Interacting Bosons in a Static Background Metric

Mainz-QVGM

O.T.Turgut
Bogazici Univ.

June 27, 2015

♠

How can we understand renormalization in a nonperturbative manner?

♠

♯ Study very simple models:

$$H = -\frac{\hbar^2}{2m}\nabla^2 - g\delta(x) \qquad (1)$$

in *two dimensions*.

♯ This problem has a *scaling symmetry*, hence if we have one bound state, we have infinitely many!

♢ H is not bounded from below!

• Making hamiltonian bounded from below (*requirement of stability*) via renormalization we break this symmetry (**Berezin and Fadeev)**.

This is an *effective theory*, there is no real delta function, if we have large momenta the potential will look different, altough we do not know its details and we are not interested in (the energy scales we work with are below this cut-off).

♯ renormalization will introduce a *scale* into the problem. Since the original one has no scale, there is no natural way to deduce energy values!

♠ the hamiltonian is symmetric but not self-adjoint, finding a *self-adjoint extension* of our Hamiltonian will make it bounded from below.

• Recall that, in momentum cut-off scheme, we choose

$$g^{-1}(\Lambda) = \int_{|k|<\Lambda} [d^2k] \frac{1}{\frac{\hbar^2}{2m}k^2 + \mu^2} \qquad (2)$$

where $-\mu^2$ corresponds to the bound state enegy!

♯ Now we may take the limit $\Lambda \to \infty$, the results become finite!

Define $R_0(x, y|E) =< x|(H_0 - E)^{-1}|y >$, where $H_0 = -\frac{\hbar^2}{2m}\nabla^2$.

$$R_0(x, y) = \frac{m}{\pi\hbar^2} K_0\left(\frac{\sqrt{-2mE}}{\hbar}|x - y|\right) \qquad (3)$$

• We can actually compute the Greens function:

$$\begin{aligned} R(x, y|E) &= R_0(x, y|E) \qquad (4) \\ &+ R_0(x, 0|E) \frac{\pi\hbar^2}{m} \frac{1}{\ln \frac{\sqrt{-E}}{\mu}} R_0(0, y|E). \end{aligned}$$

If we know the Greens function we can find the total scattering cross-section:

$$\sigma(E) = \underbrace{\frac{64\hbar}{\sqrt{2mE}}}_{from\quad dimensional\ analysis} \times \overbrace{\frac{1}{\ln^2\left(E/\mu^2\right) + \pi^2}}^{g_{eff}^2(E)} \tag{5}$$

($[\sigma(E)] \approx L$ in two dimensions)

• Experimentally, we would say g_{eff} *decreases with energy*!

We can compute the bound state wave function:

$$\psi_\nu(x) = \mu\sqrt{\frac{2m}{\pi\hbar^2}} K_0\left(\frac{\sqrt{2m}}{\hbar}\mu|x|\right), \qquad E = -\mu^2. \tag{6}$$

$$\begin{aligned} K_0(x) &= \int_0^\infty dt e^{-x\cosh t} \\ K_0(x) &\approx -\ln\left(\frac{x}{2}\right) \quad x \to 0 \\ K_0(x) &\approx \sqrt{\frac{\pi}{2x}} e^{-x} \quad x \to \infty \end{aligned}$$

Note that

$$\int d^2x |\psi_\nu(x)|^2 = 1 \tag{7}$$

Yet,

$$< \psi_\nu | \mathbf{P}^2/2m | \psi_\nu > \propto \int \frac{k^2 [d^2k]}{(\frac{\hbar^2 k^2}{2m} + \nu^2)^2} = \infty \tag{8}$$

Alternatively, we could choose,

$$\frac{1}{g(\Lambda)} = \frac{1}{g_R(M)} + \int_{|k|<\Lambda} [d^2k] \frac{1}{\frac{\hbar^2}{2m} k^2 + M^2} \tag{9}$$

where M is an arbitrary mass parameter!

We could solve for μ in terms of g_R and M.

M is arbitrary, so if the theory has physical predictions the choice of M should not matter: Renormalization group invariance,

$$\frac{\partial}{\partial M}\Big[\frac{1}{g_R(M)} + \int [d^2k] \Big(\frac{1}{\frac{\hbar^2}{2m} k^2 + M^2} - \frac{1}{\frac{\hbar^2}{2m} k^2 + \nu^2}\Big)\Big] = 0 \tag{10}$$

• For the multi center case, we may use the same approach. We may choose the coupling constants as

$$\frac{1}{g_i(\Lambda)} = \int_{|k|<\Lambda} [d^2k] \frac{1}{\frac{\hbar^2}{2m}k^2 + \mu_i^2}. \tag{11}$$

We find a matrix equation,

$$\Phi_{ij}(-\nu^2)\psi(a_j) = 0 \tag{12}$$

where

$$\Phi_{ij}(-\nu^2) = \frac{m}{\pi\hbar^2} \begin{cases} \ln(\frac{\nu}{\mu_i}) & i = j \\ -K_0(\frac{\sqrt{2m}}{\hbar}\nu|a_i - a_j|) & i \neq j \end{cases} \tag{13}$$

$\sharp$ (B. Altunkaynak and F. Erman) Abstract setting:

$$H = H_0 - \sum_i |f_i >< f_i|, \tag{14}$$

Find the Greens function which is an operator family $(H-z)^{-1}$ for $z \in \mathbf{C}$.

$\sharp$ a self-adjoint extension problem, there is a formula due to M. G. Krein.

- Think of $|f_i >= g_i^{1/2}(\epsilon)|f_i' >$,

$$(H-z)^{-1} = (H_0 - z)^{-1}$$

$$+(H_0 - z)^{-1} \sum_{i,j} |f_i' > \Phi_{ij}^{-1}(z) < f_j'|(H_0 - z)^{-1}, \tag{15}$$

we have here

$$\Phi_{ij}(z) = \begin{cases} g_i^{-1}(\epsilon) - < f_i'|(H_0 - z)^{-1}|f_i' > & i = j \\ - < f_i'|(H_0 - z)^{-1}|f_j' > & i \neq j \end{cases}. \tag{16}$$

• This allows us to formulate the problem independent of the momentum regularization!

♠ We need an independent approach for *curves* in $\mathbf{R}^3$!!!!

• Choose $|f_i' >$ to be Gaussian bump functions centered around some points a_i, approaching delta functions.

♯ A natural choice $f_i'^{\epsilon}(x) = K_\epsilon(a_i, x)$.

♠ Incidentally, this approach works for particles living on *surfaces*!

Laplace-Beltrami operator:

$$\nabla_g^2 = -\frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j), \tag{17}$$

Introduce the Heat kernel,

$$K_t(x, y) = < x|e^{-\frac{t}{\hbar}(\frac{\hbar^2}{2m}\triangle_g)}|y > \tag{18}$$

Solves the Heat equation (Euclidean Schrodinger Eqn).

$$\begin{aligned} K_t(x,y) &= K_t(y,x) \quad , \\ \frac{\partial K_t(x,x')}{\partial t} - \nabla_g^2 K_t(x,x') &= 0 \\ \lim_{t\to 0^+} K_t(x,y) &= \delta_g(x,y), \end{aligned}$$

$$\int_{\mathcal{M}} d_g x \; K_{t_1}(x,z) K_{t_2}(z,y) \; = \; K_{t_1+t_2}(x,y)$$

On a compact manifold, we have

$$K_t(x,y) \; = \; \sum_\lambda e^{-\lambda t} f_\lambda(x) f_\lambda(y) \qquad (19)$$

although we will use this expression in some cases for non-compact manifolds as well. Here,

$$\begin{aligned} \Delta_g f_\lambda &= \lambda f_\lambda \\ 0 &\leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ..... \to \infty \end{aligned}$$

For stochastically complete manifolds(for example compact manifolds):

$$\int_{\mathcal{M}} d_g x \; K_t(x,y) = 1 \qquad (20)$$

This means that the total heat content $\int d_g y\ h(y)$ is preserved.

• **Theorem**: Given a geodesically complete manifold assume that for some point x we have

$$\int_R^\infty \frac{rdr}{\ln V(x,r)} = \infty, \tag{21}$$

then the stochastic completeness holds.

• **(Yau)** A geodesically complete manifold with bounded below Ricci curvature is stochastically complete.

On a product manifold $\mathcal{M}_1 \times \mathcal{M}_2$,

$$K_t(x,y) = K_t^{(1)}(x_1,y_1) K_t^{(2)}(x_2,y_2) \tag{22}$$

Decay of the heat kernel, $\mathcal{M}$ noncompact:

$$K_t(x,y) \to 0 \text{ as } t \to \infty \tag{23}$$

$\mathcal{M}$ compact:

$$K_t(x,y) \to \frac{1}{V(\mathcal{M})} \text{ as } t \to \infty. \tag{24}$$

• Free Greens function for $\mathrm{Re}(z) < 0$,

$$(H_0 - z)^{-1} = \frac{1}{\hbar} \int_0^\infty e^{-\frac{t}{\hbar}(\frac{\hbar^2}{2m}\triangle_g - z)} dt, \qquad (25)$$

should be continued analytically to its largest set in the entire complex plane.

