Eigenvalue asymptotics for operators associated to mirror curves

based on joint works with A. Laptev and L. A. Takhtajan:

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Overview

- A functional difference operator associated to $\mathbb{F}_0$
- Eigenvalue asumptotics of this operator
- Proof method
- Which other operators can this proof be applied to?

A conjecture to construct a quantum operator for toric del Pezzo Calabi–Yau

Construction:

Associated to X is a unique curve

$$W_X(\mathrm{e}^x, \mathrm{e}^p) = 0\,.$$

The equation be quantised

$$x \to Q, p \to P = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$$

which promotes it to a functional difference operator H.

Example: Hirzebruch surface $\mathbb{F}_0$

Associated to $\mathbb{F}_0$ is

$$W_{\mathbb{F}_0}(\mathrm{e}^x, \mathrm{e}^p) = e^x + \mathrm{e}^{-x} + \mathrm{e}^p + \mathrm{e}^{-p}.$$

Quantisation leads to the operator

$$H = \mathrm{e}^Q + \mathrm{e}^{-Q} + \mathrm{e}^P + \mathrm{e}^{-P}$$

which is a functional difference operator.

Conjecture[1]

The operator $G = H^{-1}$ is trace-class and its spectrum relates to the enumerative geometry of X.

[1] A. Grassi, Y. Hatsuda and M. Mariño (2014)

The Weyl operators

Let P and Q be quantum-mechanical momentum and position operators on $L^2(\mathbb{R})$

$$(P\psi)(x) = \mathrm{i}\psi'(x)\,, \quad (Q\psi)(x) = x\psi(x)\,, \quad [P, Q] = \mathrm{i}\,.$$

The corresponding Weyl operators ($b > 0$) are defined as

$$U = \mathrm{e}^{-bP}\,, \quad V = \mathrm{e}^{2\pi bQ}\,, \quad UV = q^2 VU\,, \quad q = \mathrm{e}^{\mathrm{i}\pi b^2}$$

with domains

$$\mathrm{dom}(U) = \left\{\psi \in L^2(\mathbb{R}) :\ \mathrm{e}^{-2\pi bk}\widehat{\psi}(k) \in L^2(\mathbb{R})\right\},$$
$$\mathrm{dom}(V) = \left\{\psi \in L^2(\mathbb{R}) :\ \mathrm{e}^{2\pi bx}\psi(x) \in L^2(\mathbb{R})\right\},$$

with the Fourier transform

$$\widehat{\psi}(k) = (\mathcal{F}\psi)(k) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}kx}\psi(x)\,\mathrm{d}x\,.$$

It holds that $\mathcal{F}U\mathcal{F}^{-1} = V^{-1}$ and in coordinate representation

$$(U\psi)(x) = \psi(x + \mathrm{i}b)\,, \quad (V\psi)(x) = \mathrm{e}^{2\pi bx}\psi(x)\,.$$

The operator H_0

The symmetric operator

$$H_0 = U + U^{-1} = 2\cosh(bP)\,,$$

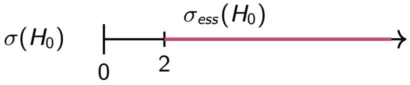
acts formally as

$$(H_0\psi)(x) = \psi(x + \mathrm{i}b) + \psi(x - \mathrm{i}b)$$

on $L^2(\mathbb{R})$. We can also interpret H_0 as a differential operator of infinite order

$$(H_0\psi)(x) = 2\sum_{n=0}^{\infty} \frac{b^{2n}}{(2n)!} P^{2n}\psi(x) = -2\sum_{n=0}^{\infty} \frac{b^{2n}}{(2n)!} \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\psi(x).$$

Since $H \geq 2$ it admits a self-adjoint Friedrichs extension.



The operator H

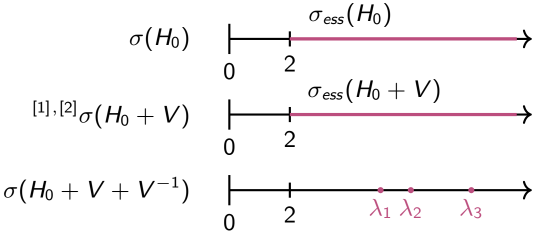
The symmetric operator

$$H = U + U^{-1} + V + V^{-1} = 2\cosh(bP) + 2\cosh(2\pi bQ)\,,$$

acts formally as

$$(H\psi)(x) = \psi(x + \mathrm{i}b) + \psi(x - \mathrm{i}b) + 2\cosh(2\pi bx)\psi(x)\,,$$

on $L^2(\mathbb{R})$. Since $H \geq 2$ it admits a self-adjoint Friedrichs extension.



