MASSIVE ANTISYMMETRIC TENSOR FIELD MODELS IN CURVED SPACE-TIME: EFFECTIVE ACTION AND PROBLEM OF QUANTUM EQUIVALENCE

I.L. Buchbinder

TSPU, Tomsk

Topical workshop "QUANTUM VACUUM AND GRAVITATION", Mainz, June 22 - 26, 2015

Aims

Review

- Structure of effective action of massive rank-2 and rank-3 antisymmetric tensor fields in curved space-time
- Quantum equivalence of these models to massive vector field model and massive scalar field model with minimal coupling to gravity respectively

Based on I.L.B, E.N. Kirillova, N.G. Pletnev, Phys.Rev. D78, 084024, 2008

- Motivations
- Model of massless rank-2 antisymmetric field. Classical equivalence to scalar field model
- Quantization
- Problem of quantum equivalence
- Massive rank-2 and rank-3 antisymmetric tensor field models in curved space-time
- Classical equivalence to massive vector and massive scalar field models respectively
- Quantization
- Problem of quantum equivalence
- Summary

- Antisymmetric tensor fields are the natural ingredients of the higher dimensional supersymmetric models
- Antisymmetric tensor fields or p-forms are naturally arisen in the low-energy limit of superstring theory

Examples:

- $D = 11$, $\mathcal{N} = 1$ Poincare supergravity:
  gravitational field $e_\mu{}^a$, real antisymmetric tensor field $B_{\alpha\beta\gamma}$,
  Majorana-Rarita-Schwinger field ψ_μ.
- $D = 10$, $\mathcal{N} = (1,1)$ Poincare supergravity (low-energy effective theory of type IIA superstring theory):
  gravitational field $e_\mu{}^a$, real antisymmetric tensor fields $B_{\alpha\beta\gamma}$, $B_{\alpha\beta}$, a real vector field B_α, a real scalar field ϕ, a Majorana-Rarita-Schwinger field ψ_α, a Majorana spinor field λ.
- $D = 10$, $\mathcal{N} = (2,0)$ Poincare supergravity (low-energy effective theory of type IIB superstring theory):
  gravitational field $e_\mu{}^a$, real antisymmetric tensor field $B_{\alpha\beta\gamma\delta}$, complex antisymmetric tensor field $B_{\alpha\beta}$, a complex scalar field τ, a Weyl-Rarita-Schwinger field of positive chirality ψ_α, a Weyl spinor field of negative chirality λ.

Motivations

After reduction from higher dimensions to four dimensions we get the antisymmetric rank-2 and rank-3 tensor fields coupled to gravitational field.

- Massless rank two antisymmetric field model and massive rank-2 and rank-3 antisymmetric tensor field models are the new (in comparison with known scalar, spinor and vector field models) four dimensional models coupled to an gravitational field.
- Massless antisymmetric rank-2 tensor field models and massive rank-2 and rank-3 antisymmetric tensor field models are the new (in comparison with the known scalar, spin and vector field models) models for study the effective action in curved space-time (Schwinger-DeWitt proper-time technique, ζ-function, divergences, exact or approximate calculations on concrete background, ...
- Non-standard (in comparison with vector field models) gauge structure. Quantization problem.
- These models are classically equivalent or dynamically dual to scalar or vector fields models. Problem of quantum equivalence.

Model of massless rank-2 antisymmetric field

V.I. Ogievetsky, I.V. Polubarinov, 1967. Notoph field.

Massless antisymmetric tensor field $B_{\alpha\beta}$ ($B_{\alpha\beta} = -B_{\beta\alpha}$) is described by the action

$$S[B] = -\frac{1}{12}\int d^4x\sqrt{-g}F^{\alpha\beta\gamma}F_{\alpha\beta\gamma}$$

where $F_{\alpha\beta\gamma}$ is the strength tensor

$$F_{\alpha\beta\gamma} = \nabla_\alpha B_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma B_{\alpha\beta}$$

Properties:

- Strength $F_{\alpha\beta\gamma}$ is totally antisymmetric.
- Action is invariant under the gauge transformations

$$B'_{\alpha\beta} = B_{\alpha\beta} + \nabla_\alpha\xi_\beta - \nabla_\beta\xi_\alpha$$

  where ξ_α is the arbitrary vector field.
- Field $B_{\alpha\beta}$ is invariant under the transformations with parameter

$$\xi_\alpha = \nabla_\alpha\xi$$

  where ξ is the arbitrary scalar field.

