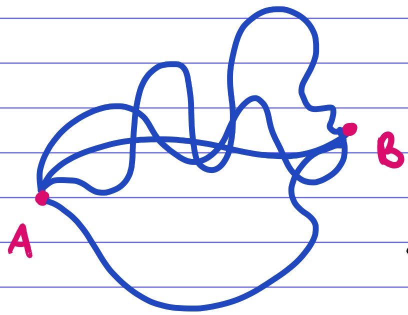
Counting Gromov-Witten invariants using localization

Miroslava Mosso Rojas

- **Feynman path integral:**



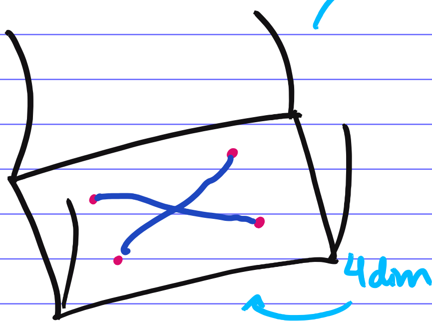
correlation function

$$= \int_{\text{space of paths}} (\text{integrand})(\text{measure})$$

- **In string theory:**

fibers of this space are 6 dim mfds, Calabi-Yau (compact)

X

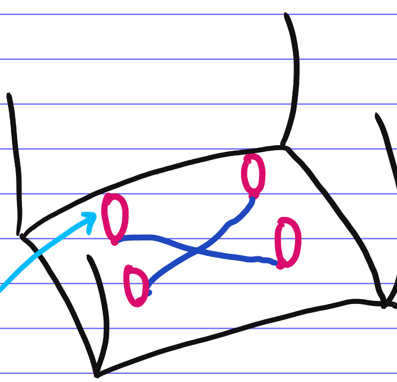


solving Einstein's eq

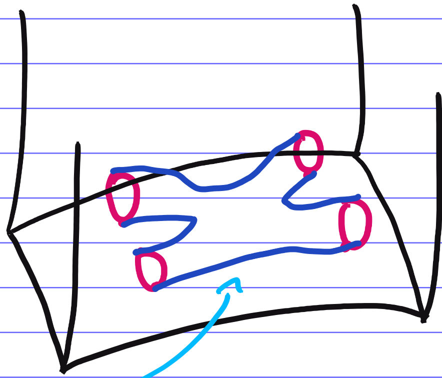
X = Ricci flat background

dim X = 10

Now, particles are extended to 1-dim curves (strings)



These paths that were curves are now surfaces Σ



Riemann surfaces Σ (worldsheet)

$$\int_{\text{family of } f\,:\,\Sigma \to X} (\text{integrand})$$

embedded

$$\Sigma \hookrightarrow X$$

To compute the integral we need to know how many of these Σ we have.

Σ is an algebraic curve

$$C \hookrightarrow X$$

ambient algebraic variety

$\mathcal{M}$ = moduli space = parameter space for objects of some kind

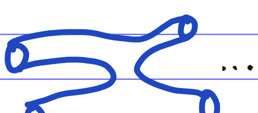
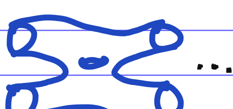
$\mathcal{M} = \left\{ \begin{array}{l} \text{moduli space of paths or algebraic} \\ \text{curves } \mathcal{C} \text{ in an ambient variety } X \end{array} \right\}$

- Gromov-Witten invariants :

  - Studying the moduli space that parametrizes algebraic curves in ambient algebraic varieties.
  - Counting these curves.
  - They are related to quantum corrections of the topological string (instantons)

- To count them we need finitely many, we impose some constraints.
- How can we classify Σ in $\mathcal{M}$?

