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Zeta functions, the Chowla-Selberg formula, and the Casimir effect

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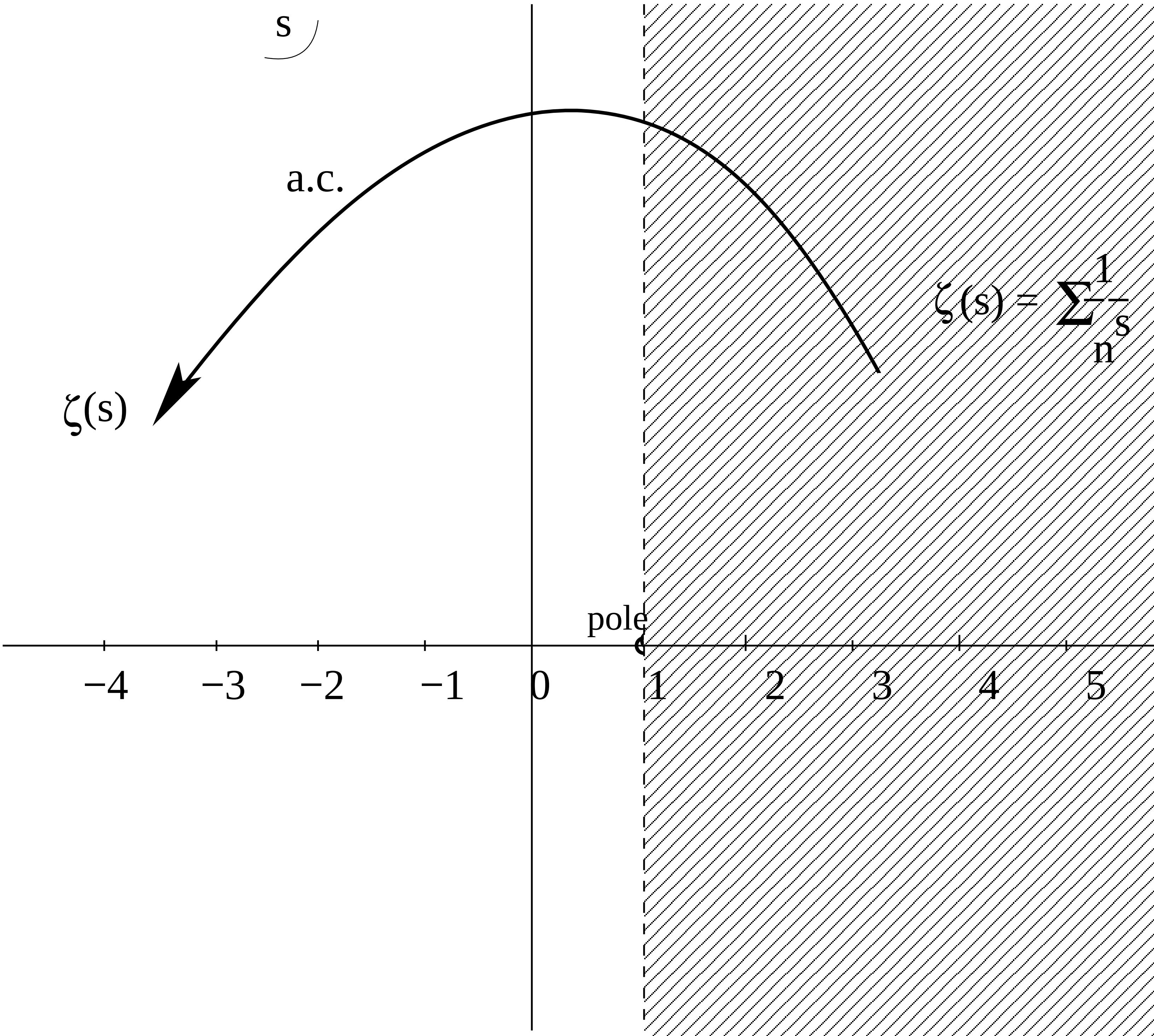
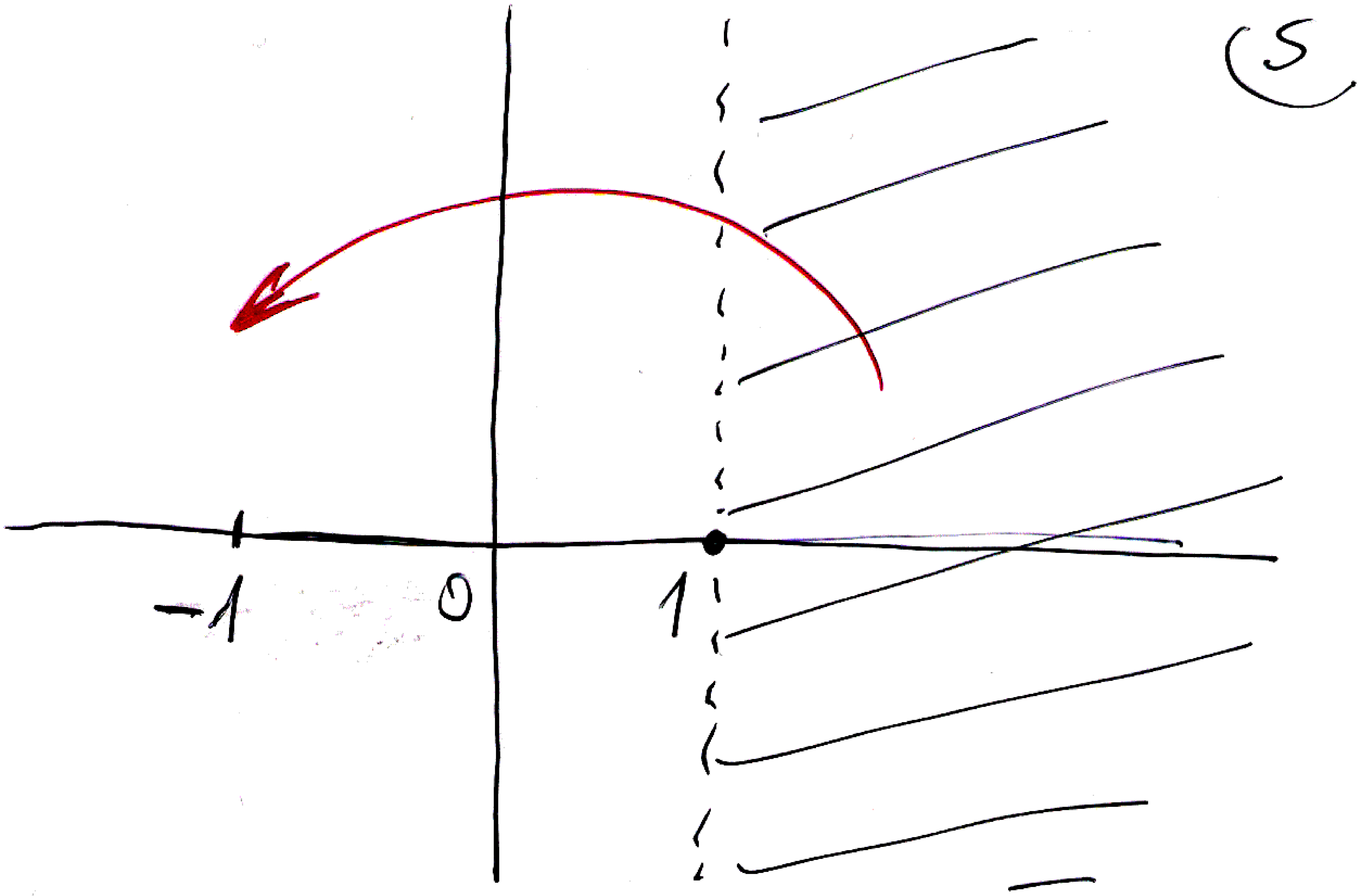
Outline

