Can effects of quantum gravity be observed in the cosmic microwave background?

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Contents

Main approaches to quantum gravity

No question about quantum gravity is more difficult than the question, "What is the question?"
(John Wheeler 1984)

- Quantum general relativity
  - Covariant approaches (perturbation theory, path integrals, ...)
  - Canonical approaches (geometrodynamics, connection dynamics, loop dynamics, ...)
- String theory
- Fundamental discrete approaches
  (quantum topology, causal sets, group field theory, ...);
  have partially grown out of the other approaches

Quantum gravitational corrections in the covariant approach

One-loop corrections to the non-relativistic potentials obtained from the scattering amplitude by calculating the non-analytic terms in the momentum transfer

- Quantum gravitational correction to the Newtonian potential

$$V(r) = -\frac{Gm_1m_2}{r}\left(1 + \underbrace{3\frac{G(m_1+m_2)}{rc^2}}_{\mathrm{GR-correction}} + \underbrace{\frac{41}{10\pi}\frac{G\hbar}{r^2c^3}}_{\mathrm{QG-correction}}\right)$$

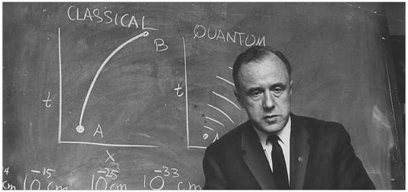
(Bjerrum-Bohr *et al.* 2003)

- Quantum gravitational effects to the Coulomb potential (scalar QED)

$$V(r) = \frac{Q_1Q_2}{r}\left(1 + 3\frac{G(m_1+m_2)}{rc^2} + \frac{6}{\pi}\frac{G\hbar}{r^2c^3}\right) + \dots$$

(Faller 2008)

Quantum geometrodynamics



- Question: what is the quantum wave equation that immediately gives Einstein's equations in the semiclassical limit?
- Answer: the Wheeler–DeWitt equation

$$\boxed{\hat{H}\Psi = 0}$$

Constraints of this type also occur in loop quantum gravity

Semiclassical (Born–Oppenheimer type) approximation

Ansatz:

$$|\Psi[h_{ab}]\rangle = C[h_{ab}]\mathrm{e}^{\mathrm{i}m_{\mathrm{P}}^2 S[h_{ab}]}|\psi[h_{ab}]\rangle$$

and expansion with respect to the Planck-mass squared.

Highest order: One evaluates $|\psi[h_{ab}]\rangle$ along a solution of the classical Einstein equations, $h_{ab}(\mathbf{x},t)$, corresponding to a solution, $S[h_{ab}]$, of the Hamilton–Jacobi equations;

$$\dot{h}_{ab} = NG_{abcd}\frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)}$$

$$\frac{\partial}{\partial t}\,|\psi(t)\rangle := \int \mathrm{d}^3x\,\dot{h}_{ab}(\mathbf{x},t)\,\frac{\delta}{\delta h_{ab}(\mathbf{x})}|\psi[h_{ab}]\rangle$$

This leads to a functional Schrödinger equation for quantized matter fields in the chosen external classical gravitational field:

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle &= \hat{H}^{\mathrm{m}}|\psi(t)\rangle \\ \hat{H}^{\mathrm{m}} &:= \int \mathrm{d}^3x\left\{N(\mathbf{x})\hat{\mathcal{H}}_{\perp}^{\mathrm{m}}(\mathbf{x}) + N^a(\mathbf{x})\hat{\mathcal{H}}_a^{\mathrm{m}}(\mathbf{x})\right\} \end{aligned}$$

$\hat{H}^{\mathrm{m}}$: matter-field Hamiltonian in the Schrödinger picture, parametrically depending on (generally non-static) metric coefficients of the curved space–time background.

