

# Free-field dispersion relations as a guide to heavy-quark physics

Stephan Dürr



University of Wuppertal  
Jülich Supercomputing Center

work with G. Koutsou and J. Weber

MITP workshop – 11 March 2023

# Dirac operator roadmap (pedestrian perspective)



Wilson Dirac operator:

$$D_W(x, y) = \sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{std}}(x, y) - \frac{a}{2} \Delta^{\text{std}}(x, y) + m_0 \delta_{x, y} - \frac{c_{\text{SW}}}{2} \sum_{\mu < \nu} \sigma_{\mu\nu} F_{\mu\nu} \delta_{x, y}$$

Brillouin Dirac operator:

$$D_B(x, y) = \sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{iso}}(x, y) - \frac{a}{2} \Delta^{\text{bri}}(x, y) + m_0 \delta_{x, y} - \frac{c_{\text{SW}}}{2} \sum_{\mu < \nu} \sigma_{\mu\nu} F_{\mu\nu} \delta_{x, y}$$

$\sigma_{\mu\nu} = \frac{i}{2} [\gamma_{\mu}, \gamma_{\nu}]$ ,  $F_{\mu\nu}$  the hermitean clover-leaf field-strength tensor, separate  $m_0, c_{\text{SW}}$

# Brillouin operator details

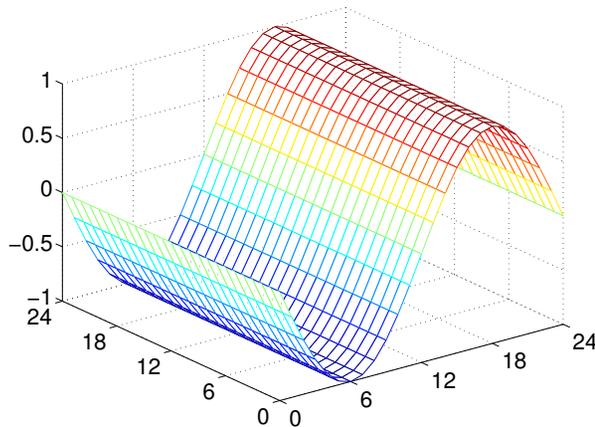
## • 3 options for (covariant) Nabla

Standard Derivative:  $\hat{\nabla}_x = i \sin(k_1)$

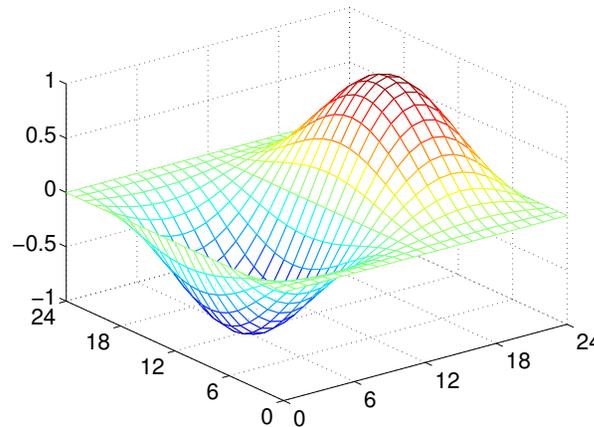
Brillouin Derivative:  $\hat{\nabla}_x = i \sin(k_1) [\cos(k_2) + 1] [\cos(k_3) + 1] [\cos(k_4) + 1] / 8$

Isotropic Derivative:  $\hat{\nabla}_x = i \sin(k_1) [\cos(k_2) + 2] [\cos(k_3) + 2] [\cos(k_4) + 2] / 27$