- The Riemann zeta function as a regularization tool.
- General scheme for Linear and Quadratic cases. Truncations.
- Spectrum only known Implicitly.
- The Chowla-Selberg formula in Number Theory.
- The Chowla-Selberg series formula (CS). Nontrivial Extensions (ECS).
- Operator Zeta Functions: ζ_A for A a ΨDO, Det's.
- Dixmier trace, Wodzicki Residue.
- Multiplicative Anomaly or Defect.


s
a.c.
ζ(s)
ζ(s) = Σ 1/n^s
pole
−4
−3
−2
−1
0
1
2
3
4
5



$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$

$\zeta(0) = -\frac{1}{2}$ or $1 + 1 + 1 + \dots \overset{?}{=} -\frac{1}{2}$

$\zeta(-1) = -\frac{1}{12}$ or $1 + 2 + 3 + \dots \overset{?}{=} -\frac{1}{12}$

⋮

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\ \mathrm{prime}}\frac{1}{1-\frac{1}{p^s}}$$

The prime number theorem {Hadamard, de la Vallée Poussin}

$$\pi(x)=\#\{\text{primes } p\le x\}\sim\frac{x}{\log x}$$

$$\xi(s)\equiv\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)=\xi(1-s)$$

(Gelbart + Miller, BAMS '03) (completed ζ f.)

(E) Entirety: $\xi(s)$ meromorphic c., $s=0,1$ poles

(FE) $\xi(s)=\xi(1-s)$

(BV) Bounded in vertical strips:

$\xi(s)+\frac{1}{s}+\frac{1}{1-s}$ bounded $-\infty<a<\mathrm{Re}\,s<b<+\infty$

Riemann (1859)

Poisson s. for.
$$\boxed{\sum_{n\in\mathbb{Z}} f(n)=\sum_{n\in\mathbb{Z}}\hat{f}(n)}$$
Fourier t.

$$\hat{f}(r)=\int_{-\infty}^{+\infty}f(x)\,e^{-2\pi i r x}\,dx\ ,\quad f\ \text{Schwartz}$$

1) FE ζ

2) $\hat{f}(r)=\frac{1}{\sqrt{t}}\,e^{-\pi r^2/t}$ $\rightarrow$ Jacobi id.

$$\theta(it)=\frac{1}{\sqrt{t}}\,\theta\left(\frac{i}{t}\right)\ ,\quad \theta(\tau)=\frac{1}{2}\sum_{n\in\mathbb{Z}}e^{\pi i n^2\tau}$$

Dirichlet $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\,n^{-s}$

Hamburger

Basic strategies

- Jacobi's identity for the $\theta-$function

$$\theta_3(z,\tau) := 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz), \qquad q := e^{i\pi\tau},\ \tau\in\mathbb{C}$$

$$\theta_3(z,\tau) = \frac{1}{\sqrt{-i\tau}}\, e^{z^2/i\pi\tau}\,\theta_3\left(\frac{z}{\tau}\Big|\frac{-1}{\tau}\right)$$

equivalently:

$$\sum_{n=-\infty}^{\infty} e^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\sum_{n=0}^{\infty} e^{-\frac{\pi^2 n^2}{t}}\cos(2\pi n z), \qquad z,t\in\mathbb{C},\ \mathrm{Re}\, t>0$$

- Higher dimensions: Poisson summ formula (Riemann)

$$\sum_{\vec{n}\in\mathbb{Z}^p} f(\vec{n}) = \sum_{\vec{m}\in\mathbb{Z}^p} \widetilde{f}(\vec{m})$$

$\widetilde{f}$ Fourier transform

[Gelbart + Miller, BAMS '03, Iwaniec, Morgan, ICM '06]

- Truncated sums $\longrightarrow$ asymptotic series

ζ: EXPLICIT CALCULATIONS

Epstein zeta functions (quadratic)

$$\zeta_E = \sum_{\vec{n} \in \mathbb{Z}^d} Q(\vec{n})^{-s}$$ Q quadratic form

Barnes zeta functions (linear)

$$\zeta_B = \sum_{\vec{n} \in \mathbb{N}^d} L(\vec{n})^{-s}$$ L affine form (coeff's $\in \mathbb{Q}^+$)

Extensions:

$\zeta_E \rightarrow Q + L$ affine

$\rightarrow \sum_{\vec{n} \in \mathbb{N}^d}$ (truncation)

$\zeta_B \rightarrow \zeta_B'(0)$ (new formulas)

$\rightarrow \sum'_{\vec{n} \in \mathbb{Z}^d}$ (by analyt. cont.)