We write $H = H_0 + W$ with the potential $W = 2\cosh(2\pi bx)$.

[1] R. Kashaev (2001)
[2] L. A. Takhtajan and L. D. Faddeev (2015)

Main results

Theorem (A. Laptev, LS and L. A. Takhtajan (2016))

The eigenvalues λ_j of the operator H have the asymptotic behaviour

$$\operatorname{Tr}(\lambda - H)_+ = \sum_{j\geq 1}(\lambda - \lambda_j)_+ = \frac{\lambda \log^2 \lambda}{(\pi b)^2} + O(\lambda \log \lambda) \quad \textit{as} \quad \lambda \to \infty.$$

These are Weyl-type results that link the asymptotical behaviour of quantum mechanical expressions to classical phase space integrals. Let

$$\sigma(k,x) = 2\cosh(2\pi bk) + 2\cosh(2\pi bx)$$

be the total symbol of H. The term $\lambda \log^2 \lambda/(\pi b)^2$ is precisely the leading term in

$$\iint_{\mathbb{R}^2} (\lambda - \sigma(k,x))_+ \, \mathrm{d}k \, \mathrm{d}x \quad \text{as} \quad \lambda \to \infty.$$

Weyl-type asymptotics

$$\lim_{\lambda\to\infty} \frac{1}{\lambda \log^2 \lambda} \sum_{j\geq 1}(\lambda - \lambda_j)_+ = \lim_{\lambda\to\infty} \frac{1}{\lambda \log^2 \lambda} \iint_{\mathbb{R}^2} (\lambda - \sigma(x,k))_+ \, \mathrm{d}k \, \mathrm{d}x \, .$$

Main results

Corollary (A. Laptev, LS and L. A. Takhtajan (2016))

The number $N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j < \lambda\}$ of eigenvalues of H less than λ satisfies

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\log^2 \lambda} = \frac{1}{(\pi b)^2}.$$

The term $\log^2 \lambda/(\pi b)^2$ is precisely the leading term of the phase volume of

$$\{(k,x) \in \mathbb{R}^2 : \sigma(k,x) \le \lambda\} \quad \text{as} \quad \lambda \to \infty.$$

Weyl-type asymptotics

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\log^2 \lambda} = \lim_{\lambda \to \infty} \frac{1}{\log^2 \lambda} \iint_{\sigma(k,x) \le \lambda} 1 \, \mathrm{d}k \, \mathrm{d}x.$$

Main results

Corollary (A. Laptev, LS and L. A. Takhtajan (2016))

The number $N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j < \lambda\}$ *of eigenvalues of* H *less than* λ *satisfies*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\log^2 \lambda} = \frac{1}{(\pi b)^2}\,.$$

This follows by 'differentiating' the Riesz mean. To argue rigorously note that for $h > 0$

$$N(\lambda) \leq \frac{1}{h}\Big(\sum_{j\geq 1}(\lambda + h - \lambda_j)_+ - \sum_{j\geq 1}(\lambda - \lambda_j)_+\Big)$$

Using the results of the theorem and the convexity of $\lambda \mapsto \lambda^2 \log \lambda$ one can show that

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\log^2 \lambda} \leq \frac{1}{(\pi b)^2}$$

by choosing h suitably depending on λ. A corresponding lower bound follows from

$$N(\lambda) \geq \frac{1}{h}\Big(\sum_{j\geq 1}(\lambda - \lambda_j)_+ - \sum_{j\geq 1}(\lambda - h - \lambda_j)_+\Big)$$

Main results

Corollary (A. Laptev, LS and L. A. Takhtajan (2016))

H^{-1} is trace-class.

This follows immediately from the asymptotics of $N(\lambda)$

$$\sum_{j\geq 1} \frac{1}{\lambda_j} = \int_2^\infty \frac{1}{\lambda}\, \mathrm{d}N(\lambda) = \frac{N(\lambda)}{\lambda}\bigg|_2^\infty + \int_2^\infty \frac{N(\lambda)}{\lambda^2}\, \mathrm{d}\lambda < \infty.$$

This provides a simple proof of the trace-class property without explicit computation of the integral kernel of H^{-1}, as done previously[1].

