Remarks on the average effective action in Functional Renormalization Group approach

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FP method
[Faddeev, Popov (1967)]

Yang-Mills action[(1954)]

$$S_{YM}(A) \;=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$$

Gauge invariance

$$\delta S_{YM} = 0, \quad \delta A^a_\mu = D^{ab}_\mu \xi^b, \quad D^{ab}_\mu = \delta^{ab}\partial_\mu + f^{acb}A^c_\mu$$

Non-unitariry of S-matrix[Feynman (1963)]

Faddeev-Popov action[(1967)]

$$S_{FP}(\Phi) = S_{YM} + S_{gf} + S_{gh} = S_{YM}(A) + \chi^a B^a + \bar{C}^a K^{ab} C^b$$

$$\Phi^A = (A^a_\mu, B^a, C^a, \bar{C}^a), \quad \varepsilon(\Phi^A) = \varepsilon_A$$

$$K^{ab} = \frac{\delta \chi^a}{\delta A^c_\mu} D^{cb}_\mu$$

FP method

For more popular gauges in Yang-Mills theories the functions χ^a are chosen as

Landau gauge

$$\chi^a = \partial^\mu A^a_\mu$$

R_ξ gauge

$$\chi^a = \partial^\mu A^a_\mu + \frac{\xi}{2} B^a$$

The FP-operator

$$K^{ab} = \partial^\mu D^{ab}_\mu = \delta^{ab} \partial^\mu \partial_\mu + f^{acb} \partial^\mu \cdot A^c_\mu$$

BRST symmetry

[Becchi,Rouet,Stora (1974),Tyutin (1975)]

$$\delta_B S_{FP}(\Phi) = 0$$

$$\begin{aligned}
\delta_B A^a_\mu(x) &= D^{ab}_\mu C^b(x)\mu \\
\delta_B C^a(x) &= \frac{1}{2} f^{abc} C^b(x) C^c(x)\mu \\
\delta_B \bar{C}^a(x) &= B^a(x)\mu \\
\delta_B B^a(x) &= 0
\end{aligned}$$

μ is a constant Grassmann parameter, $\mu^2 = 0$. Due to the Noether theorem there exists conserved charge, the BRST charge Q_B. Corresponding BRST operator, $\hat{Q}_B$, defines the physical space states, $\hat{Q}_B|phys>= 0$.

FP method

Let $\delta_B \Phi^A = \hat{s}\Phi^A \mu$. Then

Nilpotency of the BRST transformations

$$
\begin{aligned}
\hat{s}^2 A^a_\mu &= \hat{s} D^{ab}_\mu C^b = 0 \\
\hat{s}^2 \bar{C}^a &= \hat{s} B^a = 0 \\
\hat{s}^2 B^a &= 0 \\
\hat{s}^2 C^a &= \hat{s} \frac{1}{2} f^{abc} C^b C^c = 0
\end{aligned}
$$

It leads to very important property of the BRST operator $\hat{Q}_B$ to be nilpotent, $\hat{Q}_B^2 = 0$. In its turn this property allows effectively to analyze the unitarity problem with the help of KO-quartet mechanism [Kugo, Ojima (1979)].

FP method

Generating functionals of Green's functions and connected Green's functions

$$Z(J) = \int \mathcal{D}\Phi \, \exp\Big\{\frac{i}{\hbar}\Big(S_{FP}(\Phi) + J_A\Phi^A\Big)\Big\} = \exp\Big\{\frac{i}{\hbar}W(J)\Big\}$$

$J_A(x)$ are external sources to fields $\Phi^A(x)$, $\varepsilon(\Phi^A) = \varepsilon_A$.

Generating functional of vertex functions (effective action)

$$\Gamma(\Phi) = W(J) - J_A\Phi^A$$

$$\Phi^A(x) = \frac{\delta W(J)}{\delta J_A(x)}, \qquad \frac{\delta\Gamma(\Phi)}{\delta\Phi^A(x)} = -J_A(x)$$

FP method

Vacuum functional $Z(0) \equiv Z_\chi$ constructing for a given gauge $\chi^a = 0$ is

$$Z_\chi = \int \mathcal{D}\Phi \, \exp\Big\{\frac{i}{\hbar} S_{FP}(\Phi)\Big\}.$$

Consider the gauge $\chi^a + \delta\chi^a = 0$ and corresponding vacuum functional $Z_{\chi+\delta\chi}$. Use the change of the variables of integration in the form of BRST transformations with

$$\mu = -\frac{i}{\hbar} \bar{C}^a \delta\chi^a.$$

Gauge independence of vacuum functional

$$Z_{\chi+\delta\chi} = Z_\chi.$$

Recently this result has been extended on the level of arbitrary finite change of gauge [PML, Lechtenfeld (2013)]