Ex

ζ-REGULARIZ: SPECTRUM KNOWN IMPLICITLY

- Example of the ball:
  - Operator
  $$(-\Delta + m^2)$$
  on the D-dim ball $B^D = \{x \in R^D; |x| \leq R\}$
  with Dirichlet, Neumann or Robin BC
  - The zeta function
  $$\zeta(s) = \sum_k \lambda_k^{-s}$$
  - Eigenvalues implicitly obtained from
  $$\boxed{(-\Delta + m^2)\phi_k(x) = \lambda_k \phi_k(x) \quad + \quad BC}$$
  - In spherical coordinates:
  $$\phi_{l,m,n}(r,\Omega) = r^{1-\frac{D}{2}} J_{l+\frac{D-2}{2}}(w_{l,n} r) Y_{l+\frac{D}{2}}(\Omega)$$
  $J_{l+(D-2)/2}$ Bessel functions
  $Y_{l+D/2}$ hyperspherical harmonics
  - Eigenvalues $w_{l,n}$ (> 0) determined through BC
  $$\boxed{J_{l+\frac{D-2}{2}}(w_{l,n} R) = 0,} \qquad \text{for Dirichlet BC}$$

$$\frac{u}{R} J_{l+\frac{D-2}{2}}(w_{l,n}R) + w_{l,n} J'_{l+\frac{D-2}{2}}(w_{l,n}r)\,|_{r=R} = 0,$$ for Robin BC

– Now, $\lambda_{l,n} = w_{l,n}^2 + m^2$

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} d_l(D)(w_{l,n}^2 + m^2)^{-s}$$

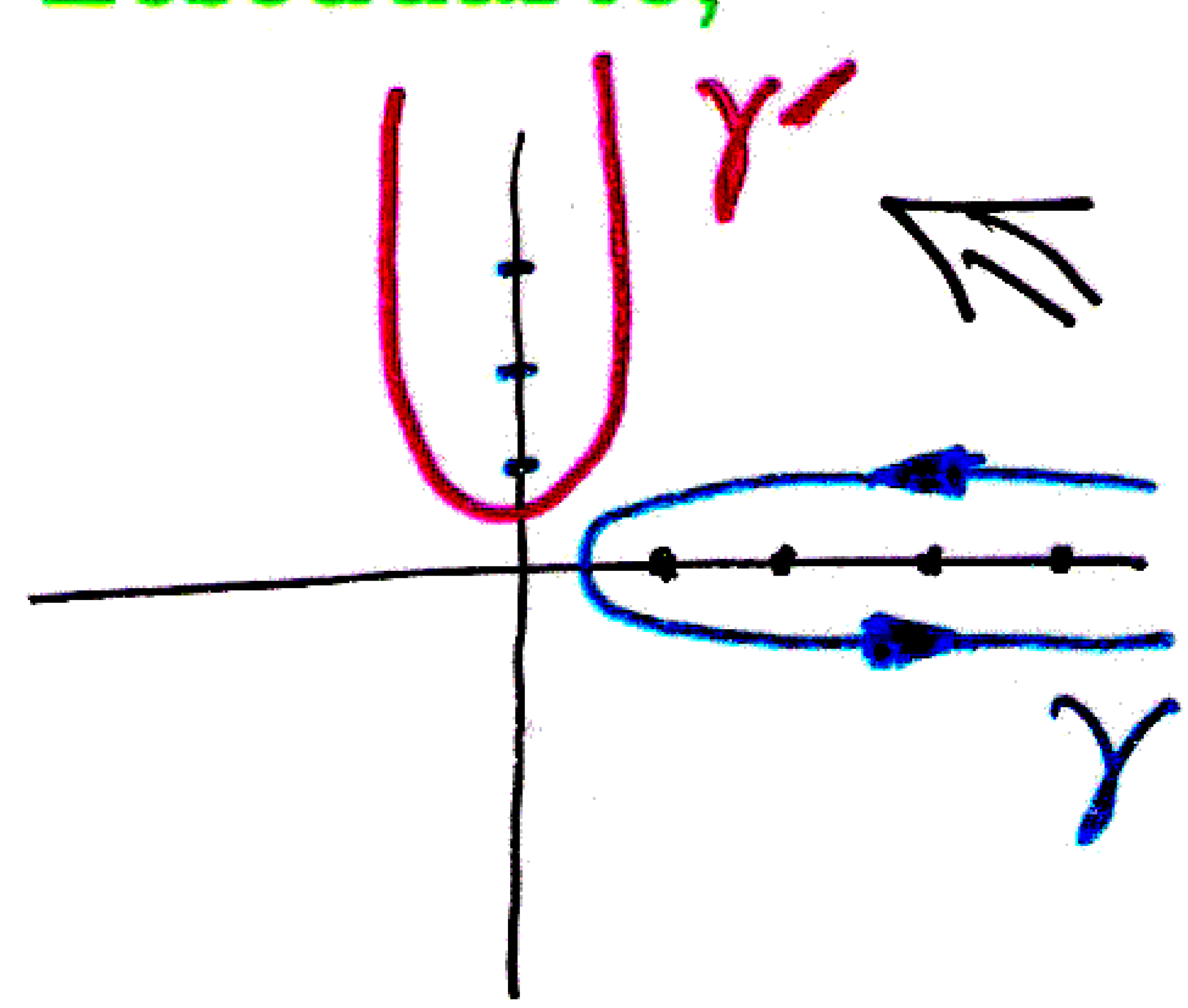
$w_{l,n}$ (> 0) is defined as the n-th root of the l-th equation, $d_l(D) = (2l + D - 2)\frac{(l+D-3)!}{l!\,(D-2)!}$

- **Procedure:**
  - Contour integral on the complex plane

$$\zeta(s) = \sum_{l=0}^{\infty} d_l(D) \int_{\gamma} \frac{dk}{2\pi i} \, (k^2+m^2)^{-s} \frac{\partial}{\partial k} \ln \Phi_{l+\frac{D-2}{2}}(kR)$$

γ runs counterclockwise and must enclose all the solutions [Ginzburg, Van Kampen, EE + I. Brevik]

- **Obtained:** [with Bordag, Kirsten, Leseduarte, Vassilievich,...]
  - Zeta functions
  - Determinants
  - Seeley [heat-kernel] coefficients



The Chowla-Selberg Formula (CS)

- M. Lerch, Sur quelques formules relatives du nombre des classes, Bull. Sci. Math. 21 (1897) 290-304
- A. Selberg and S. Chowla, On Epstein's Zeta function (I), Proc. Nat. Acad. Sci. 35 (1949) 371-74

ON EPSTEIN'S ZETA FUNCTION (I)

By S. CHOWLA AND A. SELBERG

INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

Communicated by H. Weyl, May 18, 1949

1. This paper contains a short account of results whose detailed proofs will be published later.

We define the function $Z(s)$ by

$$Z(s) = \sum{}'(am^2 + bmn + cn^2)^{-s} \tag{1}$$

where $s = \sigma + it$ (σ and t, real), $\sigma > 1$, and the summation is for all integers m, n (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further a and c are positive numbers while b is real and subject to $4ac - b^2 = \Delta > 0$.