[1] R. Kashaev and M. Mariño (2015)

Proof of the Theorem

Coherent state transform

Consider the normalised Gaussian function $g(x) = (a/\pi)^{1/4}\mathrm{e}^{-\frac{a}{2}x^2}$ with some $a > 0$. For $\psi \in L^2(\mathbb{R})$ the classical coherent state transform is given by

Coherent state transform

$$\widetilde{\psi}(k,y) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi\mathrm{i}kx} g(x-y)\psi(x)\,\mathrm{d}x\,.$$

Plancherel's theorem shows that

$$\int_{\mathbb{R}} |\widetilde{\psi}(k,y)|^2\,\mathrm{d}k = (|\psi|^2 * |g|^2)(y)\,,$$
$$\int_{\mathbb{R}} |\widetilde{\psi}(k,y)|^2\,\mathrm{d}y = (|\widehat{\psi}|^2 * |\widehat{g}|^2)(k)\,.$$

Note that $\widetilde{\psi} \in L^2(\mathbb{R}^2)$ with $\iint_{\mathbb{R}^2} |\widetilde{\psi}(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y = \|\psi\|^2$. We can also write

$$\widetilde{\psi}(k,y) = \langle e_{k,y}, \psi\rangle \quad \text{with} \quad e_{k,y}(x) = \mathrm{e}^{2\pi\mathrm{i}kx} g(x-y)\,.$$

Coherent state representation: H_0

We are trying to express $\langle \psi, H_0 \psi \rangle$ in terms of $\widetilde{\psi}$, ideally as

$$\langle \psi, H_0 \psi \rangle = \int_{\mathbb{R}} 2\cosh(2\pi bk) |\widehat{\psi}(k)|^2 \, \mathrm{d}k = \iint_{\mathbb{R}^2} h(k) |\widetilde{\psi}(k,y)|^2 \, \mathrm{d}k \, \mathrm{d}y \,.$$

Since

$$\int_{\mathbb{R}} |\widetilde{\psi}(k,y)|^2 \, \mathrm{d}y = (|\widehat{\psi}|^2 * |\widehat{g}|^2)(k)$$

we have that

$$\iint_{\mathbb{R}^2} h(k) |\widetilde{\psi}(k,y)|^2 \, \mathrm{d}k \, \mathrm{d}y = \int_{\mathbb{R}} (h * |\widehat{g}|^2)(k) |\widehat{\psi}(k)|^2 \, \mathrm{d}k \,.$$

So we need to find h such that

$$(h * |\widehat{g}|^2)(k) = 2\cosh(2\pi bk) \,.$$

This can be solved explicitly, in fact

$$(2d_1 \cosh(2\pi b \,\cdot) * |\widehat{g}|^2)(k) = 2\cosh(2\pi bk)$$

with $d_1 = \mathrm{e}^{-(\pi b^2)/a}$. As a consequence

$$\langle \psi, H_0 \psi \rangle = \iint_{\mathbb{R}^2} 2d_1 \cosh(2\pi bk) |\widetilde{\psi}(k,y)|^2 \, \mathrm{d}k \, \mathrm{d}y \,.$$

Coherent state representation: W

We are trying to express $\langle \psi, W\psi \rangle$ in terms of $\widetilde{\psi}$, ideally as

$$\langle \psi, W\psi \rangle = \int_{\mathbb{R}} 2\cosh(2\pi by)|\psi(y)|^2 \,\mathrm{d}y = \iint_{\mathbb{R}^2} w(y)|\widetilde{\psi}(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y$$

Since

$$\int_{\mathbb{R}} |\widetilde{\psi}(k,y)|^2 \,\mathrm{d}k = (|\psi|^2 * |g|^2)(y)$$

we have that

$$\iint_{\mathbb{R}^2} w(y)|\widetilde{\psi}(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y = \int_{\mathbb{R}} (w * |g|^2)(y)|\psi(y)|^2 \,\mathrm{d}y\,.$$

So we need to find w such that

$$(w * |g|^2)(y) = 2\cosh(2\pi by)\,.$$

This can be solved explicitly, in fact

$$(2d_2\cosh(2\pi b\,\cdot) * |g|^2)(y) = 2\cosh(2\pi bk)$$

with $d_2 = \mathrm{e}^{-ab^2/4}$. As a consequence

$$\langle \psi, W\psi \rangle = \iint_{\mathbb{R}^2} 2d_2\cosh(2\pi by)|\widetilde{\psi}(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y\,.$$

Coherent state representation: $H = H_0 + W$

Using the two identities

$$(2d_1 \cosh(2\pi b\,\cdot) * |\widehat{g}|^2)(k) = 2\cosh(2\pi bk)\,, \quad (2d_2 \cosh(2\pi b\,\cdot) * |g|^2)(y) = 2\cosh(2\pi bk)$$

we have thus established the following.