FP method

The effective action $\Gamma = \Gamma(\Phi)$ is the main object and depends on gauges. Consider an infinitesimal variation of gauge function $\chi^a \to \chi^a + \delta\chi^a$. The equation describing the gauge dependence of effective action Γ under variation of gauge has the form [PML,Tyutin (1981)]

$$\delta\Gamma = \frac{\delta\Gamma}{\delta\Phi^A} F^A$$

with functionals F^A depending on $\delta\chi^a$.
The main feature is that the effective action does not depend on gauge on its extremals

Gauge independence of Γ on its extremals

$$\frac{\delta\Gamma}{\delta\Phi^A} = 0, \qquad \delta\Gamma\Big|_{\frac{\delta\Gamma}{\delta\Phi}=0} = 0.$$

FRG approach
[Wetterich (1993), Reuter, Wetterich (1994)]

The main idea of the FRG is to use instead of Γ an average effective action, Γ_k, with a momentum-shell parameter k, such that

$$\lim_{k\to 0} \Gamma_k = \Gamma \,.$$

For the Yang-Mills theories it was suggested to modify the Faddeev-Popov action with the help of the specially designed regulator action S_k

$$S_k(A, C, \bar{C}) \;=\; \frac{1}{2}\, A^{a\mu} (R_{k,A})^{ab}_{\mu\nu} A^{b\nu} \,+\, \bar{C}^a (R_{k,gh})^{ab} C^b \,.$$

Regulator functions $R_{k,A}$ and $R_{k,gh}$ obey the properties

$$\lim_{k\to 0} (R_{k,A})^{ab}_{\mu\nu} \,=\, 0\,, \qquad \lim_{k\to 0} (R_{k,gh})^{ab} \,=\, 0\,.$$

BRST non-invariance

$$\delta_B\, S_k(A, C, \bar{C}) \,\neq\, 0\,.$$

The generating functionals of Green's functions, $\mathcal{Z}_k$, and connected Green's functions, $\mathcal{W}_k(J)$ is constructed in the form of the functional integral

$$\begin{aligned}\mathcal{Z}_k(J) &= \int \mathcal{D}\Phi \exp\Big\{\frac{i}{\hbar}\big[S_{FP}(\Phi) + S_k(\Phi) + J\Phi\big]\Big\} = \\ &= \exp\Big\{\frac{i}{\hbar}\mathcal{W}_k(J)\Big\}\end{aligned}$$

where, for the sake of uniformity, we used notation $S_k(\Phi)$ instead $S_k(A, C, \bar{C})$, despite S_k does not depend on fields B^a

FRG approach

The generating functional of vertex functions in the presence of regulators (the average effective action), $\Gamma_k = \Gamma_k(\Phi)$, satisfies the functional integro-differential equation

$$\exp\Big\{\frac{i}{\hbar}\,\Gamma_k(\Phi)\Big\} = \int \mathcal{D}\varphi\, \exp\Big\{\frac{i}{\hbar}\Big[S_{FP}(\Phi+\varphi) + S_k(\Phi+\varphi)\, - \\ - \frac{\delta\Gamma_k(\Phi)}{\delta\Phi}\,\varphi\Big]\Big\}\,.$$

The tree-level (zero-loop) approximation corresponds to

$$\Gamma_k^{(0)}(\Phi)\,=\,S_{FP}(\Phi)\,+\,S_k(\Phi)\,.$$

FRG approach

The FRG flow equation for Γ_k $(t = \ln k)$

$$\partial_t \Gamma_k \;=\; \partial_t S_k \;+\; i\hbar\,\Big\{\frac{1}{2}\,\partial_t (R_{k,A})^{ab}_{\mu\nu}\,\big(\Gamma''^{-1}_k\big)^{(a\mu)(b\nu)} \;+\; \partial_t (R_{k,gh})^{ab}\,\big(\Gamma''^{-1}_k\big)^{ab}\Big\}.$$