↳ ① By its topological properties (fixing the genus)

$\mathcal{M}_0 = \Big\{$  $\Big\}$, $\mathcal{M}_1 = \Big\{$  $\Big\}$

$\mathcal{M}_2, \dots, \mathcal{M}_g \rightsquigarrow$ genus (holes)

② By its homology class:

$$C \xhookrightarrow{f} X$$

embedding map

push-forward the fundamental class of $\mathcal{C}$ into X

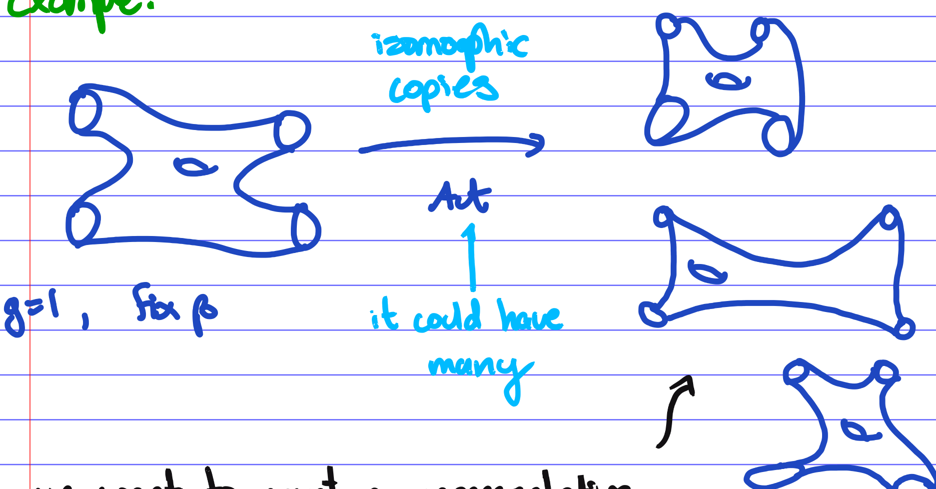
$$f_*[C] = \beta$$

A homology class

$$\mathcal{M}_{\underbrace{g,\beta}_{\text{fixed}}}$$

Σ s allow to deform in many possible ways, we might be overcounting Σs

Example:



we want to count a representative class of Σ. Count up to finite automorphisms.

G-W $= \dfrac{\# \Sigma_{g,\beta}(X)}{\left(\text{order of finite aut}\right)}$ = counting configurations of curves $\mathcal{C}$ with marked points $(p_1, \dots, p_n)$ together with maps $f : (C, p_1, \dots, p_n) \longrightarrow X$

Example:

When $g=0$ → tree level, the maps $f : \Sigma_g \longrightarrow X$ are holomorphic and correspond to instanton corrections.

- How do we compute the G-W invariants?

↳ By localization.

Computing integrals over spaces:

- To get the volume of the space we integrate the volume form.

- To get topological invariants to enumerative problems, we integrate over characteristic classes.

Idea of localization:

Represent global values as a combination of local contributions.

• **Atiyah-Bott localization:**

X smooth mfd, admits a G-action
The integral on X localizes at the fixed points

Fixed loci

inclusion

$$i : F = X^G \hookrightarrow X$$

set of fixed points p with associated normal bundles ν_p in X

equivariant class

$$\int_X \phi = \sum_{p \in F} \int_F \frac{i^* \phi}{e(\nu_p)}$$

pull-back

Euler characteristic

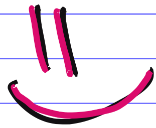
General idea:

If X has a G-action, the G-W invariants of X can be computed from the G-W invariants of the fixed point locus.

- Idea of our project:

Cohomology G-W theory	K-theory G-W
$\downarrow$	$\downarrow$
Strings	Membranes
$\downarrow$	$\downarrow$
worldsheets instantons	membrane vortex solutions
Σ_g	$S^1 \times \Sigma_g$
$f: \Sigma_g \longrightarrow X$	$f: S^1 \times \Sigma_g \longrightarrow X$

THANK YOU !



Glossary (in simple terms for everyone)

Riemann surface:- It is a 1-dim cx mfd. Which means a differentiable mfd, real 2-dim mfd with choice of cx coordinates and holomorphic transition functions between coordinate charts.