Coherent state representation of H

$$\langle \psi, H\psi \rangle = \iint_{\mathbb{R}^2} (2d_1 \cosh(2\pi bk) + 2d_2 \cosh(2\pi by))|\widetilde{\psi}(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y.$$

This allows to write the operator H in terms of its full symbol $\sigma(k,y)$ up to the two constants $d_1, d_2 < 1$.

Upper bound on the Riesz mean

From $\lambda_j = \langle \psi_j, H\psi_j \rangle$ and the coherent state representation we obtain

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \sum_{j\geq 1}\left(\lambda - \iint_{\mathbb{R}^2}\big(2d_1\cosh(2\pi bk) + 2d_2\cosh(2\pi by)\big)|\widetilde{\psi}_j(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y\right)_+ .$$

Since $\iint_{\mathbb{R}^2}|\widetilde{\psi}_j(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y = \|\psi_j\|^2 = 1$ Jensen's inequality yields

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \iint_{\mathbb{R}^2}\big(\lambda - 2d_1\cosh(2\pi bk) - 2d_2\cosh(2\pi by)\big)_+ \sum_{j\geq 1}|\widetilde{\psi}_j(k,y)|^2\,\mathrm{d}k\,\mathrm{d}y\,.$$

Using $e_{k,y}(x) = \mathrm{e}^{2\pi\mathrm{i}kx}g(x-y)$ and the fact that ψ_j form an orthonormal basis in $L^2(\mathbb{R})$,

$$\sum_{j\geq 1}|\widetilde{\psi}_j(k,y)|^2 = \sum_{j\geq 1}|\langle e_{k,y},\psi_j\rangle|^2 = \|e_{k,y}\|^2 = 1 \quad \text{for all} \quad k,y\in\mathbb{R},$$

and we arrive at the upper bound

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j)_+ &\leq \iint_{\mathbb{R}^2}\big(\lambda - 2d_1\cosh(2\pi bk) - 2d_2\cosh(2\pi by)\big)_+\,\mathrm{d}k\,\mathrm{d}y\\ &= \frac{\lambda\log^2\lambda}{(\pi b)^2} + O(\lambda\log\lambda)\,.\end{aligned}$$

Lower bound on the Riesz mean

From $\iint_{\mathbb{R}^2} |\widetilde{\psi}_j(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y = 1$ we obtain

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \sum_{j\geq 1}(\lambda-\lambda_j)_+ \iint_{\mathbb{R}^2} |\widetilde{\psi}_j(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y\,.$$

Using $\widetilde{\psi}_j(k,y) = \langle e_{k,y}, \psi_j\rangle$ we can write

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \iint_{\mathbb{R}^2} \sum_{j\geq 1}(\lambda-\lambda_j)_+ \langle e_{k,y}|\psi_j\rangle\langle\psi_j|e_{k,y}\rangle \,\mathrm{d}k\,\mathrm{d}y\,.$$

Since $\sum_{j\geq 1}\langle e_{k,y}|\psi_j\rangle\langle\psi_j|e_{k,y}\rangle = \langle e_{k,y}, e_{k,y}\rangle = \|g\|^2 = 1$ Jensen's inequality yields

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2}\Big(\lambda - \sum_{j\geq 1}\lambda_j\langle e_{k,y}|\psi_j\rangle\langle\psi_j|e_{k,y}\rangle\Big)_+ \mathrm{d}k\,\mathrm{d}y = \iint_{\mathbb{R}^2}\big(\lambda - \langle e_{k,y}, He_{k,y}\rangle\big)_+ \mathrm{d}k\,\mathrm{d}y.$$

