2} | 0 \rangle$$

Standard procedure is to compute VVR + VRR + RRR   V: virtual, R: real

Can we find a compact way?

- Transform phase-space integrations to loop integrals ZLL, Stahlhofen, '20

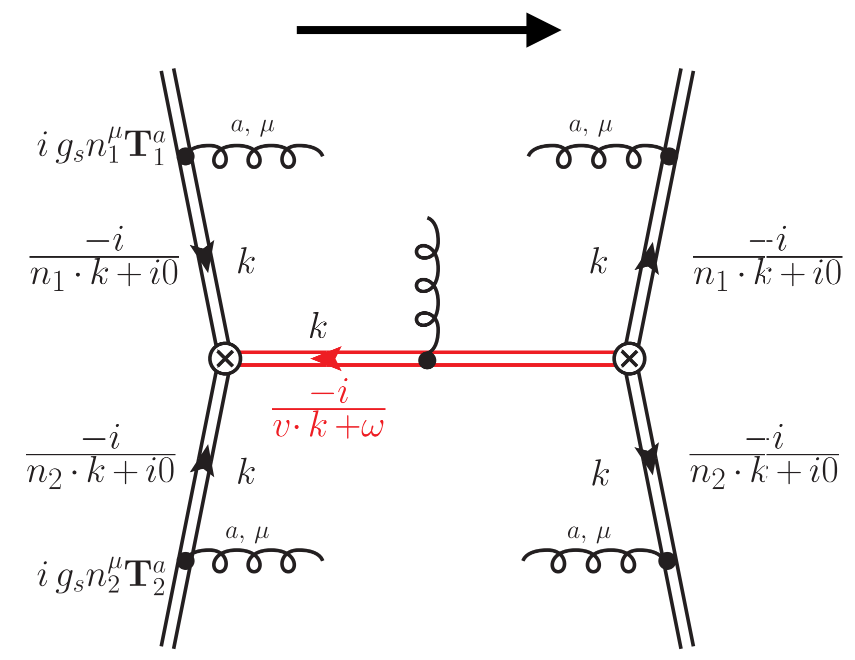
$$\boldsymbol{Y}_{n_i} = \mathrm{P} \exp\left(i g_s \int_{-\infty}^{0} dt \, n_i \cdot A^a(x + t n_i) \boldsymbol{T}_i^a \right)$$

$$\boldsymbol{Y}_v^\dagger \, \delta(\omega + i v \cdot \partial) \, \boldsymbol{Y}_v = \delta(\omega + i v \cdot D)$$

$$\delta(x) = \frac{1}{2\pi i} \left(\frac{1}{x - i0} - \frac{1}{x + i0} \right)$$ reverse unitarity

$$\boldsymbol{S}(\omega) = \mathrm{Re} \left[\mathrm{Disc}_\omega \langle 0 | \boldsymbol{Y}_{n_1}^\dagger \boldsymbol{Y}_{n_2}^\dagger \frac{1}{\omega + i v \cdot D} \boldsymbol{Y}_{n_1} \boldsymbol{Y}_{n_2} | 0 \rangle \right]$$

