Quantum Gravity, Asymptotic Safety, and the Functional Renormalization Group

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• A major success :

Classical General Relativity, based upon the Einstein-Hilbert action $S[g_{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\sqrt{g}\,(-R+2\Lambda)$ is a highly successful effective field theory on length scales $\ell \gg \ell_{Pl} \equiv G^{1/2} \approx 10^{-33}\,\text{cm}$.

• A natural question :

Is it possible to promote G.R. to a <u>fundamental</u> (microscopic) quantum theory of the gravitational interaction and spacetime structure, valid at arbitrarily small ℓ ?

• A first attempt :

The methods that work well for the electroweak + strong interactions fail:

$$g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{(8\pi G)^{1/2}\, h_{\mu\nu}}_{\text{expand}}$$

Non-renormalizable in perturbation theory !

Increasing pert. order $\Rightarrow$
increasing # of divergences, i.e. counter terms,
i.e. undetermined parameters

The Options:

- Leave the framework of Quantum Field Theory:

  LQG, Spin Foams, String Theory, ...

- Stay within (non-perturbative!) QFT:

  Asymptotic Safety

continuum approach:

Functional Renormalization Group Equation (FRGE),

Effective Average Action

statistical field theory:

Dynamical triangulations, Regge calculus, ...

The fundamental problem:

Give a meaning to ("define", "renormalize", "take the continuum limit of", ...) a functional integral over all metrics on a spacetime $\mathcal{M}$:

$$\int \mathcal{D}\hat{g}_{\mu\nu}\, e^{-S[\hat{g}_{\mu\nu}]}$$

S: diff $(\mathcal{M})$ - invariant bare action, e.g. S_{EH} + counter terms

$$\mathcal{D}\hat{g}_{\mu\nu} \equiv \prod_{x\in\mathcal{M}} \prod_{\mu,\nu} dg_{\mu\nu}(x)$$

requires regularization (UV Cutoff)

The strategy:

MR, 1996

Define and compute the functional integral indirectly by means of the associated Effective Average Action (EAA):

$\Gamma_k[g_{\mu\nu}, \cdots]$, one-parameter family of action functionals,
$0 \leq k < \infty$.

The problem, reformulated:

Construct <u>fully</u> <u>extended</u> integral curves ("RG trajectories") $k \longmapsto \Gamma_k[\cdot]$, $0 \leq k < \infty$

of an infinite dimensional flow $(\mathcal{T}, \beta)$.

$\mathcal{T}$: "theory space" $\ni A[g_{\mu\nu}, \cdots]$
specified by field contents and symmetries

β : vector field on $\mathcal{T}$ defined by the functional renormalization group equation (FRGE) satisfied by the EAA:

$$k\,\partial_k \Gamma_k = \beta(\Gamma_k)$$

1. **The Effective Average Action for quantum gravity**

2. **The Asymptotic Safety construction of fundamental theories:**

   non-perturbative renormalizability despite perturbative non-renormalizability

3. **Identifying the dynamical mechanism underlying Asymptotic Safety**

The Effective Action $\Gamma[\Phi]$

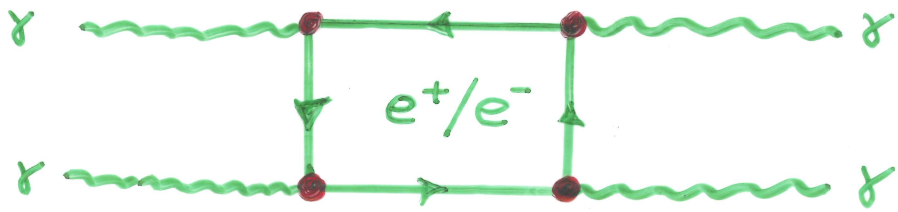
Q: How do the not directly observable vacuum fluctuations (virtual particle-/antiparticle creation, ...) manifest themselves in the dynamics of the observed particles?

A: Classical field equation $\quad \delta S[\Phi] = 0$

$\longrightarrow$ effective eq. $\quad \delta \Gamma[\Phi] = 0$

$\langle 0 | \hat{\Phi} | 0 \rangle$

E.g.:



$$\Gamma[\vec{E}, \vec{B}] = \int d^4x \left\{ \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) \right.$$

Maxwell

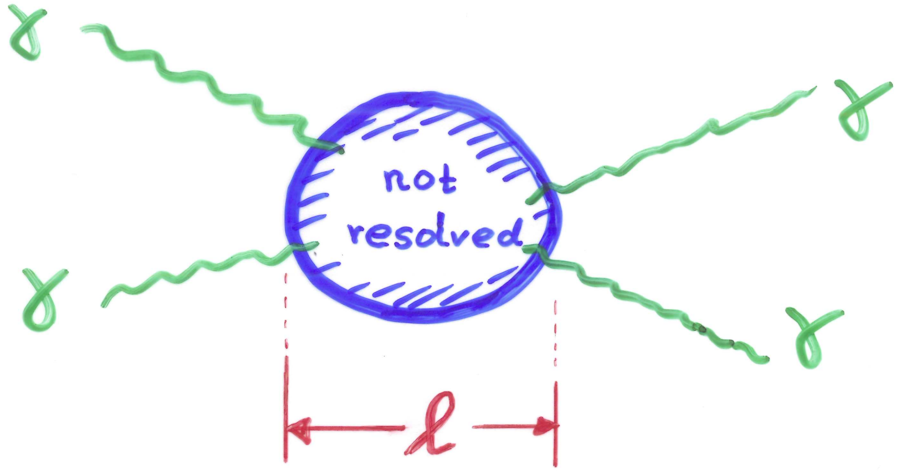
$$+ \frac{2\alpha^2}{45 m^4}\left[\left(\vec{E}^2 - \vec{B}^2\right)^2 + 7\left(\vec{E}\cdot\vec{B}\right)^2\right]$$

$$\left. + \dots \right\}$$

Heisenberg-Euler

The Effective Average Action $\Gamma_k[\Phi]$

C. Wetterich, MR, $\geq$ 1990



Q: How does the effective dynamics of the observed particles depend on the resolving power $k \equiv 1/\ell$ of the "microscope" used in the experiment?

A: Scale-dependent eff. field equation

$$\delta \Gamma_k [\Phi] = 0$$

$\Phi \equiv$ average* of fundamental field over Euclidean ball of radius $\ell = 1/k$

*: in the sense of Feynman's functional integral

$$\Rightarrow \quad \lim_{k \to 0} \Gamma_k = \Gamma$$

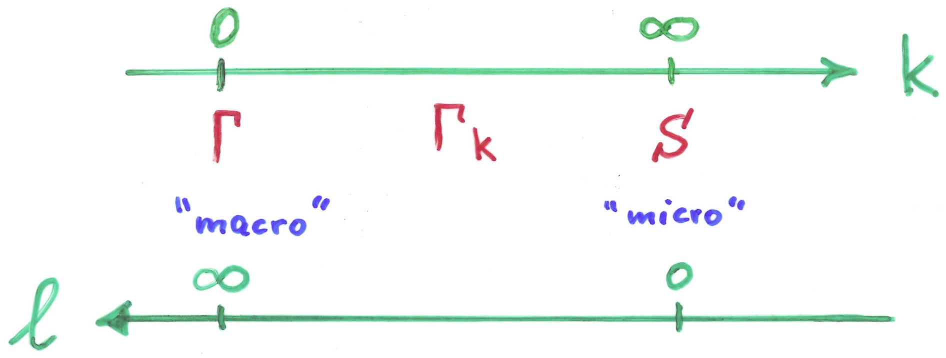
standard eff. act.

$\hat{=} \quad \ell \to \infty$

- In a theory with the classical action S, the Effective Average Action satisfies the Functional Renormalization Group Equation

$$\frac{d}{dk}\Gamma_k[\Phi] = \frac{1}{2}\,\mathrm{STr}\left[\left(\frac{\delta^2\Gamma_k[\Phi]}{\delta\Phi^2} + \mathcal{R}_k\right)^{-1}\frac{d\mathcal{R}_k}{dk}\right]$$

with initial condition $\Gamma_{k=\infty} = S$.