In the condensed notations

The Wetterich equation

$$\partial_t \bar{\Gamma}_k = i\hbar\Big\{\frac{1}{2}\,\mathsf{Tr}\Big[\partial_t R_{k,A}\,\big(\bar{\Gamma}''_k + R_{k,A}\big)^{-1}\Big]_A - \\ - \mathsf{Tr}\Big[\partial_t (R_{k,gh})\,\big(\bar{\Gamma}''_k + R_{k,gh}\big)^{-1}\Big]_C\Big\}.$$

where $\bar{\Gamma}_k = \Gamma_k - S_k$ and we took into account the anticommuting nature of the ghost fields C^A and defined

$$\mathsf{Tr}\Big[\partial_t (R_{k,gh})\big(\Gamma''^{-1}_k\big)\Big]_C \;=\; -\,\partial_t (R_{k,gh})^{ab}\big(\Gamma''^{-1}_k\big)^{ab},$$

$$\mathsf{Tr}\Big[\partial_t (R_{k,A})\big(\Gamma''^{-1}_k\big)\Big]_A \;=\; \partial_t (R_{k,A})^{ab}_{\mu\nu}\big(\Gamma''^{-1}_k\big)^{(a\mu)(b\nu)}.$$

The vacuum functional, corresponding to a given gauge function χ^a in the FP action reads

$$Z_{k,\chi} = \mathcal{Z}_k(0) = \int \mathcal{D}\Phi \, \exp\Big\{ \frac{i}{\hbar}\big(S_{FP}(\Phi) + S_k(\Phi)\big)\Big\} .$$

By construction, the regulator functions in the FRG approach do not depend on gauge χ^a and therefore the action S_k is gauge independent. Let us consider an infinitesimal variation of gauge $\chi \to \chi + \delta\chi$ and construct the vacuum functional corresponding to this gauge

$$\begin{aligned} Z_{k,\chi+\delta\chi} \; &= \; \int \mathcal{D}\Phi \; \exp\Big\{ \frac{i}{\hbar}\Big(S_{FP}(\Phi) + S_k(\Phi) + \\ &\qquad + \bar{C}^a \, \frac{\delta\,\delta\chi^a}{\delta A^c_\mu} \, D^{cb}_\mu C^b + \delta\chi^a B^a\Big)\Big\} . \end{aligned}$$

Vacuum functional

In the last functional integral we make a change of variables in the form of the BRST transformations but considering the constant Grassmann-odd parameter μ as a functional $\Lambda = \Lambda(\Phi)$. The Faddeev-Popov action, S_{FP}, is invariant under such change of variables but S_k is not invariant, with the variation given by

$$\delta S_k = A^{a\mu}(R_{k,A})^{ab}_{\mu\nu}\, D^{\nu bc} C^c \Lambda + \frac{1}{2}\,\bar{C}^a (R_{k,gh})^{ab}\, f^{bcd}\, C^c C^d \Lambda - \\ -B^a (R_{k,gh})^{ab}\, C^b \Lambda.$$

Choosing Λ in a natural way, $\Lambda = i\hbar^{-1}\,\bar{C}^a\,\delta\chi^a$, then

$$Z_{k,\chi+\delta\chi} = \int \mathcal{D}\Phi\, \exp\Big\{\frac{i}{\hbar}\big(S_{FP} + S_k + \delta S_k\big)\Big\},$$

then, for any value $k \neq 0$, one has

Gauge dependence of vacuum functional

$$Z_{k,\chi+\delta\chi} \neq Z_{k,\chi}.$$

Let us explore the gauge dependence of the generating functionals $\mathcal{Z}_k$ and Γ_k for Yang-Mills theory in the framework of the FRG approach. The derivation of this dependence is based on a variation of the gauge-fixing function, $\chi^a \to \chi^a + \delta\chi^a$, which leads to the variation of the FP action S_{FP} and consequently of the generating functionals $\mathcal{Z}_k = \mathcal{Z}_k(J)$, $\mathcal{W}_k = \mathcal{W}_k(J)$, $\Gamma_k = \Gamma_k(\Phi)$.
In terms of the average effective action[PML,Shapiro (2013)]

$$\delta\Gamma_k = \frac{\delta\Gamma_k}{\delta\Phi^A}\, F_k^A + G_k\,,$$

where $F_k^A = F_k^A(\Phi)$, $G_k = G_k(\Phi)$ are definite functionals disappearing when $\delta\chi^a$ come to zero,

$$\lim_{\delta\chi\to 0} F_k^A = 0, \quad \lim_{\delta\chi\to 0} G_k = 0\,.$$

We see that

Gauge dependence of Γ_k on its extremals

$$\frac{\delta\Gamma_k}{\delta\Phi^A} = \ 0\,, \qquad \delta\Gamma_k\Big|_{\frac{\delta\Gamma_k}{\delta\Phi}=0} \neq 0$$

This result shows that the gauge dependence represents a serious problem for the FRG approach in the standard conventional formulation.