Holomorphic map:- cx valued function that is cx differentiable in a neighbourhood of each point, it satisfies the Cauchy-Riemann eqs.

Algebraic curve:-

- It is a 1-dim cx mfd defined by a system of polynomial equations.
- An algebraic curve over a field $(\mathbb{C}, \mathbb{Q}, \mathbb{R})$ K is an equation $f(x_1, \dots, x_n) = 0$ where f is a polynomial in $x_1, \dots, x_n$ with coeff. in K

Algebraic variety

Geometric object defined by polynomial equations. It is a collection of points or solutions in cn n-dim space that satisfy a set of pol. eqs.

Instantons:- Non-perturbative sols. to the field equations of a QFT.

- They are topological configurations of the field that are tipically localized in s-t. They represent saddle points in the action functional of the field theory and can lead to tunneling-like processes between different classical vacua.

Embedding:- Injective map $f: X \longrightarrow Y$

Push-forward:- Let $\varphi: M \rightarrow N$ a sth map, given $x \in M$, the differential of φ at x is a map:

$$d\varphi_x : T_x M \longrightarrow T_{\varphi(x)} N$$
$$v \in T_x M, \quad w \in T_{\varphi(x)} N$$

it maps vectors from one mfd to another

$\phi_* v = d\phi_x v$ the image $d\phi_x v$ of a tangent vector $v \in T_x M$ under $d\phi_x$ is the pushforward of X by ϕ.

Pull-back:- $\phi: M \longrightarrow N$

$$(\phi^* \alpha)_p = \alpha_{\phi(p)} \circ d\phi_p$$

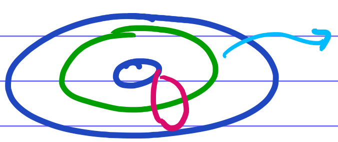
it maps differential forms from one mfd to another

α is a diff. form on N and $\phi^*\alpha$ is a new diff. form on M.

Fixed points or fixed locus: When we have a group G acting on M, the fixed locus of this action is M^G. These are points in M that are invariant under the action of the group G.

Homology class: It is an equivalence class of cycles. Used to categorize topological features of a space, it helps to count and classify holes.

exa:



since the torus is a 2-dim slce, it has 2 fundamental homology classes.

2nd homology class is a 2 dim surface that covers it including the hole.

Fundamental class: It is a homology class that represents the entire top. space.

Exa: The 2nd homology class of the torus is the fundamental one.

Characteristic class:- It is a concept that associates topological invariants to vector bundles.

- A characteristic class takes a vector bundle as input and produces an element in a cohomology group of the base space.

Vector bundle:- Geometric construction that associates a vector space with each point on a manifold

The Euler class:- It is a characteristic class of oriented vector bundles, it measures how twisted the vector bundle is: if the Euler class is 0, the vector bundle can be trivialized and has a nowhere vanishing section. If the Euler class is non-zero, the vector bundle cannot be globally trivialized, and there are topological obstructions to finding a nowhere vanishing section.

Topological invariant:- it is independent of the choice of connection or metric.

Equivariant class:- Is a generalization of ordinary cohomology classes to the setting where group actions are available.

Summary of the de Rham cohomology

- M an n-dim mfd, ω an r-form $\in \Omega^r(M)$ which is closed $d\omega = 0$ and exact $\omega = d\eta$ for some $\eta \in \Omega^{r-1}(M)$.
- The set of closed r-forms is $Z^r(M)$ and the set of exact r-forms is $B^r(M)$
- The de Rham cohomology group $H^r(M)$ is

$$H^r(M) = Z^r(M) / B^r(M)$$

Calabi-Yau mfd:- It is a compact, cx kähler mfd that satisfies the CY condition.

- Being kähler means it is cx mfd with a compatible Riemannan metric that is Hermitian and symplectic.
- The CY condition is that it is Ricci flat