- 

- "Knowing Γ is knowing everything"

$\Longrightarrow$ "Quantizing" / "solving" a theory becomes an evolution problem on the infinite-dimensional **Theory Space** $\equiv$ $\{$ all possible action functionals $A[\Phi]$ $\}$

- Interpret solution $k \mapsto \Gamma_k$ as a curve on theory space: "RG trajectory".

Ways of implementing Background Independence :

- Literally use no background → LQG, CDT
- Use background, but keep it generic : → EAA

enforce Background Independence by requiring split-symmetry.

example:

$$\underset{\text{dynamical}}{g_{\mu\nu}} = \underset{\text{backgrd.}}{\bar{g}_{\mu\nu}} + \underset{\text{fluctuation}}{h_{\mu\nu}} + \cdots$$

$$= \left(\bar{g}_{\mu\nu} + \varepsilon_{\mu\nu}\right) + \left(h_{\mu\nu} - \varepsilon_{\mu\nu}\right) + \cdots$$

split-symmetry transformation : $\begin{cases} \delta \bar{g}_{\mu\nu} = \varepsilon_{\mu\nu} \\ \delta h_{\mu\nu} = -\varepsilon_{\mu\nu} \end{cases}$

The task : Quantize $h_{\mu\nu}$-field on a family of spacetimes $(\mathcal{M}, \bar{g})$ such that the physical sector of the resulting QFT respects split-symmetry.

Our Approach:

M.R. (1996)

- Employ background field technique:

$$\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}$$

Fluctuation: Quantize in the $\bar{g}_{\mu\nu}$-background "$\forall \bar{g}_{\mu\nu}$"

Background: fixed, but arbitrary

covariant Laplacian: $\bar{D}^2$

eigenmodes: $f^{\omega}_{\mu\nu}(x)$

- Expand fluctuation:

$$-\bar{D}^2 f^{\omega}_{\mu\nu} = \omega^2 f^{\omega}_{\mu\nu}$$

$$\hat{h}_{\mu\nu}(x) = \sum_{\omega} h_{\omega} f^{\omega}_{\mu\nu}(x)$$

- Background covariant gauge fixing:

$$\int \mathcal{D}\hat{g}_{\mu\nu}\, e^{-S} \longrightarrow \int \mathcal{D}\hat{h}_{\mu\nu}\, e^{-S'}$$

$S' \equiv S$ + gauge fixing + Faddeev-Popov

$$\int \mathcal{D}\hat{h}_{\mu\nu} \equiv \prod_{\omega} \int_{-\infty}^{\infty} dh_{\omega}$$

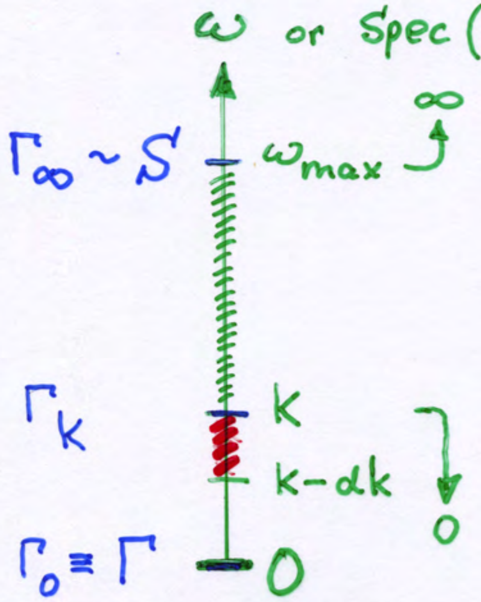
- **Regularize:**

$$\prod_{\omega=k}^{\omega=\omega_{max}} \int_{-\infty}^{\infty} dh_\omega \; e^{-S'\{h_\omega\}}$$

Dependence of the fctl. integral on IR cutoff is encoded in the **Effective Average Action:**

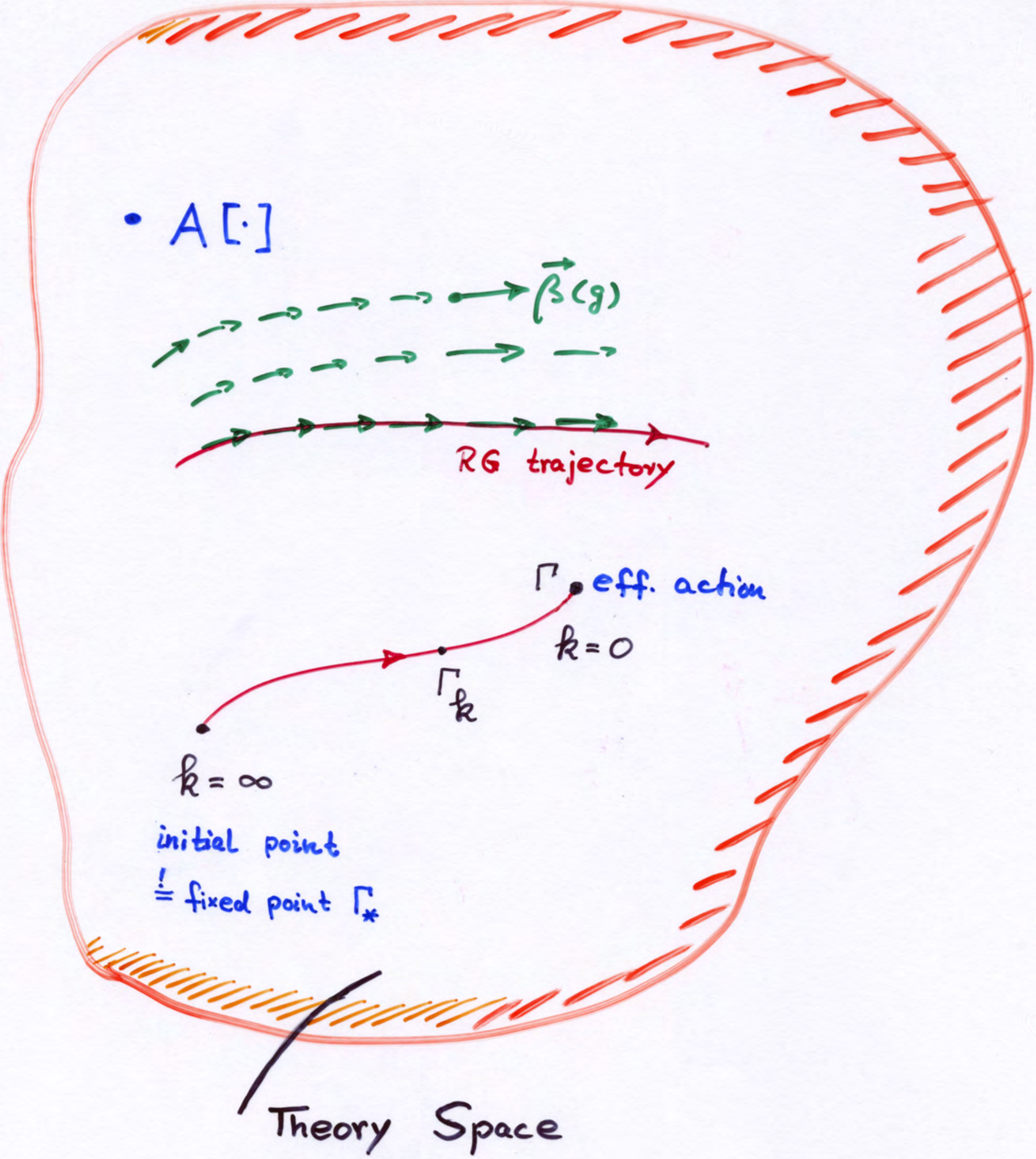
$$\Gamma_k[\underbrace{g_{\mu\nu}}_{=\langle \hat{g}_{\mu\nu}\rangle = \bar{g}_{\mu\nu} + \langle \hat{h}_{\mu\nu}\rangle}, \bar{g}_{\mu\nu}] \equiv \Gamma_k[\underbrace{h_{\mu\nu}}; \bar{g}_{\mu\nu}]$$