WKB time t controls the dynamics in this approximation

Quantum gravitational corrections

The next order in the Born–Oppenheimer approximation gives

$$\hat{H}^{\mathrm{m}} \to \hat{H}^{\mathrm{m}} + \frac{1}{m_{\mathrm{P}}^2} \times (\text{various terms})$$

(C. K. and T. P. Singh (1991); A. O. Barvinsky and C. K. (1998))

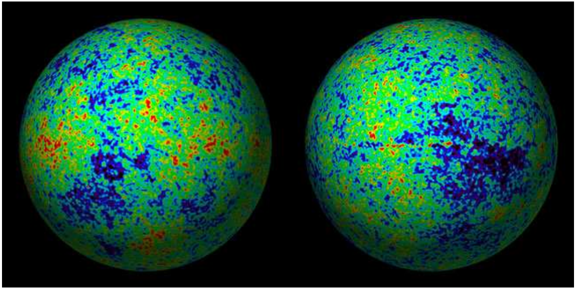
Example: Quantum gravitational correction to the trace anomaly in de Sitter space:

$$\delta\epsilon \approx -\frac{2G\hbar^2 H_{\mathrm{dS}}^6}{3(1440)^2\pi^3 c^8}$$

(C.K. 1996)

Observations

Does the anisotropy spectrum of the cosmic background radiation contain information about quantum gravity?



C.K. and M. Krämer, *Phys. Rev. Lett.*, **108**, 021301 (2012);
D. Bini, G. Esposito, C.K., M. Krämer, and F. Pessina, *Phys. Rev. D*, **87**, 104008 (2013); D. Brizuela, C.K., M. Krämer, in preparation.

Minisuperspace background

Wheeler–DeWitt equation for small fluctuations in a flat Friedmann–Lemaître universe with scale factor $a \equiv \exp(\alpha)$ and inflaton field ϕ

Choose a potential that classically obeys the slow-roll condition

$$\dot{\phi}^2 \ll |\mathcal{V}(\phi)|$$

Simple example:

$$\mathcal{V}(\phi) = \frac{1}{2} m^2 \phi^2$$

Minisuperspace Wheeler–DeWitt equation

$$\frac{1}{2}\,\mathrm{e}^{-2\alpha}\left[\frac{1}{m_{\mathrm{P}}^2}\,\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\phi^2}+2a_0^6\mathrm{e}^{6\alpha}\,\mathcal{V}(\phi)\right]\Psi_0(\alpha,\phi)=0$$

- $\hbar = c = 1$
- $m_{\mathrm{P}} = \sqrt{3\pi/2G} \approx 2.65 \times 10^{19}$ GeV
- $\phi \to \phi/\sqrt{2}\pi$
- assume below $\partial^2\Psi_0/\partial\phi^2 \ll 2a_0^6\mathrm{e}^{6\alpha}\mathcal{V}(\phi)\Psi_0$ and substitute $2\pi\sqrt{\mathcal{V}}$ by $m_{\mathrm{P}}H$, where H is the quasistatic Hubble parameter of inflation (Born–Oppenheimer type of approximation)

Introduction of inhomogeneities

For simplicity: perturbation in the scalar field only; general result below

$$\phi \to \phi(t) + \delta\phi(\mathbf{x}, t)$$

Perform a decomposition into Fourier modes with wave vector $\mathbf{k}$, $k \equiv |\mathbf{k}|$,

$$\delta\phi(\mathbf{x}, t) = \sum_{k} f_k(t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,.$$

The Wheeler-DeWitt equation including the fluctuation modes then reads (Halliwell and Hawking 1985)

$$\left[\mathcal{H}_0 + \sum_{k=1}^{\infty} \mathcal{H}_k\right] \Psi\left(\alpha, \phi, \{f_k\}_{k=1}^{\infty}\right) = 0$$

with

$$\mathcal{H}_k = \frac{1}{2}\,\mathrm{e}^{-3\alpha}\left[-\frac{\partial^2}{\partial f_k^2} + \left(k^2\,\mathrm{e}^{4\alpha} + m^2\,\mathrm{e}^{6\alpha}\right) f_k^2\right]$$

$$\mathcal{H}_k = \frac{1}{2}\,\mathrm{e}^{-3\alpha}\left[-\frac{\partial^2}{\partial f_k^2} + \left(k^2\,\mathrm{e}^{4\alpha} + m^2\,\mathrm{e}^{6\alpha}\right) f_k^2\right]$$