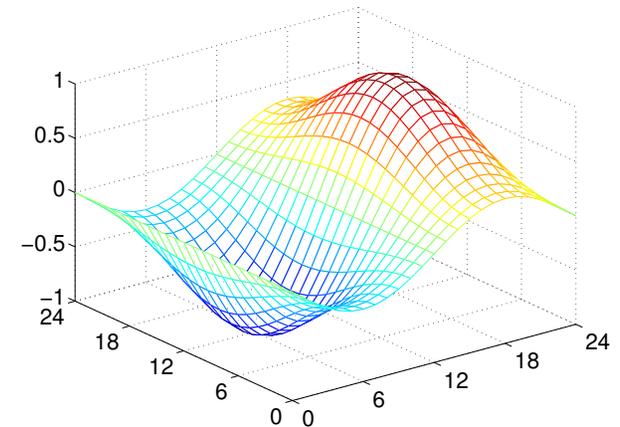
2D: der\_std for L=24



2D: der\_bri for L=24



2D: der\_iso for L=24



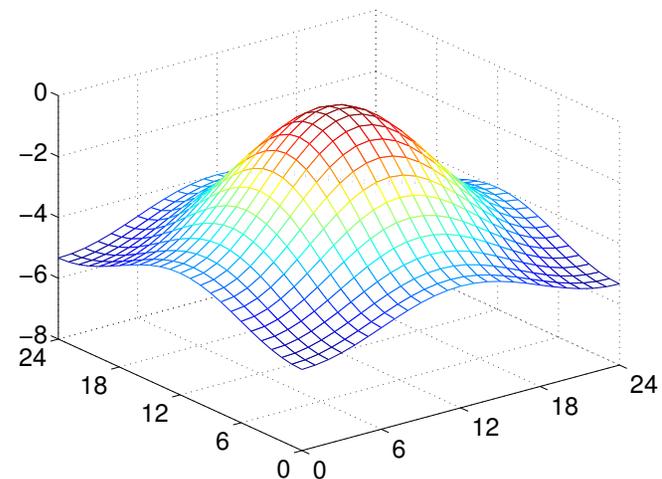
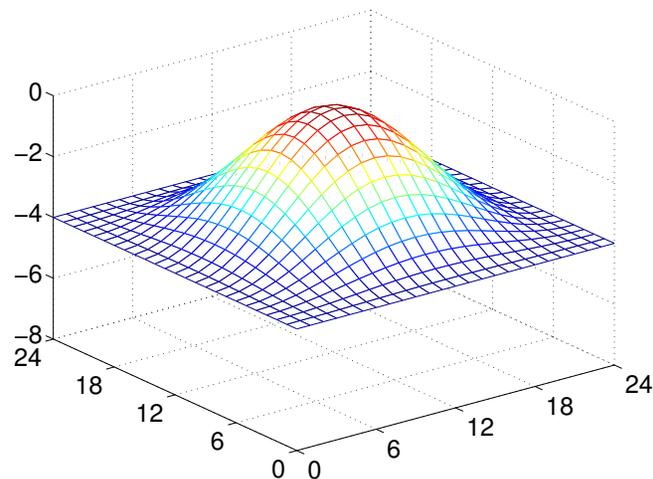
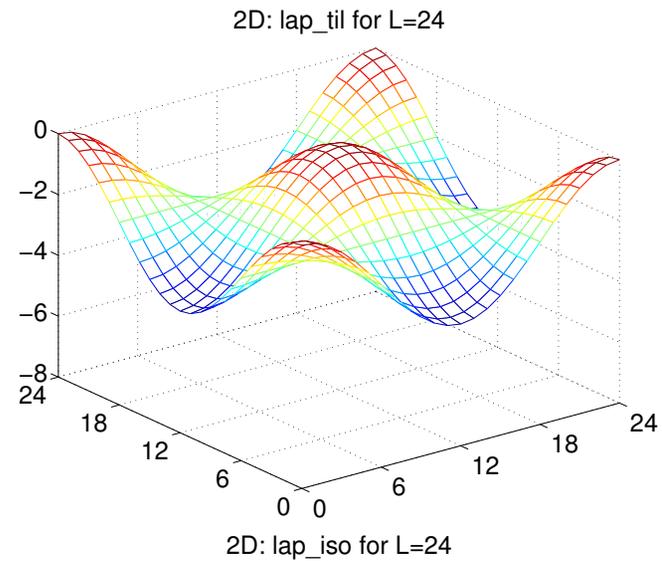
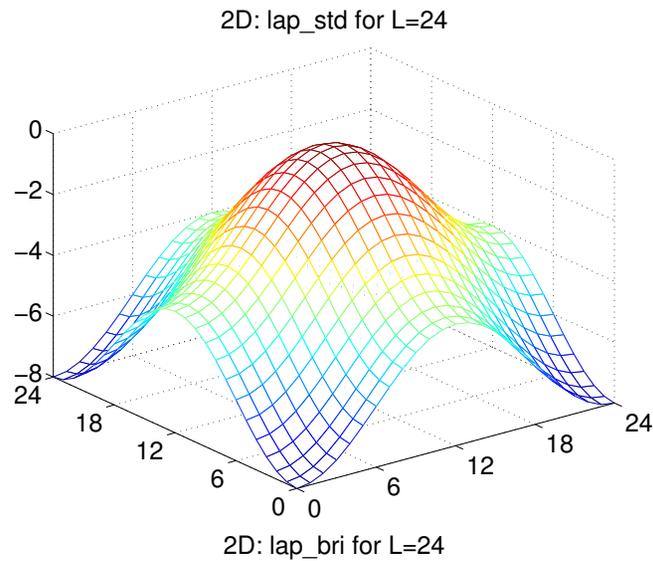
## • 4 options for (covariant) Laplacian