- **FRGE for the Eff. Average Action:**



$$\partial_k \Gamma_k = \frac{1}{2} \operatorname{Tr}\left[\left(\frac{\delta^2 \Gamma_k}{\delta g^2} + R_k\right)^{-1} \partial_k R\right]$$

- **Concrete implementation:** $\int \mathcal{D}\hat{h}\, e^{-S'}\, e^{-\int \hat{h} R_k(-\bar{D}^2)\hat{h}}$

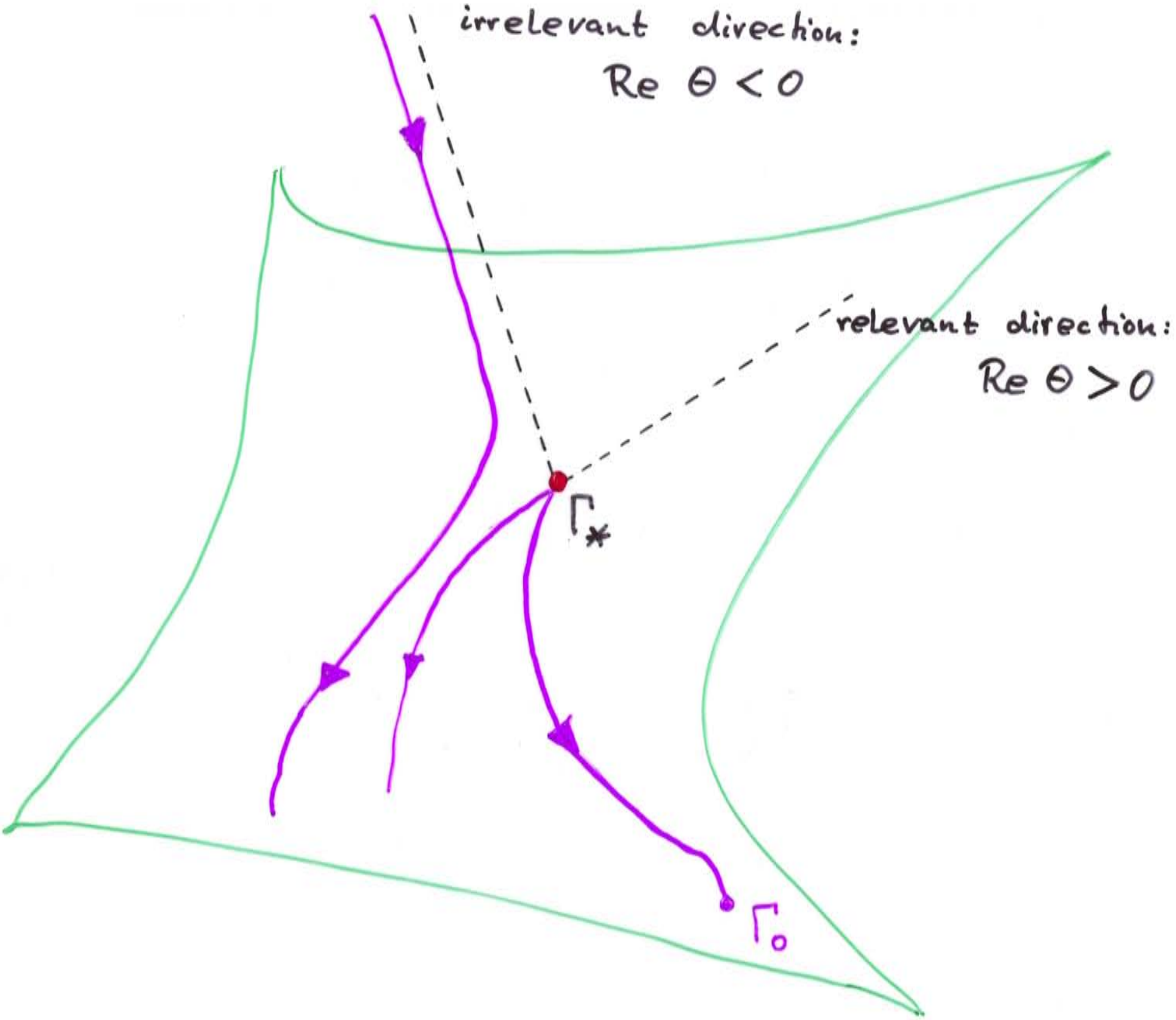

$A[\cdot]$
$\vec{\beta}(g)$
RG trajectory
Γ eff. action
$k=0$
Γ_k
$k=\infty$
initial point
$\stackrel{!}{=}$ fixed point Γ_*
Theory Space

The Asymptotic Safety idea:

- Take the infinite-cutoff limit of an UV-regularized quantum theory of gravity at a non-trivial RG fixed point with a finite dimensional UV-critical hypersurface, assuming it exists.
- The resulting continuum theory is predictive and well behaved at arbitrarily short distances.

(S. Weinberg, 1979, 2009)

The UV-critical hypersurface $\mathcal{S}_{UV}$:



$\Delta_{UV} \equiv \dim \mathcal{S}_{UV}$ = # relevant directions

= # free parameters in the a.s. quantum field theory

UV $\longrightarrow$ IR

Θ : critical exponent (neg. eigenvalue of lin. flow)

The Einstein-Hilbert Truncation

MR, 1996

Ansatz:

$$\Gamma_k = -\frac{1}{16\pi G_k}\int d^dx\,\sqrt{g}\,\left(R - 2\Lambda_k\right)$$

+classical gauge fixing and ghost terms

Running coupling constants:

Newton constant G_k, dimensionless: $g(k) = k^{d-2} G_k$

cosmological constant Λ_k, dimensionless: $\lambda(k) = k^{-2}\Lambda_k$

Insert ansatz into FRGE, "project out" monomials retained:

$$\mathrm{Tr}\,[\cdots] = (\cdots)\int\sqrt{g} + (\cdots)\int\sqrt{g}\,R + \cdots \quad \Rightarrow$$

$$\boxed{\begin{aligned} k\,\partial_k\, g(k) &= \beta_g(g,\lambda) \\ k\,\partial_k\, \lambda(k) &= \beta_\lambda(g,\lambda) \end{aligned}}$$

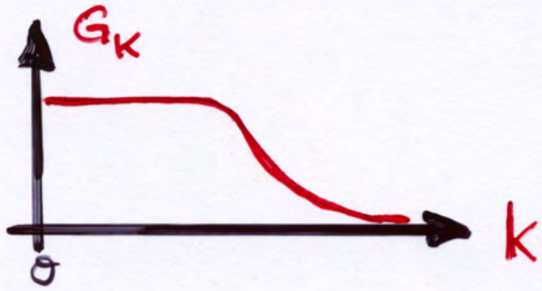
The anomalous dimension η_N :

$$\begin{cases} \partial_t g = \beta_g \equiv \big[(d-2) + \underbrace{\eta_N (g, \lambda)}_{= O(\hbar)} \big] g \\ \partial_t \lambda = \beta_\lambda \end{cases} \qquad t \equiv \ln(k)$$

with $\qquad \eta_N \equiv \dfrac{\partial_t G_k}{G_k}$

Explicit calculation : $\boxed{\eta_N < 0}$ $\Longrightarrow$

- Gravitational anti - screening:

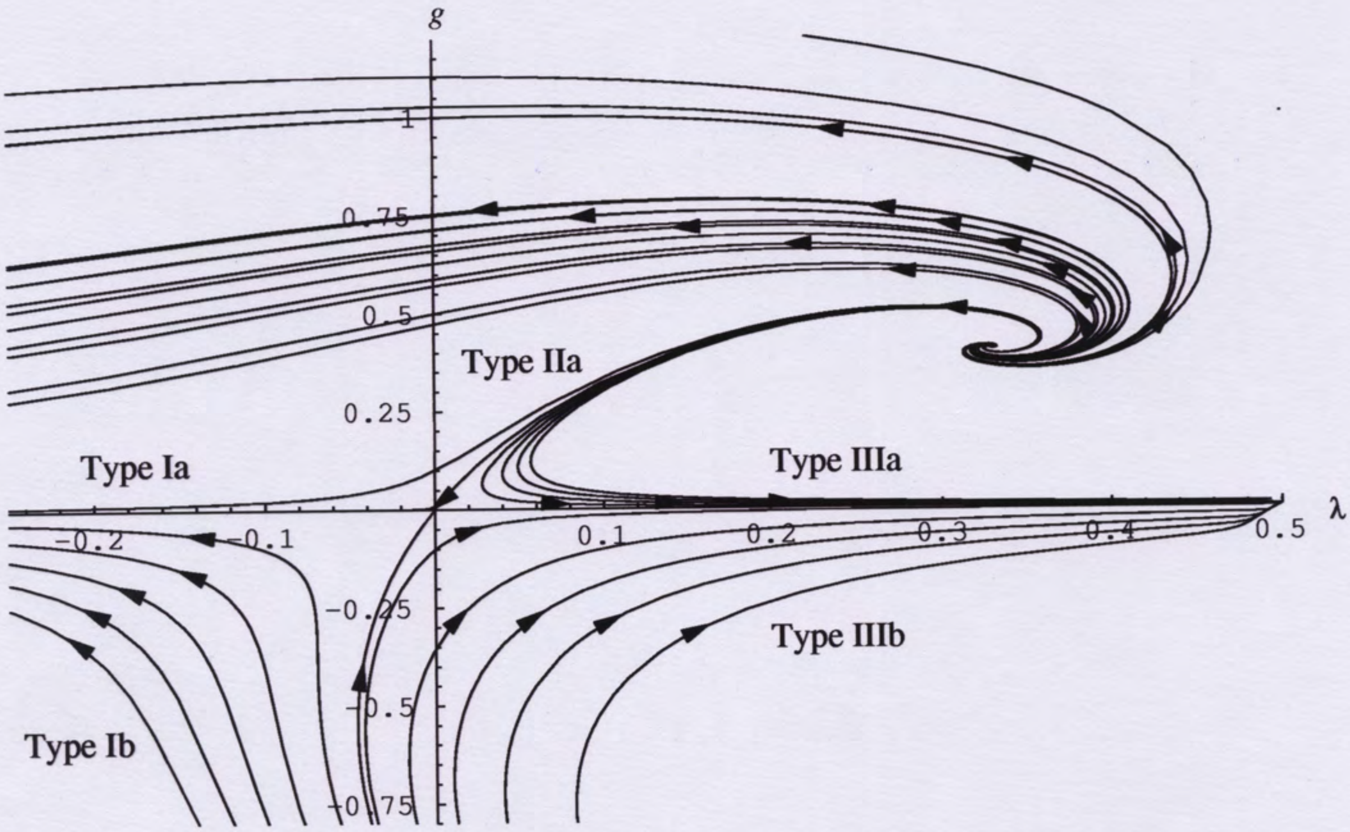


- Non-trivial fixed point:

$$\eta_N (g_*, \lambda_*) = -(d-2) \underset{d=4}{=} -2$$

Einstein - Hilbert Truncation:

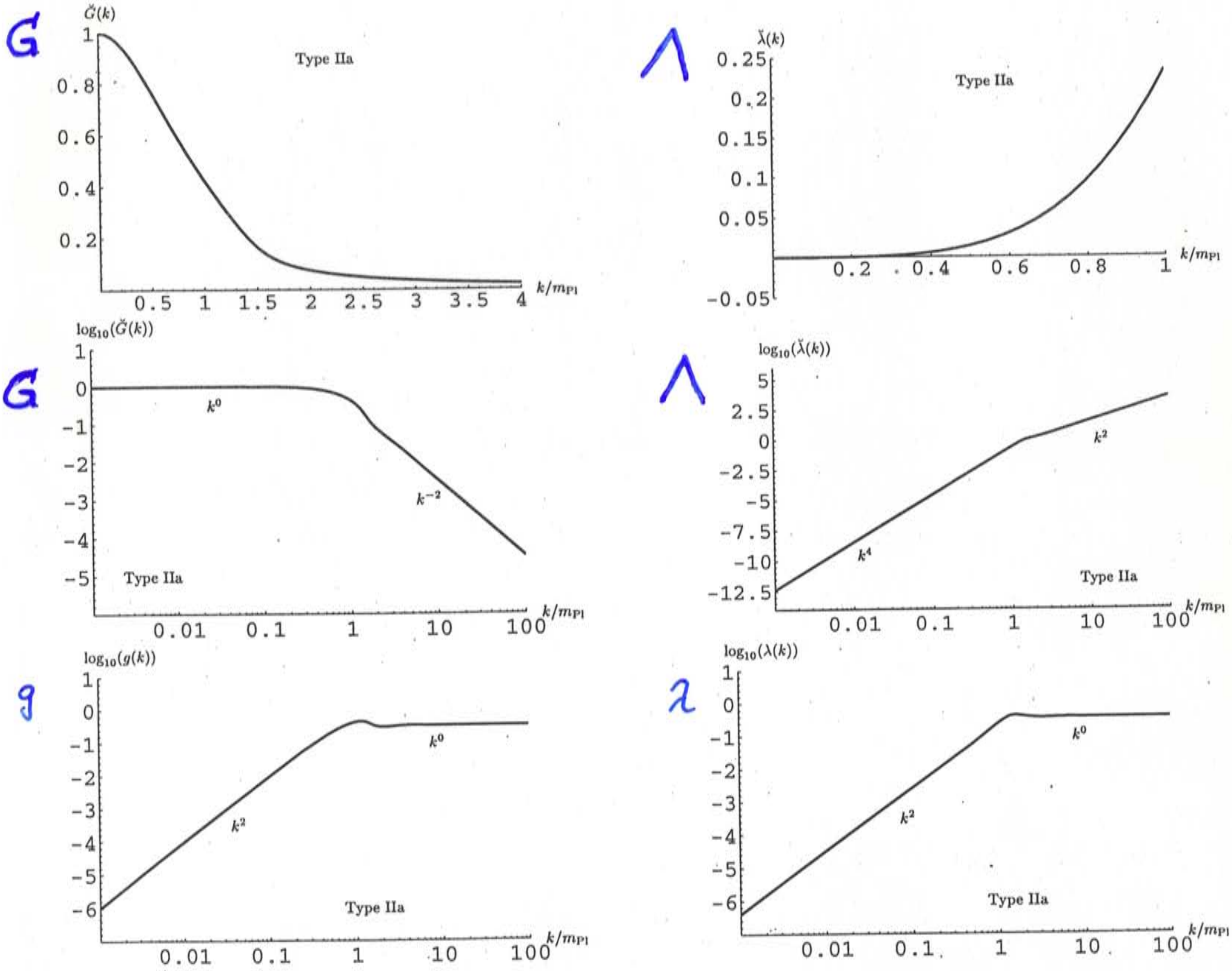
RG Flow on the $g-\lambda$ plane



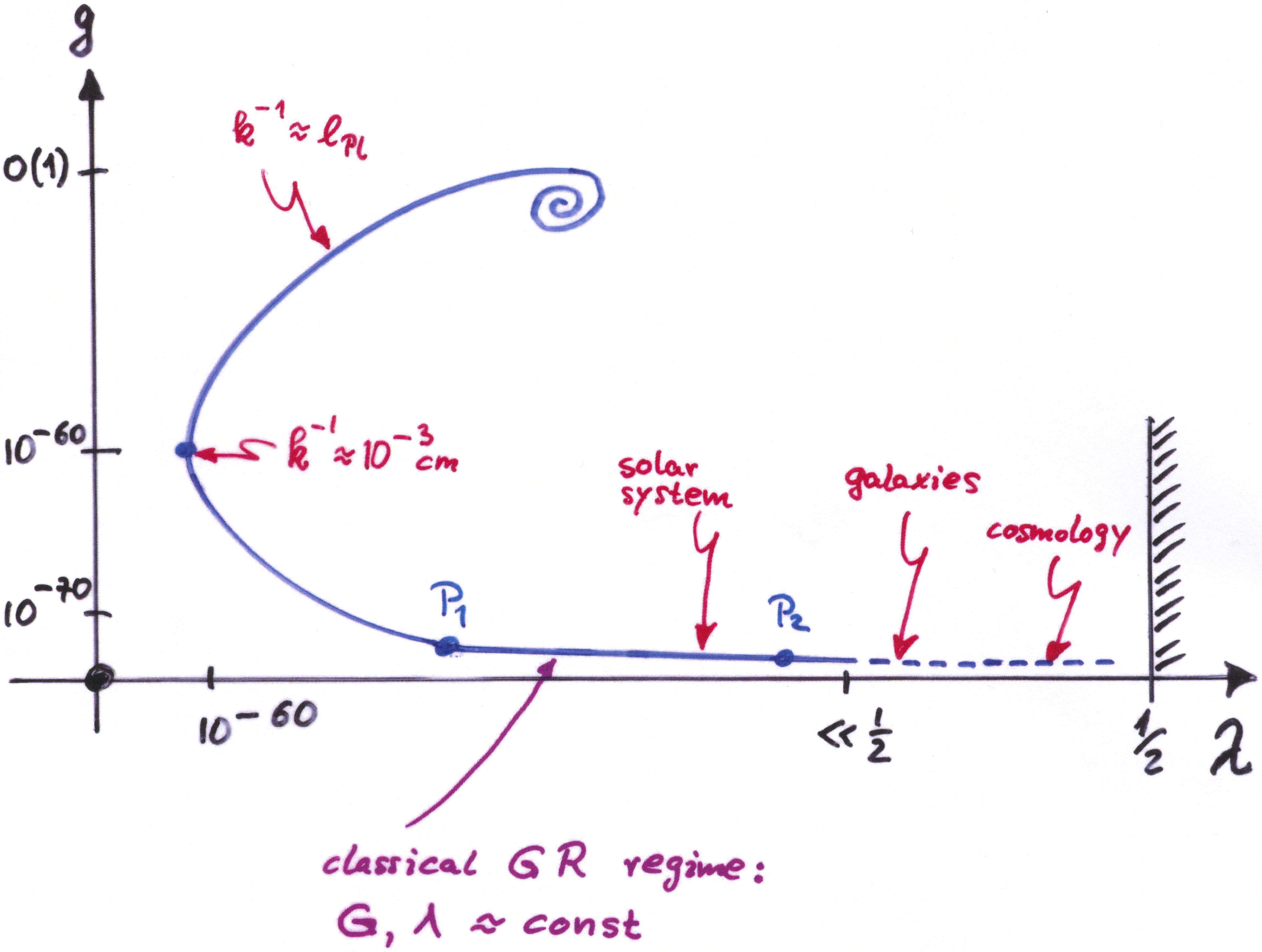
M.R., F. Saueressig, hep-th/0110054

The Separatrix:

cross-over from the non-Gaussian to the Gaussian fixed point



The RG trajectory "realized in Nature"



"Today" in cosmology:

$$\lambda_{cosmo} \equiv \frac{\Lambda_{cosmo}}{k^2_{cosmo}} = O(1) \;!!!$$

$\approx H_0^2$

$\approx H_0^2$

Properties of QEG

- **Background-independent quantization scheme:**

No special metric plays any distinguished role!

The background field method:

a) Fix arbitrary $\bar{g}_{\mu\nu}$

b) Quantize (nonlinear) fluctuations $h_{\mu\nu} \equiv \gamma_{\mu\nu} - \bar{g}_{\mu\nu}$ in the backgrd. of $\bar{g}_{\mu\nu}$

c) Adjust $\bar{g}_{\mu\nu}$ such that $\langle h_{\mu\nu} \rangle = 0$

$$\rightsquigarrow \quad g_{\mu\nu} \equiv \langle \gamma_{\mu\nu} \rangle = \bar{g}_{\mu\nu}$$

- **Fundamental action $S \approx \Gamma_*$ is a prediction:**

No special action plays any distinguished role!

The only input: field contents + symmetries $\hat{=}$ theory space

The output: $\Gamma_* = S_{\text{Einstein-Hilbert}}$ + "more"

Einstein-Hilbert action is often a reliable approximation, but not distinguished conceptually.

- Combination average action + background method successfully tested in QED and Yang-Mills theory.

- QEG reproduces successes of classical General Relativity: $\exists$ trajectories with long classical regime ($G = \text{const}$, $\Lambda = \text{const}$)

- QEG reproduces results of "QFTh. in curved spacetimes" in the classical regime: Hawking radiation, cosmological particle creation, ...

- Coexistence Asymptotic Safety $\leftrightarrow$ perturbative non-renormalizability well understood.

- Consistent quantization of gravity seems not to require "fine tuning" of matter system, special symmetries (SUSY, etc.), or unification with the other fundamental forces of Nature.

- Coupling to gravity softens/cures matter divergences:

  $$\text{QEG} + \text{QED} \rightsquigarrow e^2(k^2 \to \infty) = 0$$

  U. Harst, MR 2011

  Potentially higher degree of predictivity

- QEG spacetimes have fractal microstructure of reduced dimensionality

  O. Lauscher, MR 2002, 2005
  MR, F. Saueressig 2012

- "Phenomenology" from RG-Improvement

  - Cosmology: automatic Λ-driven inflation, scale free perturbations from NGFP, entropy production, ...

    A. Bonanno, MR (2001, 2007)
    MR, F. Saueressig (2005)

  - Black Holes: modified horizons, causal structure, final state of Hawking evaporation, ...

    A. Bonanno, MR (1999, 2000, 2006)
    MR, E. Tuiran (2006, 2011)

The physical mechanism underlying Asymptotic Safety

A. Nink, MR (2012)

The Magnetic Analogy

- **Non-relativistic electrons, Pauli eq.**

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + \mu_B \vec{B}\cdot\vec{\sigma}$$

orbital motion: Landau **dia**magnetism, $\chi_{dia} < 0$

spin alignment: Pauli spin **para**magnetism, $\chi_{para} > 0$

total susceptibility: $$\chi_{tot} = [\underbrace{3}_{\text{para}} - \underbrace{1}_{\text{dia}}]\cdot\left(\begin{smallmatrix}\text{positive}\\ \text{constant}\end{smallmatrix}\right) > 0$$

- **Relativistic electrons, Dirac eq.**

$$\not{D}^2 = \underbrace{(i\partial_\mu - eA_\mu)^2}_{\text{"dia"}} - \underbrace{\tfrac{i}{2}e\,\gamma^\mu\gamma^\nu F_{\mu\nu}}_{\text{"para"}}$$

running electric charge in QED (at 1 loop):

$$\partial_t e^2 = \beta_{e^2} = +\frac{1}{12\pi^2}[\underbrace{3}_{\text{para}} - \underbrace{1}_{\text{dia}}]\,e^4 \quad > 0$$

- Charge screening is due to the electrons' predominantly paramagnetic interaction with A_μ.
- The diamagnetic interactions drive β_{e^2} in the opposite direction.

- ## Yang-Mills gauge field fluctuations

$$S_{YM}[A^b_\mu] = \frac{1}{4}\int d^4x \, F^b_{\mu\nu} F^{b\mu\nu} \quad + \text{g.f.}$$

Expand $S_{YM}[A = \bar{A} + a]$ to order a^2:

$$\frac{1}{2}\int d^4x \; a^b_\mu \, [\underbrace{(-\bar{D}^2)^{bc}\, \delta^\mu{}_\nu}_{\text{"dia"}} + \underbrace{2ig\, \bar{F}^{bc\mu}{}_\nu}_{\text{"para"}}]\, a^{c\nu}$$

Running gauge coupling:

$$\partial_t g^2 = \beta_{g^2} = -\frac{N}{24\pi^2}\Big[\underbrace{12}_{\text{para}} \; \underbrace{-\;2}_{\text{dia}} \; \underbrace{+\;1}_{\text{ghosts}}\Big] g^4 < 0$$

- Color anti-screening and Asymptotic Freedom are due to the fluctuations' predominantly paramagnetic interaction with the background.
- The diamagnetic interactions drive β_{g^2} in the opposite (screening) direction.