Classical equivalence to massless scalar field with minimal coupling to gravity

Equations of motion:

$$\nabla_\alpha L_\beta - \nabla_\beta L_\alpha = 0$$

$$\nabla^\alpha L_\alpha = 0$$

where

$$L_\alpha = \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}\nabla^\beta B^{\gamma\delta}$$

Local solution to equations of motion:

$$L_\alpha = \nabla_\alpha \phi$$

where ϕ is a scalar field.
Equation of motion for ϕ:

$$\Box\phi = 0$$

Relation between $B_{\alpha\beta}$ and ϕ:

$$\nabla_\alpha \phi = \frac{1}{2}\sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}\nabla^\beta B^{\gamma\delta}$$

Quantization

- The theory of massless antisymmetric field $B_{\alpha\beta}$ is the gauge theory. Faddeev-Popov quantization ansatz. First step of covariant quantization is gauge fixing condition.
- The only covariant gauge is $\chi^{\alpha} = \nabla_{\beta} B^{\alpha\beta}$
- The gauge is degenerated, $\nabla_{\alpha}\chi^{\alpha} = 0$.
- Faddeev-Popov operator

$$M^{\alpha}{}_{\beta} = \frac{\delta\chi^{\alpha}}{\delta\xi^{\beta}} = -(\delta^{\alpha}{}_{\beta}\Box - \nabla_{\beta}\nabla^{\alpha})$$

- $M^{\alpha}{}_{\beta}$ is the operator in quadratic part of Maxwell Lagrangian. This operator is degenerated due to invariance of field $B_{\alpha\beta}$ under the gauge transformations with parameters $\xi_{\alpha} = \nabla_{\alpha}\xi$.
- Faddeev-Popov ansatz in its literal form does not work. Problem of definition of effective action.

Faddeev-Popov procedure for vector field:

- Gauge invariant theory with action $S[A^i]$
- Gauge transformations $\delta A^i = R^i{}_a[A]\xi^a$
- Consider the naive integral

$$\int DAe^{iS[A]}$$

- Multiply this integral with unit

$$1 = \int D\xi\Delta[A]\delta[\chi[A^\xi]]$$

- Integral over gauge group is factorized and one gets finally

$$\int DAe^{iS[A]}\Delta[A]\delta[\chi[A^\xi]]$$

Crucial point: existing the integral over gauge group

Problem of covariant quantization:
Basic element of Faddeev-Popov definition of functional integral for gauge theories is separation the integral over gauge group in naive path integral over gauge field. In the model under consideration the integral over gauge group is ill defined.
Correct path integral: A.S. Schwarz, 1978.
General idea: Application of Faddeev-Popov ansazt to integral over gauge group.
Let $F[B]$ is a (non-gauge invariant) functional of the field $B_{\alpha\beta}$ and let $B^{\xi}_{\alpha\beta}$ is the gauge transformed field, $B^{\xi}_{\alpha\beta} = B_{\alpha\beta} + \nabla_{\alpha}\xi_{\beta} - \nabla_{\beta}\xi_{\alpha}$.
The naive path integral over gauge group

$$\int D\xi F[B^{\xi}]$$

is replaced by the path integral

$$\int D\xi F[B^{\xi}] Det(\Box_0)\delta(\nabla^{\alpha}\xi_{\alpha})$$

Here $\Box_0$ is the scalar filed d'Alambertian, $Det(\Box_0)$ is the functional determinant of the differential operator and $\nabla^{\alpha}\xi_{\alpha}$ is the gauge for gauge parameters ξ_{α}.

