An Algebraic Formula for Two Loop Renormalization of Scalar Quantum Field Theory

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paper I : 2308.06315 (presented today)

$$\mathcal{L}_{KE} = \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i$$

paper II : to appear (presented on Monday)

$$\mathcal{L}_{KE} = \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j$$

Outline

- The 't Hooft formula for $\Delta\mathcal{L}^{(1)}$
- Geometric formulation
- The Background Field method at two loops
- The two-loop counterterms $\Delta\mathcal{L}^{(2)}$
- Factorizable graphs and RGEs
- Practical aspects of the calculation
- Results and application to the O(N) model

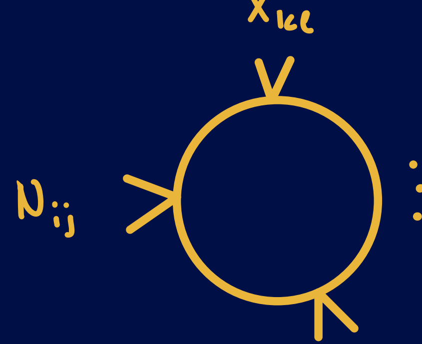
The 't Hooft formula

Start with $\mathcal{L}(\phi_i)$. Write $\phi_i \to \phi_i + \eta_i$ and expand in η_i

$\mathcal{O}(\eta^2)$ term is

$$\mathcal{L}^{(1)} = \frac{1}{2}\partial_\mu \eta_i \partial_\mu \eta_i + N^\mu_{ij}\partial_\mu \eta_i \eta_j + \frac{1}{2} X_{ij}\eta_i\eta_j$$

$\rightarrow$ generates any 1-loop diagram



With $X_{ij}[\phi]$, $N^\mu_{ij}[\phi]$ functionals of ϕ

flavor indices

Generalization to $\frac{1}{2} g_{ij}(\phi)\partial_\mu \eta^i \partial_\mu \eta^j$

$\rightarrow$ geometric formalism, ϕ^i coordinates of $\mathcal{M}$

The 't Hooft formula

We can shift $\mathcal{L}$ by a total derivative

$$\mathcal{L} \to \mathcal{L} + \partial_\mu(\eta_i Z^\mu_{ij} \eta_j) = \mathcal{L} + \partial_\mu \eta_i Z_{ij} \eta_j + \eta_i \partial_\mu Z^\mu_{ij} \eta_j + \eta_i Z^\mu_{ij} \eta_j$$

$$= \mathcal{L} + 2\partial_\mu \eta_i Z_{ij} \eta_j + \eta_i \partial_\mu Z^\mu_{ij} \eta_j$$

using $Z_{ij} = Z_{ji}$. This leads to a shift

$$N^\mu \to N^\mu + 2Z^\mu$$

$$X \to X + 2\partial_\mu Z^\mu$$

So we can assume $X_{ij} = X_{ij}$ and $N^\mu_{ij} = -N^\mu_{ji}$.

The 't Hooft formula

Complete square

$$\mathcal{L}^{(1)} = \frac{1}{2}\Big(\underbrace{(\partial_\mu + N_\mu)}_{D_\mu}\eta\Big)_i \Big(\underbrace{(\partial_\mu + N_\mu)}_{D_\mu}\eta\Big)_i + \frac{1}{2}\eta_i X_{ij}\eta_j \qquad X \to X + N^\mu N_\mu$$

with $D^\mu = \partial^\mu + N^\mu$

O(N) gauge field

$\mathcal{L}^{(1)}$ invariant under local O(N).

$\rightarrow$ $\Delta\mathcal{L}^{(1)}$ built from X, D_μ and

$$Y^{\mu\nu} = [D^\mu, D^\nu] = \partial^\mu N^\nu - \partial^\nu N^\mu + N^\mu N^\nu - N^\nu N^\mu$$

Dimensional analysis: $\Delta\mathcal{L}^{(1)} = \frac{1}{\epsilon}\left(a\, X_{ij}X_{ji} + b\, Y^{\mu\nu}_{ij} Y^{\mu\nu}_{ji}\right)$

Use generic X, N, compute $\langle X X\rangle$, $\langle N^\mu N^\nu N^\rho N^\sigma\rangle$

$$\boxed{\Delta\mathcal{L}^{(1)} = \frac{1}{16\pi^2\epsilon}\left(-\frac{1}{4}X_{ij}X_{ji} - \frac{1}{24}Y^{\mu\nu}_{ij}Y^{\mu\nu}_{ji}\right)}$$

['t Hooft, 1973]