It is well known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole s-plane, and satisfies a functional equation similar to the one satisfied by the Riemann Zeta Function. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

Deuring (*Math. Ztschr.*, **37**, 403–413 (1933)) obtained an important formula for $Z(s)$. Deuring's work led Heilbronn (*Quart. J. Maths., Oxford*, **5**, 150 (1934)) to the proof of the following famous conjecture of Gauss on the class-number of binary quadratic forms with a negative fundamental discriminant: let $h(-\Delta)$ denote the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, then

$$h(-\Delta) \to \infty \quad \text{as} \quad \Delta \to \infty \tag{2}$$

Again using the ideas of Heilbronn and Deuring, Siegel proved that

$$h(-\Delta) > \Delta^{1/2 - \epsilon} \; [\Delta > \Delta_0(\epsilon)] \tag{3}$$

which is a great advance on (2).

Our starting point is the formula:

$$Z(s) = 2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{\Gamma(s)\,\Delta^{s-1/2}}\zeta(2s-1)\Gamma(s-1/2) + Q(s) \tag{4}$$

where

$$Q(s) = \frac{\pi^s \cdot 2^{s+3/2}}{a^{1/2}\Gamma(s)\,\Delta^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2}\,\sigma_{1-2s}(n)\cos\left(\frac{n\pi b}{a}\right) \int_0^{\infty} \phi^{s-3/2} \exp\left\{-\frac{\pi n \Delta^{1/2}}{2a}(\phi + \phi^{-1})\right\} d\phi \tag{4}$$

The Chowla-Selberg Formula (CS)

- M. Lerch, Sur quelques formules relatives du nombre des classes, Bull. Sci. Math. 21 (1897) 290-304
- A. Selberg and S. Chowla, On Epstein's Zeta function (I), Proc. Nat. Acad. Sci. 35 (1949) 371-74
- S. Chowla and A. Selberg, On Epstein's Zeta function, J. reine angew. Math. (Crelle's J.) 227 (1967) 86-110

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L. Fuchs, K. Hensel, L. Schlesinger

gegenwärtig herausgegeben von

Helmut Hasse und **Hans Rohrbach**

unter Mitwirkung von

W. Brödel, M. Deuring, A. Grothendieck, P. R. Halmos, O. Haupt,
F. Hirzebruch, E. Hopf, M. Kneser, G. Köthe, K. Prachar, H. Reichardt,
P. Roquette, W. Schmeidler, L. Schmetterer, E. Stiefel

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Berlin 1967

On Epstein's Zeta-function

By *Atle Selberg* at Princeton (N. J.), and *S. Chowla* at State College (Pa.)

Introduction

This paper was written in the Spring of 1949, and a resumé appeared in the note: On Epstein's zeta Function (I), Proceedings of the National Academy of Sciences (U. S. A.), **35** (1949), 371—374.

Meanwhile, the following papers which have reference to the Proceedings paper, came to our attention:

1. *J. B. Rosser*, Real roots of real Dirichlet *L*-series, Jour. Research National Bureau of Standards, **45** (1950), 505—514.

2. *E. A. Anferteva*, On an identity of Chowla and Selberg (Russian), Izvestija Vysšik Učebnyh Zavadeniĭ Mathematika (Kazan), No. 3 (10) (1959), 13—21.

3. *P. T. Bateman* and *E. Grosswald*, On Epstein's zeta Function, Acta Arithmetica, **9** (1964), 365—373.

4. *K. Ramachandra*, Some applications of Kronecker's limit formulas, Annals of Mathematics **80** (1964), 104—148.

§ 1.

We define the function $Z(s)$ by

$$Z(s) = \Sigma' (am^2 + bmn + cn^2)^{-s} \tag{1}$$

where $s = \sigma + it$ (σ and t, real), $\sigma > 1$, and the summation is for all integers m, n (each going from $-\infty$ to $+\infty$), while the dash indicates that $m = n = 0$ is excluded from the summation; further a and c are positive numbers while b is real and subject to $4ac - b^2 = \Delta > 0$.

It is well-known that the function $Z(s)$, defined for $\sigma > 1$ by (1), can be continued analytically over the whole s-plane. The function $Z(s)$, thus defined, is a meromorphic function with a simple pole at $s = 1$.

In 1933, Deuring obtained an important formula for $Z(s)$. Deuring's work led Heilbronn to his proof of a famous conjecture of Gauss on the class number of binary quadratic forms with a negative fundamental discriminant. If $h(-\Delta)$ is the number of classes of binary quadratic forms of negative fundamental discriminant $-\Delta = b^2 - 4ac$, Gauss conjectured that

$$h(-\Delta) \to \infty \text{ as } \Delta \to \infty. \tag{2}$$

Transforming this we get

$$\sum_{j=1}^{h} \log \Delta\left(\frac{b_j + i\sqrt{|d|}}{2a_j}\right) = 6\left\{h\gamma + \log \frac{\prod_{j=1}^{h} a_j}{|d|^h}\right\} - \frac{3w}{\pi}\sqrt{|d|}\, L_d'(1).$$

Inserting here the value (obtained like (58))

$$L_d'(1) = -\frac{\pi}{\sqrt{|d|}} \sum_{m=1}^{|d|} \left(\frac{d}{m}\right) \log \Gamma\left(\frac{m}{|d|}\right) + \frac{2h\pi(\gamma + \log 2\pi)}{w\sqrt{|d|}}$$

one gets, writing $\tau_j = \dfrac{b_j + i\sqrt{|d|}}{2a_j}$,

$$\prod_{j=1}^{h} \Delta(\tau_j) = \frac{\prod_{j=1}^{h} a_j^6}{(2\pi\,|d|)^{6h}} \left\{\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{3w} \tag{2}$$

Now let $\tau = i\dfrac{K'}{K}$ be a number from the field $k(\sqrt{d})$, then from Lemma 3 we get

$$\frac{\Delta(\tau_j)}{\Delta(\tau)} = \lambda_j,$$

where λ_j are algebraic numbers. Thus (2) gives

$$\Delta(\tau) = \frac{\lambda'}{\pi^6} \left\{\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{\frac{3w}{h}}, \tag{3}$$

where λ' is an algebraic number. Finally we have from (48)

$$\Delta(\tau) = \left(\frac{2K}{\pi}\right)^{12} \cdot 2^{-8} (kk')^4 = \lambda'' \left(\frac{K}{\pi}\right)^{12},$$

where λ'' is an algebraic number. This gives, when inserted in (3)

$$K = \lambda''' \sqrt{\pi} \left\{\prod_{m=1}^{|d|} \Gamma\left(\frac{m}{|d|}\right)^{\left(\frac{d}{m}\right)}\right\}^{\frac{w}{4h}}, \tag{4}$$

which is the desired expression for K in finite terms.