Fluctuations of the metric

Expand $S_{EH}[g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}]$ + g.f. to order h^2:

$$\rightsquigarrow \quad \int d^dx \sqrt{\bar{g}} \; h_{\mu\nu} \Big[\underbrace{- K^{\mu\nu}{}_{\rho\sigma} \bar{D}^2}_{\text{"dia"}} + \underbrace{\bar{U}^{\mu\nu}{}_{\rho\sigma}}_{\text{"para"}} \Big] h^{\rho\sigma}$$

with:

$$\bar{K}^{\mu\nu}{}_{\rho\sigma} = \frac{1}{4}\left[\delta^\mu_\rho \delta^\nu_\sigma + \delta^\mu_\sigma \delta^\nu_\rho - \bar{g}^{\mu\nu}\bar{g}_{\rho\sigma}\right]$$

$$\begin{aligned} \bar{U}^{\mu\nu}{}_{\rho\sigma} = & -\frac{1}{2}\left[\bar{R}^\nu{}_\rho{}^\mu{}_\sigma + \bar{R}^\nu{}_\sigma{}^\mu{}_\rho\right] \\ & + \frac{1}{2}\left[\bar{g}^{\mu\nu}\bar{R}_{\rho\sigma} + \bar{g}_{\rho\sigma}\bar{R}^{\mu\nu}\right] - \frac{1}{4}\left[\delta^\mu_\rho \bar{R}^\nu{}_\sigma + \dots\right] \\ & + \bar{R}\; \bar{K}^{\mu\nu}{}_{\rho\sigma} \end{aligned}$$

Fluctuations drive the RG flow:

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr}\left[\Big(\underbrace{\frac{\delta^2 \Gamma_k}{\delta h \delta h}}_{= -\bar{K}\bar{D}^2 + \bar{U}} + R_k \Big)^{-1} \partial_t R_k \right] + \dots$$

$\rightsquigarrow$ clear separation of dia / para contributions

Anomalous dimension η_N (leading order):

$$\eta_N = -\underbrace{f}_{>0} g \Big[\underbrace{+12(d-1)}_{\text{para}} + \underbrace{\frac{48}{d}}_{\text{ghost-para}} \underbrace{- d(d+1)}_{\text{dia}} \underbrace{+ 4d}_{\text{ghost-dia}} \Big]$$

$$= -f g \Big[\underbrace{12(d-1) + \frac{48}{d}}_{\text{total para:}} \quad \underbrace{- d(d-3)}_{\text{total dia:}} \Big]$$

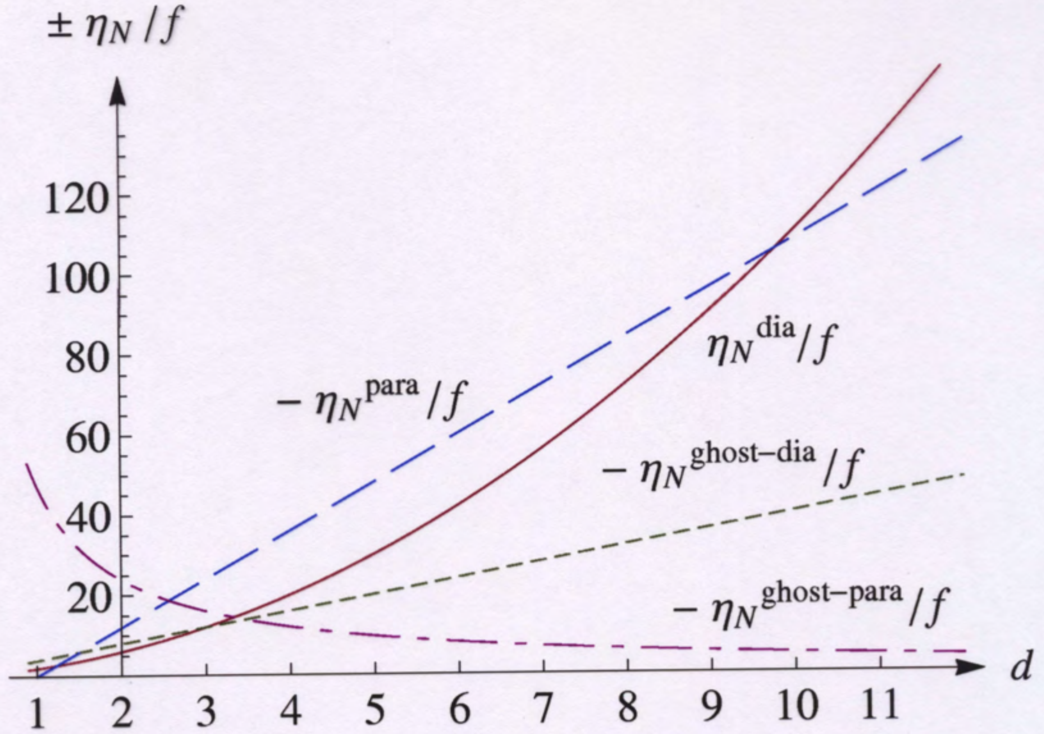
	total para:	total dia:
	positive $\forall d$	negative $\forall d > 3$ positive $\forall d < 3$
$d=4$:	$+48$	-4

"para" is 12 times stronger!

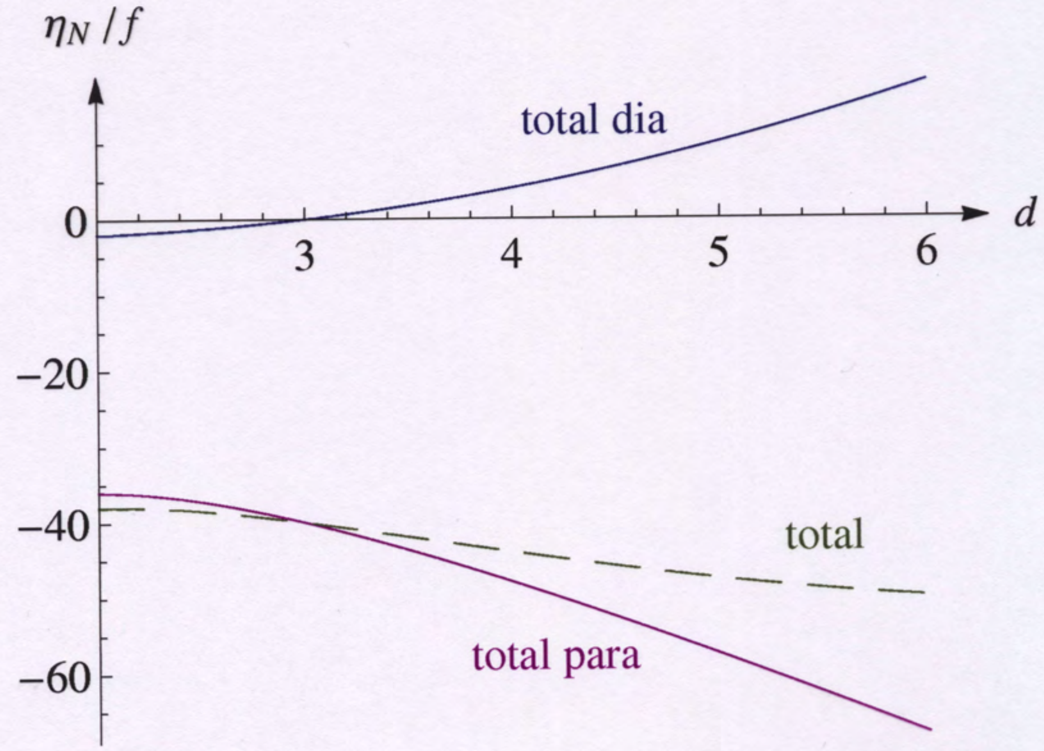
- Gravitational anti-screening and Asymptotic Safety in $d>3$ is due to the fluctuations' predominantly paramagnetic interaction with the background.

- In $d>3$, the diamag. interactions drive η_N in the opposite (screening) direction.