Forward scattering

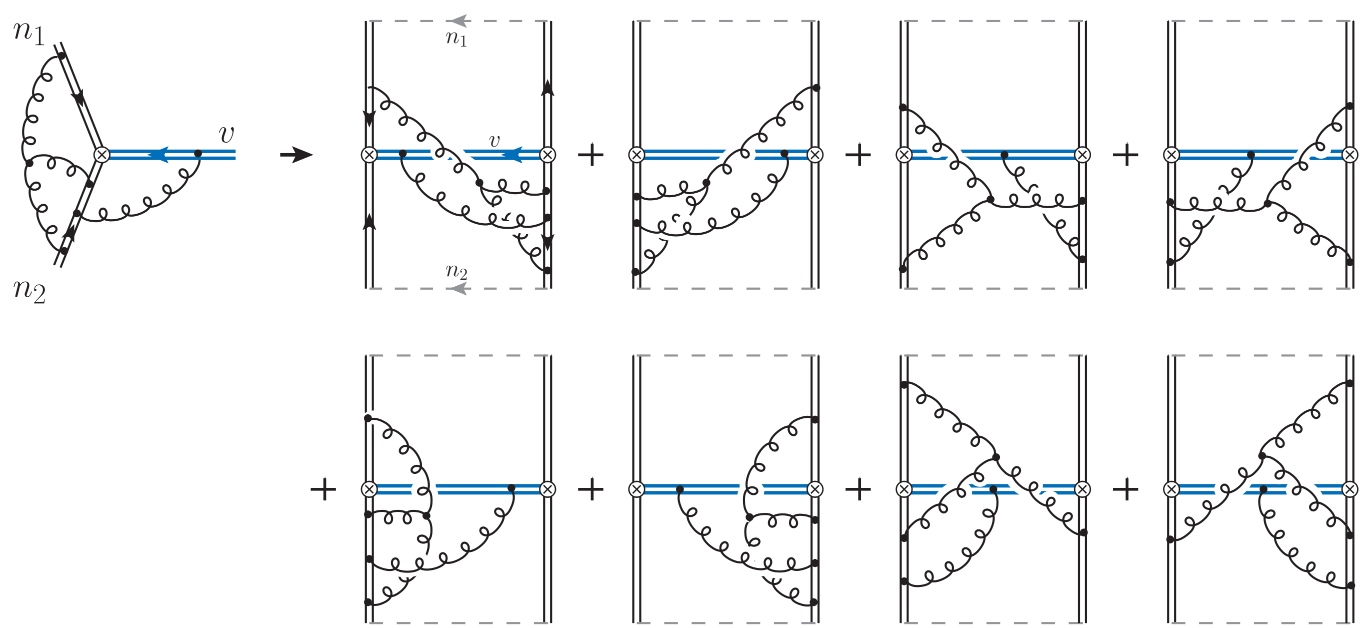


Imaginary part from branch cut in $(-n_1 \cdot n_2 - i0)^{-n\epsilon}$ cancels between left-right mirror diagrams

Calculation

- Only consider the diagrams contribute to $\mathcal{F}_{\mathrm{h}2}(r)$

  Replica trick for evaluating diagrammatic contribution to the exponent is compatible



Nogueira, '93

1. Using QGRAF to generate diagrams, 976 in total
2. Perform partial fraction, the integrals are mapped onto 30 topologies
3. IBP reduction by FIRE6 and Kira, there are 173 master integrals (MIs)
   Smirnov, Chuharev, '19 Usovitsch et al., '20
4. Using CANONICA and DlogBasis, convert DE to a canonical form
   Meyer, '17 Henn, Mistlberger, Smirnov, Wasser, '20 Henn, '13
5. Determine boundary conditions at $r = 1 \quad (v^\mu = n_1^\mu + n_2^\mu)$

Solve the DEs

- Symbol alphabet is $\{r, r-1, r-2, (r-1)\sqrt{r}, \sqrt{r(r-1)}\}$

  rationalized by $u = \sqrt{r} - 1$

- Solve the DEs order-by-order in ϵ in terms of GPL and GHPL

GPL $$G(a_1, \ldots, a_n; x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

GHPL $$G(-\rho, \vec{a}; x) = \int_0^x \frac{dt}{\sqrt{t(t+4)}} G(\vec{a}; t)$$ Aglietti, Bonciani, ’04

- Validity of the calculation
  - ✓ Gauge parameter ξ cancels out
  - ✓ GHPLs drop out, because there is no heavy quark pair threshold
  - ✓ All the poles from $1/\epsilon^5$ to $1/\epsilon$ cancels out

Result

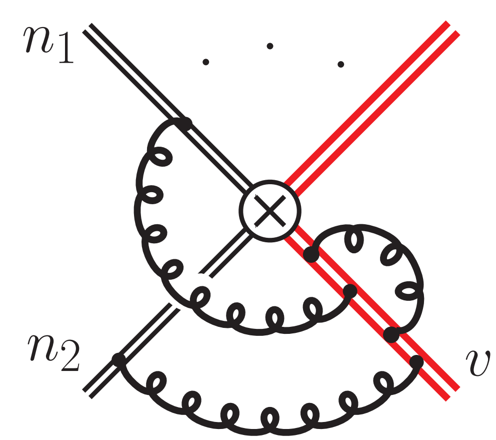
- Three-loop tripole correlation can be simplified as $\hat{r} = \sqrt{r}$