In general, it is possible to write for a positive operator H_0 a heat kernel,

$$\langle a_i | (H_0 - z)^{-1} | a_j \rangle = \frac{1}{\hbar} \int_0^\infty e^{\frac{zt}{\hbar}} K_t(a_i, a_j) \, dt \ , \qquad (26)$$

$$\Phi^{\epsilon}_{ij}(z) = \begin{cases} g_i^{-1}(\epsilon) - \int d_g x d_g y K_{\epsilon/2}(a_i, x) R_0(x, y|z) K_{\epsilon/2}(y, a_i) \\ - \int d_g x d_g y K_{\epsilon/2}(a_i, x) R_0(x, y|z) K_{\epsilon/2}(y, a_j) \end{cases}.$$

Let us look at the diagonal term,

$$g_i^{-1}(\epsilon) \; - \; \int_0^\infty \frac{dt}{\hbar} \int d_g x d_g y \; K_{\epsilon/2}(a_i, x) K_t(x, y) K_{\epsilon/2}(y, a_i) e^{tz/\hbar}.$$

Reproducing property gives us

$$g_i^{-1}(\epsilon) - \int_0^\infty \frac{dt}{\hbar} K_{t+\epsilon}(a_i, a_i) e^{tz/\hbar} \tag{27}$$

Shift of integration variable shows that

$$g_i^{-1}(\epsilon) - \int_\epsilon^\infty \frac{dt}{\hbar} K_t(a_i, a_i) e^{(t-\epsilon)z/\hbar} \tag{28}$$

• *Remark:* In general, on a manifold, as $t \to 0^+$, we have the asymptotic expansion,

$$K_t(x,x) \sim \left(4\pi\frac{\hbar t}{2m}\right)^{-1} \sum_{k=0}^{\infty} u_k(x,x) \left(\frac{\hbar t}{2m}\right)^k \ , \tag{29}$$

• The coefficients $u_k(x,x)$ are universal polynomials in the curvature tensor and its covariant derivatives. e.g.

$$\begin{aligned} u_0(x,x) &= 1 \\ u_1(x,x) &= \frac{1}{6}R \\ u_2(x,x) &= \frac{1}{360}(2R_{ijkl}R^{ijkl} + 2R_{jk}R^{jk} + 5R^2 \\ &\quad -12\Delta_g R) \end{aligned}$$

Choose,

$$g_i^{-1}(\epsilon, \mu_i) = \frac{1}{\hbar}\int_\epsilon^\infty e^{\frac{-\mu_i^2 t}{\hbar}} K_t(a_i, a_i)\, dt \ . \tag{30}$$

• As a result we find,

$$\Phi_{ij}(z) = \begin{cases} \frac{1}{\hbar}\int_0^\infty K_t(a_i,a_i)\left[e^{\frac{-\mu_i^2 t}{\hbar}} - e^{\frac{zt}{\hbar}}\right] dt & i=j \\ -\frac{1}{\hbar}\int_0^\infty e^{\frac{zt}{\hbar}} K_t(a_i,a_j)\, dt & i\neq j \end{cases} \tag{31}$$

♯ In case of $\mathbf{R}^2$, this gives us

$$\Phi_{ij}(z) = \begin{cases} \frac{m}{\pi\hbar^2}\ln[\frac{\sqrt{-z}}{\mu_i}] & i=j \\ -\frac{m}{\pi\hbar^2}K_0(\frac{\sqrt{-2mz}}{\hbar}|x-y|) & i\neq j \end{cases} \tag{32}$$

$$\begin{aligned} R(x,y|z) &= R_0(x,y|z) \\ &+ \sum_{i,j=1}^{N} R_0(x,a_i|z)\,\Phi_{ij}^{-1}(z) R_0(a_j,y|z)\,. \end{aligned}$$

• Let us go back to the multi-center case. Bound states are simple poles of $R(x, y|z)$:

$$\Phi_{ij}(E)A_j = 0 \tag{33}$$

determines bound state energies!

• All the information contained in the resolvent: How do we find the bound states?

♯ For bound states:

$$\psi_n^*(x)\psi_n(y) = \frac{1}{2\pi i}\oint_{\Gamma_n \ni \lambda_n} dz R(x, y|z) \tag{34}$$

$$
\begin{aligned}
\psi_n(x) &= \left[\int_0^\infty dt t e^{-\nu_n^2 t} \sum_{i,j} K_t(a_i, a_j) A_i^*(\nu_n) A_j(\nu_n)\right]^{-\frac{1}{2}} \\
&\times \int_0^\infty e^{-\frac{t\nu_n^2}{\hbar}} \sum_{i=1}^{N} A_i(\nu_n) K_t(a_i, x) \frac{dt}{\hbar} \, , \qquad (35)
\end{aligned}
$$

♯ how do we know that the spectrum is bounded from below? Resolvent!

General observation:

$$\Phi_{ii}(-\nu^2) \approx \ln \frac{\nu}{\mu_i}, \text{ as } \nu \to \infty,$$
$$\frac{\partial |\Phi_{ii}(-\nu^2)|}{\partial \nu} = \frac{2\nu}{\hbar^2} \int_0^\infty t K_t(a_i, a_i) e^{-(\frac{\nu^2 t}{\hbar})} dt > 0$$

and

$$|\Phi_{ij}(-\nu^2)| \approx e^{-\nu d(a_i, a_j)} \text{ as } \nu \to \infty$$
$$\frac{\partial |\Phi_{ij}(-\nu^2)|}{\partial \nu} = -\frac{2\nu}{\hbar^2} \int_0^\infty t dt K_t(a_i, a_j) e^{-(\frac{\nu^2 t}{\hbar})} < 0.$$

As we increase ν the inequality will be satisfied for some value! Thus $E_{gr} > -(\nu^*)^2$, energy is bounded from below!

• H^2 Case: Upper half-plane model, defined by the metric

$$\cosh \frac{d(x,y)}{R} = 1 + \frac{|x-y|^2}{2\, x_2\, y_2}\,, \qquad (36)$$

$$\begin{aligned} K_t(x,y) &= \frac{\sqrt{2}}{(4\pi \left[\frac{\hbar}{2mR^2}\right] t)^{3/2}} \frac{e^{-\frac{\hbar}{2mR^2}\frac{t}{4}}}{R^2} \\ &\quad \int_{\frac{d(x,y)}{R}}^{\infty} \frac{r e^{-\frac{r^2}{4}\frac{2mR^2}{\hbar}\frac{1}{t}}}{\sqrt{\cosh r - \cosh \frac{d(x,y)}{R}}}\, dr \end{aligned}$$

$$\Phi_{ii}(z) = \frac{m\sqrt{2}}{2\pi\hbar^2}\left[\psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{z}{\mu_R^2}}\right) - \psi\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\mu_i^2}{\mu_R^2}}\right)\right] \qquad (37)$$

$$\begin{aligned} \Phi_{ij}(z) &= -\frac{m}{2\pi\hbar^2}\int_{\frac{d_{ij}}{R}}^{\infty} \frac{e^{-\frac{1}{2}r\sqrt{1-\frac{4z}{\mu_R^2}}}}{\sqrt{\cosh r - \cosh \frac{d_{ij}}{R}}}\, dr, \\ &= \frac{\sqrt{2}}{4\pi R^2 \mu_R^2} Q_{1/2\sqrt{1-4z/\mu_R^2}-1/2}(\cosh \frac{d(a_i,a_j)}{R}) \end{aligned}$$

• a comparison theorem. Instead of $-\nu^2$ use E!

$$\omega_m(E) = (A^m(E), \Phi(E) A^m(E)). \tag{38}$$

By the Feynman-Helmann theorem,

$$\begin{aligned}
\frac{\partial \omega_m(E)}{\partial E} &= (A^m(E), \frac{\partial \Phi(E)}{\partial \nu} A^m(E)) \\
\frac{\partial \Phi_{ij}(E)}{\partial E} &= -\int_0^\infty dt t K_t(a_i, a_j) e^{Et}
\end{aligned}$$

the integral is finite in two and three dimensions due to the short time behaviour of the heat kernel.

$$\begin{aligned}
\frac{\partial \omega_m(E)}{\partial E} &= -\int_0^\infty dt\ te^{Et} \sum_{i,j} K_t(a_i, a_j) A_i^{m*}(E) A_j^m(E) \\
&= -\int_0^\infty dt\ te^{Et} \\
&\quad \times \int d_g x |\sum_i K_{t/2}(a_i, x) A_i^m(E)|^2 < 0.
\end{aligned}$$

$\sharp$ uniqueness of the ground state: The rigorous proof of non-degeneracy and positivity of the ground state in standard quantum mechanics is nontrivial (see for example *Reed and Simon*).

Perron - Frobenius theorem: If $M \in M_N$ and if we suppose that all $M_{ij} < 0$, and $M_{ii} > 0$, then let ω_0 be the smallest eigenvalue of M;

(1) There is an $x \in \mathbf{C}^N$ with all $x_i > 0$ and $M_{ij} x_j = \omega_0 x_i$;

(2) ω_0 is an algebraically (and hence geometrically) simple eigenvalue of M;

$\sharp$ Φ_{ij} obeys this condition!

Ground state corresponds to $\omega_0(E) = 0$, $E_{gr} = -\nu_*^2$, the smallest eigenvalue of Φ_{ij}. Because of the flow of the eigenvalues,

$$\frac{\partial \omega_k(E)}{\partial E} < 0. \tag{39}$$

This means that the positive eigenvector A_i corresponds to the ground state energy so we prove that the eigenvector A_i is strictly positive.

$$\psi_{\nu^*}(x) = \left[\sum_{i,j=1}^{N} A_i(\nu^*) \int_0^\infty \frac{t \mathrm{d}t}{\hbar^2} K_t(a_i, a_j) e^{-\frac{t\nu^{*2}}{\hbar}} A_j(\nu^*)\right]^{-\frac{1}{2}} \times \int_0^\infty \frac{\mathrm{d}t}{\hbar} \underbrace{e^{-\frac{t\nu^{*2}}{\hbar}}}_{>0} \sum_{i=1}^{N} \underbrace{A_i(\nu^*)}_{>0} \underbrace{K_t(a_i, x)}_{>0} > 0 \, ,$$

Hence, we prove that despite the singular character of the interaction, the ground state is still non-degenerate and unique.

• **Remark:**A general bound for the ground state energy for the multi-delta functions could be proved using the upper and lower bounds given above (Erman and Turgut 2010).

♠ what is the symmetry of this problem?