New approach
[Lavrov, Shapiro (2013)]

Our approach is based on an idea of the composite fields [Cornwell, Jackiw, Tomboulis, (1974)] to formulate the FRG framework for gauge theories. The idea is to use such fields to implement regulator functions. Consider the regulator functions

$$
\begin{aligned}
L^1_k(x) &= \frac{1}{2}\,A^{a\mu}(x)(R_{k,A})^{ab}_{\mu\nu}(x)A^{b\nu}(x)\,, \quad (R_{k,A})^{ab}_{\mu\nu} = (R_{k,A})^{ba}_{\nu\mu},\\
L^2_k(x) &= \bar{C}^a(x)(R_{k,gh})^{ab}(x)C^b(x)\,, \qquad (R_{k,gh})^{ab} = -(R_{k,gh})^{ba}.
\end{aligned}
$$

Now we introduce external scalar sources $\Sigma_1(x)$ and $\Sigma_2(x)$ and construct the generating functional of Green's functions for Yang-Mills theories with composite fields

$$
\mathcal{Z}_k(J,\Sigma) = \int \mathcal{D}\Phi \exp\Big\{\frac{i}{\hbar}\big[S_{FP}(\Phi) + J\Phi + \Sigma_i L^i_k(\Phi)\big]\Big\} = \exp\Big\{\frac{i}{\hbar}\mathcal{W}_k(J,\Sigma)\Big\}
$$

where $\Sigma_i L^i_k(\Phi) = \int dx[\Sigma_1(x)L^1_k(x) + \Sigma_2(x)L^2_k(x)$.

Using the explicit structure of the regulator Lagrangians and the definitions $\mathcal{Z}_k, \mathcal{W}_k$ we deduce the relations

$$\frac{\delta \mathcal{Z}_k}{\delta \Sigma_i} = \frac{\hbar}{2i} \frac{\delta^2 \mathcal{Z}_k}{\delta J_B \delta J_A} (L_k^{i\,\prime\prime})_{AB} (-1)^{\varepsilon_B},$$

or,

$$\frac{\delta \mathcal{W}_k}{\delta \Sigma_i} = \frac{\hbar}{2i} \left[\frac{\delta^2 \mathcal{W}_k}{\delta J_B \delta J_A} + \frac{i}{\hbar} \frac{\delta \mathcal{W}_k}{\delta J_B} \frac{\delta \mathcal{W}_k}{\delta J_A} \right] (L_k^{i\,\prime\prime})_{AB} (-1)^{\varepsilon_B}.$$

$$(L_k^{i\,\prime\prime})_{AB} = \overrightarrow{\partial}_A L_k^i(\Phi) \overleftarrow{\partial}_B, \quad i = 1, 2, \quad \partial_A = \frac{\delta}{\delta \Phi^A}$$

are constant supermatrices.

New approach

The effective action $\Gamma_k = \Gamma_k(\Phi; F)$ is introduced with the help of double Legendre transformations

$$\Gamma_k(\Phi;F) \,=\, \mathcal{W}_k(J;\Sigma) \,-\, J_A\Phi^A \,-\, \Sigma_i\big[L^i_k(\Phi) + \frac{\hbar}{2}F^i\big]\,,$$

where

$$\Phi^A = \frac{\delta\mathcal{W}_k}{\delta J_A}\,, \qquad \frac{\hbar}{2}F^i = \frac{\delta\mathcal{W}_k}{\delta\Sigma_i} - L^i_k\Big(\frac{\delta\mathcal{W}_k}{\delta J}\Big)\,, \qquad i = 1, 2.$$

$$\frac{\delta\Gamma_k}{\delta\Phi^A} = -J_A - \Sigma_i\frac{\delta L^i_k(\Phi)}{\delta\Phi^A}\,, \qquad \frac{\delta\Gamma_k}{\delta F^i} = -\frac{\hbar}{2}\Sigma_i\,.$$

New approach

Let us introduce the full sets of fields $\mathcal{F}^{\mathcal{A}}$ and sources $\mathcal{J}_{\mathcal{A}}$ according to

$$\mathcal{F}^{\mathcal{A}} = (\Phi^A, \hbar F^i)\,, \qquad \mathcal{J}_{\mathcal{A}} = (J_A, \Sigma_i).$$

From the condition of solvability of equations

$$\frac{\delta \mathcal{F}^{\mathcal{C}}(\mathcal{J})}{\delta \mathcal{J}_{\mathcal{B}}}\,\frac{\delta_l \mathcal{J}_{\mathcal{A}}(\mathcal{F})}{\delta \mathcal{F}^{\mathcal{C}}} \;=\; \delta^{\mathcal{B}}_{\mathcal{A}}\,.$$