$$
\begin{aligned}
\mathcal{F}_{\mathrm{h2}}(r) =& 128\Big[H_{-1,0,0,0}(\hat{r}) + H_{-1,1,0,0}(\hat{r}) + H_{1,-1,0,0}(\hat{r}) - H_{1,0,0,0}(\hat{r})\Big] \\
&+ 128(\zeta_2 + \zeta_3)\Big[H_{1,0}(\hat{r}) - H_{-1,0}(\hat{r})\Big] + 96(\zeta_3 + \zeta_4)\Big[H_{-1}(\hat{r}) - H_1(\hat{r})\Big] \\
&+ 128\zeta_2\Big[H_{-2,0}(\hat{r}) - H_{2,0}(\hat{r}) + H_{-1,0,0}(\hat{r}) - H_{1,0,0}(\hat{r})\Big] \\
&+ 256\Big[H_{1,2,0,0}(\hat{r}) + H_{2,0,0,0}(\hat{r}) - H_{-2,0,0,0}(\hat{r}) + H_{-1,-2,0,0}(\hat{r}) \\
&\quad - H_{-1,2,0,0}(\hat{r}) - H_{1,-2,0,0}(\hat{r}) - H_{-1,0,0,0,0}(\hat{r}) + H_{1,0,0,0,0}(\hat{r})\Big] \\
&+ 48(2\zeta_2\zeta_3 + \zeta_5)
\end{aligned}
$$

weight 4

- HPLs are sufficient to describe the result
- Returns to f in massless case when $r \to 0$
- Does NOT have a uniform transcendental weight $2L-1$, unlike f and F_4

Boomerang webs



Gardi, Harley, Lodin, Palusa, Smillie, White, Yeomans, '21

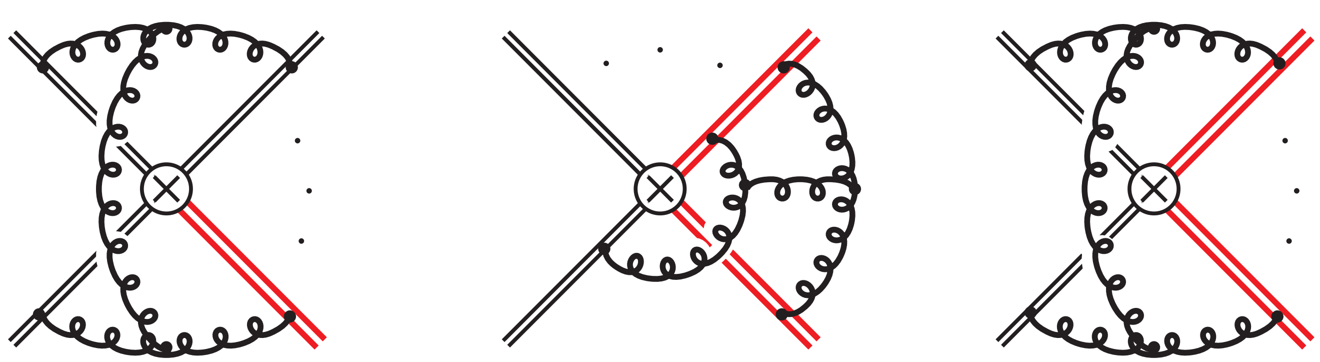
Summary

- Complete the general structure of three-loop anomalous dimensions for QCD amplitudes with a massive and an arbitrary number of external legs

- Obtain the relations in two-particle collinear and small-mass limits

- Calculate the contribution from the tripole correlation between one massive and two massless legs

- Wish list



Thanks for your attention!