Using again Plancherel's theorem we note that

$$\langle e_{k,y}, H_0 e_{k,y}\rangle = (2\cosh(2\pi b\,\cdot) * |\widehat{g}|^2)(k) = \frac{1}{d_1} 2\cosh 2\pi bk$$

$$\langle e_{k,y}, We_{k,y}\rangle = (W * |g|^2)(y) = \frac{1}{d_2} 2\cosh 2\pi by\,.$$

Lower bound on the Riesz mean

From $\iint_{\mathbb{R}^2} |\widetilde{\psi}_j(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y = 1$ we obtain

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \sum_{j\geq 1}(\lambda-\lambda_j)_+ \iint_{\mathbb{R}^2} |\widetilde{\psi}_j(k,y)|^2 \,\mathrm{d}k\,\mathrm{d}y\,.$$

Using $\widetilde{\psi}_j(k,y) = \langle e_{k,y}, \psi_j\rangle$ we can write

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \iint_{\mathbb{R}^2} \sum_{j\geq 1}(\lambda-\lambda_j)_+ \langle e_{k,y}|\psi_j\rangle\langle\psi_j|e_{k,y}\rangle \,\mathrm{d}k\,\mathrm{d}y\,.$$

Since $\sum_{j\geq 1}\langle e_{k,y}|\psi_j\rangle\langle\psi_j|e_{k,y}\rangle = \langle e_{k,y}, e_{k,y}\rangle = \|g\|^2 = 1$ Jensen's inequality yields

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2}\Big(\lambda - \sum_{j\geq 1}\lambda_j\langle e_{k,y}|\psi_j\rangle\langle\psi_j|e_{k,y}\rangle\Big)_+ \mathrm{d}k\,\mathrm{d}y = \iint_{\mathbb{R}^2}\big(\lambda - \langle e_{k,y}, He_{k,y}\rangle\big)_+ \mathrm{d}k\,\mathrm{d}y.$$

We arrive at the lower bound

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2}\left(\lambda - \frac{2}{d_1}\cosh(2\pi bk) - \frac{2}{d_2}\cosh(2\pi by)\right)_+ \mathrm{d}k\,\mathrm{d}y$$

$$= \frac{\lambda\log^2\lambda}{(\pi b)^2} + O(\lambda\log\lambda)\,.$$

Generalisation to $X = \mathbb{F}_0$

The proof generalises to

$$H(\zeta) = U + U^{-1} + V + \zeta V^{-1}$$

with a parameter $\zeta > 0$.

Theorem (A. Laptev, LS and L. A. Takhtajan (2016))

The eigenvalues λ_j of the operator $H(\zeta)$ have the asymptotic behaviour

$$\mathrm{Tr}(\lambda - H)_+ = \sum_{j\geq 1}(\lambda - \lambda_j)_+ = \frac{\lambda \log^2 \lambda}{(\pi b)^2} + O(\lambda \log \lambda) \quad \textit{as} \quad \lambda \to \infty.$$

Corollary (A. Laptev, LS and L. A. Takhtajan (2016))

The number $N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j < \lambda\}$ of eigenvalues of $H(\zeta)$ less than λ satisfies

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\log^2 \lambda} = \frac{1}{(\pi b)^2}\,.$$

In particular $H(\zeta)^{-1}$ is trace-class.

Where can this proof be applied?

Recap of the proof idea

For the upper bound we derived a coherent state representation. This required to solve

$$(h * |\widehat{g}|^2)(k) = 2\cosh(2\pi bk)\,, \quad (w * |g|^2)(y) = 2\cosh(2\pi by)$$

which allowed us to obtain the upper bound

$$\sum_{j\geq 1}(\lambda - \lambda_j)_+ \leq \iint_{\mathbb{R}^2} \big(\lambda - h(k) - w(y)\big)_+ \, \mathrm{d}k\, \mathrm{d}y\,.$$

For the lower bound, we computed

$$\langle e_{k,y}, H_0 e_{k,y}\rangle = (2\cosh(2\pi b\,\cdot) * |\widehat{g}|^2)(k)\,, \quad \langle e_{k,y}, W e_{k,y}\rangle = (W * |g|^2)(y)$$

which allowed us to obtain the lower bound

$$\sum_{j\geq 1}(\lambda - \lambda_j)_+ \geq \iint_{\mathbb{R}^2} \Big(\lambda - (2\cosh(2\pi b\,\cdot) * |\widehat{g}|^2)(k) - (W * |g|^2)(y)\Big)_+ \, \mathrm{d}k\, \mathrm{d}y.$$

Generalisation to Seiberg–Witten curve of $\mathcal{N} = 2$ Yang–Mills theory

The operator[1]

$$H = U + U^{-1} + x^{2N}$$

is given by the following formal functional-difference expression

$$(H\psi)(x) = \psi(x + \mathrm{i}b) + \psi(x - \mathrm{i}b) + x^{2N}\psi(x)\,.$$

and its spectrum can be proven to be discrete.