One can express $\mathcal{J}_{\mathcal{A}}$ as a function of the fields in the form

$$\mathcal{J}_{\mathcal{A}} \;=\; \Big(-\,\frac{\delta \Gamma_k}{\delta \Phi^A} \;+\; \frac{2}{\hbar}\frac{\delta \Gamma_k}{\delta F^i}\,\frac{\delta L^i_k(\Phi)}{\delta \Phi^A}\,, -\frac{2}{\hbar}\frac{\delta \Gamma_k}{\delta F^i}\Big)$$

and, therefore,

$$\frac{\delta_l \mathcal{J}_{\mathcal{B}}(\mathcal{F})}{\delta \mathcal{F}^{\mathcal{A}}} = -(G^{''}_k)_{\mathcal{A}\mathcal{B}}\,, \qquad \frac{\delta \mathcal{F}^{\mathcal{B}}(\mathcal{J})}{\delta \mathcal{J}_{\mathcal{A}}} = -(G^{''-1}_k)^{\mathcal{A}\mathcal{B}}\,.$$

New approach

Now we can present the new relation on the level of average effective action in a closed form

$$-iF^i = \mathcal{W}_k^{AB}(L_k^{i\,\prime\prime})_{BA}(-1)^{\varepsilon_A} = \mathsf{sTr}\,\mathcal{W}_k^{AC}(L_k^{i\,\prime\prime})_{CB},$$

where

$$\mathcal{W}_k^{AB} = -\Big((\Gamma_k^{\prime\prime})_{AB} - \frac{2}{\hbar}\Gamma_{k,\,i}(L_k^{i\,\prime\prime})_{AB} - \big(\overrightarrow{\partial}_A\Gamma_{k,\,i}\big)(\Gamma_k^{-1})^{ij}\big(\Gamma_{k,\,j}\overleftarrow{\partial}_B\big)\Big)^{-1}.$$

New approach

It is very important to discuss the structure of supermatrices $(L^{i}_{k}{}'')_{AB}$ which itself have no maximal rank but for any combination

$$\Sigma_i (L^{i}_{k}{}'')_{AB} = \begin{pmatrix} \Sigma_1 (R_{k,\,A})^{ab}_{\mu\nu} & 0 & 0 \\ 0 & 0 & \Sigma_2 (R_{k,\,gh})^{ba} \\ 0 & \Sigma_2 (R_{k,\,gh})^{ab} & 0 \end{pmatrix} .$$

there exists the inverse supermatrix

$$(L_k{}''^{-1})^{AB} = \begin{pmatrix} (R^{-1}_{k,\,A})^{\mu\nu}_{ab} & 0 & 0 \\ 0 & 0 & (R^{-1}_{k,\,gh})_{ba} \\ 0 & (R^{-1}_{k,\,gh})_{ab} & 0 \end{pmatrix} ,$$

where

$$(R_{k,\,A})^{ac}_{\mu\alpha} (R^{-1}_{k,\,A})^{\alpha\nu}_{cb} = \delta^a_b \delta^\nu_\mu , \quad (R_{k,\,gh})^{ac} (R^{-1}_{k,\,gh})_{cb} = \delta^a_b .$$

New approach

We obtain an useful relation

$$\Sigma_i (L_k^{i\,\prime\prime})_{AC} (L_k{}^{\prime\prime -1})^{CB} = \begin{pmatrix} \Sigma_1 \delta^a_b \delta^\nu_\mu & 0 & 0 \\ 0 & \Sigma_2 \delta^a_b & 0 \\ 0 & 0 & \Sigma_2 \delta^a_b \end{pmatrix}.$$

The gauge dependence of Γ_k is described by the equation

$$\delta\Gamma_k = \Gamma_{k,A} H^A + \Gamma_{k,i} G^i, \quad \Gamma_{k,A} = \frac{\delta \Gamma_k}{\delta \Phi^A}, \quad \Gamma_{k,i} = \frac{\delta \Gamma_k}{\delta F^i}$$

where $H^A = H^A(\Phi, F)$, $G^i = G^i(\Phi, F)$ obey the properties

$$\lim_{\delta\chi \to 0} H^A = 0, \quad \lim_{\delta\chi \to 0} G^i = 0.$$

Gauge independence

$$\delta\Gamma_k \Big|_{\Gamma_{k,A}=0, \Gamma_{k,i}=0} = 0$$

Loop expansion

Our starting point in loop expansion is the equation for the effective action $\bar{\Gamma}_k(\Phi; F) = \Gamma_k(\Phi; F) - S_{FP}(\Phi)$