Ansatz:

$$\Psi\left(\alpha, \phi, \{f_k\}_{k=1}^{\infty}\right) = \Psi_0(\alpha, \phi) \prod_{k=1}^{\infty} \widetilde{\Psi}_k(\alpha, \phi, f_k)\,.$$

The components $\Psi_k(\alpha, \phi, f_k) := \Psi_0(\alpha, \phi)\widetilde{\Psi}_k(\alpha, \phi, f_k)$ obey

$$\frac{1}{2}\,\mathrm{e}^{-3\alpha}\left[\frac{1}{m_{\mathrm{P}}^2}\,\frac{\partial^2}{\partial\alpha^2} + \mathrm{e}^{6\alpha}\,m_{\mathrm{P}}^2\,H^2 - \frac{\partial^2}{\partial f_k^2} + W_k(\alpha) f_k^2\right]\Psi_k(\alpha, \phi, f_k) = 0$$

with

$$W_k(\alpha) := k^2\,\mathrm{e}^{4\alpha} + m^2\,\mathrm{e}^{6\alpha}\,,$$

Born–Oppenheimer approximation

Following the general scheme, we make the ansatz

$$\Psi_k(\alpha, f_k) = \mathrm{e}^{\mathsf{i}\, S(\alpha, f_k)}$$

and expand $S(\alpha, f_k)$ in terms of powers of m_{P}^2,

$$S(\alpha, f_k) = m_{\mathrm{P}}^2\, S_0 + m_{\mathrm{P}}^0\, S_1 + m_{\mathrm{P}}^{-2}\, S_2 + \ldots$$

We insert this ansatz into the full Wheeler–DeWitt equation and compare consecutive orders of m_{P}^2.

- $\mathcal{O}(m_{\rm P}^4)$: S_0 is independent of f_k
- $\mathcal{O}(m_{\rm P}^2)$: S_0 obeys the Hamilton–Jacobi equation

$$\left[\frac{\partial S_0}{\partial \alpha}\right]^2 - V(\alpha) = 0\,, \quad V(\alpha) := \mathrm{e}^{6\alpha} H^2$$

  solved by $S_0(\alpha) = \pm \mathrm{e}^{3\alpha} H/3$

- $\mathcal{O}(m_{\rm P}^0)$: Write $\psi_k^{(0)}(\alpha, f_k) \equiv \gamma(\alpha)\,\mathrm{e}^{\mathrm{i}\,S_1(\alpha, f_k)}$ and impose a condition on $\gamma(\alpha)$ that makes it equal to the standard WKB prefactor. After introducing the 'WKB time' according to

$$\frac{\partial}{\partial t} := -\,\mathrm{e}^{-3\alpha} \frac{\partial S_0}{\partial \alpha}\, \frac{\partial}{\partial \alpha}\,,$$

  one finds that each $\psi_k^{(0)}$ obeys a Schrödinger equation,

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\psi_k^{(0)} = \mathcal{H}_k \psi_k^{(0)}.$$

Solution of the uncorrected Schrödinger equation

Ansatz:

$$\psi_k^{(0)}(t, f_k) = \mathcal{N}_k^{(0)}(t)\, \mathrm{e}^{-\frac{1}{2}\,\Omega_k^{(0)}(t)\, f_k^2}$$