References (in the order of appearance in the text)

[1] *S. Chowla*, Quart. J. of Maths. (Oxford) 5 (1934), 304—307.

[2] *S. Chowla*, Acta Arithmetica **1** (1935), 113—114.

[3] *Deuring*, Math. Ztschr. **37** (1933), 403—413.

[4] *Fricke-Klein*, Modulfunktionen, Leipzig 1890.

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[6] *Heilbronn*, Acta Arithmetica **2** (1936), 212—213.

[7] *Heilbronn* and *Linfoot*, Quart. J. of Maths. (Oxford) 5 (1934), 293—301.

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- S. Iyanaga and Y. Kawada, Eds., Encyclopedic Dictionary of Mathematics, Vol. II (The MIT Press, Cambridge, 1977), pp. 1378-79
- B.H. Gross, On the periods of abelian integrals and a formula of Chowla and Selberg, Inv. Math. 45 (1978) 193-211
- P. Deligne, Valeurs de fonctions L et periodes d'integrales, PSPM 33 (1979) 313-346

History

- Lerch (1897):

$$\sum_{\lambda=1}^{|D|} \left(\frac{D}{\lambda}\right) \log\Gamma\left(\frac{\lambda}{D}\right) = h\log|D| - \frac{h}{3}\log(2\pi) - \sum_{(a,b,c)} \log a + \frac{2}{3}\sum_{(a,b,c)} \log\left[\theta_1'(0|\alpha)\theta_1'(0|\beta)\right]$$

D discriminant, $\theta_1' \sim \eta^3$

h class number of binary quadratic forms (a, b, c)

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- Eta evaluations Dedekind eta function for Im $(\tau) > 0$

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q := e^{2\pi i\tau}$$

It is a 24-th root of the discriminant func $\Delta(\tau)$ of an elliptic curve $\mathbb{C}/L$ from a lattice $L = \{a\tau + b \,|\, a, b \in \mathbb{Z}\}$

$$\Delta(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}$$

Properties & Recent Results

- ⟹ The C-S formula gives the value of a product of eta functions
- ⟹ If there is only one form in the class, it yields the value of a single eta function in terms of gamma functions
- ⟹ Long series of improvements: Kaneko (90), Nakajima and Taguchi (91), Williams et al. (95)
- ⟹ In the last years the C-S formula has been 'broken' to isolate the eta functions:
  Williams, van Poorten, Chapman, Hart

- R. Chapman and W.B. Hart, Evaluation of the Dedekind eta function, Can. Math. Bull. (2005)
- W.B. Hart, PhD Thesis, 2004 (Macquarie U., Sidney)
- M. Suzuki, An analogue of the Chowla-Selberg formula for several automorphic L-functions, arXiv:math/0606096 (2006)
- T. Yang, The Chowla-Selberg formula and the Colmez conjecture, Canad. J. Math. 62 (2010), pp. 456-472

Extended CS Series Formulas (ECS)

- Consider the zeta function (Re$s > p/2, A > 0,$ Re$q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[\frac{1}{2}\left(\vec{n}+\vec{c}\right)^T A \left(\vec{n}+\vec{c}\right) + q\right]^{-s} = \sum_{\vec{n}\in\mathbb{Z}^p}{}' \left[Q\left(\vec{n}+\vec{c}\right) + q\right]^{-s}$$

prime: point $\vec{n} = \vec{0}$ to be excluded from the sum
(inescapable condition when $c_1 = \cdots = c_p = q = 0$)

$$Q\left(\vec{n}+\vec{c}\right) + q = Q(\vec{n}) + L(\vec{n}) + \bar{q}$$

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- Case $q \neq 0$ (Re$q > 0$)

$$\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2} q^{p/2-s}}{\sqrt{\det A}} \frac{\Gamma(s-p/2)}{\Gamma(s)} + \frac{2^{s/2+p/4+2}\pi^s q^{-s/2+p/4}}{\sqrt{\det A}\,\Gamma(s)}$$

$$\times \sum_{\vec{m}\in\mathbb{Z}^p_{1/2}}{}' \cos(2\pi\vec{m}\cdot\vec{c}) \left(\vec{m}^T A^{-1}\vec{m}\right)^{s/2-p/4} K_{p/2-s}\left(2\pi\sqrt{2q\,\vec{m}^T A^{-1}\vec{m}}\right)$$

[ECS1]

Extended CS Series Formulas (ECS)

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[ECS1]

- Pole: $s = p/2$ Residue:

$$\text{Res}_{s=p/2}\,\zeta_{A,\vec{c},q}(s) = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}(\det A)^{-1/2}$$

- Gives (analytic cont of) multidimensional zeta function in terms of an exponentially convergent multiseries, valid in the whole complex plane

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- K_ν modified Bessel function of the second kind and the subindex 1/2 in $\mathbb{Z}^p_{1/2}$ means that only half of the vectors $\vec{m} \in \mathbb{Z}^p$ participate in the sum. E.g., if we take an $\vec{m} \in \mathbb{Z}^p$ we must then exclude $-\vec{m}$ [simple criterion: one may select those vectors in $\mathbb{Z}^p \backslash \{\vec{0}\}$ whose first non-zero component is positive]

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- Case $c_1 = \cdots = c_p = q = 0$ [true extens of CS, diag subcase]

$$\zeta_{A_p}(s) = \frac{2^{1+s}}{\Gamma(s)} \sum_{j=0}^{p-1} (\det A_j)^{-1/2} \left[\pi^{j/2} a_{p-j}^{j/2-s}\, \Gamma\left(s - \frac{j}{2}\right) \zeta_R(2s-j) + 4\pi^s a_{p-j}^{\frac{j}{4}-\frac{s}{2}} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{j/2-s} \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{s/2-j/4} K_{j/2-s}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right)\right]$$

[ECS3d]

QFT in s-t with non-comm toroidal part

- $D-$dim non-commut manifold: $M = \mathbb{R}^{1,d} \bigotimes \mathbb{T}_\theta^p, \quad D = d + p + 1$
  $\mathbb{T}_\theta^p$ a $p-$dim non-commutative torus: $[x_j, x_k] = i\theta\sigma_{jk}$
  σ_{jk} a real, nonsingular, antisymmetric matrix of ± 1 entries
  θ the non-commutative parameter.