Geometric formulation

Higher-dimensional operators with two derivatives

$$\mathcal{L} = \frac{1}{2} g_{ij}(\phi)\,(D_\mu \phi)^i (D_\mu \phi)^j - V(\phi)$$

The formula

$$\boxed{\Delta \mathcal{L}^{(1)} = -\frac{1}{4\epsilon} X^i{}_j X^j{}_i - \frac{1}{24\epsilon} (Y_{\mu\nu})^i{}_j (Y^{\mu\nu})^j{}_i}$$

before: $i \overset{\eta}{\text{———}} j = \dfrac{i\delta_{ij}}{p^2}$

after: $i \overset{\eta}{\text{———}} j = \dfrac{i g^{ij}}{p^2}$

still holds but now

$$X_{ij} = -R_{ikjl}(D_\mu \phi)^k (D_\mu \phi)^l - \nabla_i \nabla_j V$$

$$Y^{\mu\nu}{}_{ij} = R_{ijkl}(D^\mu \phi)^k (D^\nu \phi)^l + \nabla_j t_{\alpha i} F^\alpha_{\mu\nu}$$

[Alonso, Jenkins, Manohar, 2016]

[Helset, Jenkins, Manohar, 2023]

Example: Renormalization of ϕ^4 theory

E.g. $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 - \Lambda$

$$X = \frac{\delta^2 \mathcal{L}}{\delta\phi\,\delta\phi} = -m^2 - \frac{\lambda}{2}\phi^2 \,, \quad Y^{\mu\nu} = 0$$

So the counterterms at one loop are

$$\Delta\mathcal{L}^{(1)} = -\frac{1}{4\epsilon}X^2 = -\frac{1}{4\epsilon}\left(-m^2 - \frac{\lambda}{2}\phi\right)^2$$

$$= -\frac{m^4}{4\epsilon} - \frac{\lambda m^2}{4\epsilon}\phi^2 - \frac{\lambda^2}{16\epsilon}\phi^4$$

$Z_\Lambda \qquad Z_m \qquad Z_\lambda \qquad$ and $Z_\phi = 1$

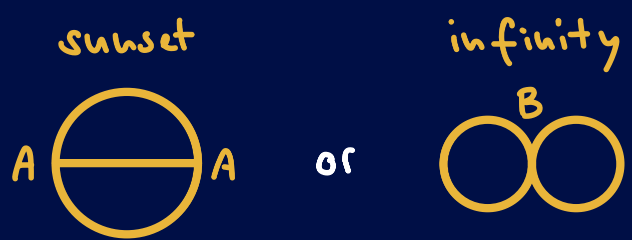
Instead of calculating 

The Background Field Method at two loops

Expand up to $O(\eta^4)$ and up to $O(\partial\eta^2)$

$$\mathcal{L}^{(2)}(\phi) = A_{ijk}\,\eta^i\eta^j\eta^k + A^{\mu}_{ijk}\,D_\mu\eta^i\eta^j\eta^k + A^{\mu\nu}_{ijk}\,D_\mu\eta^i D_\nu\eta^j\,\eta^k$$
$$+ B_{ijkl}\,\eta^i\eta^j\eta^k\eta^l + B^{\mu}_{ijkl}\,D_\mu\eta^i\eta^j\eta^k\eta^l + B^{\mu\nu}_{ijkl}\,D_\mu\eta^i D_\nu\eta^j\,\eta^k\eta^l$$

Generic two-loop diagram is



e.g.

- A_{ijk}, B_{ijkl} are completely symmetric
- $\mathcal{L}^{(2)} \longrightarrow \mathcal{L}^{(2)} + D_\mu\left(C^{\mu}_{ijk}\,\eta^i\eta^j\eta^k\right) + D_\mu\left(F^{\mu}_{ijkl}\,\eta^i\eta^j\eta^k\eta^l\right)$

to make $A^{\mu}_{(ijk)} = B^{\mu}_{(ijkl)} = 0$. Can't do this for $A^{\mu\nu}$, $B^{\mu\nu}$.

The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

Dimensional analysis

$$[D_\mu] = 1 \quad [X] = [Y] = 2 \quad [A] = 1 \quad [A^\mu] = 0 \quad [A^{\mu\nu}] = -1$$

$$[B] = 0 \quad [B^\mu] = -1 \quad [B^{\mu\nu}] = -2$$

Ansatz for $\Delta\mathcal{L}^{(2)}$

$$\Delta\mathcal{L}^{(2)} = AAD^2 + AAX + AAY + A^\mu AD^3 + A^\mu ADX + \dots$$

$$+ BD^4 + BXD^2 + BYD^2 + BXX + BXY + BYY + B^\mu D^5 + \dots$$

Any term can have multiple independent contractions

The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

AA	$D^2 \quad X \quad Y$
$A^\mu A$	$D^3 \quad XD \quad YD$
$A^\mu A^\mu$	$D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY$
$A^{\mu\nu} A$	$D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY$
$A^{\mu\nu} A^\mu$	$D^5 \quad XD^3 \quad YD^3 \quad X^2D \quad XYD \quad YYD$
$A^{\mu\nu} A^{\mu\nu}$	$D^6 \quad XD^4 \quad YD^4 \quad X^2D^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY$

