Renormalization of (Scalar) EFTs using a Geometrical Approach

Aneesh Manohar

(work done with Elizabeth Jenkins, Luca Naterop, Julie Pagès)

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Non-linear Sigma Models

Non-linear sigma models describe pion dynamics.

$SU(2)$ chiral perturbation theory: fields live on scalar manifold $\mathcal{M} = SU(2) = S^3$.

Different parameterizations give the same S-matrix elements

$$(\pi, \sqrt{f^2 - \pi \cdot \pi}) \qquad U = e^{2i\pi/f} \qquad U = u^2$$

The dynamics is governed by $\mathcal{M}$. Curvature is $1/f^2$.

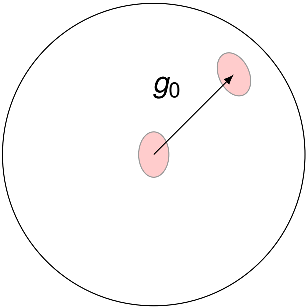
Independence under field redefinitions

Chisholm, NP 26 (1961) 469

Kamefuchi, O'Raifertaigh, Salam, NP 28 (1961) 529

Politzer, NPB 172 (1980) 349

Sigma model all points are equivalent. One can map a neighborhood of $g = \mathbb{1}$ to a neighborhood of g_0 by the map $g \mapsto g_0 g$.



SMEFT vs HEFT

Alonso, Jenkins, AM, PLB 754 (2016) 355, JHEP 08 (2016) 101 (arXiv version)

SM has spontaneous symmetry breaking $SU(2) \times U(1)_Y \to U(1)_{\text{EM}}$.

If you assume custodial symmetry, the breaking is $SO(4) \to SO(3)$, and $SO(4)/SO(3) \sim S^3$.

Three Goldstone bosons which are eaten by the $W^{\pm}, Z$, and at tree-level, $M_W = M_Z \cos\theta_W$.

SM Higgs doublet:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} i\varphi_1 + \varphi_2 \\ \text{h} - i\varphi_3 \end{bmatrix}$$

SMEFT

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} i\varphi_1 + \varphi_2 \\ \mathrm{h} - i\varphi_3 \end{bmatrix}$$

Usually use

$$\mathrm{h} = v + h$$

but could equally well use

$$v^2 = \mathrm{h}^2 + \boldsymbol{\varphi} \cdot \boldsymbol{\varphi}$$

or any of the parameterizations of the $SU(2)$ sigma model.

The theory is still renormalizable, and does not generate operators of arbitrary high dimension, even using the exponential parametrization.

HEFT

Arose from studying models where EW symmetry was broken by strong dynamics.

Know we have the Goldstone boson space S^3.

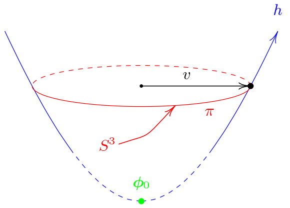
Typically, these models do not have a light scalar field

Assume that we have a field h plus a $SU(2) \times U(1) \to U(1)$ sigma model — write down χPT with an additional scalar field h.

SM has relations which arise from (h, φ) combining to form H, which are lost in HEFT — relations between radial and angular directions.

What is the difference between SMEFT and HEFT, and how do you tell experimentally?

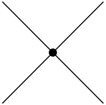
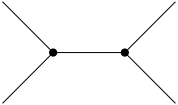
$\mathcal{M}$ is a four-dimensional space with a radial direction h and angular direction S^3.



Riemann curvature tensor

$$R_{abcd} = \mathfrak{R}_4(h)\widehat{R}_{abcd} \qquad R_{ahbh} = \mathfrak{R}_{2h}(h)\widehat{R}_{ab}$$

with a, b, c, d angular directions. $\widehat{R}_{abcd}$ and $\widehat{R}_{ab}$ are the curvature tensors of S^3 with radius v. In perturbation theory, probe small fluctuations about the vacuum, so $\mathfrak{R}(h=0) = \mathfrak{r}$.

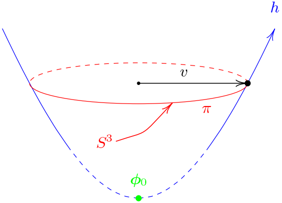
$$R \sim \partial\Gamma + \Gamma\,\Gamma$$

$$\begin{aligned}\mathcal{A}\,(W_L W_L \to W_L W_L) &= \frac{s+t}{v^2}\mathfrak{r}_4\,,\\ \mathcal{A}\,(W_L W_L \to hh) &= -\frac{2s}{v^2}\mathfrak{r}_{2h}\,.\end{aligned}$$

Amplitudes grow with energy, and scale of new physics is

$$\Lambda \sim \frac{4\pi v}{\sqrt{\mathfrak{r}}}$$

SM is flat, so $\mathfrak{R}$ = 0, and no bad high-energy behavior. Difference is not "linear vs non-linear" but flat vs. curved.