This leads to

$$\begin{aligned} \dot{\mathcal{N}}_k^{(0)}(t) &= -\frac{\mathrm{i}}{2}\,\mathrm{e}^{-3\alpha}\,\mathcal{N}_k^{(0)}(t)\,\Omega_k^{(0)}(t), \\ \dot{\Omega}_k^{(0)}(t) &= \mathrm{i}\,\mathrm{e}^{-3\alpha}\Big[-\big(\Omega_k^{(0)}(t)\big)^2 + W_k(t)\Big]. \end{aligned}$$

The solution can be given in terms of Bessel functions. Using $m^2 \ll H^2$ (realistic for inflationary models) as well as the boundary condition that the Minkowski vacuum is obtained for $k \to \infty$, it reads

$$\Omega_k^{(0)}(\xi) = \frac{k^3}{H^2\xi}\,\frac{1}{\xi - \mathrm{i}} + \mathcal{O}\left(\frac{m^2}{H^2}\right)$$

with $\xi(t) := k/(Ha(t))$

Quantum gravitational corrections

$\mathcal{O}(m_{\rm P}^{-2})$: decompose $S_2(\alpha, f_k)$ as

$$S_2(\alpha, f_k) \equiv \varsigma(\alpha) + \eta(\alpha, f_k)$$

and demand that $\varsigma(\alpha)$ be the standard second-order WKB correction. The wave functions

$$\psi_k^{(1)}(\alpha, f_k) := \psi_k^{(0)}(\alpha, f_k)\, \mathrm{e}^{\mathrm{i}\, m_{\rm P}^{-2}\, \eta(\alpha, f_k)}$$

then obey the quantum gravitationally corrected Schrödinger equation

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\psi_k^{(1)} = \mathcal{H}_k \psi_k^{(1)} - \frac{\mathrm{e}^{3\alpha}}{2 m_{\rm P}^2 \psi_k^{(0)}} \left[\frac{\left(\mathcal{H}_k\right)^2}{V} \psi_k^{(0)} + \mathrm{i}\frac{\partial}{\partial t}\left(\frac{\mathcal{H}_k}{V}\right)\psi_k^{(0)} \right] \psi_k^{(1)}$$

Solution of the corrected Schrödinger equation

Ansatz:

$$\begin{aligned}\psi_k^{(1)}(t,f_k) &= \left(\mathcal{N}_k^{(0)}(t)+\frac{1}{m_{\mathrm{P}}^2}\,\mathcal{N}_k^{(1)}(t)\right)\\ &\times \exp\left[-\frac{1}{2}\left(\Omega_k^{(0)}(t)+\frac{1}{m_{\mathrm{P}}^2}\,\Omega_k^{(1)}(t)\right)f_k^2\right]\end{aligned}$$

Inserting this into the corrected Schrödinger equation leads to

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\,\Omega_k^{(1)}(\xi)=\frac{2\,\mathrm{i}\,\xi}{\xi-\mathrm{i}}\,\Omega_k^{(1)}(\xi)+\frac{3\,\xi^3}{2}\,\frac{2\xi-\mathrm{i}}{(\xi-\mathrm{i})^3}$$

This equation can be solved analytically (special functions).

Unperturbed power spectrum

Use of gauge-invariant variables

$$\mathrm{d}s^2 = a^2(\eta)\Big\{-(1-2A)\,\mathrm{d}\eta^2 + 2\,(\partial_i B)\,\mathrm{d}x^i\mathrm{d}\eta + [(1-2\psi)\,\delta_{ij} + 2\partial_i\partial_j E]\,\mathrm{d}x^i\mathrm{d}x^j\Big\}$$

$$\delta\phi^{(\mathrm{gi})}(\eta,\mathbf{x}) := \delta\phi + \phi'\,(B - E')$$

$$\Phi_{\mathrm{B}}(\eta,\mathbf{x}) := A + \frac{1}{a}\,[a\,(B - E')]'$$

Mukhanov–Sasaki variable:

$$v(\eta,\mathbf{x}) := a\left[\delta\phi^{(\mathrm{gi})} + \phi'\frac{\Phi_{\mathrm{B}}}{\mathcal{H}}\right],$$

where $\mathcal{H} := a'/a = Ha$, and $H = \dot{a}/a$ is the standard Hubble parameter.