• Consider metric rescalings

$$g(.,.) \mapsto \gamma^{-2} g(.,.) \tag{40}$$

Then

$$\nabla_g^2 \mapsto \gamma^2 \nabla_g \qquad \delta_g(x,a) \mapsto \gamma^2 \delta_g(x,a) \tag{41}$$

So a similar argument leads to infinite bound state energies! To save the hamiltonian the symmetry is broken.

♯ Alternative renormalization:

$$\frac{1}{\lambda^R(M)} = \frac{1}{\lambda_i(\epsilon)} - \int_\epsilon^\infty \mathrm{d}t \, \frac{e^{-M^2 t}}{4\pi t} + \Sigma_i \,, \tag{42}$$

where M is the arbitrary *renormalization scale*.

Σ_i's refer to the relative strengths of delta interactions in this renormalization scheme: $\Phi^R_{11}(-\mu_1^2) = 0$, hence $\Sigma_1 = 0$. Choose $\Sigma_i(\mu_i)$ such that $\Phi^R_{ii}(-\mu_i^2) = 0$ can be satisfied.

$$
\begin{aligned}
\Phi^R_{ii}(E) &= \frac{1}{\lambda_R(M)} - \Sigma_i \\
&- \int_0^\infty dt (K_t(a_i, a_i; g) e^{tE} - \frac{e^{-M^2 t}}{4\pi t}) \\
\Phi^R_{ij}(E) &= - \int_0^\infty \mathrm{d}t \; K_t(a_i, a_j; g) e^{tE} \;\; i \neq j.
\end{aligned}
$$

The renormalization condition is given by

$$M \frac{d\Phi^R_{ij}(M, \lambda_R(M), E; g)}{dM} = 0 \tag{43}$$

or equivalently,

$$(M \frac{\partial}{\partial M} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R}) \Phi^R_{ij}(M, \lambda_R(M), E; g) = 0 \tag{44}$$

where

$$\beta(\lambda_R) = M \frac{\partial \lambda_R}{\partial M} \tag{45}$$

we can find β function exactly

$$\beta(\lambda_R) = -\frac{\lambda_R^2}{2\pi} < 0 \, . \tag{46}$$

The heat kernel scales as,

$$K_t(x, y; g) = \gamma^{-D} K_{\gamma^{-2} t}(x, y; \gamma^{-2} g) \tag{47}$$

Under metric and energy rescalings:

$$\Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) = \Phi^R_{ij}(\gamma^{-1} M, \lambda_R(M), E; g)\ . \tag{48}$$

Let us see this:

$$\Phi^R_{ii}(\gamma^2 E; \gamma^{-2} g) = \frac{1}{\lambda_R(M)} - \Sigma_i$$
$$- \int_0^\infty dt (K_t(a_i, a_i; \gamma^{-2} g) e^{t\gamma^2 E} - \frac{e^{-M^2 t}}{4\pi t})$$
$$= \frac{1}{\lambda_R(M)} - \Sigma_i$$
$$- \int_0^\infty dt (\gamma^2 K_{\gamma^2 t}(a_i, a_i; \gamma^{-2}\gamma^2 g) e^{t\gamma^2 E} - \frac{e^{-M^2 t}}{4\pi t})$$
$$= \frac{1}{\lambda_R(M)} - \Sigma_i$$
$$- \int_0^\infty dt (K_t(a_i, a_i; g) e^{tE} - \frac{e^{-M^2 \gamma^{-2} t}}{4\pi t})$$

Hence we have

$$\gamma \frac{d}{d\gamma} [\Phi^R_{ij}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g)$$
$$- \Phi^R_{ij}(\gamma^{-1} M, \lambda_R(M), E; g)] = 0.$$

This leads to the renormalization group equation for $\Phi^R_{ij}(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g)$

$$\gamma\frac{d}{d\gamma}\Phi^R_{ij}(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g) + M\frac{\partial}{\partial M}\Phi^R_{ij}(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g) = 0$$

or

$$[\gamma\frac{d}{d\gamma}-\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}]\Phi^R_{ij}(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g) = 0\ . \tag{49}$$

If we postulate the following functional form for the principal matrix

$$\Phi^R_{ij}(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g) = f(\gamma)\Phi^R_{ij}(M,\lambda_R(\gamma M),E;g) \tag{50}$$

we obtain an ordinary differential equation for the function f

$$\gamma\frac{df(\gamma)}{d\gamma} = 0\ . \tag{51}$$

This gives the solution $f(\gamma) = 1$ using the initial condition at $\gamma = 1$. Therefore, we get

$$\Phi^R_{ij}(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g) = \Phi^R_{ij}(M,\lambda_R(\gamma M),E;g) \tag{52}$$

which means that there is no anamolous scaling. After integrating

$$\beta(\lambda_R) = \bar{M}\frac{\partial \lambda_R(\bar{M})}{\partial \bar{M}} = -\frac{\lambda_R^2(\bar{M})}{2\pi} \tag{53}$$

between $\bar{M} = M$ to $\bar{M} = \gamma M$ we can find the flow equation for the coupling constant

$$\lambda_R(\gamma M) = \frac{\lambda_R(M)}{1 + \frac{1}{2\pi}\lambda_R(M)\ln\gamma} \tag{54}$$

One can explicitly check the above scaling relation if the coupling constant evolves accordingly. Recall that heat kernel scales as,

$$K_t(x, y; g) = \alpha^{-D} K_{\gamma^{-2}t}(x, y; \gamma^{-2}g) \tag{55}$$

$$\begin{aligned}
\Phi_{ii}^R(M, \lambda_R(\gamma M), E; g) &= \frac{1}{\lambda_R(M)} + \frac{1}{2\pi}\ln\gamma \\
&\quad - \int_0^\infty \mathrm{d}t \; (K_t(a_i, a_i; g)e^{tE} - \frac{e^{-M^2 t}}{4\pi t}) - \Sigma_i \\
&= \frac{1}{\lambda_R(M)} + \frac{1}{2\pi}\ln\gamma \\
&\quad - \int_0^\infty \mathrm{d}t \Big(K_t(a_i, a_i; g)e^{tE} - \frac{e^{-M^2 t}}{4\pi t} + \frac{e^{-M^2\gamma^{-2}t}}{4\pi t} \\
&\quad - \frac{e^{-M^2\gamma^{-2}t}}{4\pi t}\Big) - \Sigma_i \\
&= \frac{1}{\lambda_R(M)} - \Sigma_i
\end{aligned}$$

$$- \int_0^\infty \mathrm{d}t \left(K_t(a_i, a_i; g) e^{tE} - \frac{e^{-M^2\gamma^{-2}t}}{4\pi t}\right)$$

and then using the scaling property of heat kernel we get

$$\begin{aligned}
& \frac{1}{\lambda_R(M)} - \Sigma_i \\
& - \int_0^\infty \mathrm{d}t \Big(\gamma^{-2} K_{\gamma^{-2}t}(a_i, a_i; \gamma^{-2} g) e^{tE} - \frac{e^{-M^2\gamma^{-2}t}}{4\pi t}\Big) \\
= & \frac{1}{\lambda_R(M)} - \Sigma_i \\
& - \int_0^\infty \mathrm{d}s \left(K_s(a_i, a_i; \gamma^{-2} g) e^{s\gamma^2 E} - \frac{e^{-M^2 s}}{4\pi s}\right) \\
= & \Phi^R_{ii}(M, \lambda_R(M), \gamma^2 E; \gamma^{-2} g) \; .
\end{aligned}$$

Off diagonal term can be directly checked using just the scaling property of heat kernel.

• *An interesting extension is possible* Curves!

$$-\frac{\hbar}{2m}\nabla_g^2\psi(x) - \frac{\lambda}{L}\int_\Gamma d_g s \delta_g(x,\gamma(s)) \int_\Gamma d_g s' \psi(\gamma(s'))$$

$$= E\psi(x) \tag{56}$$

• We introduce a family of functions supported on curves,

$$\Gamma_i^\epsilon(x) = \int_{\Gamma_i} d_g s K_{\epsilon/2}(x,\gamma_i(s))\,. \tag{57}$$

Note that as $\epsilon \to 0^+$ we get a delta function supported on the curve.

We can rewrite a regularized Schrödinger equation for this family,

$$(H_0 - E)|\psi\rangle = \sum_i \frac{\lambda_i}{L_i}|\Gamma_i^\epsilon\rangle\langle\Gamma_i^\epsilon|\psi\rangle \tag{58}$$

• As done before. we find the resolvent,

$$(H-E)^{-1} = (H_0-E)^{-1}$$

$$+\frac{1}{\sqrt{L_i L_j}}(H_0-E)^{-1}|\Gamma_i^{\epsilon}\rangle \Phi_{ij}^{-1}\langle\Gamma_j^{\epsilon}|(H_0-E)^{-1} \tag{59}$$

where Φ_{ij} refers to the principal operator,

$$\Phi_{ij} = \begin{cases} \frac{1}{\lambda_i} - \frac{1}{L_i}\langle\Gamma_i^{\epsilon}|(H_0-E)^{-1}|\Gamma_i^{\epsilon}\rangle \\ -\frac{1}{\sqrt{L_i L_j}}\langle\Gamma_i^{\epsilon}|(H_0-E)^{-1}|\Gamma_j^{\epsilon}\rangle \end{cases} . \tag{60}$$

$$\Phi_{\epsilon}(E) = \frac{1}{\lambda(\epsilon)} - \frac{1}{L}\int_{\mathcal{M}\times\mathcal{M}} d_g^3x\, d_g^3y \int_{\Gamma\times\Gamma} d_g s d_g s' \times \int_0^{\infty} \frac{dt}{\hbar} e^{Et/\hbar} K_{\epsilon/2}(\gamma(s),x) K_t(x,y) \times K_{\epsilon/2}(y,\gamma(s')) \tag{61}$$

$$\Phi_R(E) \;=\; \frac{1}{L}\int_{\Gamma\times\Gamma} d_g s d_g s' \int_0^{\infty} \frac{dt}{\hbar}[e^{-\mu^2 t/\hbar} - e^{Et/\hbar}] \times K_t(\gamma(s),\gamma(s')). \tag{62}$$

satisfies the following eigenvalue equation for the kth eigenvalue $\omega^{(k)}(E)$,

$$\Phi_{ij}(E)A_j^{(k)} = \omega^{(k)}(E)A_i^{(k)}, \qquad (63)$$

$A^{(k)}$ being the kth eigenvector, and there is a summation over the repeated index j. The derivative of the principal operator with respect to the energy E, after which we rewrite in a convenient form, reads

$$\begin{aligned} \frac{\partial \omega^{(k)}(E)}{\partial E} &= -\int_{\mathcal{M}} d_g^3 x \int_0^\infty \frac{dt}{\hbar}\frac{t}{\hbar} e^{Et/\hbar} \\ &\times \left| \sum_i \frac{1}{\sqrt{L_i}} \int_{\Gamma_i} ds K_t\left(\gamma_i(s), x\right) A_i^{(k)} \right|^2 < 0. \end{aligned} \qquad (64)$$

It is obvious that the expression above is strictly negative. Therefore, all eigenvalues are decreasing functions of energy. This tells that the ground state energy must correspond to the zero of the lowest eigenvalue of the principal operator.