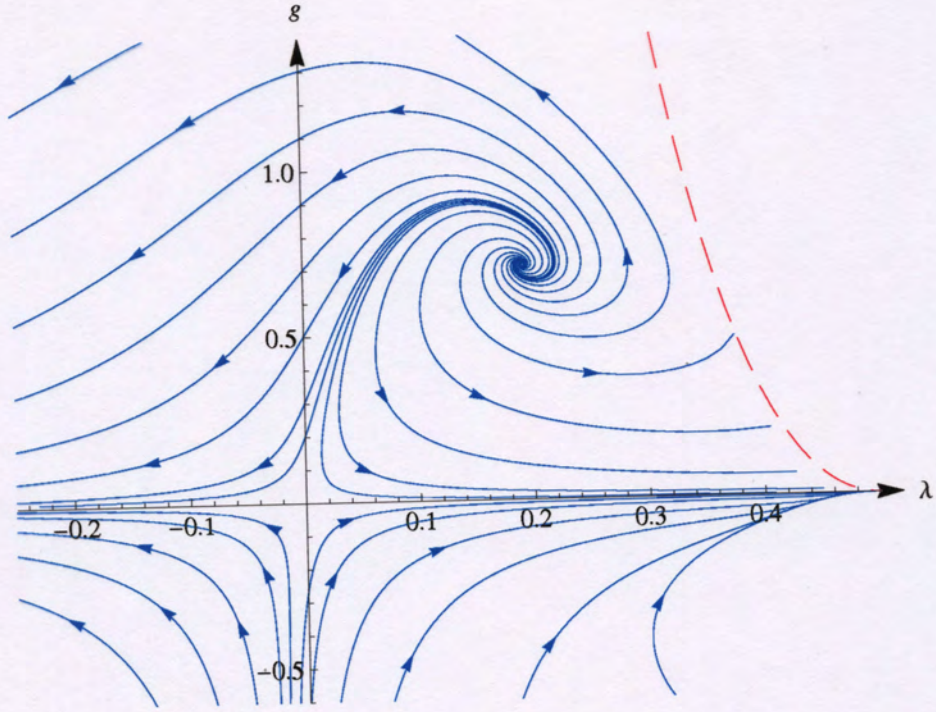
Very similar to Yang-Mills theory!



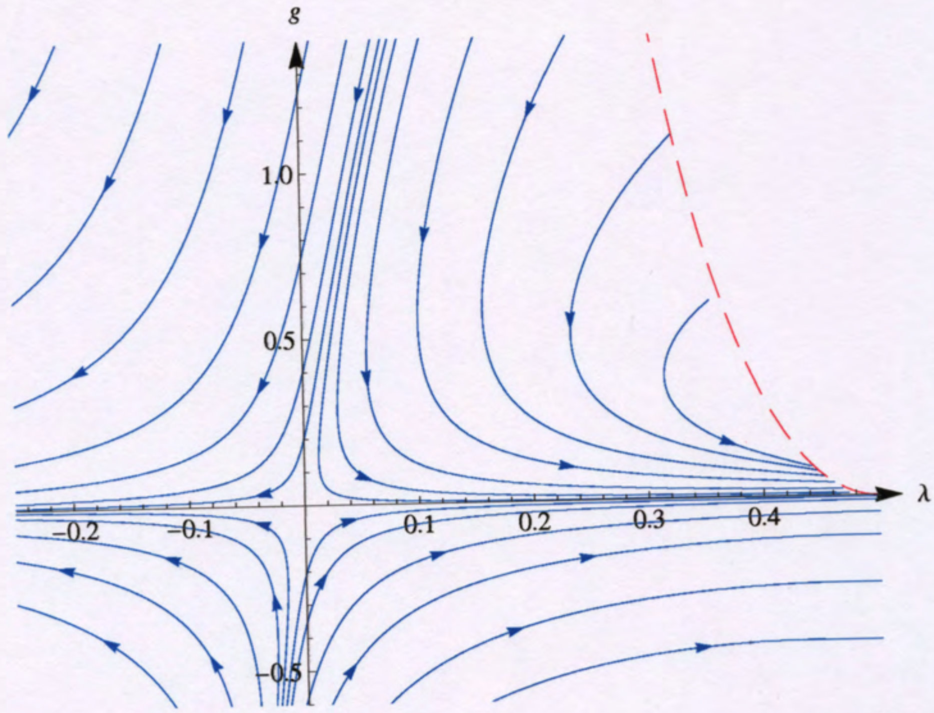
$\pm\,\eta_N\,/f$
120
100
80
60
40
20
$-\,\eta_N{}^{\text{para}}\,/f$
$\eta_N{}^{\text{dia}}/f$
$-\,\eta_N{}^{\text{ghost-dia}}/f$
$-\,\eta_N{}^{\text{ghost-para}}/f$
1 2 3 4 5 6 7 8 9 10 11
d



$\eta_N\,/f$
total dia
0
3 4 5 6
d
−20
−40
−60
total
total para



Flow diagram obtained from the *total paramagnetic contributions* to η_N alone.



Flow diagram taking into account the *total diamagnetic terms* in η_N only.

Paramagnetic Dominance

- In a large class of well understood physical systems quantum fluctuations are governed by non-minimal differential operators $\Delta_A + \mathcal{F}(A)$ which give rise to antagonistic dia- and para-type interactions. The para-type interactions "win" and determine the qualitative properties of the system.
- QEG seems to belong to this class !
- <u>The emerging picture of spacetime:</u>
  - Paramagnetic coupling $\sim h \bar{R} h$ is <u>ultra-local</u>, analogous to $\bar{\Psi}\,(\vec{\sigma}\cdot\vec{B})\,\Psi$
  - Spin orientation effects dominate over orbital motion $\sim h \bar{D}^2 h$

$\Longrightarrow$ <u>Analogy:</u> Spin system with magnetic moments sitting at fixed lattice points, interacting with their mean field.

(Rather than a gas of itinerant electrons !)

Asymptotic safety on the lattice: The Nonlinear O(N) Sigma Model

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We study the non-perturbative renormalization group flow of the nonlinear O(N) sigma model in two and three spacetime dimensions using a scheme that combines an effective local Hybrid Monte Carlo update routine, blockspin transformations and a Monte Carlo demon method. In two dimensions our results verify perturbative renormalizability. In three dimensions, we determine the flow diagram of the theory for various N and different truncations and find a non-trivial fixed point, which indicates non-perturbative renormalizability. It is related to the well-studied phase transition of the O(N) universality class and characterizes the continuum physics of the model. We compare the obtained renormalization group flows with recent investigations by means of the Functional Renormalization Group.

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I. INTRODUCTION

The renormalization of coupling parameters due to quantum fluctuations is a characteristic feature of any quantum field theory and many different methods have been developed to study this interesting property. While most of these methods rely on a perturbative treatment of the theories, the investigation of strongly coupled or strongly correlated systems without small expansion parameter, like e.g. the theory of strong interaction, requires a non-perturbative approach. One non-perturbative and very flexible method is the *Functional Renormalization Group* (FRG) introduced by K. Wilson [1]. In a particularly useful implementation of the functional renormalization group, one studies the flow of the effective average action Γ_k w.r.t. the momentum scale k, which interpolates between the bare action at the UV-cutoff Λ, and the full effective action in the IR, $\Gamma_{k\to 0} = \Gamma$ [2]. With the help of this powerful non-perturbative approach one can explore theories which are non-renormalizable in perturbation theory, i.e. in the vicinity of a Gaußian fixed point, but are renormalizable in a non-perturbative setting. In such asymptotically save theories the running of the couplings in the UV is controlled by a non-trivial fixed point with a finite number of relevant directions. The most important theory where this so-called *asymp-*

running coupling such as the renormalized correlation functions or the Schwinger functional [8]. In an alternative recent approach one tries to directly integrate out momentum shells on the lattice by using Fourier Monte Carlo simulation [9]. In the present work we make use of the well-known *Monte Carlo Renormalization Group* method (MCRG) [10–13]. It is based on the idea of blockspin transformations and can be applied to theories with fermionic or gauge fields [14]. By applying successive blockspin transformations, real-space RG-transformations are performed and a renormalization trajectory is calculated. However, since every RG step typically reduces the linear extent of the lattice by a factor of $b = 2$, exponentially large lattices are needed in order to obtain sufficiently long trajectories that get close enough to the fixed point regime [15]. Even worse, a standard method to determine the effective couplings relies on the matching of correlation functions on the initial and blocked lattices and requires expensive scanning runs for the parameters of the bare action at the largest lattice used [16]. In order to circumvent these problems we employ the *demon method* [17–19] which allows us to efficiently compute RG trajectories at a fixed lattice volume.