Application of Faddeev-Popov procedure with modified integration over gauge group.
Final result for path integral:

$$e^{i\Gamma_B[g_{\mu\nu}]} = Det(\Box_1)Det^{-\frac{3}{2}}(\Box_0)\int DBe^{i(S[B]+S_{GF}[B])}$$

Here $S[B]$ is the classical action and $S_{GF}[B] = -\frac{1}{2}\int d^4x\sqrt{-g}\chi^\alpha\chi_\alpha$ is the gauge fixing term.
After calculating the integral:

$$\Gamma_B[g_{\mu\nu}] = \Gamma_\phi[g_{\mu\nu}] + \frac{i}{2}(Trln(\Box_2) - 2Trln(\Box_1) + 2Trln(\Box_0))$$

Here $\Box_2$ is the antisymmetric field d'Alambertian, $\Box_1$ is the vector field d'Alambertian and $\Gamma_\phi[g_{\mu\nu}] = \frac{i}{2}Trln(\Box_0)$ is the effective action of real scalar field minimally coupled to gravity.

$$\Box_2 B_{\alpha\beta} = \nabla^\mu\nabla_\mu B_{\alpha\beta} - 2R^\gamma{}_{[\alpha}B_{\beta]\gamma} - 2R^\gamma{}_{[\alpha\beta]}{}^\delta B_{\gamma\delta}$$

$$\Box_1 V_\alpha = \nabla^\mu\nabla_\mu V_\alpha - R_\alpha{}^\beta V_\beta$$

$$\Box_0\phi = \nabla^\mu\nabla_\mu\phi$$

Problem: whether the effective action of the field $B_{\alpha\beta}$ coincides with effective action of the field ϕ? Whether $\Gamma_B = \Gamma_\phi$?

$$\Gamma_B[g_{\mu\nu}] = \Gamma_\phi[g_{\mu\nu}] + X$$

$$X = \frac{i}{2}(Trln(\Box_2) - 2Trln(\Box_1) + 2Trln(\Box_0))$$

Possibilities to get the quantum equivalence:

- Whether $X = 0$?
- Whether $\frac{\delta X}{\delta g_{\alpha\beta}(x)} = 0$?
- X is a topological invariant.

M.T Grisaru, N.K. Nielsen, W. Siegel, D. Zanon, 1984;
I.L.B, S.M. Kuzenko, 1988

Idea of proof

- Non-degenerate change of variables

$$K_\mu = \nabla_\mu \Phi + \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\alpha\beta}\nabla^\nu B^{\alpha\beta}$$

$$V_\mu = \nabla_\mu \Psi + \nabla^\nu B_{\mu\nu}$$

Jacobian: $D(K,V|\Phi,\Psi,B) = Det(\Box_0)Det^{\frac{1}{2}}(\Box_2)$

- Inverse change of variable

$$B_{\mu\nu} = \frac{1}{\Box_2}\nabla_{[\mu}V_{\nu]} + \sqrt{-g}\epsilon_{\mu\nu\alpha\beta}\frac{1}{\Box_2}\nabla^\alpha K^\beta$$

$$\Phi = \frac{1}{\Box_0}\nabla^\mu K_\mu, \Psi = \frac{1}{\Box_0}\nabla^\mu V_\mu$$

Jacobian: $D(\Phi,\Psi,B|K,V) = Det^{-1}(\Box_1)$

- $D(K,V|\Phi,\Psi,B)D(\Phi,\Psi,B|K,V) = 1$. It is equivalent to $X = 0$ up to integrals of total divergences.

Massive rank-2 antisymmetric tensor field model
P. Townsend, 1981.

- Field $B_2 = (B_{\alpha\beta})$, $B_{\alpha\beta} = -B_{\beta\alpha}$
- Action

$$S[B_2] = \int d^4x \sqrt{-g(x)}(-\frac{1}{12}F^{\alpha\beta\gamma}(B)F_{\alpha\beta\gamma}(B) + \frac{1}{4}m^2 B^{\alpha\beta}B_{\alpha\beta})$$

- $F_{\alpha\beta\gamma}(B) = \nabla_\alpha B_{\beta\gamma} + \nabla_\beta B_{\gamma\alpha} + \nabla_\gamma B_{\alpha\beta}$
- Kinetic term is gauge invariant. Massive term violates this symmetry.