If we look at the ground state wave function, we see that each term in it is positive,

$$\psi_{gr}(x) = \frac{1}{\mathcal{N}} \sum_i \frac{1}{\sqrt{L_i}} \int_0^\infty \frac{dt}{\hbar} \overbrace{e^{E_{gr} t/\hbar}}^{>0} \times \int_{\Gamma_i} ds \, \overbrace{K_t\left(x, \gamma_i(s)\right)}^{>0} \overbrace{A_i^{(0)}}^{>0} . \quad (65)$$

The ground state is, hence, proven to be positive, and as a result unique.

•An extension is possible, the semi-relativistic version (*joint work with C. Dogan*):

$$H_\epsilon = H_0 + H^I_\epsilon$$

where

$$H_0 = \int d_g x : \phi(x)(-\nabla_g^2 + m^2)\phi(x) :$$

$$H^I_\epsilon = -\sum_{i=1}^{N} g_i^2 \phi^{(-)}(a_i)\phi^{(+)}(a_i)$$

$$\begin{aligned} \phi^{(-)}(x) &= \sum_\sigma \frac{a^\dagger_\sigma}{\sqrt{\omega_\sigma}} f_\sigma(x) \\ \omega_\sigma^2 &= \sigma^2 + m^2 \end{aligned}$$

$\phi^{(+)}(x)$ is the Hermitian conjugate.

we will use the orthofermion algebra technique developed by Rajeev. Introduce fictitous operators χ_i and $\chi_i^\dagger$ named angels. These operators commute with the bosonic creation and annihilation operators and satisfy the relations given below.

$$\begin{aligned} \chi_i \chi_j^\dagger &= \delta_{ij} \Pi_0 \\ \chi_i \chi_j &= 0 \\ \sum_i \chi_i^\dagger \chi_i &= \Pi_1 \end{aligned} \tag{66}$$

Π_0 and Π_i are projection operators onto the spaces with no angels and with one angel respectively. The physical space will be the direct product of the space of angels and the bosonic one.

The new operator in matrix form is as follows

$$G - E\Pi_0 = \begin{pmatrix} (H_0 - E)\Pi_0 & \sum_{i=1}^N \int d_g x K_\epsilon(a_i, x) \phi^{(-)}(x) \chi_i \\ \sum_{j=1}^N \int d_g y K_\epsilon(a_j, y) \phi^{(+)}(y) \chi_j^\dagger & \sum_{k=1}^N \frac{1}{g_k^2(\epsilon)} \chi_k^\dagger \chi_k \end{pmatrix} \equiv$$

with the resolvent or the Green's function defined as

$$(G - E\Pi_0)^{-1} \equiv \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} \tag{68}$$

The projection of this Green's function matrix on to the no angel subspace can be written in two alternative ways:

$$\begin{aligned} \alpha &= (a - b^\dagger d^{-1} b)^{-1} = (H - E)^{-1} \\ &= a^{-1} + a^{-1} b^\dagger \Phi^{-1} b a^{-1} \end{aligned} \tag{69}$$

The characteristic matrix Φ is given by

$$\Phi \equiv d - b a^{-1} b^\dagger \tag{70}$$

• The first relation for α, by the properties of the angel operators, shows that the projection of the resolvent of the new operator onto the no angel subspace reproduces the Green's function of the original Hamiltonian.

$$\begin{aligned} \Phi_\epsilon &= \sum_{i=1}^{N} \frac{1}{g_i^2} \chi_i^\dagger \chi_i - \sum_{i,j} \sum_{\sigma} \int d_g x K_{\epsilon/2}(a_i, x) f_\sigma(x) \\ &\times \int d_g y K_{\epsilon/2}(a_j, y) f_\sigma^*(y) \frac{1}{\omega_\sigma} \cdot \frac{\chi_i^\dagger \chi_j}{(H_0 - E + \omega_\sigma)} \\ &- \sum_{i,j} \int d_g x K_{\epsilon/2}(a_j, x) \sum_{\lambda,\sigma} \frac{f_\lambda^*(x) a_\lambda^\dagger}{\sqrt{\omega_\lambda}} \\ &\times \frac{\chi_i^\dagger \chi_j}{(H_0 - E + \omega_\sigma + \omega_\lambda)} \int d_g y K_{\epsilon/2}(a_i, y) \frac{f_\sigma(y) a_\sigma}{\sqrt{\omega_\sigma}} \end{aligned}$$

♯ Subordination:

$$e^{-sA} = \frac{s}{2\sqrt{\pi}} \int_0^\infty e^{-s^2/(4u) - uA^2} \frac{du}{u^{3/2}} \qquad (71)$$

We find for the principal operator:

$$\Phi_\epsilon = \sum_{i=1}^{N} \frac{1}{g_i^2}\chi_i^\dagger\chi_i - \frac{1}{2\sqrt{\pi}}\sum_{i,j}\int_0^\infty \frac{du}{\sqrt{u}}\ e^{-um^2} \times \underbrace{\int d_g x \int d_g y K_{\epsilon/2}(a_i,x)K_{\epsilon/2}(a_j,y)K_u(x,y)}_{K_{u+\epsilon}(a_i,a_j)} \times \int_0^\infty ds\ s e^{-s^2/4}\left[\frac{1-e^{sE\sqrt{u}}}{(-E)}\right]\chi_i^\dagger\chi_j$$

$$\Phi_\epsilon = \sum_{i=1}^{N}\Big[\frac{1}{g_i^2} - \frac{1}{\sqrt{\pi}}\int_0^\infty ds e^{-s^2/4} \times \underbrace{\int_\epsilon^\infty du}_{\text{cut}-\text{off}}\, e^{sE\sqrt{u}}\ e^{-um^2}K_u(a_i,a_i)\Big]\chi_i^\dagger\chi_i - \frac{1}{\sqrt{\pi}}\sum_{\substack{i,j\\(i\neq j)}}\int_0^\infty ds\ e^{-s^2/4}\int_0^\infty du e^{sE\sqrt{u}} \times e^{-um^2}K_u(a_i,a_j)\chi_i^\dagger\chi_j$$

$$\lim_{\epsilon\to 0}\int_\epsilon^\infty du e^{sE\sqrt{u}}\ e^{-um^2}K_u(a_i,a_i) \approx -\frac{1}{4\pi}\ln(\epsilon) \tag{72}$$

$$\frac{1}{g_i^2(\mu_i,\epsilon)} = \frac{1}{\sqrt{\pi}}\int_0^\infty ds\ e^{-s^2/4} \times \int_\epsilon^\infty du \underbrace{e^{s\mu_i\sqrt{u}}}_{\text{a mass scale}} e^{-um^2} K_u(a_i,a_i).$$

$$\begin{aligned}\Phi &= \frac{1}{\sqrt{\pi}}\sum_{i=1}^{N}\int_0^\infty ds e^{-s^2/4}\int_0^\infty du e^{-um^2} K_u(a_i,a_i) \\ &\quad \times \left[e^{\sqrt{u}\mu_i s} - e^{\sqrt{u}Es}\right]\chi_i^\dagger\chi_i \\ &- \frac{1}{\sqrt{\pi}}\sum_{\substack{i,j\\(i\neq j)}}\int_0^\infty ds e^{-s^2/4}\int_0^\infty du e^{-um^2} \\ &\quad \times K_u(a_i,a_j) e^{\sqrt{u}Es}\chi_i^\dagger\chi_j\end{aligned}$$

After $\epsilon \mapsto 0^+$, we have

$$b = \sum_j \phi^{(+)}(a_j)\chi_j^\dagger \tag{73}$$

$$\begin{aligned}
&[a^{-1} + a^{-1}b^\dagger \Phi^{-1} b a^{-1}] \left(|0> \bigotimes |\Psi(1)>_B\right) \\
&\propto [1 + a^{-1}b^\dagger \Phi^{-1} b] \left(|0> \bigotimes |\Psi'(1)>_B\right) \\
&\propto \left(|0> \bigotimes |\Psi'(1)>_B\right) \\
&\quad + a^{-1}b^\dagger \Phi^{-1} \left(\sum_{i=1}^N c_i|\chi_i> \bigotimes |0>_B\right)
\end{aligned}$$

$$\Phi^{-1}(E^*) \left(\sum_{i=1}^N c_i|\chi_i> \bigotimes |0>_B\right) \mapsto \infty \tag{74}$$

defines the energy eigenvalue $E^* < m$.

$$[\chi_i^\dagger \Phi_{ij}(E)\chi_j]^{-1} = \chi_i^\dagger [\Phi^{-1}(E)]_{ij}\chi_j \tag{75}$$

Thus what matters is the solution to

$$\Phi(E)_{ij} R_j(E) = \omega(E) R_i(E) \tag{76}$$

for $\omega(E^*) = 0$!