B	$D^4 \quad XD^2 \quad YD^2 \quad XX \quad XY \quad YY$
B^μ	$D^5 \quad XD^3 \quad YD^3 \quad XXD \quad XYD \quad YYD$
$B^{\mu\nu}$	$D^6 \quad XX \quad XXD^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY$

Each corresponds to a Green's function

The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

AA	$D^2 \quad X \quad \cancel{Y}$
$A^\mu A$	$\cancel{D^3} \quad XD \quad YD$
$A^\mu A^\mu$	$D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY$
~~$A^{\mu\nu} A$~~	~~$D^4 \quad XD^2 \quad YD^2 \quad X^2 \quad XY \quad YY$~~
~~$A^{\mu\nu} A^\mu$~~	~~$D^5 \quad XD^3 \quad YD^3 \quad X^2 D \quad XYD \quad YYD$~~
~~$A^{\mu\nu} A^{\mu\nu}$~~	~~$D^6 \quad XD^4 \quad YD^4 \quad X^2D^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY$~~

B	$\cancel{D^4} \quad \cancel{XD^2} \quad \cancel{YD^2} \quad XX \quad \cancel{XY} \quad \cancel{YY}$
B^μ	$\cancel{D^5} \quad \cancel{XD^3} \quad \cancel{YD^3} \quad XXD \quad XYD \quad \cancel{YYD}$
$B^{\mu\nu}$	$\cancel{D^6} \quad XX \quad XXD^2 \quad XYD^2 \quad YYD^2 \quad XXX \quad XXY \quad XYY \quad YYY$

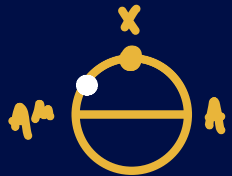
E.g. $A_{ijk} A_{ijk} Y^{\mu\mu} = 0$, $A^\mu_{ijk} A_{ijk} D^3 = 0$ (D_μ preserves symmetry)

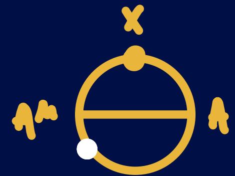
E.g. $\langle BXD^2 \rangle$: 

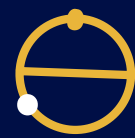
scaleless & power divergent

The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

Possible flavor contractions in $(A^\mu A X D)$



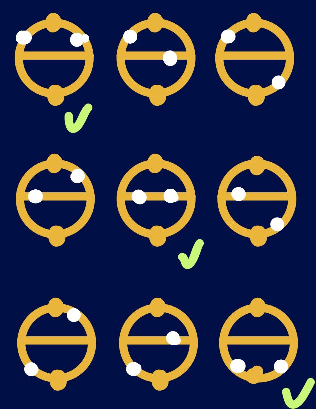


but  $+ 2$  $= 0$

$A^\mu_{ijk} A_{jkl} X_{il}$ $\quad$ $A^\mu_{ijk} A_{ijl} X_{kl}$ $\quad$ $(A^\mu_{ijk} + A^\mu_{jki} + A^\mu_{kij}) A_{ijl} X_{kl} = 0$

Flavor contractions in $(A^\mu A^\mu X Y)$

2 independent contractions



3 independent contractions

The two-loop counterterms $\Delta\mathcal{L}^{(2)}$

Need to calculate 21 Green's functions.

E.g. want $B^\mu X \psi D$ via $(B^\mu X N)$

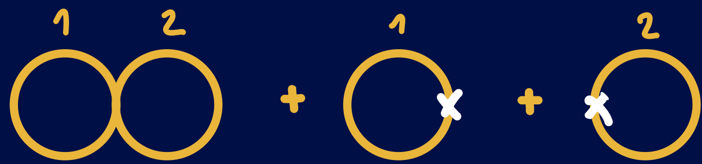
but both $(B^\mu X D^3)$ and $(B^\mu X \psi D)$ contribute to $(B^\mu X N)$

$\rightarrow$ Gauge invariance checks

E.g. $A^\mu A^\nu D^4$, $A^\mu A^\nu \psi D^2$ can both be determined from $A^\mu A^\nu \psi \psi$