Massive rank-3 antisymmetric tensor field model
P. Townsend, 1981.

- Field $B_3 = (B_{\alpha\beta\gamma})$, $B_{\alpha\beta\gamma} = B_{[\alpha\beta\gamma]}$
- Action

$$S[B_3] = \int d^4x\sqrt{-g(x)}(-\frac{1}{48}F^{\alpha\beta\gamma\delta}(B)F_{\alpha\beta\gamma\delta}(B) + \frac{1}{12}m^2 B^{\alpha\beta\gamma}B_{\alpha\beta\gamma})$$

- $F_{\alpha\beta\gamma\delta}(B) = \nabla_\alpha B_{\beta\gamma\delta} - \nabla_\beta B_{\gamma\delta\alpha} + \nabla_\gamma B_{\delta\alpha\beta} - \nabla_\delta B_{\alpha\beta\gamma}$
- Kinetic term is gauge invariant under the transformations

$$B_{\mu\nu\rho} \to B^{\xi}_{\mu\nu\rho} = B_{\mu\nu\rho} + \nabla_\mu \xi_{\nu\rho} + \nabla_\nu \xi_{\rho\mu} + \nabla_\rho \xi_{\mu\nu}$$

  with a tensor gauge parameter $\xi_{\mu\nu} = -\xi_{\nu\mu},$. This parameter is defined up to a gauge transformation $\xi'_{\mu\nu} = \xi_{\mu\nu} + \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$ with the vector gauge parameter ξ_μ. In its turn, the parameter ξ_μ is defined up to gauge transformation $\xi'_\mu = \xi_\mu + \nabla_\mu \xi$ with the scalar gauge parameter ξ. This means from general point of view that the gauge generators are linearly dependent. Massive term violates this symmetry.

Equivalence of the massive B_2 field model to the massive vector field model

Consider of the theory of the fields $V_\mu, A_\mu, B_{\mu\nu}$ with action

$$S[V,A,B] = \int d^4x\sqrt{-g}(\frac{1}{2}V^\mu V_\mu + \frac{1}{4}m^2 B^{\mu\nu}B_{\mu\nu} + mA^\mu(V_\mu - \sqrt{-g}\epsilon_{\mu\alpha\beta\gamma}\nabla^\alpha B^{\beta\gamma}))$$

Equations of motion

$$V_\mu = -mA_\mu$$

$$V_\mu = L_\mu(B) = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\alpha\beta\gamma}\nabla^\alpha B^{\beta\gamma}$$

$$mB_{\mu\nu} = \frac{1}{2}\sqrt{-g}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}(A)$$

- If to eliminate the fields V_μ, A_μ one gets the equations of motion for massive antisymmetric tensor field $B_{\mu\nu}$
- If to eliminate the field $B_{\mu\nu}$ one gets the equations of motion for massive vector field A_μ

The theory of the massive rank-2 antisymmetric tensor field is classically equivalent to theory of massive vector field

Equivalence of the massive B_3 field model to the massive scalar field model

- Consider of the theory of the fields ϕ and $B_{\alpha\beta\gamma}$ with action

$$S[\phi, B_3] = \int d^4x \sqrt{-g(x)} (\frac{1}{8} \nabla_\alpha \phi \frac{1}{\sqrt{-g}} \epsilon^{\alpha\beta\gamma\delta} B_{\beta\gamma\delta} - \frac{1}{2} m^2 \phi^2 + \frac{1}{6} m^2 B^{\alpha\beta\gamma} B_{\alpha\beta\gamma})$$

- If to eliminate the field ϕ from the equation of motion and substitute the result into action one gets the action of massive rank-3 antisymmetric tensor field.
- If to eliminate the field $B_{\alpha\beta\gamma}$ from the equations of motion and substitute the result into action, one gets the action of massive scalar field minimally coupled to gravity.