$$
\exp\Big\{\frac{i}{\hbar}\bar{\Gamma}_k(\Phi;F)\Big\} = \exp\Big\{-\frac{i}{2}\Sigma_i F^i\Big\} \times
$$
$$
\times \int \mathcal{D}\varphi \exp\Big\{\frac{i}{\hbar}\Big[\frac{1}{2}\,\varphi^A\Big((S''_{FP}(\Phi))_{AB} + \Sigma_i\,(L^{i}_{k}{}'')_{AB}\Big)\varphi^B -
$$
$$
-(\bar{\Gamma}_k(\Phi;F)\overleftarrow{\partial}_A)\,\varphi^A + S_{int}(\Phi,\varphi)\Big]\Big\}
$$

where

$$
\begin{aligned}
S_{int}(\Phi,\varphi) \;&=\; S_{FP}(\Phi+\varphi) - S_{FP}(\Phi) - (S_{FP}(\Phi)\overleftarrow{\partial}_A)\varphi^A - \\
&\quad -\frac{1}{2}\,\varphi^A(S''_{FP}(\Phi))_{AB}\varphi^B,
\end{aligned}
$$

We assume the effective action in the form

$$\bar{\Gamma}_k(\Phi; F) = \hbar\,\Gamma_k^{(1)}(\Phi; F) + \Gamma_{k\,2}(\Phi; F)\,.$$

Here $\Gamma_k^{(1)}(\Phi; F)$ is the one-loop effective action for the set of fields Φ^A taking into account composite fields F^i. The term $\Gamma_{k\,2}(\Phi; F)$ includes all the two-particle irreducible vacuum graphs in a theory with vertices determined by $S_{int}(\Phi, \varphi)$ and propagators set equal to F^i. Note that $\Gamma_{k\,2}(\Phi; F)$ by itself is of order $\hbar^2$.

To calculate the one-loop contribution $\Gamma^{(1)}_k(\Phi,F)$ we have to omit in functional integral all terms of order more than φ^2. Then we have

$$\exp\Big\{i\Gamma^{(1)}_k(\Phi;F)\Big\}=\exp\Big\{-\frac{i}{2}\Sigma_i F^i\Big\}\times$$
$$\times\int\mathcal{D}\varphi\exp\Big\{\frac{i}{2\hbar}\,\varphi^A\Big((S''_{FP}(\Phi))_{AB}+\Sigma_i\,(L^{i}_k{}'')_{AB}\Big)\varphi^B\Big\}$$

$$\Gamma^{(1)}_k(\Phi;F)-\Gamma^{(1)}_{k,\,i}F^i=\frac{i}{2}\,\mathsf{sTr}\,\ln\Big((S''_{FP}(\Phi))_{AB}-2\Gamma^{(1)}_{k,\,i}(L^{i}_k{}'')_{AB}\Big)$$

$$\Big\{\Big((S''_{FP}(\Phi))_{AB}-2\Gamma^{(1)}_{k,\,j}(L^{j}_k{}'')_{AB}\Big)^{-1}(L^{i}_k{}'')_{BA}\Big\}(-1)^{\varepsilon_A}=-iF^i\,,$$

Loop expansion

$$\Gamma^{(1)}_{k,j}(L^{j}_{k}{}'')_{AB} = -\frac{i}{2} n_j (F^j)^{-1} (L^{j}_{k}{}'')_{AB} + \frac{1}{2}(S''_{FP}(\Phi))_{AB}$$

$$n_1 = \mathsf{Tr}\,\delta^{\mu}_{\nu}\delta^{b}_{a}, \quad n_2 = -2\,\mathsf{Tr}\,\delta^{b}_{a}.$$

$$\Gamma^{(1)}_{k,j} = -\frac{i}{2} m_j (F^j)^{-1} + \frac{1}{2m_j}\,\mathsf{sTr}\,(S''_{FP}(\Phi))_{AC}(L_k{}''^{-1})^{CB},$$

where in the first term there is no summation over index j and $n_1 = m_1, n_2 = -2m_2$.

$$\Gamma^{(1)}_{k}(\Phi;F) = \frac{1}{2m_j}\,\mathsf{sTr}\,(S''_{FP}(\Phi))_{AC}(L_k{}''^{-1})^{CB}\,F^j + \\ + \frac{i}{2}\,\mathsf{sTr}\,\ln\left(i n_j (F^j)^{-1}(L^{j}_{k}{}'')_{AB}\right)$$