Replica Trick

- Soft exponent can be extracted from   Gardi, Laenen, Stavenga, White '10

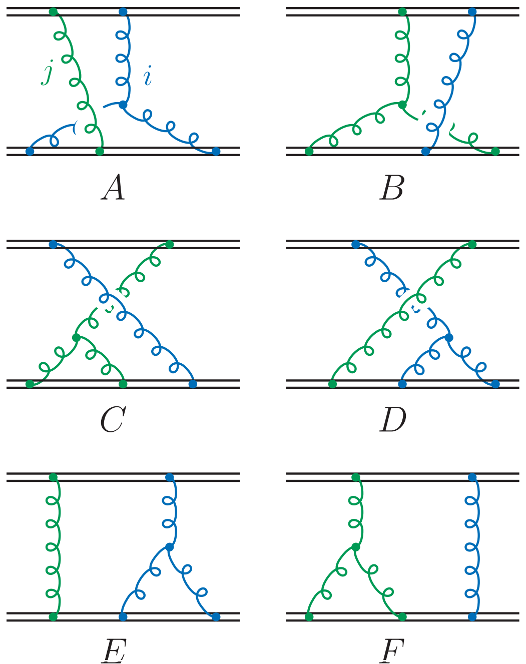
$$\ln \boldsymbol{S} = \lim_{N\to 0} \frac{\boldsymbol{S}^N - 1}{N}$$

Then consider a theory with N non-interacting copies of gauge fields

$$\mathcal{R}\,\mathbf{P}\exp\left[i\sum_{j=1}^{N}\int ds_k\, n_k\cdot A^{k(j)}(s_k n_k)\right] \quad \text{with} \quad \mathcal{R}\left[\boldsymbol{T}_k^{(i)}\,\boldsymbol{T}_k^{(j)}\right] = \begin{cases} \boldsymbol{T}_k^{(i)}\,\boldsymbol{T}_k^{(j)} & i\le j \\ \boldsymbol{T}_k^{(j)}\,\boldsymbol{T}_k^{(i)} & i > j \end{cases}$$

replica ordering operator

Diagram	replica	color	multiplicity	Diagram	replica	color	multiplicity
A	$i=j$	$C(A)$	N	D	$i=j$	$C(D)$	N
	$i>j$	$C(E)$	$N(N-1)/2$		$i>j$	$C(E)$	$N(N-1)/2$
	$i<j$	$C(F)$	$N(N-1)/2$		$i<j$	$C(F)$	$N(N-1)/2$
B	$i=j$	$C(B)$	N	E	$i=j$	$C(E)$	N
	$i>j$	$C(F)$	$N(N-1)/2$		$i>j$	$C(E)$	$N(N-1)/2$
	$i<j$	$C(E)$	$N(N-1)/2$		$i<j$	$C(F)$	$N(N-1)/2$
C	$i=j$	$C(C)$	N	F	$i=j$	$C(F)$	N
	$i>j$	$C(F)$	$N(N-1)/2$		$i>j$	$C(F)$	$N(N-1)/2$
	$i<j$	$C(E)$	$N(N-1)/2$		$i<j$	$C(E)$	$N(N-1)/2$



$$\sum_{D'} R_{DD'}C(D') = \left(f^{bde}f^{cae}\boldsymbol{T}_1^a\boldsymbol{T}_1^b\boldsymbol{T}_2^c\boldsymbol{T}_2^d,\ f^{bde}f^{cae}\boldsymbol{T}_1^a\boldsymbol{T}_1^b\boldsymbol{T}_2^c\boldsymbol{T}_2^d,\ \frac{C_A^2}{4}\boldsymbol{T}_1^a\boldsymbol{T}_2^a,\ \frac{C_A^2}{4}\boldsymbol{T}_1^a\boldsymbol{T}_2^a,\ 0,\ 0\right)^T$$

Boundary Conditions of the MIs

- The scale dependence of integrals is trivial at boundary $r = 1$
  - Mellin-Barnes representation + PSLQ
  - Sector decomposition
  - Dimensional Recurrence Relations (DRR) is adopted

Difficult to compute 3-loop integrals analytically

von Manteuffel, Panzer, Schabinger, '14 '15

$$G_i(d+2) \xrightarrow{\text{DRR}} \sum_j c_j(d) F_j(d) \xrightarrow{\text{IBP}} \sum_j \tilde{c}_j(d) G_j(d)$$

MIs

DRR: LiteRed (Lee, '12)