For an upper bound we note that a solution to

$$(w * |g|^2)(y) = y^{2N}$$

is given by

$$w(y) = y^{2N} + q(y)$$

with a polynomial q of order $2N - 2$. We thus obtain the

$$\sum_{j\geq 1}(\lambda - \lambda_j)_+ \leq \iint_{\mathbb{R}^2} \left(\lambda - 2d_1 \cosh(2\pi bk) - y^{2N} - q(y)\right)_+ \mathrm{d}k\,\mathrm{d}y$$
$$= \frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}} \log\lambda + O(\lambda^{\frac{2N+1}{2N}})\,.$$

[1] A. Grassi and M. Mariño (2019)

Generalisation to Seiberg–Witten curve of $\mathcal{N} = 2$ Yang–Mills theory

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$$(H\psi)(x) = \psi(x + \mathrm{i}b) + \psi(x - \mathrm{i}b) + x^{2N}\psi(x)\,.$$

and its spectrum can be proven to be discrete.

For a lower bound we can compute that

$$\langle e_{k,y}, |\cdot|^{2N} e_{k,y}\rangle = (|\cdot|^{2N} * |g|^2)(y) = y^{2N} + p(y)$$

with a polynomial p of order $2N - 2$. We thus obtain

$$\begin{aligned}\sum_{j\geq 1}(\lambda - \lambda_j)_+ &\geq \iint_{\mathbb{R}^2} \left(\lambda - \frac{2}{d_1}\cosh(2\pi bk) - y^{2N} - p(y)\right)_+ \mathrm{d}k\,\mathrm{d}y \\ &= \frac{2}{\pi b}\frac{2N}{2N+1}\lambda^{\frac{2N+1}{2N}}\log\lambda + O(\lambda^{\frac{2N+1}{2N}})\,.\end{aligned}$$

[1] A. Grassi and M. Mariño (2019)

Generalisation to Seiberg–Witten curve of $\mathcal{N} = 2$ Yang–Mills theory

The operator

$$H = U + U^{-1} + x^{2N} + r(x)$$

with $r(x) = O(x^{2N-\varepsilon})$ is given by the following formal functional-difference expression

$$(H\psi)(x) = \psi(x + \mathrm{i}b) + \psi(x - \mathrm{i}b) + x^{2N}\psi(x) + r(x)\psi(x)\,.$$

and its spectrum can be proven to be discrete.

Theorem (A. Laptev, LS and L. A. Takhtajan (2019))

The eigenvalues λ_j of the operator H satisfy

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^{\frac{2N+1}{2N}}\log\lambda} \sum_{j\geq 1}(\lambda - \lambda_j)_+ = \frac{2}{\pi b}\frac{2N}{2N+1}\,,$$

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{1}{2N}}\log\lambda} = \frac{2}{\pi b}\,.$$

Again, these results can be identified as Weyl-type asymptotics and H^{-1} is trace-class. Recently[2] the results were extended to potentials $W(x) = |x|^p \mathrm{e}^{|x|^\beta}$.

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[1] Y. W. Qiu (2023)

Generalisation to $X = \mathbb{P}_2$

The operator[1]

$$H = U + V + q^{-mn} U^{-m} V^{-n}$$

is given by the following formal functional-difference expression

$$(H\psi)(x) = \psi(x + \mathrm{i}b) + \mathrm{e}^{2\pi bx}\psi(x) + q^{mn}\mathrm{e}^{-2\pi nbx}\psi(x - m\mathrm{i}b)$$

and its spectrum can be proven to be discrete.