A simple example

Now we illustrate the problem of gauge dependence using a simple example. To this end we consider the effective action $\bar{\Gamma}_k(\Phi; F)$ up to first order in $\hbar$,

$$\bar{\Gamma}_k(\Phi; F) = \hbar \Gamma_k^{(1)}(\Phi; F)$$

Note that in consistent gauge theories the effective action does not depend on gauge on its extremals. First, we check the gauge dependence of effective action $\bar{\Gamma}_k(\Phi; F)$. Consider the quantum equations of motion $\Gamma_{k,j}^{(1)}(\Phi; F) = 0$

$$-\frac{i}{2} n_j (F^j)^{-1} (L_k^{j\,\prime\prime})_{AB} + \frac{1}{2} (S''_{FP}(\Phi))_{AB} = 0\,,$$
$$-\frac{i}{2} m_j (F^j)^{-1} + \frac{1}{2m_j} \,\mathsf{sTr}\, (S''_{FP}(\Phi))_{AC} ({L_k}^{\prime\prime -1})^{CB} = 0$$

A simple example

$$\Gamma_k^{(1)}(\Phi; F) = \frac{i}{2}\,\mathsf{sTr}\,\ln S''_{FP}(\Phi)\,.$$

In this approximation the average effective action coincides with the one-loop answer for effective action in a given Yang-Mills theory. It is well-known fact that it does not depend on the gauge when the fields Φ^A satisfy the quantum equations of motion.

The average effective action, $\Gamma_k^{(1)}(\Phi)$, in the standard FRG approach reads

$$\Gamma_k^{(1)}(\Phi) = \frac{i}{2}\,\mathsf{sTr}\,\ln\big(S''_{FP}(\Phi) + S''_k(\Phi)\big)$$

This action depends on gauge even on its extremals. To illustrate this feature explicitly, we restrict ourselves to the case of electromagnetic field in the flat space-time.

A simple example

The classical action of the model is

$$S_0(A) = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, .$$

We choose the gauge fixing function in the form

$$\chi(A,B) = \frac{1}{\sqrt{1+\lambda}}\,\partial^\alpha A_\alpha + B$$

Integrating over field B yields the gauge fixing action

$$S_{gf}(A) = -\frac{1}{2(1+\lambda)}\int d^4x\,(\partial^\alpha A_\alpha)^2\,.$$

The action for ghosts reads

$$S_{gh}(C) = \frac{1}{\sqrt{1+\lambda}}\int d^4x\,\bar{C}(\partial^\alpha\partial_\alpha)C\,.$$

A simple example

The effective action of the model is

$$\Gamma(\Phi) = S(\Phi) + i\hbar\Gamma^{(1)}(\lambda), \quad S(\Phi) = S_0(A) + S_{gf}(A) + S_{gh}(C),$$
$$\Phi = (A_\mu, C, \bar{C}),$$

where

$$\Gamma^{(1)}(\lambda) = \frac{1}{2}\,\mathsf{Tr}\,\ln\left(\Box\delta^\alpha_\beta - \frac{\lambda}{1+\lambda}\partial^\alpha\partial_\beta\right) - \,\mathsf{Tr}\,\ln\left(\frac{1}{\sqrt{1+\lambda}}\,\Box\right).$$

Dependence of effective action $\Gamma(\Phi)$ on gauge parameter λ is described by the relation

$$\delta\Gamma(\Phi) = \frac{\delta\Gamma(\Phi)}{\delta\Phi}\delta\Phi + i\hbar\frac{\partial\Gamma^{(1)}(\lambda)}{\partial\lambda}\delta\lambda.$$

A simple example

Using the quantum equations of motion

$$\frac{\delta\Gamma(\Phi)}{\delta\Phi} = 0,$$

we see that all dependence on λ comes from $\Gamma^{(1)}(\lambda)$. In turn

$$\begin{aligned}
\Gamma^{(1)}(\lambda) &= \Gamma^{(1)}(0) + \frac{1}{2}\,\mathsf{Tr}\,\ln\left(\delta^{\alpha}_{\beta} - \frac{\lambda}{1+\lambda}\frac{\partial^{\alpha}\partial_{\beta}}{\Box}\right) - \ln\frac{1}{\sqrt{1+\lambda}}\,\mathsf{Tr}\,\mathbf{1} \\
&= \Gamma^{(1)}(0) + \frac{1}{2}\ln\frac{1}{1+\lambda}\,\mathsf{Tr}\,\frac{\partial^{\alpha}\partial_{\beta}}{\Box} - \ln\frac{1}{\sqrt{1+\lambda}}\,\mathsf{Tr}\,\mathbf{1} = \Gamma^{(1)}(0)
\end{aligned}$$

$$\delta\Gamma(\Phi)\Big|_{\frac{\delta\Gamma}{\delta\Phi}=0} = 0$$

The same result is valid for the average effective action in the new FRG approach.