Factorizable graphs

Factorizable graphs only give $1/\epsilon^2$ poles

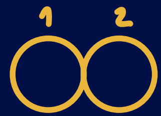
$\underline{I}^{sub} =$ 

$$\overset{MS}{=} \left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right) + \left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(-\frac{I_{2\infty}}{\epsilon}\right) + \left(-\frac{I_{1\infty}}{\epsilon}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right)$$

$$= -\frac{I_{1\infty} I_{2\infty}}{\epsilon^2} + I_{1f} I_{2f}$$

Factorizable graphs

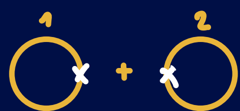
k, ℓ loop momenta, p_i, q_i external. Evaluate the loops separately:

$$I = $$ 

$$= I_1^{\{\alpha\}}(k, \{p\})\, I_2^{\{\alpha\}}(\ell, \{q\})$$

$$= \left(\frac{1}{\epsilon} I_{1\infty}^{\{\alpha\}}(\{p\}) + I_{1f}^{\{\alpha\}}(\{p\})\right)\left(\frac{1}{\epsilon} I_{2\infty}^{\{\alpha\}}(\{q\}) + I_{2f}^{\{\alpha\}}(\{q\})\right)$$

The subtraction of subdivergences is

$$I_{sub} = $$ 

$$= \left(\frac{1}{\epsilon} I_{1\infty}^{\{\alpha\}}(\{p\}) + I_{1f}^{\{\alpha\}}(\{p\})\right)\left(-\frac{1}{\epsilon} I_{2\infty}^{\{\alpha\}}(\{q\})\right)$$

$$+ \left(\frac{1}{\epsilon} I_{1\infty}^{\{\alpha\}}(\{p\})\right)\left(\frac{1}{\epsilon} I_{2\infty}^{\{\alpha\}}(\{q\}) + I_{2f}^{\{\alpha\}}(\{q\})\right)$$

So the subdivergence subtracted two-loop integral is

$$I_{tot} = I + I_{sub} = -\frac{1}{\epsilon^2} I_{1\infty}^{\{\alpha\}}(\{p\})\, I_{2\infty}^{\{\alpha\}}(\{q\}) + I_{1f}^{\{\alpha\}}(\{p\})\, I_{2f}^{\{\alpha\}}(\{q\})$$

Factorizable graphs

Formula for subtracted 2-loop integral

$$I + I_{sub} = -\frac{1}{\epsilon^2} I_{1\infty}^{\{1\}} I_{2\infty}^{\{1\}} + I_{1f}^{\{1\}} I_{2f}^{\{1\}}$$

→ predicts divergence

→ $\frac{1}{\epsilon}$ pole cancels, two-loop CT is purely $\frac{1}{\epsilon^2}$

→ does not affect RGE

But what if additional factor of ϵ in numerator?

Factorizable graphs

Case 1: ϵ generated by an individual loop. $\eta_{\alpha\alpha} = d = 4-2$

 $= \eta^{\alpha}_{\alpha}\left(\frac{I_\infty}{\epsilon} + I_f\right) = \frac{4}{\epsilon} I_\infty + d I_f - 2 I_\infty$

from $(D_\mu \eta D_\nu \eta)$

$$= \underline{d}\left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right)$$

$$+ \underline{d}\left(\frac{I_{1\infty}}{\epsilon} + I_{1f}\right)\left(-\frac{I_{2\infty}}{\epsilon}\right) + \underline{4}\left(-\frac{I_{1\infty}}{\epsilon}\right)\left(\frac{I_{2\infty}}{\epsilon} + I_{2f}\right) = -\frac{4 I_{1\infty} I_{2\infty}}{\epsilon^2} + \text{finite}$$

$\rightarrow$ minimal subtraction gives no $\frac{1}{\epsilon}$ pole. Nice.

Factorizable graphs

Case 2: ϵ generated after combining both loops

E.g. if $I_1^{\alpha\beta} = \eta^{\alpha\beta} I_1$, $I_2^{\alpha\beta} = \eta^{\alpha\beta} I_2$

$$= -\frac{1}{\epsilon^2} I_1^{\alpha\beta} I_{2\,\alpha\beta} = -\frac{\overbrace{\eta^{\alpha}{}_{\alpha}}^{4-2\epsilon}}{\epsilon^2} I_1 I_2$$

$\rightarrow$ factorizable topologies generate $\frac{1}{\epsilon}$ poles.