For an upper bound a direct computation yields the coherent state representation

$$\begin{aligned}\langle \psi, q^{-mn} U^{-m} V^{-n} \psi \rangle &= \langle V^{-n/2}\psi, U^{-m} V^{-n/2}\psi \rangle \\ &= d_1^{m^2} d_2^{n^2} \iint_{\mathbb{R}^2} \mathrm{e}^{2\pi b(mk - ny)} |\widetilde{\psi}(k, y)|^2 \, \mathrm{d}k \, \mathrm{d}y\end{aligned}$$

which allows us to obtain the upper bound

$$\begin{aligned}\sum_{j \geq 1} (\lambda - \lambda_j)_+ &\leq \iint_{\mathbb{R}^2} \left(\lambda - d_1 \mathrm{e}^{-2\pi bk} - d_2 \mathrm{e}^{2\pi by} - d_1^{m^2} d_2^{n^2} \mathrm{e}^{2\pi b(mk - ny)}\right)_+ \mathrm{d}k \, \mathrm{d}y \\ &= \frac{(m + n + 1)^2}{2mn} \lambda \log^2 \lambda + O(\lambda \log \lambda) \,.\end{aligned}$$

[1] A. Grassi, Y. Hatsuda and M. Mariño (2014)

Generalisation to $X = \mathbb{P}_2$

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$$H = U + V + q^{-mn} U^{-m} V^{-n}$$

is given by the following formal functional-difference expression

$$(H\psi)(x) = \psi(x + \mathrm{i}b) + \mathrm{e}^{2\pi bx}\psi(x) + q^{mn}\mathrm{e}^{-2\pi nbx}\psi(x - m\mathrm{i}b)$$

and its spectrum can be proven to be discrete.

For a lower bound a direct computation yields

$$\begin{aligned} \langle e_{k,y}, q^{-mn} U^{-m} V^{-n} e_{k,y} \rangle &= \langle V^{-n/2} e_{k,y}, U^{-m} V^{-n/2} e_{k,y} \rangle \\ &= \frac{1}{d_1^m d_2^n} \mathrm{e}^{2\pi b(mk - ny)} \end{aligned}$$

which allows us to obtain the lower bound

$$\begin{aligned} \sum_{j \geq 1} (\lambda - \lambda_j)_+ &\geq \iint_{\mathbb{R}^2} \left(\lambda - \frac{1}{d_1} \mathrm{e}^{-2\pi bk} - \frac{1}{d_2} \mathrm{e}^{2\pi by} - \frac{1}{d_1^{m^2} d_2^{n^2}} \mathrm{e}^{2\pi b(mk - ny)} \right)_+ \mathrm{d}k \, \mathrm{d}y \\ &= \frac{(m + n + 1)^2}{2mn} \lambda \log^2 \lambda + O(\lambda \log \lambda) \,. \end{aligned}$$

[1] A. Grassi, Y. Hatsuda and M. Mariño (2014)

Generalisation to $X = \mathbb{P}_2$

The operator

$$H = U + V + q^{-mn} U^{-m} V^{-n}$$

is given by the following formal functional-difference expression

$$(H\psi)(x) = \psi(x + \mathrm{i}b) + \mathrm{e}^{2\pi bx}\psi(x) + q^{mn}\mathrm{e}^{-2\pi nbx}\psi(x - m\mathrm{i}b)$$

and its spectrum can be proven to be discrete.

Theorem (A. Laptev, LS and L. A. Takhtajan (2016))

The eigenvalues λ_j of the operator H satisfy

$$\lim_{\lambda\to\infty} \frac{1}{\lambda \log^2 \lambda} \sum_{j\geq 1} (\lambda - \lambda_j)_+ = \frac{c_{m,n}}{(\pi b)^2}\,,$$

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\log^2 \lambda} = \frac{c_{m,n}}{(\pi b)^2}\,,$$

with $c_{m,n} = \dfrac{(m+n+1)^2}{2mn}$.

Again, these results can be identified as Weyl-type asymptotics and H^{-1} is trace-class.

Further generalisation

Consider the general operator

$$H = \sum_{m,n \in \mathbb{Z}} a_{m,n} q^{-mn} U^{-m} V^{-n}$$

with $a_{m,n} \geq 0$ and $a_{m,n} = 0$ for all but finitely many $m, n \in \mathbb{Z}$. If H contains at least one negative and one positive power of both U and V, it can be proven to have discrete spectrum. Furthermore its inverse H^{-1} is trace-class.

Theorem

Under the above assumptions H has discrete spectrum and its Riesz mean and eigenvalue counting function satisify Weyl-type asymptotics. In particular H^{-1} is trace-class.

Thank you for your attention!