SMEFT if there is an $O(4)$ invariant fixed point on $\mathcal{M}$.

Alonso, Jenkins, AM, JHEP 08 (2016) 101

Additional work by

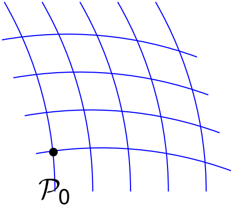
Cohen, Craig, Lu, Sutherland, JHEP 03 (2021) 237, JHEP 12 (2021) 003

Scale of new physics and validity of the EFT expansion.

Radiative Corrections

$\pi\pi$ scattering: compute the off-shell amplitude and counterterms, and find that they are not covariant (i.e do not respect the $G \to H$ symmetry).

Reason is that π does not have good symmetry transformation properties.

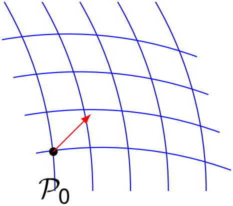


On a general curved manifold $\phi \in \mathcal{M}$, ϕ^i are coordinates, and are *not* tensors.

Tangent vectors

$$V^i = \frac{\mathrm{d}\phi^i}{\mathrm{d}\lambda}$$

are tensors and are associated with a point on $\mathcal{M}$. [Tangent bundle]



Under a change of coordinates (field redefinitions) $\phi \to \phi'$

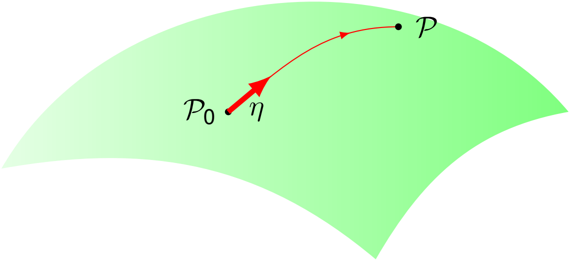
$$V'^{\,i} = \frac{\partial \phi'^{\,i}}{\partial \phi^j} V^j$$

Second and higher derivatives are not tensors, and do not transform covariantly.

For $\pi\pi$ scattering at one-loop, need π^4 vertices, i.e. expansion beyond linear order, which leads to non-covariance of off-shell amplitudes.

The S-matrix is covariant.

Solution is to use Riemann normal coordinates:



A geodesic starting at $\mathcal{P}_0$ with velocity η that ends at $\mathcal{P}$ in unit time. Then the coordinates η of $\mathcal{P}$ are a tensor at $\mathcal{P}_0$.

$$g_{ij}(\mathcal{P}_0) = \delta_{ij} \qquad \Gamma^i_{jk}(\mathcal{P}_0) = 0 \qquad g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} \phi^k \phi^l + \ldots$$

Lagrangian up to two derivatives

$$L = \frac{1}{2} g_{ij}(\phi)(D_\mu \phi)^i (D^\mu \phi)^j - V(\phi)$$

$$(D_\mu \phi)^i = \partial_\mu \phi^i + t^i_\alpha(\phi) A^\alpha_\mu$$

$(D_\mu \phi)^i$ transforms as a vector, so g_{ij} transforms as a metric. Usually

$$t^i_\alpha(\phi) = (T_\alpha)^i{}_j \phi^j$$

In SMEFT to dim 6

$$L = \frac{1}{2} D_\mu H^\dagger D^\mu H + C_{HD}(D_\mu H^\dagger H)(H^\dagger D_\mu H) - C_{H\Box} \partial_\mu (H^\dagger H) \partial^\mu (H^\dagger H)$$
$$- \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2 + C_H (H^\dagger H)^3$$

[Gauge fields included through covariant derivatives]

SMEFT metric

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} i\phi_1 + \phi_2 \\ \phi_4 - i\phi_3 \end{bmatrix}$$

$$g_{ij} = \delta_{ij} - 2C_{H\Box}\phi_i\phi_j + \frac{1}{2}C_{HD}\mathcal{H}_{ij}(\phi)$$

where

$$\mathcal{H}_{ij}(\phi) = \phi_i\phi_j + \begin{bmatrix} \phi_2^2 & -\phi_1\phi_2 & -\phi_2\phi_4 & \phi_2\phi_3 \\ -\phi_1\phi_2 & \phi_1^2 & \phi_1\phi_4 & -\phi_1\phi_3 \\ -\phi_2\phi_4 & \phi_1\phi_4 & \phi_4^2 & -\phi_3\phi_4 \\ \phi_2\phi_3 & -\phi_1\phi_3 & -\phi_3\phi_4 & \phi_3^2 \end{bmatrix}.$$