- Interest recently, in connection with $M-$theory & string theory [Connes,Douglas,Seiberg,Cheung,Chu,Chomerus,Ardalan, . . .]

- Unified treatment: only one zeta function, nature of field (bosonic, fermionic) as a parameter, together with # of compact, noncompact, and noncommutative dimensions

$$\zeta_\alpha(s) = \frac{V\,\Gamma(s-(d+1)/2)}{(4\pi)^{(d+1)/2}\,\Gamma(s)} \sum_{\vec{n}\in\mathbb{Z}^p}{}' Q(\vec{n})^{(d+1)/2-s}\left[1+\Lambda\theta^{2-2\alpha}Q(\vec{n})^{-\alpha}\right]^{(d+1)/2-s}$$

$\alpha = 2$ bos, $\alpha = 3$ ferm, $V = \text{Vol}\,(\mathbb{R}^{d+1})$ of non-compact part

$Q(\vec{n}) = \sum_{j=1}^p a_j n_j^2$ a diag quadratic form, $R_j = a_j^{-1/2}$ compactif c radii

- After some calculations,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \left(-\Lambda\theta^{2-2\alpha}\right)^l \zeta_{Q,\vec{0},0}\left(s+\alpha l - \frac{d+1}{2}\right)$$

for all radii equal to R, with $I(\vec{n}) = \sum_{j=1}^{p} n_j^2$,

$$\zeta_\alpha(s) = \frac{V}{(4\pi)^{(d+1)/2} R^{d+1-2s}} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-\frac{d+1}{2})}{l!\,\Gamma(s)} \left(-\Lambda\theta^{2-2\alpha}\right)^l \zeta_E\left(s+\alpha l - \frac{d+1}{2}\right)$$

where we use the notation $\zeta_E(s) := \zeta_{I,\vec{0},0}(s)$
e.g., the Epstein zeta function for the standard quadratic form

- Rich pole structure: pole of Epstein zf at $s = p/2 - \alpha k + (d+1)/2 = D/2 - \alpha k$, combined with poles of Γ, yields a rich pattern of singul for $\zeta_\alpha(s)$

- Classify the different possible cases according to the values of d and $D = d + p + 1$. We obtain, at $s = 0$:

For $d = 2k$ $\begin{cases} \text{if } D \neq \dot{\overline{2\,\alpha}} \Longrightarrow \zeta_\alpha(0) = 0 \text{ if } D = \\ \dot{\overline{2\,\alpha}} \Longrightarrow \zeta_\alpha(0) = \text{finite} \end{cases}$

$$\text{For } d = 2k-1 \quad \begin{cases} \text{if } D \neq \dot{\overline{2\alpha}} \left\{ \begin{array}{ll} \text{finite, for } l \leq k \\ 0, \quad \text{for } l > k \end{array} \right\} \Longrightarrow \zeta_\alpha(0) = \text{finite} \\ \\ \text{if } D = 2\alpha l \left\{ \begin{array}{ll} \text{pole,} & \text{for } l \leq k \\ \text{finite,} & \text{for } l > k \end{array} \right\} \Longrightarrow \zeta_\alpha(0) = \text{pole} \end{cases}$$

– Pole structure of the zeta function $\zeta_\alpha(s)$, at $s = 0$, according to the different possible values of d and D ($\dot{\overline{2\alpha}}$ means multiple of 2α)

$\Longrightarrow$ Explicit analytic continuation of $\zeta_\alpha(s)$, $\alpha = 2, 3$, & specific pole structure

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)}\sum_{l=0}^{\infty}\frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)}(-2^{\alpha}\Lambda\theta^{2-2\alpha})^{l}\sum_{j=0}^{p-1}(\mathrm{det}A_j)^{-\frac{1}{2}}$$

$$\times\Big[\pi^{j/2}a_{p-j}^{-s-\alpha l+(d+j+1)/2}\Gamma(s+\alpha l-(d+j+1)/2)\zeta_R(2s+2\alpha l-d-j-1)$$

$$+\,4\pi^{s+\alpha l-(d+1)/2}a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4}\sum_{n=1}^{\infty}\sum_{\vec{m}_j\in\mathbb{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l}$$

$$\times\left(\vec{m}_j^t A_j^{-1}\vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l}\left(2\pi n\sqrt{a_{p-j}\vec{m}_j^t A_j^{-1}\vec{m}_j}\right)\Big]$$

$$\zeta_\alpha(s) = \frac{2^{s-d}\,V}{(2\pi)^{(d+1)/2}\Gamma(s)} \sum_{l=0}^{\infty} \frac{\Gamma(s+l-(d+1)/2)}{l!\,\Gamma(s+\alpha l-(d+1)/2)} (-2^\alpha \Lambda \theta^{2-2\alpha})^l \sum_{j=0}^{p-1} (\det A_j)^{-\frac{1}{2}}$$

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$$+ 4\pi^{s+\alpha l-(d+1)/2} a_{p-j}^{-(s+\alpha l)/2-(d+j+1)/4} \sum_{n=1}^{\infty} \sum_{\vec{m}_j \in \mathbb{Z}^j}{}' n^{(d+j+1)/2-s-\alpha l}$$

$$\left. \times \left(\vec{m}_j^t A_j^{-1} \vec{m}_j\right)^{(s+\alpha l)/2-(d+j+1)/4} K_{(d+j+1)/2-s-\alpha l}\left(2\pi n \sqrt{a_{p-j} \vec{m}_j^t A_j^{-1} \vec{m}_j}\right) \right]$$

p \ D	even	odd
odd	(1a) pole / f nite ($l \geq l_1$)	(2a) pole / pole
even	(1b) double pole / pole ($l \geq l_1, l_2$)	(2b) pole / double pole ($l \geq l_2$)