Unperturbed power spectrum

$$\begin{aligned}
\Xi(\mathbf{r}) &:= \langle \psi_{\mathbf{k}} \left| \hat{v}(\eta, \mathbf{x}) \hat{v}(\eta, \mathbf{x}+\mathbf{r}) \right| \psi_{\mathbf{k}} \rangle \\
&= \int \prod_{\mathbf{k}} \mathrm{d}v_{\mathbf{k}}\, \psi^*_{\mathbf{k}}(v_{\mathbf{k}}) v(\eta, \mathbf{x}) v(\eta, \mathbf{x}+\mathbf{r}) \psi_{\mathbf{k}}(v_{\mathbf{k}}) \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}\,\mathbf{p} \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{r}} \frac{1}{2\Re\Omega^{(0)}_{\mathbf{p}}} \\
&= \frac{1}{2\pi^2} \int_0^{+\infty} \frac{\mathrm{d}p}{p} \frac{\sin(pr)}{pr} p^3 \left| y_{\mathbf{p}} \right|^2
\end{aligned}$$

With

$$\Omega^{(0)}_{\mathbf{k}}(\eta) = -\,\mathrm{i}\, \frac{y'_{\mathbf{k}}(\eta)}{y_{\mathbf{k}}(\eta)},$$

one can define

$$\mathcal{P}^{(0)}_v(k) := \frac{k^3}{2\pi^2} \left| y_{\mathbf{k}} \right|^2.$$

- de Sitter case:

$$\mathcal{P}_{\mathsf{S}}^{(0)}(k) = \left.\frac{(t_{\mathrm{P}} H_0)^2}{\pi\epsilon}\right|_{k=H_0 a},$$

- Slow-roll case:

$$\mathcal{P}_{\mathsf{S}}^{(0)}(k) = \left.\frac{(t_{\mathrm{P}} H)^2}{\pi\epsilon}\left[1 - 2\epsilon + (2\epsilon - \delta)\left(4 - 2\gamma_{\mathsf{E}} - 2\ln(2)\right)\right]\right|_{k=Ha}$$

This coincides with the standard result.

Note that this is already a quantum-gravitational effect (tree level)!

Modification of the power spectrum

- de Sitter case:

$$\mathcal{P}_{\mathsf{S}}^{(1)}(k) = \frac{G H_0^2}{\pi\,\epsilon}\left[1 - \frac{H_0^2}{m_{\mathrm{P}}^2}\,\frac{0.32347}{k^3} + \mathcal{O}\left(\frac{H_0^4}{m_{\mathrm{P}}^4}\right)\right].$$

- Slow-roll case:

$$\mathcal{P}_{\mathsf{S}}^{(1)}(k) = \mathcal{P}_{\mathsf{S}}^{(0)}(k)\left[1 + \Delta_{\mathsf{S};\epsilon,\delta}^{\mathsf{WDW}}(k) + \mathcal{O}\left(\frac{H_0^4}{m_{\mathrm{P}}^4}\right)\right],$$

where

$$\Delta_{\mathsf{S};\epsilon,\delta}^{\mathsf{WDW}}(k) := -\frac{H_0^2}{m_{\mathrm{P}}^2}\,\frac{1}{k^3}\,(0.32347 + 4.8331\,\epsilon - 2.9492\,\delta)\Big|_{k=Ha}$$

We note that we have a violation of (near) scale invariance and a suppression of power on the largest scales!