$$\begin{aligned}\psi_{E^*}(x) &= \pi^{-1/4} \sum_{i=1}^{N} R_i(E^*) \int_0^\infty ds\ e^{-s^2/4} \\ &\quad \int_0^\infty du\ e^{-um^2} e^{sE^*\sqrt{u}} K_u(a_i, x) \\ &\quad \times \Big[\sum_{j,k} R_j^*(E^*) R_k(E^*) \int_0^\infty ds\ e^{-s^2/4} \\ &\quad \times \int_0^\infty du \sqrt{u}\ e^{-um^2} e^{sE^*\sqrt{u}} K_u(a_k, a_j) \Big]^{-1/2}\end{aligned}$$

- on $\mathbf{R^2}$ we have,

$$\begin{aligned}\Phi_{ii} &= \frac{1}{2\pi} \ln \left(\frac{m-E}{m-\mu_i} \right) \\ \Phi_{ij} &= \frac{-1}{2\pi} \int_0^\infty \frac{ds}{\sqrt{s^2+1}}\ \exp\left[-d_{ij} \left(m\sqrt{s^2+1} - Es \right) \right]\end{aligned}$$

Many body theory of bosons(OTT and F. Erman, J.Phys A 2013)

$$H = H_0 + H_I \; , \tag{77}$$

where

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2m} \int_{\mathcal{M}} \mathrm{d}_g^2 x \; \phi_g^\dagger(x) \, \nabla_g^2 \, \phi_g(x) \; , \\ H_I &= -\frac{\lambda}{2} \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \; \phi_g^\dagger(x') \, \phi_g^\dagger(x) \\ & \quad \times \delta_g^2(x, x') \, \phi_g(x) \, \phi_g(x') \end{aligned} \tag{78}$$

Regularized version:

$$\begin{aligned} H^\epsilon = H_0 - \frac{\lambda(\epsilon)}{2} & \int_{\mathcal{M}^5} \mathrm{d}_g^2 x_1 \mathrm{d}_g^2 x_1' \mathrm{d}_g^2 x_2 \mathrm{d}_g^2 x_2' \mathrm{d}_g^2 y \\ & \times \phi_g^\dagger(x_1) \phi_g^\dagger(x_2) K_\epsilon(x_1, y; g) K_\epsilon(x_2, y; g) \\ & \times K_\epsilon(x_1', y; g) K_\epsilon(x_2', y; g) \phi_g(x_1') \phi_g(x_2'). \end{aligned}$$

Using very similar ideas and employing again an orthofermion algebra(proposed by Rajeev for the flat case)

$$
\begin{aligned}
\chi_g(x)\chi_g^\dagger(y) &= \delta_g^{(2)}(x,y)\Pi_0, \\
\chi_g(x)\chi_g(y) = 0 &= \chi_g^\dagger(x)\chi_g^\dagger(y),
\end{aligned}
$$

where

$$
\Pi_1 = \int_{\mathcal{M}} \mathrm{d}_g^2 x \, \chi_g^\dagger(x)\chi_g(x), \quad \Pi_0 = 1 - \Pi_1 \qquad (79)
$$

are the projection operators onto the 1-angel and no-angel states, respectively.

$$
\begin{aligned}
\Phi(E) = &\int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \chi_g^\dagger(x) \int_0^\infty \mathrm{d}t [\frac{e^{-t\mu^2}}{4\pi t/m} \delta_g^2(x, x') \\
&\quad - K_t^2(x, x'; g) e^{-t(H_0 - E)}] \chi_g(x') \\
&\quad - \frac{1}{2} \int_{\mathcal{M}^2} \mathrm{d}_g^2 x \, \mathrm{d}_g^2 x' \; \chi_g^\dagger(x) \\
&\times \Big[\int_{\mathcal{M}^4} \mathrm{d}_g^2 x_1 \, \mathrm{d}_g^2 x_2 \, \mathrm{d}_g^2 x_1' \, \mathrm{d}_g^2 x_2' \\
&\quad \times \phi_g^\dagger(x_1') \, \phi_g^\dagger(x_2') \int_0^\infty \mathrm{d}t \; K_t(x_1, x; g) \, K_t(x_2, x; g) \\
&\times K_t(x', x_1'; g) \, K_t(x', x_2'; g) \, e^{-t(H_0 - E)} \phi_g(x_1) \, \phi_g(x_2) \\
&\quad + 4 \int_{\mathcal{M}^2} \mathrm{d}_g^2 x_1 \, \mathrm{d}_g^2 x_2 \; \phi_g^\dagger(x_1) \\
&\times \int_0^\infty \mathrm{d}t \; K_t(x_2, x; g) \, K_t(x', x; g) \, K_t(x', x_1; g) \\
&\quad \times e^{-t(H_0 - E)} \phi_g(x_2) \Big] \chi_g(x') \; . \qquad (80)
\end{aligned}
$$

• renormalization is much more complicated, requires a careful study of the singular structure! Above operator is well-defined!

♯ How do we know that we define a bound state system? The variational principle for *the first eigenvalue* $\omega_0(E)$ of $\Phi(E)$ in *the two-boson sec-*

tor. On a compact manifold we choose the orthofermion wave function as constant, $\frac{1}{\sqrt{V(\mathcal{M})}}$.

$$|\Psi^{var}\rangle = |0\rangle \otimes \frac{1}{\sqrt{V(\mathcal{M})}} \int_{\mathcal{M}} \mathrm{d}_g^3 x \; \chi^\dagger(x)|0\rangle \; . \quad (81)$$

Since $\Phi(E)$ is normal ordered, all the parts which contain bosonic creation and annihilation operators will vanish. The only term which survives sets an upper bound for $\omega_0(E)$. Hence,

$$\begin{aligned} \omega_0(E) &\leq \langle \Psi^{var}|\Phi(E)|\Psi^{var}\rangle \\ &\leq \int_0^\infty \mathrm{d}t \; \Big[\frac{e^{-\mu^2 t}}{8\pi t} \\ &- \frac{1}{V(\mathcal{M})} \int_{\mathcal{M}} \mathrm{d}_g^2 x \; K_{2t}(x,x;g) e^{-|E|t}\Big]. \end{aligned}$$

Compactness of the manifold implies that it is complete as a Riemannian manifold and it has a Ricci tensor bounded from below which we formally write $Rc \geq \kappa$. As a result of the theorem proven by **J. Cheeger and S.-T. Yau**, the heat kernel has the following lower bound

$$K_t(x,y;g) \geq K_t^\kappa(d_g(x,y)) \; , \quad (82)$$

where K_t^κ is the heat kernel of the simply connected complete two dimensional manifold of constant sectional curvature κ. In particular, we choose $K_t^\kappa(d_g(x,y))$ as the heat kernel of the two dimensional Hyperbolic manifold $\mathbf{H^2}$ for $\kappa = -1/R^2$, where R is the corresponding length scale.

$$K_{2t}(x,x) \geq \frac{R\sqrt{2}}{(8\pi t)^{3/2}}\, e^{-t/2R^2} \int_0^\infty \mathrm{d}s\, \frac{s\, e^{-s^2 R^2/8t}}{\sqrt{\cosh s - 1}}.$$

We can then show by a careful estimate of the integral,

$$\omega_0(E) \;\leq\; \frac{1}{8\pi} \ln\left(\frac{|E| + \frac{R^2}{2}}{\mu^2}\right) + \frac{1}{2\pi}\frac{1}{|E| + \frac{R^2}{2}} \tag{83}$$

For large values of μ^2 there always exists a unique $E_* < 0$ such that

$$\frac{1}{8\pi} \ln\left(\frac{|E_*| + \frac{R^2}{2}}{\mu^2}\right) = -\frac{1}{2\pi}\frac{1}{|E_*| + \frac{R^2}{2}}\,. \tag{84}$$

As we will emphasize below, one has,

$$\frac{\partial \omega_0}{\partial E} < 0\,, \tag{85}$$

thus to get the true zero $E_{gr}^{(2)}$ of $\omega_0(E)$, we must further decrease E (or increase $|E|$) so that we will have a well-defined expression of μ^2 in terms of two-particle binding energy $E_{gr}^{(2)} < E_* < 0$

♯ Mean field approach: $\Phi(E)$'s lowest eigenfunction may be approximated by a product form for large number of bosons, that is,

$$\widehat{u}_0(x_1,\cdots,x_{n-2}) = u_0(x_1)\cdots u_0(x_{n-2}) \qquad (86)$$

with the normalization

$$\begin{aligned} ||u_0||^2 \;\; = \;\; \int_{\mathcal{M}} \mathrm{d}_g^2 x \; |u_0(x)|^2 = 1 \\ \int_{\mathcal{M}} \mathrm{d}_g^2 x \; |\psi_0(x)|^2 = 1 \, . \end{aligned} \qquad (87)$$

♯ Why lowest eigenvalue? Again, using Feynmann-Helmann, after some estimates, similar to delta-function case but harder, we prove that

$$\frac{\partial \omega_k(E)}{\partial E} < 0. \qquad (88)$$

Hence, lowest eigenvalue of $\Phi(E)$ cuts the E-axis first, as they flow with E and that solution gives the ground state energy.