↓

Related to a set of finite integrals $F_i(d = n - 2\epsilon)$ with $n = 4, 6, 8, \ldots$

evaluated by HyperInt   Panzer, '15

increase the dimension to D=6 → decrease IR poles

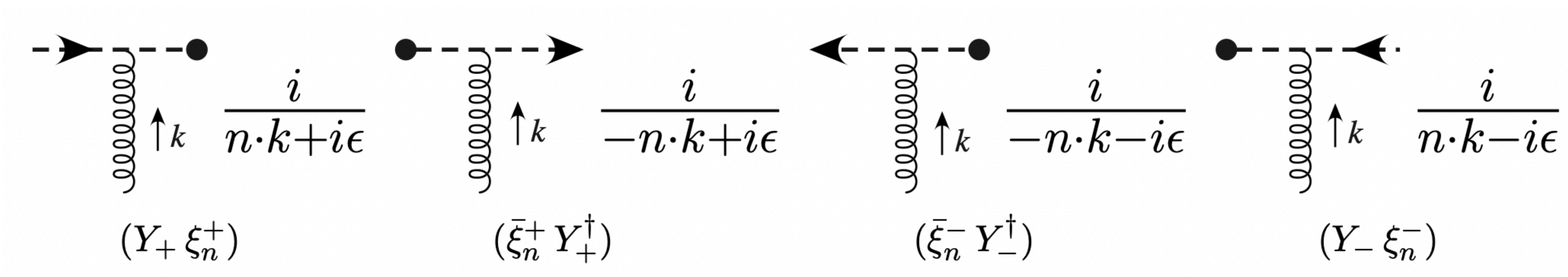
increase the power of propagators → decrease UV poles

Each MI can be expressed by a linear combination of finite integrals

Backup

- Eikonal $i\varepsilon$ prescriptions for incoming/outgoing quarks and anti-quarks

Figure from Arnesen, Kundu, Stewart, 0508214



incoming quark | outgoing quark | incoming antiquark | outgoing antiquark

$$\boldsymbol{Y}_+(x) = \bar{\mathrm{P}}\exp\left[-ig_s\int_{-\infty}^{0} dt\, n\cdot A^a(x+tn)\boldsymbol{T}^a\right]$$

$$\boldsymbol{Y}_-(x) = \mathrm{P}\exp\left[+ig_s\int_{0}^{\infty} dt\, n\cdot A^a(x+tn)\boldsymbol{T}^a\right]$$

$$\boldsymbol{Y}_\pm^\dagger = (\boldsymbol{Y}_\mp)^\dagger$$

Phase Space to Loop Integration

- Soft function is defined at cross section level

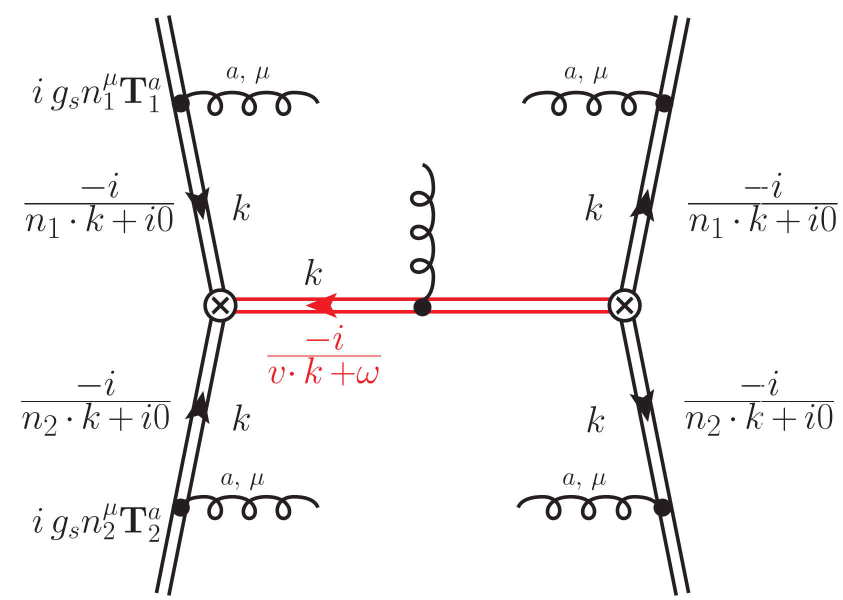
$$\boldsymbol{S}(\omega) = \langle 0|\overline{\mathrm{T}}\left[\boldsymbol{Y}^\dagger_{n_1}\boldsymbol{Y}^\dagger_{n_2}\boldsymbol{Y}^\dagger_{v}\right]\delta(\omega - v\cdot\hat{p})\mathrm{T}\left[\boldsymbol{Y}_{n_1}\boldsymbol{Y}_{n_2}\boldsymbol{Y}_{v}\right]|0\rangle$$