Tensor perturbations

- Uncorrected spectrum:

$$\mathcal{P}_{\mathsf{T}}^{(0)}(k) = \frac{16G\,H^2}{\pi}\left[1 - 2\epsilon + \epsilon\left(4 - 2\gamma_{\mathsf{E}} - 2\ln(2)\right)\right]\Big|_{k=Ha}$$

- Corrected spectrum (slow-roll):

$$\mathcal{P}_{\mathsf{T}}^{(1)}(k) = \mathcal{P}_{\mathsf{T}}^{(0)}(k)\left[1 + \Delta_{\mathsf{T};\epsilon}^{\mathsf{WDW}}(k) + \mathcal{O}\left(\frac{H_0^4}{m_{\mathrm{P}}^4}\right)\right],$$

where

$$\Delta_{\mathsf{T};\epsilon}^{\mathsf{WDW}}(k) := -\frac{H_0^2}{m_{\mathrm{P}}^2}\,\frac{1}{k^3}\,(0.32347 + 1.8838\,\epsilon)\Big|_{k=Ha}$$

r-parameter

$$\boxed{r^{(1)} \simeq 16\,\epsilon\left[1 + (\delta - \epsilon)\left(1.4593 - 2.9492\,\frac{H_0^2}{m_{\mathrm{P}}^2}\,\frac{1}{k^3}\right)\right]}$$

Correction is of higher order!

Observations

Following G. Calcagni (2013), we can write

- Spectral index:

$$n_{\rm s} - 1 := \frac{{\rm d}\log\mathcal{P}}{{\rm d}\log k} \approx 2\delta - 4\epsilon - 3\Delta_{\sf S}^{\sf WDW}(k)$$

- Running of spectral index:

$$\alpha_{\rm s} := \frac{{\rm d}n_s}{{\rm d}\log k} \approx 2(5\epsilon\delta - 4\epsilon^2 - \iota^2) + 9\Delta_{\sf S}^{\sf WDW}(k)$$

ϵ, δ, ι: slow-roll parameters

If $r < 0.11$, one has

$$\frac{H_0}{m_{\rm P}} \lesssim 3.5 \times 10^{-6}$$

and thus (taking the reference scale to be the pivot scale $k_0 \sim 2 \times 10^{-3}$ Mpc^{-1}):

$$\left| \Delta^{\rm WDW}_{{\rm S};\epsilon,\delta=0}(k_0) \right| \lesssim 4.0 \times 10^{-12}$$

Comparing this with the PLANCK 2015 results,

$$n_{\rm S} = 0.968 \pm 0.006 \quad \text{and} \quad \alpha_{\rm S} = -0.0065 \pm 0.0076,$$

we see that the correction terms are too small to be seen. Cosmic variance may forbid to improve these errors.

Comparison with loop quantum cosmology

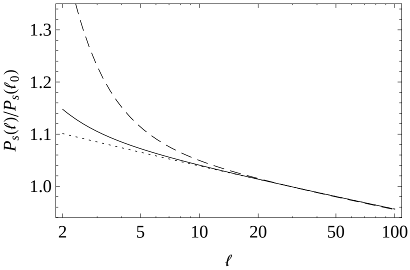


Figure: Primordial power spectrum for a certain model of loop quantum cosmology (upper curve). The dotted line is the classical case, and the solid line is the experimental upper bound. From: M. Bojowald, G. Calcagni, and S. Tsujikawa, *Phys. Rev. Lett.*, **107**, 211302 (2011).

Loop quantum cosmology predicts an enhancement of power at large scales.

(See talk by Ivan Agullo)

Summary and Outlook

- Concrete prediction from a conservative approach to quantum gravity; consistent with existing observational limits
- Suppression of power on largest scales
- k^{-3}-dependence
- In the present case, the effect is too small to be observable (main limit for accuracy: cosmic variance)
- Similar results (but different in detail) from a modified scheme put forward by Kamenshchik, Tronconi, and Venturi (2013–2015)
- Quantum gravitational corrections for galaxy–galaxy correlation functions?
- More general initial states?
- More general models of inflations?