• If we assume that $E_{gr} \approx f(n) >> 1$, then there is a simplification, the true ground state becomes, in terms of the eigenvector $\widehat{u}_0(y_1,...y_{n-2})$ of $\Phi(E)$,

$$
\begin{aligned}
|\Psi_0\rangle \approx \frac{1}{\sqrt{2}} \int_{\mathcal{M}^n} \mathrm{d}_g^2 y_1 \cdots \mathrm{d}_g^2 y_n \\
\frac{1}{n!} \sum_{\sigma \in [1 \cdots n]} \int_0^\infty \mathrm{d}t \; e^{-t|E_{gr}|} K_t(y_{\sigma(1)}, y_{\sigma(2)}; g) \\
\times \, \hat{u}_0(y_{\sigma(3)}, \cdots, y_{\sigma(n)}) \psi_0(y_{\sigma(2)}) \\
\times \left(- \frac{\partial \omega_0(E)}{\partial E} |_{E_{gr}} \right)^{-1/2} |y_1 \cdots y_n\rangle \; .
\end{aligned}
$$

It is important to notice that $|\Psi_0\rangle$ is not in the domain of H_0. The solution takes a kind of convolution of the wave functions in the domain of H_0 with the bound state wave function which is outside of this domain. We use

$$\langle \hat{u}_0 | \Phi(E_{gr}) | \hat{u}_0 \rangle = 0 \, . \tag{89}$$

• In the spirit of mean-field we assume $E_{gr} >> 1$ as $n >> 1$, hence we may use an asymptotic expansion. The kinetic term becomes, in the limit $|E_{gr}| \to \infty$

$$\begin{aligned}
&\int_0^\infty \mathrm{d}t \, \Big[\frac{e^{-t\mu^2}}{8\pi t} \\
&- \int_{\mathcal{M}} \mathrm{d}_g^2 x \, |\psi_0(x)|^2 \frac{e^{-t|E_{gr}|}}{8\pi t} \big(\int_{\mathcal{M}} \mathrm{d}_g^2 x' \, |u_0(x')|^2 \big)^n \Big] \\
&= \int_0^\infty \mathrm{d}t \, \Big[\frac{e^{-t\mu^2}}{8\pi t} - \frac{e^{-t|E_{gr}|}}{8\pi t} \Big] \\
&= \frac{1}{8\pi} \ln(|E_{gr}|/\mu^2),
\end{aligned}$$

We can rewrite the "potential part" by making a change of variable $t = t'/|E_{gr}|$ as

$$\begin{aligned}
&\frac{n^2}{2} \int_0^\infty \frac{\mathrm{d}t'}{|E_{gr}|} \Big| \int_{\mathcal{M}} \mathrm{d}_g^2 x \, |u_0(x)|^2 \psi_0(x) \Big|^2 e^{-t'} \\
\times \; & \Big[(1 - \frac{t'}{|E_{gr}|} K[u_0])^{|E_{gr}|} \Big]^{\frac{n}{|E_{gr}|}} \\
&+ 2n \int_0^\infty \frac{\mathrm{d}t'}{|E_{gr}|} \, \Big| \int_{\mathcal{M}} \mathrm{d}_g^2 x \, u_0^*(x) \psi_0(x) \Big|^2 e^{-t'} \\
\times \; & \Big[(1 - \frac{t'}{|E_{gr}|} K[u_0])^{|E_{gr}|} \Big]^{\frac{n}{|E_{gr}|}} \, .
\end{aligned} \tag{90}$$

where

$$K[u_0] = \int_{\mathcal{M}} \mathrm{d}_g^2 x \, |\nabla_g u_0(x)|^2 \,. \tag{91}$$

Moreover, we can think of terms in the square brackets as an exponential when $|E_{gr}| \to \infty$ so that

$$\begin{aligned} & \frac{n^2}{2} \int_0^\infty \frac{\mathrm{d}t'}{|E_{gr}|} \Big| \int_{\mathcal{M}} \mathrm{d}_g^2 x \, |u_0(x)|^2 \psi_0(x) \Big|^2 e^{-t' - \frac{t'n}{|E_{gr}|} K[u_0]} \\ + \;\; & 2n \int_0^\infty \frac{\mathrm{d}t'}{|E_{gr}|} \Big| \int_{\mathcal{M}} \mathrm{d}_g^2 x u_0^*(x) \psi_0(x) \Big|^2 e^{-t' - \frac{t'n}{|E_{gr}|} K[u_0]} . \end{aligned}$$

set the normalized wave function of the angel to saturate the Cauchy-Schwartz inequality:

$$\psi_0(x) = \frac{|u_0(x)|^2}{\left(\int_{\mathcal{M}} \mathrm{d}_g^2 x \, |u_0(x)|^4 \right)^{1/2}} \,. \tag{92}$$

Sobolev Inequality in 2-dimensions(**Aubin**):

$$\begin{aligned} \Big(\int_{\mathcal{M}} \mathrm{d}_g^2 x |f(x)|^2 \Big)^{1/2} \;\; \leq \;\; & A(0) \int_{\mathcal{M}} \mathrm{d}_g^2 x \, |f(x)| \\ & + K(2,1) \int_{\mathcal{M}} \mathrm{d}_g^2 x \, |\nabla_g f(x)| \end{aligned}$$

get

$$|E_{gr}| \ln(|E_{gr}|/\mu^2) \approx n^2 A^2(0) \frac{(1+\beta z)^2}{1+\alpha z^2} \tag{93}$$

where $\alpha = 1/|E_{gr}|$, $\beta = 2K(2,1)/A(0)\sqrt{n}$ and $z = \sqrt{nK[u_0]}$.

$$K[u_0] = \int_{\mathcal{M}} d_g^2 x |\nabla_g u_0|^2. \tag{94}$$

Maximize with respect to z and find,

$$E_{gr} \approx -\mu^2 e^{4K^2(2,1)n} \tag{95}$$

♠ Contrast this approach with the 1-dimensional boson model. In that case the answer is known exactly, which agrees to the leading order with the mean-field result:

$$E_g r \approx -\frac{\lambda^2}{48} n^3. \tag{96}$$

If we use the same approach we have for two dimensions–and do not use the fact that 1-dimension is flat and the heat equation is known, we find

$$\frac{1}{\lambda} - \frac{1}{2\sqrt{2|E_{gr}|}} = \frac{n^2}{2|E_{gr}|} \frac{1}{(1 + \frac{nK[u_0]}{|E_{gr}|})} \int \mathrm{d}x \, |u_0(x)|^4 + \frac{2n}{|E_{gr}|} \frac{1}{(1 + \frac{nK[u_0]}{|E_{gr}|})}. \tag{97}$$

Sobolev inequality in 1-dimension gives

$$\begin{aligned} &\int \mathrm{d}x|u_0|^4 \\ &\leq S_{1,4}^{-2}\Big(\int \mathrm{d}x\left|\frac{\mathrm{d}u_0}{\mathrm{d}x}\right|^2\Big)^{1/2}\Big(\int \mathrm{d}x|u_0|^2\Big)^{3/2} \\ &= \frac{1}{\sqrt{3}}K^{1/2}[u_0], \end{aligned} \tag{98}$$

Keeping the leading order term on both sides, we obtain

$$\frac{1}{\lambda} \leq \frac{n^2}{2\sqrt{3}|E_{gr}|}\frac{K^{1/2}[u_0]}{(1+\frac{nK[u_0]}{|E_{gr}|})}. \tag{99}$$

Let us define the variables $z = nK[u_0]$ and $\alpha = 1/|E_{gr}|$, and then find the upper bound to the right hand side. This occurs at $z = 1/\alpha$ so we get

$$E_{gr} \geq -\frac{\lambda^2}{48}n^3 \,, \tag{100}$$

- Renormalization group!

We define $\lambda_R(M)$ in terms of the bare coupling constant $\lambda(\epsilon)$

$$\frac{1}{\lambda_R(M)} = \frac{1}{\lambda(\epsilon)} - \int_\epsilon^\infty \mathrm{d}t \frac{e^{-M^2 t}}{8\pi t} \,, \tag{101}$$

where M is the renormalization scale. For the Φ operator, we then demand,

$$M \frac{\mathrm{d}\Phi^R(M, \lambda_R(M), E; g)}{\mathrm{d}M} = 0 \tag{102}$$

or

$$(M \frac{\partial}{\partial M} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R}) \Phi^R(M, \lambda_R(M), E; g) = 0 \tag{103}$$

where

$$\beta(\lambda_R) = M \frac{\partial \lambda_R}{\partial M} \tag{104}$$

we can find β function exactly

$$\beta(\lambda_R) = -\frac{\lambda_R^2}{4\pi} < 0. \tag{105}$$

This result is exactly the same as the one in flat spaces given in (Bergmann), so our problem is asymptotically free, too.

The scaling transformation of the metric $g \mapsto \gamma^{-2}g$, there is a unitary operator, such that the creation and annihilation operators transform like

$$\begin{aligned} U(\gamma)\phi_g(x)U^\dagger(\gamma) &= \gamma^{-1}\phi_{\gamma^{-2}g}(x) \\ U(\gamma)\chi_g(x)U^\dagger(\gamma) &= \gamma^{-1}\chi_{\gamma^{-2}g}(x) \end{aligned} \tag{106}$$

$$\begin{aligned} U^\dagger(\gamma)\Phi^R(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g)\,U(\gamma) \\ = \Phi^R(\gamma^{-1}M,\lambda_R(M),E;g). \end{aligned} \tag{107}$$

we get

$$\begin{aligned} U^\dagger(\gamma)\Phi^R(M,\lambda_R(M),\gamma^2E;\gamma^{-2}g)U(\gamma) \\ = \Phi^R(M,\lambda_R(\gamma M),E;g)\ , \end{aligned} \tag{108}$$

which means that there is no anamolous scaling.

We have the flow equation for the coupling constant,

$$\lambda_R(\gamma M) = \frac{\lambda_R(M)}{1+\frac{1}{4\pi}\lambda_R(M)\ln\gamma}. \tag{109}$$

This would allow us to check the above renormalization group relation *exactly*. **It holds, non-perturbatively!**

•BEC in a nontrivial geometry. (Joint work with **L. Akant, E. Ertugrul, and F. Tapramaz**)

• Let (S, g) be a d dimensional Riemannian manifold with metric g and nonnegative Ricci curvature

$$\mathbf{Ric}_S \geq 0. \tag{110}$$

Let M be a connected open submanifold of S with compact closure and smooth convex boundary ∂M.