- Transform phase-space integrations to loop integrals

ZLL, Stahlhofen, '20

$$\boldsymbol{Y}_{i,+}(x) = \bar{\mathrm{P}}\exp\left[-ig_s\int_{-\infty}^{0} dt\, n_i\cdot A^a(x+tn_i)\boldsymbol{T}^a\right]$$

$$\boldsymbol{Y}_{i,-}(x) = \mathrm{P}\exp\left[+ig_s\int_{0}^{\infty} dt\, n_i\cdot A^a(x+tn_i)\boldsymbol{T}^a\right]$$



$$\boldsymbol{S}(\omega) = \frac{1}{2\pi}\mathrm{Re}\left[\boldsymbol{\mathcal{S}}(\omega+i0) - \boldsymbol{\mathcal{S}}(\omega-i0)\right]$$

$$\boldsymbol{\mathcal{S}}(\omega) = \int_0^\infty dt\, e^{i\omega t}\langle 0|\mathrm{T}\left[\boldsymbol{Y}^\dagger_{n_1}(tv)\boldsymbol{Y}^\dagger_{n_2}(tv)\mathrm{P}\exp\left[ig\int_0^t ds\, v\cdot A^c(sv)\boldsymbol{T}^c_v\right]\boldsymbol{Y}_{n_1}(0)\boldsymbol{Y}_{n_2}(0)\right]|0\rangle$$

outgoing — incoming

Result

- Three-loop tripole correlation can be simplified as $\hat{r} = \sqrt{r}$

$$
\begin{aligned}
\mathcal{F}_{\rm h2}(r) =& 128\Big[H_{-1,0,0,0}(\hat{r}) + H_{-1,1,0,0}(\hat{r}) + H_{1,-1,0,0}(\hat{r}) - H_{1,0,0,0}(\hat{r})\Big] \\
&+ 128(\zeta_2+\zeta_3)\Big[H_{1,0}(\hat{r}) - H_{-1,0}(\hat{r})\Big] + 96(\zeta_3+\zeta_4)\Big[H_{-1}(\hat{r}) - H_1(\hat{r})\Big] \\
&+ 128\zeta_2\Big[H_{-2,0}(\hat{r}) - H_{2,0}(\hat{r}) + H_{-1,0,0}(\hat{r}) - H_{1,0,0}(\hat{r})\Big] \\
&+ 256\Big[H_{1,2,0,0}(\hat{r}) + H_{2,0,0,0}(\hat{r}) - H_{-2,0,0,0}(\hat{r}) + H_{-1,-2,0,0}(\hat{r}) \\
&\qquad - H_{-1,2,0,0}(\hat{r}) - H_{1,-2,0,0}(\hat{r}) - H_{-1,0,0,0,0}(\hat{r}) + H_{1,0,0,0,0}(\hat{r})\Big] \\
&+ 48(2\zeta_2\zeta_3+\zeta_5)
\end{aligned}
$$

- Does $\mathcal{F}_{\rm h2}(r_{ijI})$ have an imaginary part?

$$
r_{ijI} \equiv \frac{v_I^2\,(n_i\cdot n_j)}{2\,(v_I\cdot n_i)(v_I\cdot n_j)} = -e^{\beta_{ij}-\beta_{Ii}-\beta_{Ij}}
$$

$0 < r_{ijI} < 1$ in full kinematic region

$$
F_{\rm h2}(0,\alpha_s) = 3f(\alpha_s) = \left(\frac{\alpha_s}{4\pi}\right)^3 48(2\zeta_2\zeta_3+\zeta_5) + \mathcal{O}(\alpha_s^4)
$$