– General pole structure of $\zeta_\alpha(s)$, for the possible values of D and p being odd or even. Magenta, type of behavior corresponding to lower values of l; behavior in blue corresponds to larger values of l

Pseudodifferential Operator (ΨDO)

- A ΨDO of order m $\qquad$ M_n manifold
- Symbol of A: $a(x,\xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^n) \subset C^\infty$ functions such that for any pair of multi-indices α, β there exists a constant $C_{\alpha,\beta}$ so that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

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Definition of A (in the distribution sense)

$$Af(x) = (2\pi)^{-n} \int e^{i<x,\xi>} a(x,\xi) \hat{f}(\xi)\, d\xi$$

- f is a smooth function

$$f \in \mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}^n);\ \text{sup}_x |x^\beta \partial^\alpha f(x)| < \infty,\ \ \forall \alpha, \beta \in \mathbb{N}^n \right\}$$

- $\mathcal{S}'$ space of tempered distributions
- $\hat{f}$ is the Fourier transform of f

ΨDOs are useful tools

The symbol of a ΨDO has the form:

$$a(x,\xi) = a_m(x,\xi) + a_{m-1}(x,\xi) + \cdots + a_{m-j}(x,\xi) + \cdots$$

being $a_k(x,\xi) = b_k(x)\,\xi^k$

$a(x,\xi)$ is said to be elliptic if it is invertible for large $|\xi|$ and if there exists a constant C such that $|a(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m}$, for $|\xi| \geq C$

– An elliptic ΨDO is one with an elliptic symbol

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– – ΨDOs are basic tools both in Mathematics & in Physics – –

1. Proof of uniqueness of Cauchy problem [Calderón-Zygmund]

2. Proof of the Atiyah-Singer index formula

3. In QFT they appear in any analytical continuation process —as complex powers of differential operators, like the Laplacian [Seeley, Gilkey, ...]

4. Basic starting point of any rigorous formulation of QFT & gravitational interactions through μlocalization (the most important step towards the understanding of linear PDEs since the invention of distributions)

[K Fredenhagen, R Brunetti, . . . R Wald '06, R Haag EPJH35 '10]

Existence of ζ_A for A a ΨDO

1. A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}^+$
2. A acts on the space of smooth sections of
3. E, n-dim vector bundle over
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(a) The zeta function is defined as:

$$\zeta_A(s) = \text{tr } A^{-s} = \sum_j \lambda_j^{-s}, \quad \text{Re } s > \frac{n}{m} := s_0$$

$\{\lambda_j\}$ ordered spect of A, $s_0 = \dim M / \text{ord } A$ abscissa of converg of $\zeta_A(s)$

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(b) $\zeta_A(s)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at $s = 0$), provided the principal symbol of A, $a_m(x, \xi)$, admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \text{Arg}\, \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\text{Spec}\, A \cap L_\theta = \emptyset$ (the Agmon-Nirenberg condition)

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(d) The only possible singularities of $\zeta^A(s)$ are poles at

$$s_j = (n-j)/m, \qquad j = 0, 1, 2, \ldots, n-1, n+1, \ldots$$

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H ΨDO operator $\{\varphi_i, \lambda_i\}$ spectral decomposition

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Definition: zeta function of H $\zeta_H(s) = \sum_{i\in I} \lambda_i^{-s} = \mathrm{tr}\ H^{-s}$

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Determinant: Ray & Singer, '67

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C. Soulé et al, Lectures on Arakelov Geometry, CUP 1992; A. Voros,...

Properties

- The definition of the determinant $\det_\zeta A$ only depends on the homotopy class of the cut
- A zeta function (and corresponding determinant) with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators ofcomplex order $m \in \mathbb{C}\backslash\mathbb{Z}$ (they do not admit spectral cuts), has been obtained [Kontsevich and Vishik]
- Asymptotic expansion for the heat kernel:

$$\text{tr } e^{-tA} = \sum\nolimits'_{\lambda \in \text{Spec } A} e^{-t\lambda}$$

$$\sim \alpha_n(A) + \sum\nolimits_{n \neq j \geq 0} \alpha_j(A) t^{-s_j} + \sum\nolimits_{k \geq 1} \beta_k(A) t^k \ln t, \quad t \downarrow 0$$

$$\alpha_n(A) = \zeta_A(0), \quad \alpha_j(A) = \Gamma(s_j)\, \text{Res}_{s=s_j} \zeta_A(s), \; s_j \notin -\mathbb{N}$$

$$\alpha_j(A) = \frac{(-1)^k}{k!} \left[\text{PP}\, \zeta_A(-k) + \psi(k+1)\, \text{Res}_{s=-k} \zeta_A(s)\right], \qquad s_j = -k, \; k \in \mathbb{N}$$

$$\beta_k(A) = \frac{(-1)^{k+1}}{k!} \text{Res}_{s=-k} \zeta_A(s), \quad k \in \mathbb{N}\backslash\{0\}$$

$$\text{PP}\, \phi := \lim_{s \to p} \left[\phi(s) - \frac{\text{Res}_{s=p}\, \phi(s)}{s-p}\right]$$

The Dixmier Trace

- In order to write down an action in operator language one needs a functional that replaces integration
- For the Yang-Mills theory this is the Dixmier trace
- It is the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum:

$$\sigma_N(T) := \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N), \qquad \mu_0 \geq \mu_1 \geq \cdots$$

- Definition of the Dixmier trace of T:

$$\mathrm{Dtr}\, T = \lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T)$$

provided that the Cesaro means $M(\sigma)(N)$ of the sequence in N are convergent as $N \to \infty$ [remember: $M(f)(\lambda) = \frac{1}{\ln \lambda} \int_1^\lambda f(u) \frac{du}{u}$]

- The Hardy-Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$ [Connes]

$$\mathrm{Dtr}\, T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s)$$

The Wodzicki Residue

- The Wodzicki (or noncommutative) residue is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$
- Only trace one can define in the algebra of ΨDOs (up to multipl const)
- Definition: res $A = 2\,\mathrm{Res}_{s=0}\,\mathrm{tr}(A\Delta^{-s})$, Δ Laplacian
- Satisfies the trace condition: res $(AB) =$ res (BA)
- Important!: it can be expressed as an integral (local form)

$$\text{res } A = \int_{S^*M} \mathrm{tr}\; a_{-n}(x,\xi)\; d\xi$$

with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral: Adler-Manin residue)