The theory of the massive rank-3 antisymmetric tensor field is classically equivalent to the theory of massive scalar field

Gauge invariant formulation with help of Stückelberg fields

The theories of massive rank-2 and rank-3 antisymmetric fields models are not gauge theories and therefore there should be no quantization problem. Effective actions are defined by conventional path integrals

$$e^{i\Gamma_{B_2}[g_{\mu\nu}]} = \int DB_2 e^{iS[B_2]}$$

$$e^{i\Gamma_{B_3}[g_{\mu\nu}]} = \int DB_3 e^{iS[B_3]}$$

However we face the problem how to evaluate the effective actions.

- In general the effective action in quadratic theories with set of the fields Φ and action $S[\Phi]$ is given by the expression of the form $\Gamma \sim Trln(\mathcal{H})$ where $\mathcal{H} = \frac{\delta^2 S[\Phi]}{\delta\Phi\delta\Phi}$.
- The systematic methods for calculating the effective action like the Schwinger-DeWitt proper-time technique are applicable only if the operator $\mathcal{H}$ is minimal. That means that the second derivatives in operator $\mathcal{H}$ form the $\Box$, $\mathcal{H} = \mathbf{1}\Box +$ lower derivatives.
- But in the theories where the kinetic term of Lagrangian is gauge invariant and massive term violates this gauge invariance the operator $\mathcal{H}$ does not have the minimal form.

Gauge invariant formulation of massive electrodynamics

- Consider the model

$$S[A] = \int d^4x\sqrt{-g}(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2A^\mu A_\mu)$$

Kinetic term is gauge invariant, the massive term violates this symmetry. Operator associated with action is non-minimal, $g_{\mu\nu}(\Box - m^2) - \nabla_\mu\nabla_\nu$.

- Consider another model

$$S[A,C] = \int d^4x\sqrt{-g}(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2(A^\mu - \frac{1}{m}\nabla^\mu C)(A_\mu - \frac{1}{m}\nabla_\mu C))$$

Action is invariant under the transformations $\delta A_\mu = \nabla_\mu\xi$, $\delta C = -m\xi$. This is the gauge theory. In the gauge $C = 0$ one gets the model with action $S[A]$. In the gauge $\nabla^\mu A_\mu = 0$ one gets the equivalent model with minimal operator. After gauge fixing the Lagrangian takes the form

$$\frac{1}{2}A^\mu(g_{\mu\nu}(\Box - m^2) - R_{\mu\nu})A^\nu - mA^\mu\nabla_\mu C - \frac{1}{2}C\Box C.$$

Operator associated with this Lagrangian is minimal. Price for this is the Stückelberg field C.

Aim: reformulate the models of massive rank-2 and rank-3 antisymmetric tensor fields in gauge invariant form with help of the Stückelberg fields.

Model of massive rank-2 antisymmetric tensor field.

- Model with action $S[B_2]$ is equivalent to model with action $S[B_2, C_1]$, containing the vector Stückelberg field C_μ,

$$S[B_2, C_1] = \int d^4x\sqrt{-g(x)}(-\frac{1}{12}F^{\mu\nu\lambda}(B)F_{\mu\nu\lambda}(B)+\frac{1}{4}m^2(B^{\mu\nu}+\frac{1}{m}F^{\mu\nu}(C))^2)$$

  where $F_{\mu\nu}(C) = \nabla_\mu C_\nu - \nabla_\nu C_\mu$.
- Action is invariant under the gauge transformations of the field B_2 and under the shift of the field C_μ,

$$C'_\mu = C_\mu - m\xi_\mu.$$

  Also the action is invariant under the gauge transformations of the Stückelberg vector field

$$C'_\mu = C_\mu + \nabla_\mu\Lambda, \quad B'_{\mu\nu} = B_{\mu\nu}$$

- In the gauge $C_\mu = 0$ one gets the initial action $S[B_2]$, in the gauge $\nabla_\mu B^{\mu\nu} = 0$ one gets the action corresponding to the minimal operator.