But suppose we split

$$\mathcal{L}_{eff} = \bar{\mathcal{L}}_{eff} + \hat{\mathcal{L}}_{eff}$$

Now

$\bar{\mathcal{L}}_{eff}$ generates $\bar{\eta}^{\alpha}{}_{\alpha} = 4$ $\longrightarrow$ no effect on RGE

$\hat{\mathcal{L}}_{eff}$ generates $\hat{\eta}^{\alpha}{}_{\alpha} = -2\epsilon$ $\longrightarrow$ no effect on RGE when we deviate from MS

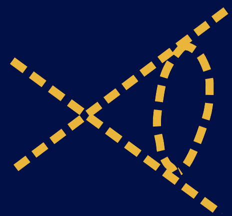
[Dugan, Grinstein, 1991]

Factorizable graphs

Therefore factorizable 2-loop diagrams do not affect RGEs.

Example 1:

no effect

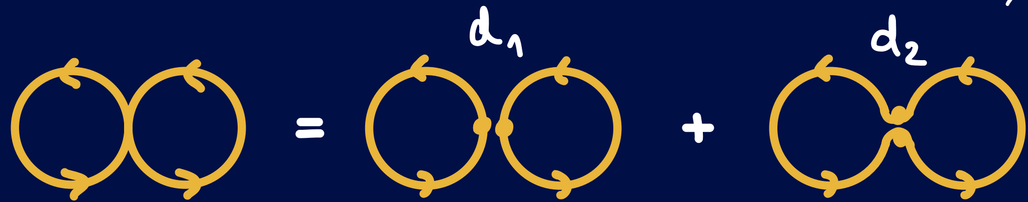


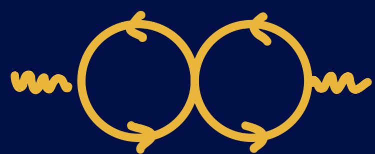
affects RGE

[Machacek, Vaughn, 1985]

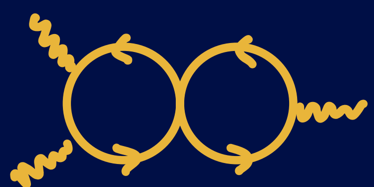
Example 2:

d_1 + Fierz - evanescent





$\longrightarrow$ no φ^4 coefficients in $\gamma^{(2)}_{g_i}$, $\gamma^{(2)}_{\theta_{QCD}}$



$\longrightarrow$ no φ^4 coefficients in $\gamma^{(2)}_{G}, \gamma^{(2)}_{\tilde{G}}, \gamma^{(2)}_{W}, \gamma^{(2)}_{\tilde{W}}$

⋮

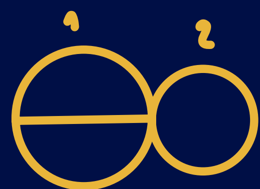
[Bern, Parra-Martinez, Sawyer, 2020]

Factorizable graphs

Argument generalizes to arbitrary loop order

E.g.  + ... CT ... $= \dfrac{I_{1\infty} I_{2\infty} I_{3\infty}}{\epsilon^3}$ + finite

E.g.  + ... CT ... $= \dfrac{(-1)^{n+1}}{\epsilon^n} \prod_i I_{i\infty}$

E.g.  + ... CT ... $= -\left(\dfrac{I^2_{1\infty}}{\epsilon^2} + \dfrac{I^1_{1\infty}}{\epsilon} \right) \left(\dfrac{I^1_{2\infty}}{\epsilon} \right)$ + finite

With $I^i_{1\infty}$ the subdivergence subtracted (local) divergences of diag 1.

$\rightarrow$ predict subtracted divergences of factorizable graphs

$\boxed{\text{Factorizable Graphs do not contribute to RGEs}}$

Factorizable graphs

At 3 loops:


1


2


3


4


5


6

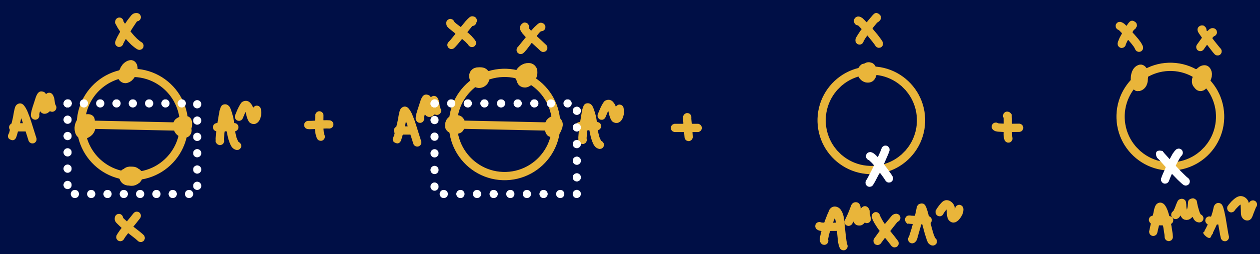

7


8

do not affect RGE

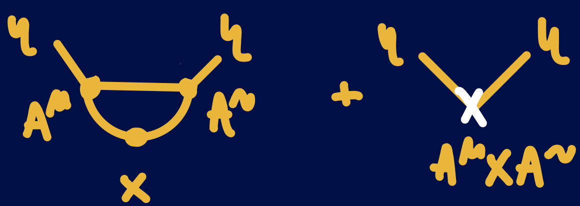
Non-factorizable graphs (AA)