Standard Laplacian:  $\hat{\Delta} = 2 \cos(k_1) + 2 \cos(k_2) + 2 \cos(k_3) + 2 \cos(k_4) - 8$

Tilted Laplacian:  $\hat{\Delta} = 2 \cos(k_1) \cos(k_2) \cos(k_3) \cos(k_4) - 2$

Brillouin Laplacian:  $\hat{\Delta} = 4 \cos^2(k_1/2) \cos^2(k_2/2) \cos^2(k_3/2) \cos^2(k_4/2) - 4$

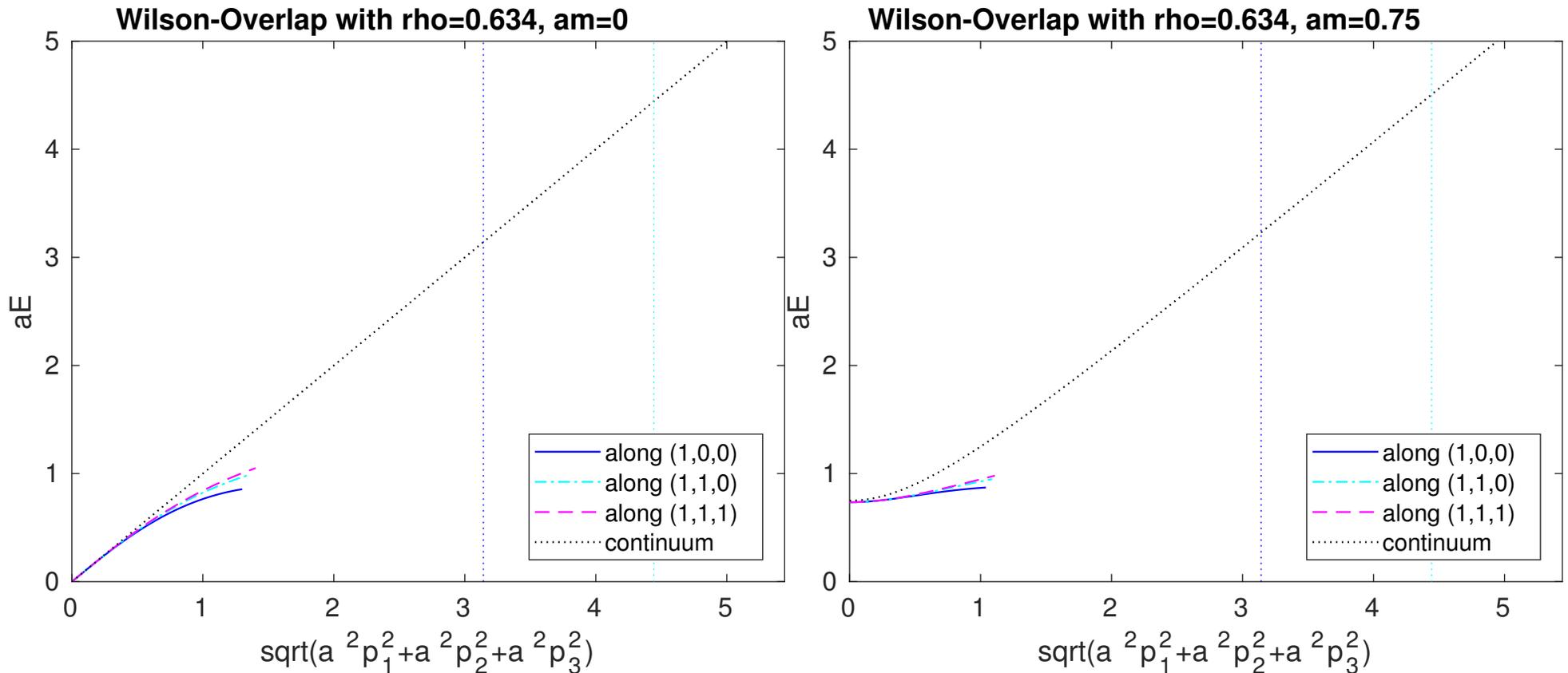
Isotropic Laplacian:  $\hat{\Delta} = [2c_1 c_2 c_3 c_4 + 7c_1 c_2 c_3 + \dots + 20c_1 c_2 + \dots + 25c_1 + \dots - 250] / 54$