♯ $\{f_\sigma\}$ $(\sigma = 0, 1, 2, \ldots)$ be a complete orthonormal set of real (standing wave) square-integrable eigenfunctions of $-\Delta$ on M, obeying the Neumann boundary conditions

$$-\Delta\, f_\sigma = \epsilon_\sigma f_\sigma, \quad \hat{n} \cdot \nabla f_\sigma|_{\partial K} = 0. \tag{111}$$

Here $\hat{n}$ is the outward looking unit normal to ∂M. The eigenvalues can be ordered as

$$\epsilon_0 = 0 < \epsilon_1 \leq \ldots \to \infty. \tag{112}$$

The ground state is

$$f_0 = \frac{1}{\sqrt{V}}, \tag{113}$$

with eigenvalue $\nu_0 = 0$. Here V is the volume of M. Connectedness of M implies the uniqueness of the ground state and the existence of the fundamental gap $\epsilon_1 > 0$.

For the Neumann heat kernel on a manifold M with a nonnegative Ricci curvature and diameter D_M one has the following estimates of Li and Yau:

$$\frac{1}{(4\pi t)^{d/2}} V \leq \mathrm{Tr}\, e^{\Delta t} \leq \tilde{C}(d) g(t). \tag{114}$$

Here $\tilde{C}(d)$ is a positive constant which depends only on the dimension d and

$$g(t) = \begin{cases} \left(\frac{D_M}{\sqrt{t}}\right)^d & \text{if } \sqrt{t} \leq D_M, \\ 1 & \text{if } \sqrt{t} \geq D_M. \end{cases}$$

A direct consequence is the eigenvalue bound of **Li-Yau**

$$\epsilon_\sigma \geq \frac{C(d)}{D_M^2} \sigma^{2/d}, \tag{115}$$

where $C(d)$ is a positive constant which depends only on the dimension.

One also has the following upper bound of Colbois and Maerten for the eigenvalues

$$\epsilon_\sigma \leq B(d) \left(\frac{\sigma}{V}\right)^{2/d}. \qquad (116)$$

Here $B(d)$ is a positive constant which depends only on the dimension.

• We will assume that the gas obeys Neumann boundary conditions on ∂M.

The thermal averages in the grand-canonical ensemble are given by

$$\langle O \rangle = \frac{\mathrm{Tr}\, O\, e^{-\beta\, H}}{\mathrm{Tr}\, e^{-\beta\, H}}. \qquad (117)$$

Here

$$H = \int d\mu_g \phi^\dagger(x)(h-\mu)\,\phi(x) + H_I \qquad (118)$$

and μ is the chemical potential.

• **Assumption:** the volume and the diameter of our box M satisfies

$$D_M = O(V^{1/d}) \quad \text{as} \quad V \to \infty. \qquad (119)$$

Thus we have a finite limit,

$$A = \lim_{V \to \infty} \frac{D_M^d}{V}. \tag{120}$$

♯ Remark: by the Bishop-Gromov volume comparison the geodesic balls in such a space satisfy:

$$\text{Vol}(B_r(p)) \leq \omega_n r^n \tag{121}$$

where ω_n is volume of the unit ball in Euclidean space. It is known (**Perelman, Cheeger-Colding**) that if we have

$$\text{Vol}(B_r(p)) \geq (1 - \epsilon(n))\omega_n r^n \tag{122}$$

for a calculable specific constant $\epsilon(n)$ then the manifold is *diffeomorphic* to $\mathbf{R^n}$.

The many-body Hamiltonian with a hard-core repulsive potential

$$H' = \int d\mu_g \, [\phi^\dagger(x) h \phi(x) + \frac{u_0}{2} \phi^\dagger(x)\phi^\dagger(x)\phi(x)\phi(x)] \tag{123}$$

here $u_0 > 0$. It is convenient to include the chemical potential in the Hamiltonian and define

$$H = H' - \mu N. \tag{124}$$

• We can prove the exactness of the c-number substitution in the thermodynamic limit folowing **Lieb et al.**.

• Our starting point will be the expansion of the field operator $\phi(x)$ around the background

$$\phi_0(x) = \sqrt{N_0} f_0 = \sqrt{\frac{N_0}{V}} = \sqrt{n_0} \tag{125}$$

as

$$\phi(x) = \phi_0(x) + \eta(x), \tag{126}$$

with

$$\eta(x) = \sum_{\sigma \neq 0} a_\sigma f_\sigma. \tag{127}$$

• Assuming the quantum fluctuations to be small, we can approximate H' as:

$$H'_{eff} = \frac{u_0 n_0^2 V}{2} + \sum_{\sigma \neq 0} [(\epsilon_\sigma + 2u_0 n_0) a_\sigma^\dagger a_\sigma + \frac{u_0 n_0}{2}(a_\sigma^2 + a_\sigma^{\dagger\,2})]. \tag{128}$$

Similarly, the number operator is given by

$$\begin{aligned} N &= \int d\mu_g(x)\, \phi^\dagger(x)\phi(x) \\ &= \int d\mu_g(x)\, [\phi_0^2 + \phi_0(\eta^\dagger(x) + \eta(x)) + \eta^\dagger(x)\eta(x)] \\ &= n_0 V + \sum_{\sigma \neq 0} a_\sigma^\dagger a_\sigma. \end{aligned} \tag{129}$$

or

$$H'_{eff} - \mu N = \frac{u_0 n_0^2 V}{2} - \mu n_0 V + \sum_{\sigma \neq 0} [(\epsilon_\sigma + 2u_0 n_0 - \mu) a_\sigma^\dagger a_\sigma + \frac{u_0 n_0}{2}(a_\sigma^2 + a_\sigma^{\dagger 2})]. \quad (130)$$

For a fixed n_0 the thermodynamic pressure

$$\frac{1}{V} \ln \mathrm{Tr}' e^{-\beta(H'_{eff} - \mu N)} \quad (131)$$

is maximized at the zeroth order in η by

$$\mu = n_0 u_0. \quad (132)$$

Here Tr$'$ means that the trace is taken over the states with no quanta in the f_0 mode. With this value of μ we get

$$H'_{eff} - \mu N = -\frac{u_0 n_0^2 V}{2} + \sum_{\sigma \neq 0} [(\epsilon_\sigma + u_0 n_0) a_\sigma^\dagger a_\sigma + \frac{u_0 n_0}{2}(a_\sigma^2 + a_\sigma^{\dagger 2})].$$

Therefore the thermal averages can be calculated using the effective Hamiltonian

$$H_{eff} = \sum_{\sigma\neq 0} [(\epsilon_\sigma + u_0 n_0) a_\sigma^\dagger a_\sigma + \frac{u_0 n_0}{2}(a_\sigma^2 + a_\sigma^{\dagger\, 2})], \tag{133}$$

which can be diagonalized by the Bogoliubov transformation

$$a_\sigma = (\sinh \xi_\sigma)\, b_\sigma^\dagger + (\cosh \xi_\sigma)\, b_\sigma, \tag{134}$$

where

$$\begin{aligned} \lambda_\sigma &:= \epsilon_\sigma + u_0 n_0 = \omega_\sigma \cosh 2\xi_\sigma, & (135)\\ u_0 n_0 &= -\omega_\sigma \sinh 2\xi_\sigma, & (136)\\ \omega_\sigma &:= \sqrt{(\epsilon_\sigma + u_0 n_0)^2 - (u_0 n_0)^2}. & (137) \end{aligned}$$

The last equation is the curved space analog of the Bogoliubov dispersion relation.

The resulting diagonal Hamiltonian is

$$\begin{aligned} H_{eff} &= \sum_{\sigma\neq 0} [\lambda_\sigma \cosh 2\xi_\sigma + u_0 n_0 \sinh 2\xi_\sigma]\, b_\sigma^\dagger b_\sigma + E_0 \\ &= \sum_{\sigma\neq 0} \omega_\sigma b_\sigma^\dagger b_\sigma + E_0, \end{aligned} \tag{138}$$

where E_0 is the ground state energy

$$
\begin{aligned}
E_0 &= -\frac{1}{2}\sum_{\sigma=1}^{\infty}(\lambda_\sigma - \omega_\sigma) \\
&= -\frac{1}{2}\sum_{\sigma=1}^{\infty}\frac{(u_0 n_0)^2}{2}\frac{1}{\epsilon_\sigma} + O(\frac{1}{\epsilon_\sigma^2}). \quad (139)
\end{aligned}
$$

Now using the lower eigenvalue bound (115) we see that for $k \geq 2$ and $d = 3, 2$

$$
\sum_{\sigma=1}^{\infty}\frac{1}{\epsilon_\sigma^k} \quad (140)
$$

is convergent. On the other hand, we see that the $k = 1$ sum is divergent for $d = 3, 2$.

• since u_0 should be taken as the bare coupling, we have a renormalized ground state energy:

$$
\begin{aligned}
E_0 &= -\frac{1}{2}\sum_{\sigma=1}^{\infty}[(\epsilon_\sigma + u_0 n_0) \\
&- \sqrt{\epsilon_\sigma^2 + 2u_0 n_0 \epsilon_\sigma} - \frac{(u_0 n_0)^2}{2\epsilon_\sigma}]. \quad (141)
\end{aligned}
$$

Remark : This renomalization can be understood as the renormalization of the bare coupling in equation (35), if we do it order by order in perturbation theory keeping renormalized coupling and the number of excited particles in the same order.