Model of massive rank-3 antisymmetric tensor field.

- Model with action $S[B_3]$ is equivalent to model with action $S[B_3, C_2]$, containing the rank-2 antisymmetric tensor Stückelberg field,

$$S[B_3, C_2] = \int d^4x\sqrt{-g}(-\frac{1}{48}F^{\mu\nu\rho\sigma}(B)F_{\mu\nu\rho\sigma}(B)+\frac{m^2}{12}(B^{\mu\nu\rho}+\frac{1}{m}F^{\mu\nu\rho}(C))^2)$$

  where $F_{\mu\nu\rho}(C)$ is the strength tensor for rank-2 antisymmetric tensor field $C_{\mu\nu}$.
- Action is invariant under the gauge transformations of the field B_3 and under the shift of the field $C_{\mu\nu}$,

$$C'_{\mu\nu} = C_{\mu\nu} - m\xi_{\mu\nu}.$$

  Also the action is invariant under the gauge transformations of the Stückelberg tensor field

$$C'_{\mu\nu} = C_{\mu\nu} + \nabla_\mu\Lambda_\nu - \nabla_\nu\Lambda_\nu, \quad B'_{\mu\nu\rho} = B_{\mu\nu\rho}$$

- In the gauge $C_{\mu\nu} = 0$ one gets the initial action $S[B_3]$. In the gauge $\nabla_\mu B^{\mu\nu\rho} = 0$, $\nabla_\mu C^{\mu\nu} = 0$ one gets the action corresponding to minimal operator.

After the models under consideration are formulated with help of Stückelberg fields, they can be quantized as the gauge theories.

From general point of view such models belong to a class of gauge theories with linearly dependent generators. In principle all such theories can be quantized by Batalin-Vilkovisky method (I. Batalin, G. Vilkovisky, 1983).

However the quantization of theories with quadratic actions and Abelian linearly dependent gauge generators can be carried out much more simpler by multi-step applications of the Faddeev-Popov procedure (I.L.B, S.M. Kuzenko, 1988).

Quantization of massless rank-2 antisymmetric tensor field models on the base of the multi-step applications of the Faddeev-Popov procedure has been illustrated in my talk for the example of massless rank-2 antisymmetric tensor field model. Omitting the calculations one formulates only the final results for the effective actions.

Quantization and effective action

Effective action ${\Gamma_2}^{(m)}[g_{\mu\nu}]$ of the massive rank-2 antisymmetric tensor field model

$$\Gamma_2^{(m)}[g_{\mu\nu}] = \frac{i}{2}[\mathsf{Tr}\ln(\Box_2 + m^2) - \mathsf{Tr}\ln(\Box_1 + m^2) + \mathsf{Tr}\ln(\Box_0 + m^2)]$$

Effective action ${\Gamma_1}^{(m)}[g_{\mu\nu}]$ of the massive vector field

$$\Gamma_1^{(m)}[g_{\mu\nu}] = \frac{i}{2}[\mathsf{Tr}\ln(\Box_1 + m^2) - \mathsf{Tr}\ln(\Box_0 + m^2)]$$

Effective action $\Gamma_3^{(m)}[g_{\mu\nu}]$ of the massive rank-3 antisymmetric tensor field

$$\Gamma_3^{(m)}[g_{\mu\nu}] = \frac{i}{2}[\mathsf{Tr}\ln(\Box_3+m^2)-\mathsf{Tr}\ln(\Box_2+m^2)+\mathsf{Tr}\ln(\Box_1+m^2)-\mathsf{Tr}\ln(\Box_0+m^2)]$$

Effective action of $\Gamma_0^{(m)}[g_{\mu\nu}]$ of the massive scalar field minimally coupled to gravity

$$\Gamma_0^{(m)}[g_{\mu\nu}] = \frac{i}{2}\mathsf{Tr}\ln(\Box_0 + m^2)$$

Here the $\Box_3, \Box_2, \Box_1$ and $\Box_0$ are the d'Alembertians acting on rank-p antisymmetric tensor fields.