With $C_{H\Box}$ term only to dim 6:

$$\Gamma^i_{jk} = \frac{1}{2} g^{ir} (g_{rj,k} + g_{rk,j} - g_{jk,r}) = -2 C_{H\Box} \delta_{kj} \phi_i$$

$$R_{ijkl} = -2 C_{H\Box} \left(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right)$$

Radiative Corrections

Second derivatives are not tensors. So have to extend the formalism for higher derivative operators.

Cheung, Helset, Parra-Martinez, PRD 106 (2022) 045016

Cohen, Craig, Lu, Sutherland, PRL 130 (2023) 041603

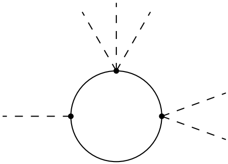
Craig, Yu-Tse, Lee, 2307.15742

Alminawi, Brivio, Davighi, 2308.00017

$$\delta\phi^i = \eta^i - \frac{1}{2}\Gamma^i_{jk}\eta^j\eta^k + \ldots$$

$$S[\phi + \delta\phi] = S[\phi] + \frac{\delta S}{\delta\phi^i}\left(\eta^i - \frac{1}{2}\Gamma^i_{jk}\eta^j\eta^k\right) + \frac{\delta^2 S}{\delta\phi^i\delta\phi^j}\eta^i\eta^j + \mathcal{O}(\eta^3)\,,$$

Usually use $\delta\phi = \eta$ and then compute the one-loop graphs:



Covariant expansion is a field redefinition of the quantum field, which does not change the *S*-matrix.

't Hooft Formula

't Hooft, NPB 62 (1973) 444. Discussed in Luca's talk in detail.
Quadratic terms in $\eta = (\eta^1, \dots \eta^N)$ can be written as

$$L = \frac{1}{2}(D_\mu \eta)^T (D^\mu \eta)^T + \frac{1}{2}\eta^T X \eta$$

$$D_\mu \eta = \partial_\mu \eta + N_\mu \eta$$

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu]$$

where $N_\mu(\phi)$, $X(\phi)$, $Y_{\mu\nu}(\phi)$ are functions of background fields, and we can add total derivatives to make $N_\mu^T = -N_\mu$.

Theory has an $O(N)$ symmetry. Up and down indices are the same.

One loop counterterm is

$$L_{\text{c.t.}} = \frac{1}{16\pi^2 \epsilon} \text{Tr}\left[-\frac{1}{4}X^2 - \frac{1}{24}Y_{\mu\nu}^2\right]$$

EFTs

’t Hooft’s formula requires the kinetic term to be canonical. So cannot directly be applied to EFTs.

In SMEFT, C_{HD} and $C_{H\Box}$ generate $\phi^2(\partial\eta)^2$ terms, which are not included in the ’t Hooft formula.

Cannot make a field redefinition of ϕ to turn g_{ij} into δ_{ij}. The obstruction to this is the Riemann curvature tensor R_{ijkl} constructed from g_{ij}.

Can solve the problems using an expansion in Riemann normal coordinates.

Quadratic in η terms in Riemann Normal Coordinates

The quadratic in η terms are:

$$L = \frac{1}{2} g_{ij} \left(\mathscr{D}_\mu \eta\right)^i \left(\mathscr{D}^\mu \eta\right)^j - \frac{1}{2} R_{ikjl} \left(D_\mu \phi\right)^k \left(D^\mu \phi\right)^l \eta^i \eta^j - \frac{1}{2} \left(\nabla_i \nabla_j V\right) \eta^i \eta^j$$

with

$$\left(\mathscr{D}_\mu \eta\right)^i = \left(\partial_\mu \eta^i + \Gamma^i_{kj} \partial_\mu \phi^k \eta^j\right) + A^\beta_\mu \left(t^i_{\beta\,,j} + \Gamma^i_{jk}\, t^k_\beta\right) \eta^j$$

- $(\mathscr{D}_\mu \eta)$ is gauge and coordinate covariant, and transforms as a coordinate vector.
- Covariant derivatives of the potential, not ordinary derivatives
- Still have a non-trivial kinetic term

However, because everything is a tensor, we can go to a local inertial frame (Cartan frame, i.e. use vielbeins).