In the present work we apply the MCRG method in combination with the demon method to calculate the global flow dia-

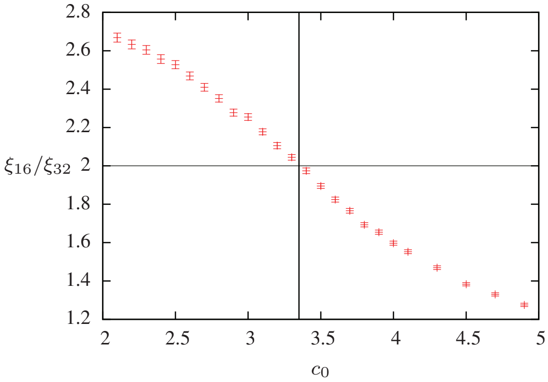

FIG. 8. The ratio of correlation lengths obtained by blocking a 32^3 lattice down to 16^3 using different optimization constants c_0. A value of $\xi_{16}/\xi_{32} = 2$ is expected to minimize truncation errors and we read off the optimal value $c_0^{\text{opt}} = 3.35$ for $N = 3$.

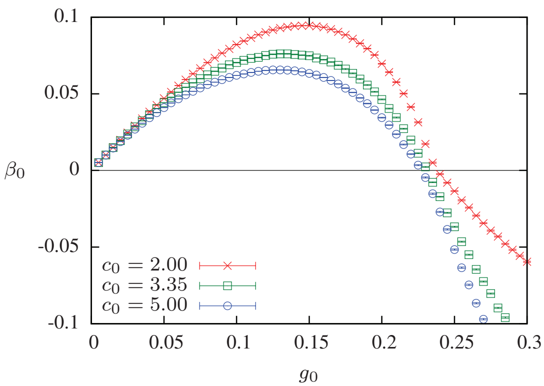

FIG. 9. The β function for the $1 \to 1$ truncation in three dimensions and $N = 3$ is shown for different values of c_0.

flow to the disordered phase in the IR which is controlled by the Gaußian fixed point at $g_0 = 0$, while systems with bare coupling $g_0 > g_0^*$ flow to the completely ordered phase described by $g_0 = \infty$ or $1/g_0 = 0$. These two fixed points correspond to the expected low-temperature fixed point at infinite coupling (absolute order) and the expected high-temperature fixed point at zero coupling (absolute disorder). The critical hypersurface is reduced to a single point g_0^* in this truncation and the operator S_0 corresponds to a *relevant* direction of the RG flow.

Using the information provided by thermodynamical observables like e.g. the susceptibility of the order parameter, we can determine the *critical point* g_0^c where the correlation length of the system diverges at infinite volume. In general theory space, it is the point of intersection between the critical hypersurface and the line where $g_i = 0$ except g_0. A lattice simulation starting at g_0^c in the UV will flow along the critical line into the non-trivial fixed point and observables measured on this ensemble reflect the macroscopic physics at this point. Please note that g_0^c need not be identical to g_0^* due to truncation errors that affect the value for g_0^*. Of course, without truncation errors the fixed point is located at the critical surface. We now proceed to discuss higher-order truncations which take additional operators into account and provide a more complete picture of the flow of the effective action.

B. Higher-order truncations

In the preceding sections we have seen that near the non-trivial fixed point the operator S_0 defines a relevant direction. In this section we include more operators in the effective action in order to find the total number of relevant directions. Figure 10 (upper panel) shows the global flow diagram for the truncation using two operators $S = g_0 N S_0 + g_1 N S_1$, both for ensemble generation as well as in the demon method ($2 \to 2$ truncation). The blockspin transformation is optimized in

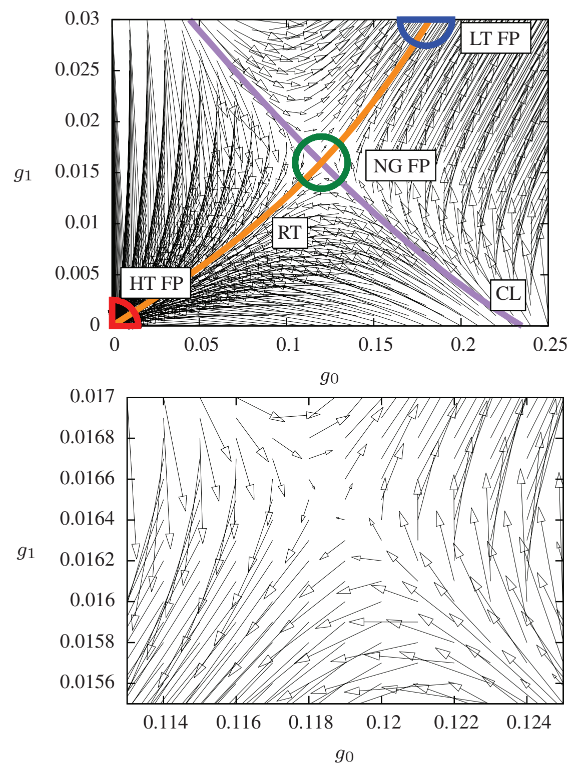

FIG. 10. The flow diagram using the $2 \to 2$ truncation in three dimensions and $N = 3$ clearly shows a non-Gaußian fixed point (NG FP) in the center of the plot in the upper panel. The critical line (CL) and renormalized trajectory (RT) intersect at the NG FP. The lower panel shows the vicinity of the NG FP. The RG parameters for this flow diagram are $c_0 = 3.1$ and $c_1 = 2.5$.

the same way as for the action with a single parameter. Our choice for the parameters is $c_0 = 3.1$ and $c_1 = 2.5$ and it leads to a correlation length ratio of around 2 in the vicinity of the fixed point. Note that this choice for the parameters is not unique if we only tune the correlation length to the desired value. In general we have to consider higher correlation functions as well. Below we will also discuss other choices

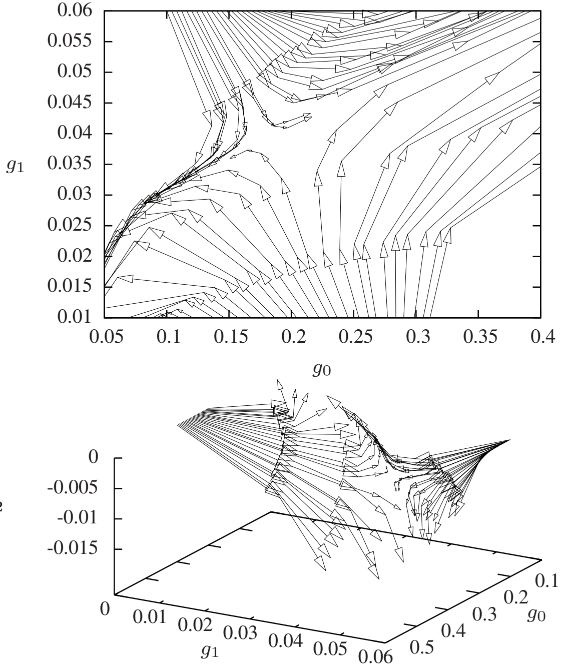

FIG. 11. Using a shooting technique, the RG trajectories for the $3 \to 3$ truncation with operators S_0, S_1 and S_2 reveal an analogous structure to the $2 \to 2$ case. The projection on the g_0-g_1 axis in the upper panel shows only a single relevant direction at the non-Gaußian fixed point. The lower panel shows that the trajectories first approach the fixed point regime and afterwards flow along the renormalized trajectory to the respective IR fixed points.