## ● Selection Procedure

All 12 options live on  $[-1:1]^4$  hypercube (81 sites in 4D, ultralocal). Test them systematically (eigenvalues, dispersion relation). Combination  $(\Delta^{\text{bri}}, \nabla^{\text{iso}})$  wins.

# Dispersion relation for overlap with Wilson kernel



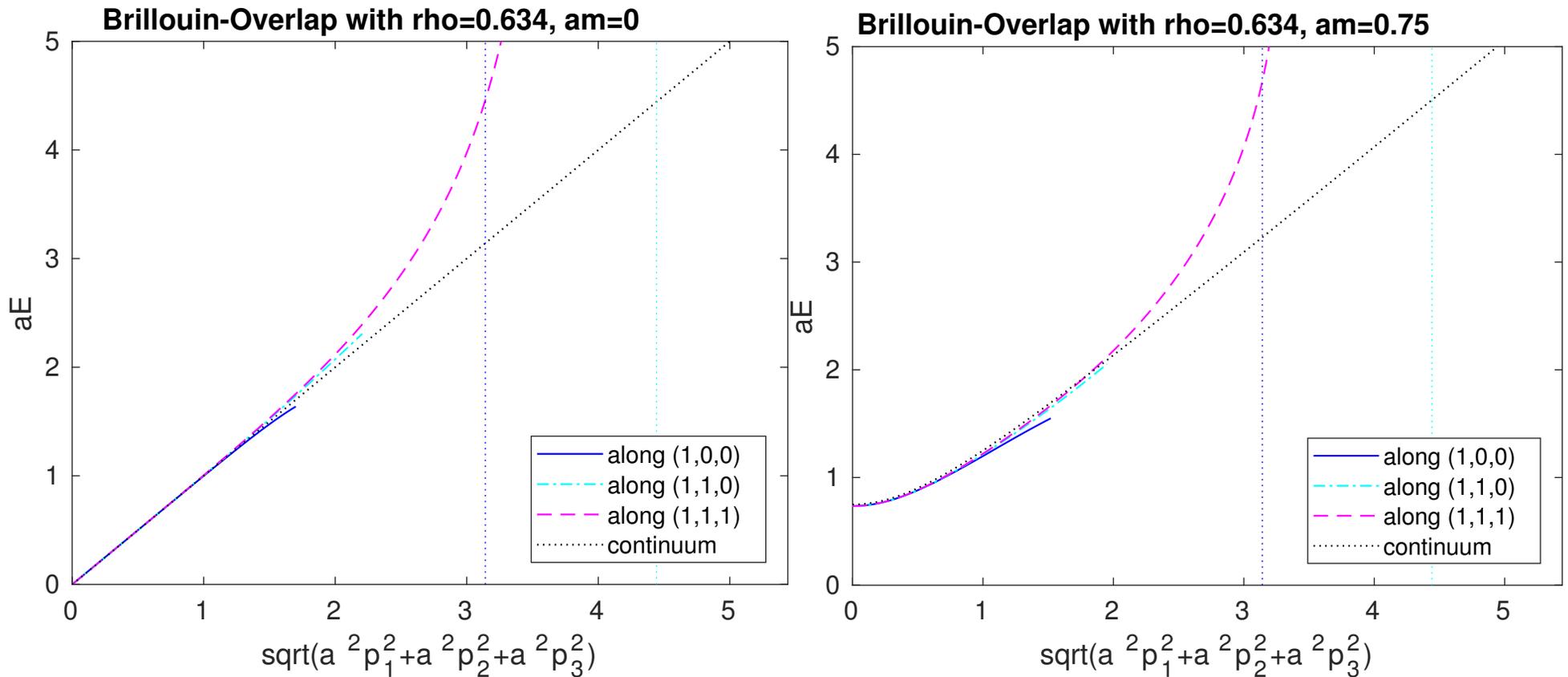
⊖ strong deviation from continuum for any  $a|\mathbf{p}| > 1$