AA graphs produce the $\frac{1}{\epsilon}$ poles which give RGEs. E.g. $A^\mu A^\nu X X$:



Requires additional 1-loop counterterms

$$\Delta \mathcal{L}_Q^{(1)} \supset A^\mu X A^\nu \eta\eta + A^\mu A^\nu \eta\eta D^2$$



= finite

Non-factorizable graphs (AA)

Can find $\Delta\mathcal{L}_Q^{(1)}$ algebraically

$$\mathcal{L}^{(2)} = A_{ijk}\,\eta^i\eta^j\eta^k \,, \quad \text{shift again:} \quad \eta_i \to \eta_i + x_i$$

$$\mathcal{L}^{(2)} = A_{ijk}(\eta^i + x^i)(\eta^j + x^j)(\eta^k + x^k)$$

$$= \mathcal{O}(x^0) + \mathcal{O}(x^1) + 3A_{ijk}\,x^i x^j \eta^k + \ldots$$

Apply the 't Hooft formula again:

$$X_{ij}[\phi,\eta] = 6A_{ijk}[\phi]\,\eta^k$$

$$\Delta\mathcal{L}_Q^{(1)} \supset -\frac{1}{4\epsilon} X_{ij}X_{ji} = -\frac{9}{\epsilon} A_{ijk}A_{ije}\,\eta^k\eta^e$$

Determined all $\Delta\mathcal{L}_Q^{(1)}$ this way, checked cancellation

Practical calculation

- Start with first nonvanishing Green's function
- Generate diagrams with qgraf
- Calculate UV divergences using

$$\frac{1}{(q+p)^2 - M^2} = \frac{1}{q^2 - m^2} + \frac{M^2 - p^2 - 2p\cdot q - m^2}{q^2 - m^2} \frac{1}{(q+p)^2 - M^2}$$

- Identify divergent subgraphs $\rightarrow$ add terms to $\Delta\mathcal{L}_Q^{(1)}$
- Evaluate 1-loop integrals with package-X
- Make sure non-local terms cancel, using FORM
- Map local divergences to $\Delta\mathcal{L}^{(2)}$

[Nogueira, 1993] [Chetyrkin, Misiak, Muenz, 1997]

[Patel, 2015] [Vermaseren, 2000]

Result

$$\boxed{\begin{aligned}\Delta\mathcal{L}^{(2)} = -\frac{3}{4\epsilon} D_\mu A_{ijk} D_\mu A_{ijk} + \left(\frac{9}{2\epsilon^2} - \frac{9}{2\epsilon}\right) A_{ijk} A_{ijl} X_{kl} + \frac{3}{\epsilon^2} B_{ijkl} X_{ij} X_{kl} \\ + \text{ terms with } A^4,\ B^4,\ B^{\mu\nu}\end{aligned}}$$

the 2-loop counterterm for any scalar theory

(with no more than 2 derivatives)

Application to O(N) model

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi_i)^2 - \frac{m^2}{2}\phi_i^2 - \frac{\lambda}{4}(\phi_i^2)^2$$

$$\frac{\delta \mathcal{L}}{\delta \phi_i} = -m^2 \phi_i - \lambda\, \phi\cdot\phi\, \phi_i \,, \qquad X_{ij} = \frac{\delta^2 \mathcal{L}}{\delta\phi_i \delta\phi_j} = -m^2 \delta_{ij} - \lambda\left(2\phi_i\phi_j + \phi\cdot\phi\, \delta_{ij}\right)$$

$$\underline{\Delta\mathcal{L}^{(1)}} = -\frac{1}{4\epsilon} X_{ij}X_{ji} = -\frac{1}{4\epsilon} N m^2 - \frac{1}{2}(N+2)\lambda m^2 (\phi\cdot\phi) - \frac{1}{4}(N+8)\lambda^2(\phi\cdot\phi)^2$$

$$A_{ijk} = \frac{1}{3!}\frac{\delta^3 \mathcal{L}}{\delta\phi_i\delta\phi_j\delta\phi_k} = -\frac{1}{3}\lambda\left(\delta_{ij}\phi_k + \delta_{ik}\phi_j + \delta_{jk}\phi_i\right)$$