A simple example

Calculation of the one-loop average effective action of the model within the standard FRG method gives

$$\Gamma_k(\Phi) = S(\Phi) + S_k(\Phi) + i\hbar\,\Gamma_k^{(1)}(\lambda),$$

where the regulator action, $S_k(\Phi)$, is

$$S_k(\Phi) = \frac{1}{2}\int d^4x\, A^\alpha (R_{k,\,A})_{\alpha\beta} A^\beta + \int d^4x \bar{C}\; R_{k,gh}\; C,$$

and the one-loop contribution, $\Gamma_k^{(1)}(\lambda)$, reads

$$\Gamma_k^{(1)}(\lambda) \;\; = \;\; \frac{1}{2}\,\mathrm{Tr}\,\ln\left(\Box\delta^\alpha_\beta - \frac{\lambda}{1+\lambda}\partial^\alpha\partial_\beta + (R_{k,\,A})^\alpha_\beta\right) - \\ - \mathrm{Tr}\,\ln\left(\frac{1}{\sqrt{1+\lambda}}\Box + R_{k,\,gh}\right)$$

A simple example

As in previous case using the quantum equations of motion we again find that the gauge dependence of average effective action $\Gamma_k(\Phi)$ on its extremals comes essentially from $\Gamma_k^{(1)}(\lambda)$ which can be presented in the form

$$\Gamma_k^{(1)}(\lambda) = \Gamma^{(1)}(\lambda) + \frac{1}{2}\,\mathsf{Tr}\,\ln\left(1 - G^{\alpha}_{\gamma}(\lambda)\,(R_{k,A})^{\gamma}_{\beta}\right) - \\ - \mathsf{Tr}\,\ln\left(1 + \sqrt{1+\lambda}\,\frac{R_{k,gh}}{\Box}\right).$$

$\Gamma^{(1)}(\lambda) = \Gamma^{(1)}(0)$ does not depend on λ, and $G^{\alpha}_{\gamma}(\lambda)$ is the Green's function

$$\left(\Box\delta^{\alpha}_{\gamma} - \frac{\lambda}{1+\lambda}\partial^{\alpha}\partial_{\gamma}\right)G^{\gamma}_{\beta}(\lambda) = -\delta^{\alpha}_{\beta}\,, \quad G^{\gamma}_{\beta}(\lambda) = -\frac{\delta^{\gamma}_{\beta}}{\Box} + \frac{\lambda}{1+2\lambda}\frac{\partial^{\gamma}\partial_{\beta}}{\Box^2}$$

A simple example

The last two terms explicitly depend on the gauge fixing parameter λ. Using the following property of cutoff functions $R_k(p) \to 0$ when $k \to 0$, we can approximate the trace of logarithm by linear term

$$\Gamma_k^{(1)}(\lambda) \approx \Gamma^{(1)}(0) + \frac{1}{2}\,\mathsf{Tr}\left(\frac{(R_{k,\,A})^{\alpha}_{\beta}}{\Box} - \frac{\lambda}{1+2\lambda}\frac{\partial^{\alpha}\partial_{\gamma}(R_{k,\,A})^{\gamma}_{\beta}}{\Box^2}\right) - \sqrt{1+\lambda}\,\mathsf{Tr}\left(\frac{R_{k,\,gh}}{\Box}\right).$$

It is clear that

$$\frac{\partial\Gamma_k^{(1)}(\lambda)}{\partial\lambda} \neq 0,$$

and one meets the gauge dependence of average effective action within the standard FRG approach even on its extremals.

Conclusion

- The study of gauge dependence of Green's functions within the FRG approach was given.
- It was shown the gauge dependence of average effective action even on its extremals. In particular, vacuum expectation values of gauge invariant operators such as $F^a_{\mu\nu}F^{a\mu\nu}$ do depend on gauge.
- A new FRG approach being free of the gauge dependence problem has been proposed.
- Loop expansion of effective action with composite fields in new FRG approach has been formulated. Explicit form of effective action at the leading order in $\hbar$ has been found.
- Gauge dependence of effective actions constructed within FP method, FRG and new FRG approaches have been illustrated using a simple example of Abelian vector field.

Thank you
for attention!