$$
\begin{aligned}
g_{ij}(\phi) &= e_i^a(\phi)\, e_j^b(\phi)\delta_{ab} \\
(\mathscr{D}_\mu\eta)^a &= e_i^a\,(\mathscr{D}_\mu\eta)^i \\
R_{abcd} &= e_a^i e_b^j e_c^k e_d^l R_{ijkl} \\
\nabla_a\nabla_b V &= e_a^i e_b^j \nabla_i\nabla_j V
\end{aligned}
$$

$$
L = \frac{1}{2}\,(\mathscr{D}_\mu\eta)^a\,(\mathscr{D}^\mu\eta)^a - \frac{1}{2}R_{acbd}\,(D_\mu\phi)^c\,(D^\mu\phi)^d\,\eta^a\eta^b - \frac{1}{2}\,(\nabla_a\nabla_b V)\,\eta^a\eta^b
$$

- In 't Hooft form
- N_μ is automatically antisymmetric since the connection in the Cartan frame (spin-connection) is antisymmetric.
- Up and down indices are the same since metric is δ_{ab}.

$$L_{\text{c.t.}} = \frac{1}{16\pi^2\epsilon}\left[-\frac{1}{4}X_{ab}X^{ba} - \frac{1}{24}[Y_{\mu\nu}]_{ab}[Y_{\mu\nu}]^{ba}\right]$$

But all indices are contracted, so we have a scalar quantity which can be evaluated in any coordinate system:

$$X_{ab}X^{ba} = X_{ij}X^{ji}$$

so we do not actually have to make any transformations to the Cartan frame.

$$\begin{aligned} X_{ij} &= -R_{ikjl}\,(D_\mu\phi)^k\,(D^\mu\phi)^l - \left(\nabla_i\nabla_j V\right) \\ [Y_{\mu\nu}]_{ij} &= R_{ijkl}\,(D_\mu\phi)^k\,(D_\nu\phi)^l \end{aligned}$$

Raise indices using g^{ij}.

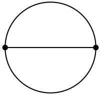
Result holds for EFTs with operators up to two derivatives, but arbitrary high dimension. Do not get result in standard basis.

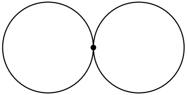
In a renormalizable theory, the kinetic term is canonical so $g_{ij} = \delta_{ij}$ and $R_{ijkl} = 0$.

If one includes operators of dim 6 or higher, have $R_{ijkl} \neq 0$. Then the one-loop correction induces four-derivative terms, two-loops induces 6 derivative terms, etc.

Two Loops

Luca Naterop's Talk: Generalization of the 't Hooft formula to two loops





$$
\begin{aligned}
\mathcal{L} = {} & A_{abc}\eta^a\eta^b\eta^c + A^{\mu}_{a|bc}(D_\mu\eta)^a\eta^b\eta^c + A^{\mu\nu}_{ab|c}(D_\mu\eta)^a(D_\nu\eta)^b\eta^c \\
& + B_{abcd}\eta^a\eta^b\eta^c\eta^d + B^{\mu}_{a|bcd}(D_\mu\eta)^a\eta^b\eta^c\eta^d + B^{\mu\nu}_{ab|cd}(D_\mu\eta)^a(D_\nu\eta)^b\eta^c\eta^d \,.
\end{aligned}
$$

where

$$
A^{\mu}_{a|bc}(\text{completely symmetric}) = 0, \quad B^{\mu}_{a|bcd}(\text{completely symmetric}) = 0
$$

and $A^{\mu\nu}_{ab|c} \to 0$, which simplified the two loop computation.

$O(N)$ symmetry greatly simplifies result.

66 terms:

$$\mathcal{L}_{\text{c.t.}} = \frac{1}{(16\pi^2)^2}\left[a_{2,1} A_{abc} A_{abd} X_{cd} + \ldots\right] \qquad a_{2,1} = \frac{9}{2\epsilon^2} - \frac{9}{2\epsilon}$$

Use the Riemann normal coordinate expansion to order η^4 and go to the Cartan frame:

- Tensors automatically have the correct symmetry properties using the symmetries of the Riemann curvature tensor including the Bianchi identity
- $A^{\mu\nu}_{ab|c} = 0$.
- All tensors in terms of R_{abcd}, its covariant derivative $\nabla_e R_{abcd}$ and covariant derivatives of the potential

Note that an expansion of $(\phi \cdot \phi)(\partial_\mu \phi \cdot \partial^\mu \phi)$ using $\phi \to \phi + \eta$ will generate $A^{\mu\nu}_{ab|c}$. So the geometric method is much simpler.

One-Loop

$O(N)$ model, χPT, SMEFT.