$$B_{ijkl} = \frac{1}{4!}\frac{\delta^4 \mathcal{L}}{\delta\phi_i\delta\phi_j\delta\phi_k\delta\phi_l} = -\frac{1}{12}\lambda\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\underline{\Delta\mathcal{L}^{(2)}} = -\frac{3}{4\epsilon} D_\mu A_{ijk} D_\mu A_{ijk} + \left(\frac{9}{2\epsilon^2} - \frac{9}{2\epsilon}\right) A_{ijk}A_{ijl}X_{kl} + \frac{3}{\epsilon^2} B_{ijkl} X_{ij} X_{kl}$$

$$= -\frac{1}{4\epsilon}(2+N)\lambda^2 (D_\mu\phi)^2 - \frac{3}{2\epsilon}(2+N)\lambda^2 m^2 \phi\cdot\phi$$

$$- \frac{1}{2\epsilon}(22+5N)\lambda^3(\phi\cdot\phi)^2 + \dots \frac{1}{\epsilon^2} \text{ poles}$$

$\longrightarrow$ extract Z_ϕ, Z_m, Z_λ. Agrees with 2-loop SM RGE.

Conclusions

- Geometry + BF method allows algebraic renormalization
- We extend the approach to two loops
- Our results give $\Delta\mathcal{L}^{(2)}$ for scalar EFT with ≤ 2 derivatives
- Results are most efficiently used in geometric formalism

  $\longrightarrow$ talk by Aneesh on Monday

- Found formula for subdivergence subtraction of factorizable graphs
- Predicts some zeros in RGEs at arbitrary loop order

?

Backup: O(N) model & Geometry

E.g. take O(N) model and add $\mathcal{O}_{H\Box}$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)\cdot(\partial_\mu \phi) + \frac{m_H^2}{4}\phi\cdot\phi - \frac{\lambda}{4}(\phi\cdot\phi)^2 - c_{H\Box}(\phi\cdot\partial_\mu\phi)(\phi\cdot\partial_\mu\phi)$$

Read off $g_{ij}(\phi) = \delta_{ij} - 2c_{H\Box}\phi_i\phi_j$, $V(\phi) = -\frac{m_H^2}{4}\phi\cdot\phi + \frac{\lambda}{4}(\phi\cdot\phi)^2$

Do some algebra on $\mathcal{M}$

$$\Gamma^k_{ij} = \frac{1}{2}g^{kn}\left(\partial_i g_{jn} + \partial_j g_{in} - \partial_n g_{ij}\right) = -2c_{H\Box}\phi^k\delta_{ij}$$

$$R_{ijkl} = \ldots = 2c_{H\Box}\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right)$$

Find

$$\Delta\mathcal{L}^{(1)} = \underset{\substack{\uparrow \\ Z_\Lambda}}{-\frac{m_H^4}{4\epsilon}} \underset{\substack{\uparrow \\ Z_\phi}}{-\frac{3c_{H\Box}m_H^2}{2\epsilon}(\partial_\mu\phi)^2} + \underset{\substack{\uparrow \\ Z_{m_H}}}{\frac{6m_H^2\lambda - 5c_{H\Box}m_H^4}{4\epsilon}\phi\cdot\phi} \underset{\substack{\uparrow \\ Z_\lambda}}{-\frac{3\lambda^2}{\epsilon}(\phi\cdot\phi)^2} + \underset{\substack{\uparrow \\ Z_{c_{H\Box}}}}{\frac{11c_{H\Box}m_H^2}{2\epsilon}(\phi\cdot\partial_\mu\phi)^2}$$

Backup: Factorizable diagrams

$$R(G) = \bar{R}(G) + \Delta(G)$$

$$\bar{R}(G_1 G_2) = \underbrace{R(G_1 G_2)}_{\text{finite}} - \underbrace{\Delta(G_1 G_2)}_{\Delta(G_1)\,\Delta(G_2)}$$

E.g. 

$+ \ldots CT \ldots = \dfrac{a_1 a_2 a_3}{\epsilon^3} + \text{finite}$

E.g. 