- If dim $M = n = -$ ord A (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and $\mathrm{Res}_{s=1}\zeta_A(s) = \frac{1}{n}\,\text{res } A^{-1}$
- The Wodzicki residue makes sense for ΨDOs of arbitrary order. Even if the symbols $a_j(x,\xi)$, $j < m$, are not coordinate invariant, the integral is, and defines a trace

Singularities of ζ_A

- A complete determination of the meromorphic structure of some zeta functions in the complex plane can be also obtained by means of the Dixmier trace and the Wodzicki residue

- Missing for full descript of the singularities: residua of all poles

- As for the regular part of the analytic continuation: specific methods have to be used (see later)

- Proposition. Under the conditions of existence of the zeta function of A, given above, and being the symbol $a(x,\xi)$ of the operator A analytic in ξ^{-1} at $\xi^{-1}=0$:

  $$\text{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\,\text{res}\,A^{-s_k} = \frac{1}{m}\int_{S^*M}\text{tr}\,a_{-n}^{-s_k}(x,\xi)\,d^{n-1}\xi$$

- Proof. The homog component of degree $-n$ of the corresp power of the principal symbol of A is obtained by the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1}=0$:

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial\xi^{-1}}\right)^k \left[\xi^{n-k}a^{(k-n)/m}(x,\xi)\right]\Big|_{\xi^{-1}=0}\,\xi^{-n}$$

Multipl or N-Comm Anomaly, or Defect

- Given A, B, and AB ψDOs, even if ζ_A, ζ_B, and ζ_{AB} exist, it turns out that, in general,

$$\det{}_\zeta(AB) \neq \det{}_\zeta A \, \det{}_\zeta B$$

$$\det_\zeta(AB) \overset{?}{=} \det_\zeta A \, \det_\zeta B$$

$$\log \det_\zeta = \mathrm{tr}_\zeta \log \,, \quad \det_\zeta = e^{\mathrm{tr}_\zeta \log}$$

$$\det_\zeta(AB) \overset{1}{=} e^{\mathrm{tr}_\zeta \log(AB)} \overset{2}{=} e^{\mathrm{tr}_\zeta(\log A + \log B)}$$

$$\overset{3}{=} e^{\mathrm{tr}_\zeta \log A + \mathrm{tr}_\zeta \log B} =$$

$$\overset{4}{=} e^{\mathrm{tr}_\zeta \log A} \, e^{\mathrm{tr}_\zeta \log B} =$$

$$\overset{5}{=} \det_\zeta A \cdot \det_\zeta B$$

$[A, B] = 0$ assumed!

Which step is wrong?

tr_ζ is <u>no</u> trace at all

$$\mathrm{tr}_\zeta (A_1 + A_2) \neq \mathrm{tr}_\zeta A_1 + \mathrm{tr}_\zeta A_2$$

recall

$$\mathrm{tr}_\zeta A = \zeta_A(s=-1) = \sum_n \lambda_n^{-s} \Big|_{s=-1}$$

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- The multiplicative (or noncommutative) anomaly (defect) is defined as

$$\delta(A, B) = \ln \left[\frac{\det_\zeta(AB)}{\det_\zeta A \;\det_\zeta B} \right] = -\zeta'_{AB}(0) + \zeta'_A(0) + \zeta'_B(0)$$

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- Wodzicki formula

$$\delta(A,B) = \frac{\text{res}\left\{ [\ln \sigma(A,B)]^2 \right\}}{2 \,\text{ord}\, A \,\text{ord}\, B \,(\text{ord}\, A + \text{ord}\, B)}$$

where $\sigma(A,B) = A^{\text{ord}\, B} B^{-\text{ord}\, A}$

Consequences of the Multipl Anomaly

- In the path integral formulation

$$\int [d\Phi] \ \exp\left\{-\int d^Dx \left[\Phi^\dagger(x)(\quad)\Phi(x) + \cdots\right]\right\}$$

Gaussian integration: $\longrightarrow \quad \det(\quad)^{\pm}$

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A & \\ & B \end{pmatrix}$$

$\det(AB)$ or $\det A \cdot \det B$?

- In a situation where a superselection rule exists, AB has no sense (much less its determinant): $\Longrightarrow \det A \cdot \det B$
- But if diagonal form obtained after change of basis (diag. process), the preserved quantity is: $\Longrightarrow \det(AB)$

Lecture Notes in Physics 855

Emilio Elizalde

Ten Physical Applications of Spectral Zeta Functions

Second Edition

Springer

Elizalde
Odintsov
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Zeta Regularization Techniques with Applications

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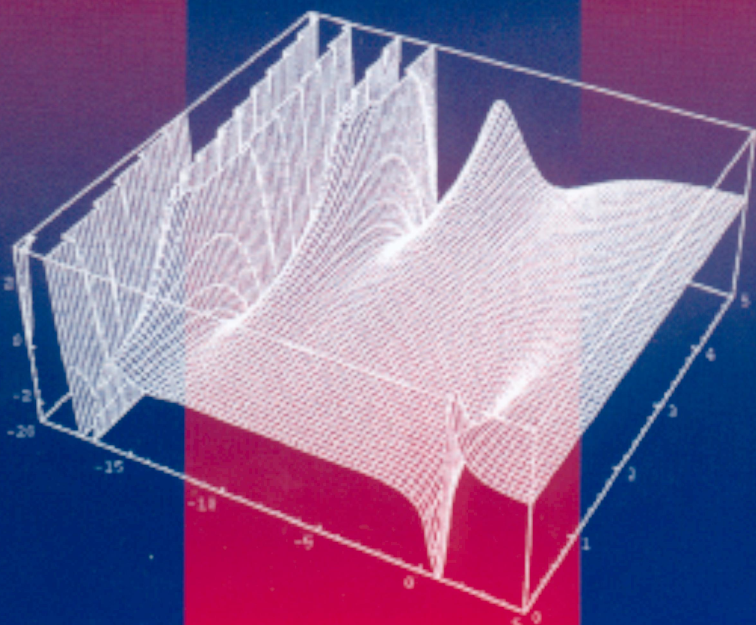

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