In χPT, the Riemann curvature tensor is

$$R_{abcd} = \frac{1}{F^2} f_{abg} f_{cdg} + \mathcal{O}(\pi^2)$$

and $V = 0$. The one-loop terms are order p^4, and we get the correct one-loop counterterms for $SU(N)$.

Gasser, Leutwyler, Ann Phys 158 (1984) 142, NPB 250 (1985) 465

Bijnens, Colangelo, Ecker, Ann Phys 280 (2000) 100

geo-SMEFT: Helset, Martin, Trott, JHEP 03 (2020) 163

SMEFT One-Loop: Bosons

Can include gauge bosons by combining scalar and gauge fields into a unified space.

Helset, Jenkins, AM, PRD 106 (2022) 11

other approaches: Buchalla, Celis, Krause, Toelstede, 1904.07840

Alonso, West, 2207.02050, Alonso, Kanshin, Saa, PRD 97 (2018) 035010

Helset, Jenkins, AM, JHEP 02 (2023) 063

Include SM $m_H^2, g_1, g_2, g_3, \lambda$

$$\text{dim } 6 \quad {}^6C_{H^6},\ {}^6C_{H^4\Box},\ {}^6C_{H^4D^2},\ {}^6C_{G^2H^2},\ {}^6C_{W^2H^2},\ {}^6C_{B^2H^2},\ {}^6C_{WBH^2},$$

$$\text{dim } 8 \quad {}^8C_{H^8},\ {}^8C^{(1)}_{H^6D^2},\ {}^8C^{(2)}_{H^6D^2},\ {}^8C^{(1)}_{G^2H^4},\ {}^8C^{(1)}_{W^2H^4},\ {}^8C^{(3)}_{W^2H^4},\ {}^8C^{(1)}_{B^2H^4},\ {}^8C^{(1)}_{WBH^4}.$$

agrees for terms common to both computations with

Das Bakshi, Chala, Díaz-Carmona, Guedes, EPJ+ 137 (2022) 973

Das Bakshi, Díaz-Carmona, JHEP 06 (2023) 123

Compute running of H^6, H^4D^2, X^2H^2 ; $X^4, H^8, H^6D^2, H^4D^4, X^2H^4, X^3H^2, X^2H^2D, XH^4D^2$.

dim 6 $\propto$ dim 6, dim 8 $\propto$ dim 8, (dim 6)^2.

Can get anomalous dimension of H^4D^4. Have to reduce results back to standard basis.

SMEFT One-Loop: Fermions

Alonso, Kanshin, Saa, PRD 97 (2018) 035010

Finn, Karamitsos, Pilaftsis, EPJC 81 (2021) 572

Gattus, Pilaftsis, 2307.01126

SMEFT to dim 8

Assi, Helset, AM, Pagès, Shen, 2307.03187

common terms (almost) agree with

Das Bakshi, Chala, Díaz-Carmona, Guedes, EPJ+ 137 (2022) 973

Das Bakshi, Díaz-Carmona, JHEP 06 (2023) 123

Can include ψ^4 operators. $\psi^4 \to \psi^2\eta^2$, so background ψ^2 is a bosonic operator. 6 possible ways to pick η^2, so a lot of Fierz relations.

What remains is loops in which some internal lines are fermions and some are bosons.

Two Loops

$\boxed{O(N) \text{ EFT including dim 6 terms.}}$

$$L = \frac{1}{2}(\partial_\mu \phi \cdot \partial_\mu \phi) - \frac{1}{2} m^2 (\phi \cdot \phi) - \frac{1}{4}\lambda(\phi \cdot \phi)^2$$
$$+ C_{\phi^6}(\phi \cdot \phi)^3 + C_E(\phi \cdot \phi)(\partial_\mu \phi \cdot \partial_\mu \phi) - C_{\phi\Box}\partial_\mu(\phi \cdot \phi)\partial^\mu(\phi \cdot \phi)$$

Cao, Herzog, Melia, Nepveu, JHEP 09 (2021) 014

Dim 6 real scalar (N=1) to 5 loops, and dim 6 complex scalar (N=2) to 4 loops.

$\boxed{\chi\text{PT to two loops}}$

Bijnens, Colangelo, Ecker, Ann Phys 280 (2000) 100

$\boxed{\text{SMEFT to two loops}}$

Two Loops

Should be extendable to fermions and gauge bosons.

't Hooft (1973): Gauge bosons A_μ like 4 scalars, and fermions

$$\frac{1}{\not{p} - m} = \frac{\not{p} + m}{p^2 - m^2}$$

like a boson with modified vertices.