$+ \ldots CT \ldots = (-1)^{n+1} \dfrac{\prod_i a_i}{\epsilon^n}$

E.g. 
$+ \ldots CT \ldots = -\left(\dfrac{1}{\epsilon^2} + \dfrac{1}{\epsilon}\right)\left(\dfrac{1}{\epsilon}\right) + \text{finite}$

Backup: Extraction of UV divergences

$$\int d^D k_1\, d^D k_2 \,\frac{k_1^{\mu_1}\cdots k_1^{\mu_{n_1}} k_2^{\nu_1}\cdots k_2^{\nu_{n_2}}}{(k_1^2)^{b_1}(k_2^2)^{b_2}((k_1+k_2)^2)^{b_3}}$$

$$D_1 = 4 + n_1 - 2b_1 - 2b_3$$

$$D_2 = 4 + n_2 - 2b_2 - 2b_3$$

$$D_3 = 4 + n_1 + n_2 - 2b_1 - 2b_2$$

$$D_{OA} = 8 + n_1 + n_2 - 2b_1 - 2b_2 - 2b_3$$

towards tadpoles

$1^0 2^0 3^0 OA^0 + 1^{--} 2^0 3^{--} OA^{--}$

$$\frac{1}{(q+p)^2 - M^2} = \frac{1}{q^2 - m^2} + \frac{M^2 - p^2 - 2qp - m^2}{q^2 - m^2}\,\frac{1}{(q+p)^2 - M^2}$$

towards $(1\text{-loop})^2$

$1^0 2^{++} 3^{--} OA^0 + 1^{-} 2^{++} 3^0 OA^0$

$$\frac{1}{(k_1+q_2)^2 - M^2} = \frac{1}{k_1^2 - m^2} + \frac{M^2 - q_2^2 - 2k_1\cdot q_2 - m^2}{k_1^2 - m^2}\,\frac{1}{(k_1+q_2)^2 - M^2}$$

tadpole integrals:

$$J^{(2)}_{n_1 n_2 n_3} := \tilde{\mu}^{4\epsilon} \int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\,\frac{1}{[k_1^2 - M^2]^{n_1}[k_2^2 - M^2]^{n_2}[(k_1+k_2)^2 - M^2]^{n_3}}$$

Backup: 2-loop RGEs

$\frac{1}{\epsilon^n}$ pole @ ℓ loops

$$\mathcal{L} = \sum_i C_i^{\text{bare}} O_i \,, \quad C_i^{\text{bare}} = \mu^{n_i \epsilon}(C_i + \delta_i) \,, \quad \delta_i = \sum_{\ell=1}^{\infty} \sum_{n=0}^{\ell} \frac{a^{(\ell,n)}}{\epsilon^n}$$

Using $\mu \frac{d}{d\mu} C_i^{\text{bare}} = 0$ and topological identities

$$\dot{C}_i^{(1)} = 2 a_i^{(1,1)}$$

$$\dot{C}_i^{(2)} = 4 a_i^{(2,1)} - 2 a_j^{(1,0)} \frac{\partial a_i^{(1,1)}}{\partial C_j} - 2 a_j^{(1,1)} \frac{\partial a_i^{(1,0)}}{\partial C_j}$$

where $\dot{C} = \mu \frac{d}{d\mu} C$.

→ depends on finite renormalizations

→ scheme dependence in RGEs @ two loops

Backup : Permutation symmetries

Why can we assume $N_{ij} = -N_{ji}$? Start with generic $\tilde{N}^\mu$:

$$0 = \partial_\mu \left(\tilde{N}^\mu_{ij} \phi_i \phi_j \right) = \partial_\mu \tilde{N}^\mu_{ij} \phi_i \phi_j + \tilde{N}^\mu_{ij} \partial_\mu \phi_i \phi_j + \hat{N}^\mu_{ij} \phi_i \partial_\mu \phi_j$$

$$\mathcal{L} \supset N^\mu \partial_\mu \phi \phi + \frac{1}{2} X \phi \phi +$$

Backup: Permutation symmetries

Covariant derivatives

$$\nabla_a T_{ij\ldots} = \partial_a T_{ij\ldots} - \Gamma^k_{ai} T_{kj\ldots} - \Gamma^k_{aj} T_{ik\ldots} - \ldots$$

preserve tensor symmetries.

Proof for $T_{ji} = T_{ij}$:

$$\nabla_a T_{ji} = \partial_a T_{ji} - \Gamma^k_{aj} T_{ki} - \Gamma^k_{ai} T_{jk} = \partial_a T_{ij} - \Gamma_{aj} T_{ik} - \Gamma_{ai} T_{kj} = \nabla_a T_{ij}$$

General proof: Switch to RNC where $g_{ij} = \delta_{ij}$ and $\Gamma^i_{jk} = 0$.

Then $\nabla_a T^{i_1 \ldots i_n}_{j_1 \ldots j_n} = \partial_a T^{i_1 \ldots i_n}_{j_1 \ldots j_n}$ $\rightarrow$ statement trivial

$\rightarrow$ also holds for multiterm symmetries like $A^\mu_{(ijk)} = 0$