Quantum equivalence

Aim: to study the quantities

$$\Delta\Gamma^{(1)} = \Gamma_2^{(m)}[g_{\mu\nu}] - \Gamma_1^{(m)}[g_{\mu\nu}]$$

$$\Delta\Gamma^{(2)} = \Gamma_3^{(m)}[g_{\mu\nu}] - \Gamma_0^{(m)}[g_{\mu\nu}]$$

$$\Delta\Gamma^{(1)} = \frac{i}{2}\left[\mathsf{Tr}\ln(\Box_2 + m^2) - 2\mathsf{Tr}\ln(\Box_1 + m^2) + 2\mathsf{Tr}\ln(\Box_0 + m^2)\right]$$

$$\Delta\Gamma^{(2)} = \frac{i}{2}[\mathsf{Tr}\ln(\Box_3 + m^2) - \mathsf{Tr}\ln(\Box_2 + m^2) + \mathsf{Tr}\ln(\Box_1 + m^2) - 2\mathsf{Tr}\ln(\Box_0 + m^2)]$$

One can prove that in flat space both these quantities are exactly zeros.

We will show that they are exactly zeros in curved space under appropriate definition of functional determinants of the differential operators.

Definition of functional determinants

- The fields ϕ, A_μ, B_2, B_3 are considered as the p -forms $(p = 0, 1, 2, 3)$.
- Euclidean formulation
- The operators $\Box_p$ are the Laplacians acting on p -forms
- Effective action is defined in terms of generalized $\zeta-$ functions associated with the operators $-\Box_p + m^2$ in the form

$$\zeta_p(s, m) = \sum_{\lambda_i \neq 0} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-tm^2} \mathrm{Tr}(e^{t\Box_p} - \mathcal{P}_p)$$

  where $\mathcal{P}_p$ is the projector operator onto the space of the zero modes of the operator $\Box_p$ (S. Rosenberg, Laplacians on Riemannian Manifolds, 1977).
- Effective action associated with the operator $-\Box_p + m^2$ is written as follows

$$\mathrm{lnDet}(-\Box_p + m^2) = -(\zeta_p'(0, m) + \mathrm{ln}(\mu^2)\zeta_p(0, m))$$

  where μ is the arbitrary mass scale parameter. The effective action defined this way is finite for any p.

In the framework of the above definition

$$\Delta\Gamma^{(1)} = (\zeta_2'(0,m) - 2\zeta_1'(0,m) + 2\zeta_0'(0,m))$$
$$+\ln(\mu^2)(\zeta_2(0,m) - 2\zeta_1(0,m) + 2\zeta_0(0,m))$$

and

$$\Delta\Gamma^{(2)} = (\zeta_3'(0,m) - \zeta_2'(0,m) + \zeta_1'(0,m) - 2\zeta_0'(0,m))$$
$$+\ln(\mu^2)(\zeta_3(0,m) - \zeta_2(0,m) + \zeta_1(0,m) - 2\zeta_0(0,m)).$$

Hodge duality between p -forms and $4-p$ -forms leads to identities between the operators $\Box_p$ with different p. It allows us to prove the identity

$$\zeta_p(s,m) = \zeta_{(4-p)}(s,m).$$

Using this identity one gets

$$\Delta\Gamma^{(2)} = -\Delta\Gamma^{(1)}.$$

Therefore it is sufficient to study only $\Delta\Gamma^{(1)}$.