♠ Let $a=(u_0 n_0)^2/2$

♯ This expression written in terms of heat kernel becomes:

$$(\frac{1}{at}I_1(at)e^{-at}-\frac{1}{2})\mathrm{Tr}\, e^{\Delta t} \sim \frac{1}{at}\frac{at}{2}(at)\frac{1}{t^{3/2}} \quad \text{as} \quad t\to 0^+. \tag{142}$$

We observe here, $\mathrm{Tr}\, e^{\Delta t}=O(t^{3/2})$, and $I_1(t)\sim e^t/\sqrt{t}$, there is no divergence as $t\to\infty$. However, as $t\to 0^+$ which corresponds to the ultraviolet properties, we have $I_1(t)\sim t/2$ we get

$$\frac{1}{at}I_1(at)e^{-at}\mathrm{Tr}\, e^{\Delta t} \sim \frac{1}{at}\frac{at}{2}\frac{1}{t^{3/2}} \quad \text{as} \quad t\to 0^+, \tag{143}$$

so subtraction of 1/2 is essential. Note that the subtraction does not lead to an infrared divergence, i.e. an ultraviolet divergence in the t variable, thanks to $t^{-3/2}$ behavior of the heat kernel.

Let us recall the formula;

$$\frac{I_1(t)}{t} = \frac{2}{\pi}\int_0^1 dx\sqrt{1-x^2}\cosh(tx) \qquad (144)$$

moreover we have

$$\int_0^1 dx\sqrt{1-x^2} = \frac{\pi}{4} \qquad (145)$$

As a result we rewrite the ground state energy as

$$\begin{aligned} E_g &= \frac{u_0 n_0^2 V}{2} - \frac{aV}{\pi}\int_0^\infty dt\int_0^1 dx\sqrt{1-x^2} \\ &\qquad \times(\cosh(tx)e^{-t}-1)\frac{1}{V}\mathrm{Tr}\, e^{\Delta t/a} \end{aligned}$$

or reorganizing this as we have a new form for the Lee-Yang formula:

$$\begin{aligned} E_g &= \frac{u_0 n_0^2 V}{2} + \frac{aV}{2\pi}\int_0^\infty dt\int_0^1 dx\sqrt{1-x^2} \\ &\qquad \times(\underbrace{1-e^{-t(1-x)}}_{>0}+\underbrace{1-e^{-t(1+x)}}_{>0})\frac{1}{V}\mathrm{Tr}\, e^{\Delta t/a} \end{aligned}$$

since $0 \le x \le 1$.

♯ **Remark:** This allows us to use the upper and lower bounds of the heat kernel for now to get bounds on the ground state energy!

♯Assuming $V \to \infty$ and *using the physical values for the energy eigenvalues*, i.e. *set the flat space heat kernel* corrected with a factor of $\hbar^2/2m$, we find:

$$\frac{E_g}{N} = \frac{2\pi\sigma n_0}{m}[1 + \frac{128}{15\pi^{1/2}}(n_0\sigma^3)^{1/2}], \qquad (146)$$

where, $u_0 = 4\pi\sigma\hbar^2/m$, in terms of the scattering cross section, the well-known result of Lee and Yang is recovered.

♯ We will now study the depletion coefficient in $3d$.

♯ depletion for $T = 0$:

$$n_e = \frac{1}{V}\sum_{\sigma\neq 0}\langle a_\sigma^\dagger a_\sigma\rangle \qquad (147)$$

is expressed in terms of quasi-particle states as

$$\begin{aligned} n_e &= \frac{1}{V}\sum_{\sigma\neq 0}[\sinh^2\xi_\sigma + \cosh 2\xi_\sigma\langle b_\sigma^\dagger b_\sigma\rangle \\ &\qquad + \frac{1}{2}\sinh 2\xi_\sigma\langle b_\sigma^{\dagger\,2} + b_\sigma^2\rangle] \\ &= \frac{1}{2V}\sum_{\sigma\neq 0}[\frac{\lambda_\sigma}{\omega_\sigma}\coth\frac{\beta\omega_\sigma}{2} - 1]. \qquad (148) \end{aligned}$$

The zero temperature limit is

$$n_e = \frac{1}{2V} \sum_{\sigma \neq 0} [\frac{\lambda_\sigma}{\sqrt{\lambda_\sigma^2 - (u_0 n_0)^2}} - 1]. \qquad (149)$$

equivalently,

$$\frac{\lambda_\sigma}{\sqrt{\lambda_\sigma^2 - (u_0 n_0)^2}} = 1 + u_0 n_0 \int_0^\infty dt\, e^{-\lambda_\sigma t} I_1(u_0 n_0 t).$$

Thus,

$$n_e = \frac{u_0 n_0}{2} \int_0^\infty dt\, \frac{1}{V} \mathrm{Tr}' e^{-ht}\, e^{-u_0 n_0 t}\, I_1(u_0 n_0 t).$$

• In the limit, $V, D_M \to \infty$, $D_M^d / V \to A$ we get an upper bound:

$$n_e \leq \frac{u_0 n_0}{2} \frac{A}{C^{d/2}} \Gamma\left(\frac{d}{2} + 1\right) \int_0^\infty dt\, \frac{1}{t^{d/2}} e^{-u_0 n_0 t} I_1(u_0 n_0 t). \qquad (150)$$

The integral is convergent for $d = 3$.

Similarly we have a lower bound:

$$n_e \geq \frac{u_0 n_0}{2} \frac{1}{B^{d/2}} \Gamma\left(\frac{d}{2} + 1\right) \int_0^\infty dt\, \frac{1}{t^{d/2}}\, e^{-u_0 n_0 t} I_1(u_0 n_0 t). \qquad (151)$$

As a result we get:

$$(u_0 n_0)^{d/2} \alpha \gamma_d \leq n_e \leq (u_0 n_0)^{d/2} \beta \gamma_d, \tag{152}$$

where

$$\alpha = \frac{1}{2} \frac{1}{B^{d/2}} \Gamma\left(\frac{d}{2}+1\right), \quad \beta = \frac{1}{2} \frac{A}{C^{d/2}} \Gamma\left(\frac{d}{2}+1\right), \tag{153}$$

and

$$\gamma_d = \int_0^\infty \frac{ds}{s^{d/2}} e^{-s} I_1(s). \tag{154}$$

Thus we get, as in the flat case,

$$\frac{n_e}{n_0} = O(u_0^{d/2} n_0^{d/2-1}). \tag{155}$$

The smallness of the parameter $u_0^{d/2} n_0^{d/2-1}$ can now be used as a criterion for the validity of the Gross-Pitaevskii equation.

• Depletion of the condensate at finite temperatures:

$$n_e(T) = n_e(0) + \tilde{n}_e(T), \tag{156}$$

$$
\begin{aligned}
\tilde{n}_e(T) &= \sum_{k=1}^{\infty} [(\frac{1}{V}\mathrm{Tr}' e^{k\beta\,\Delta}) e^{-k\beta u_0 n_0} \\
&+ u_0 n_0 \int_0^{\infty} dt\, (\frac{1}{V}\mathrm{Tr}' e^{\Delta\sqrt{t^2+k^2\beta^2}}) \\
&\qquad \times e^{-u_0 n_0\sqrt{t^2+k^2\beta^2}} I_1(u_0 n_0 t)].
\end{aligned} \tag{157}
$$

We now use heat kernel bounds that we alluded before, as a result *the first expression* becomes bounded by

$$
C_1 \sum_{k=1}^{\infty} \frac{1}{(k\beta)^{3/2}} + C_1 \frac{(u_0 n_0)^{1/2}}{\beta}. \tag{158}
$$

Apply the subordination identity for the exponent:

$$
e^{-b\sqrt{x}} = \frac{b}{2\sqrt{\pi}} \int_0^{\infty} \frac{ds}{s^{3/2}}\, e^{-\frac{b^2}{4s}}\, e^{-sx}, \tag{159}
$$

and find that *the second expression* becomes,

$$
\begin{aligned}
& C_2 u_0 n_0 \int_0^{\infty} dt \sum_{k=1}^{\infty} \frac{1}{(t^2+k^2\beta^2)^{3/4}} \\
& \times \int_0^{\infty} \frac{ds}{s^{3/2}} e^{-\frac{1}{4s} - s(u_0 n_0)^2 (t^2+k^2\beta^2)}\, I_1(u_0 n_0 t).
\end{aligned}
$$

♯ the terms of the sum are monotonically decreasing as the summand increases, hence the integral gives an upper bound which we estimate separately;

$$\sum_{k=1}^{\infty} \frac{1}{(t^2 + k^2\beta^2)^{3/4}} e^{-(u_0 n_0)^2 s k^2 \beta^2}$$
$$< \; C_3 \frac{1}{s^{1/4}(u_0 n_0)^{1/2}\beta^{1/2}} \frac{1}{\beta^{1/2} t}. \qquad (160)$$

After some algebra and recognizing the modified bessel function, we rewrite,

$$\frac{C_5 (u_0 n_0)^{1/2}}{\beta} \int_0^\infty \frac{dt}{t} (u_0 n_0 t)^{3/4} K_{3/4}(u_0 n_0 t)\, I_1(u_0 n_0 t). \qquad (161)$$

Note that in the integral $(u_0 n_0)$ completely scales out. Hence we find that

$$\tilde{n}_e(T) < C_1 \sum_{k=1}^{\infty} \frac{1}{(k\beta)^{3/2}} + C_6 \frac{(u_0 n_0)^{1/2}}{\beta}, \quad (162)$$

the last piece of which will go to zero as $u_0 \to 0^+$ and moreover the full expression will go to zero as $\beta \to \infty$.

$\sharp$ As a by product, we can take the $V \to \infty$ limit *within Bogoliubov theory* rigorously, thanks to the estimate, for *Neumann heat kernel* and Lipshitz boundaries,

$$\left| K_t(x,x) - \frac{1}{(4\pi t)^{3/2}} \right| \leq C_{1/2} \left(\frac{\sqrt{t}}{\partial(x)} \right)^{1/2} \frac{e^{-\partial^2(x)/t}}{(4\pi t)^{3/2}},$$

here $\partial(x)$ denotes the distance from x to the boundary.

We may now prove that

$$|E_{gr}^{\text{box}} - E_{gr}^{0}| < \frac{C}{L^{1/4}} \tag{163}$$

for a box of size L. It will be interesting to improve this and generalize for convex bodies with bounded extrinsic curvature.

♯ Thanks for your patience and it is great to be here!!!!