Quantum equivalence

Applications of the identities for the effective action

- Identity

$$\sum_{p=0}^{4}(-1)^p p\,\zeta_p(s,m) = 2(\zeta_2(s,m) - 2\zeta_1(s,m) + 2\zeta_0(s,m))$$

- $\Delta\Gamma^{(1)}$ in terms of this identity

$$\Delta\Gamma^{(1)} = \frac{1}{2}[\sum_{p=0}^{4}(-1)^p p\,\zeta'_p(0,m) + \ln(\mu^2)\sum_{p=0}^{4}(-1)^p p\,\zeta_p(0,m)]$$

- Expansion for mass dependent ζ -function

$$\zeta_p(s,m) = \sum_{n=0}^{\infty}\frac{(-m^2)^n\Gamma(n+s)}{n!\Gamma(s)}\zeta_p(s+n,0).$$

- Identity

$$\sum_{p=0}^{4}(-1)^p p\,\zeta_p(s,m) = \sum_{n=0}^{\infty}\frac{(-m^2)^n\Gamma(n+s)}{n!\Gamma(s)}\sum_{p=0}^{4}(-1)^p p\,\zeta_p(s+n,0)\ .$$

Quantum equivalence

Ray and Singer identity (D. Ray, I. Singer, 1971, 1973).

$$\sum_{p=0}^{4}(-1)^p p\, \zeta_p(s) = 0\ .$$

Identity for mass dependent ζ - functions

$$\sum_{p=0}^{4}(-1)^p p\, \zeta_p(s,m) = 0\ .$$

General result

$$\Delta\Gamma^{(1)} = 0\ , \quad \Delta\Gamma^{(2)} = 0\ .$$

Effective action for massive second rank antisymmetric field model exactly coincides with effective action for massive vector field model. These theories are quantum equivalent.
Effective action for massive third rank antisymmetric field model exactly coincides with effective action for massive scalar field model with minimal coupling to gravity. These theories are quantum equivalent.

Effective action on the base of the another definition of ζ -function.

Usually the one-loop effective action in quantum field theory is defined with help of Schwinger-DeWitt technique. It is equivalent to define ζ -function including the zero-modes in the form

$$\tilde{\zeta}_p(s,m) = \sum_{\lambda_i} \lambda_i^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} e^{-tm^2} \mathrm{Tr}(e^{t\Box_p})$$

One can show that the relation $\Delta\Gamma^{(2)} = -\Delta\Gamma^{(1)}$ still takes place. However the relation $\sum_{p=0}^{4}(-1)^p p\, \zeta_p(s,m) = 0$ is not valid now.
Let $\tilde{\Gamma}$ is the effective action defined with help of $\tilde{\zeta}$ -function. Then the corresponding $\Delta\tilde{\Gamma}^{(1)}$ has the form

$$\Delta\tilde{\Gamma}^{(1)} = \ln(\frac{\mu^2}{m^2})\frac{1}{16\pi^2}\int d^4x\sqrt{-g(x)}[b_2 - m^2 b_1 + \frac{m^4}{2}b_0] \ ,$$

where

$$b_n = a_n^{(2)} - 2a_n^{(1)} + 2a_n^{(0)}, \quad n = 0,1,2$$

The DeWitt coefficients $a_n^{(p)}, n = 0,1,2; p = 01,1,2$ are known from literature.

Calculations of the DeWitt coefficients

- $b_0 = 0$
- $b_1 = 0$
  Difference of effective actions for classically equivalent theories is mass independent
-
$$b_2 = \frac{1}{2}[R^2_{\mu\nu\lambda\rho} - 4R^2_{\mu\nu} + R^2].$$

As a result

$$\Delta\tilde{\Gamma}^{(1)} = \ln(\frac{\mu^2}{m^2})\chi,$$

where χ is the Gauss-Bonnet topological invariant.

The effective actions of classically equivalent theories, defined in terms of ζ -function without zero-modes and in terms of ζ -function with zero-modes, are equal to each other up to the topological invariant. The expectation values of the corresponding energy-momentum tensors coincide

Summary

- Gauge invariant formulation of massive second and third ranks antisymmetric tensor field models with Stückelberg fields.
- Quantization of the models and structure of the effective actions in terms of functional determinants of appropriate differential operators acting on $p-$ forms.
- Definitions of the effective actions in terms of generalized $\zeta-$ functions.
- Proof of quantum equivalence of massive second rank antisymmetric tensor field model to massive vector field model and massive third rank tensor field model to massive scalar field model with minimal coupling.

THANK YOU VERY MUCH!