⊖ strong rotational symmetry breaking for any  $a|\mathbf{p}| > 1$

⊕ mild effect of  $am \ll 1$  for “magic value”  $\rho = 0.634$  (neg. mass of  $D_W$ )

See [arXiv:1701.00726] for details.

# Dispersion relation for overlap with Brillouin kernel



- ⊕ mild deviation from continuum up to  $a|\mathbf{p}| \simeq 1.5$
- ⊕ mild rotational symmetry breaking up to  $a|\mathbf{p}| \simeq 1.5$
- ⊕ mild effect of  $am \ll 1$  for “magic value”  $\rho = 0.634$  (neg. mass of  $D_W$ )

See [arXiv:1701.00726] for details.

# Cut-off effects for Wilson and Brillouin operators

Wilson operator:

$$\begin{aligned}(aE)^2 - (a\mathbf{p})^2 &= \left[ (am)^2 - (am)^3 + \frac{11}{12}(am)^4 - \frac{5}{6}(am)^5 + \dots \right] \\ &+ \left[ -\frac{2}{3}(am)^2 + \frac{7}{6}(am)^3 \right] (a\mathbf{p})^2 \\ &+ \left[ -\frac{2}{3} + \frac{am}{2} \right] \left( \sum_{i<j} a^4 p_i^2 p_j^2 + \sum_i (ap_i)^4 \right) + O(a^6)\end{aligned}$$

Brillouin operator:

$$\begin{aligned}(aE)^2 - (a\mathbf{p})^2 &= \left[ (am)^2 - (am)^3 + \frac{11}{12}(am)^4 - \frac{5}{6}(am)^5 + \dots \right] \\ &+ \left[ 0 + \frac{1}{12}(am)^3 \right] (a\mathbf{p})^2 \\ &+ \left[ 0 + \frac{am}{12} \right] \left( \sum_{i<j} a^4 p_i^2 p_j^2 + \sum_i (ap_i)^4 \right) + O(a^6)\end{aligned}$$

Fermilab reinterpretation manages to get rid of  $-(am)^3 + \dots$  part at  $\mathbf{p} = 0$ .

# Cut-off effects for overlap with Wilson and Brillouin kernel

Overlap operator with Wilson kernel:

$$\begin{aligned}(aE)^2 - (a\mathbf{p})^2 &= \left[ (am)^2 - \frac{2\rho^2 - 6\rho + 3}{6\rho^2} (am)^4 + \dots \right] \\ &+ \left[ -\frac{2}{3} (am)^2 + 0 \right] (a\mathbf{p})^2 \\ &+ \left[ -\frac{2}{3} + 0 \right] \left( \sum_{i<j} a^4 p_i^2 p_j^2 + \sum_i (ap_i)^4 \right) + O(a^6)\end{aligned}$$

Overlap operator with Brillouin kernel:

$$\begin{aligned}(aE)^2 - (a\mathbf{p})^2 &= \left[ (am)^2 - \frac{2\rho^2 - 6\rho + 3}{6\rho^2} (am)^4 + \dots \right] \\ &+ \left[ 0 + 0 \right] (a\mathbf{p})^2 \\ &+ \left[ 0 + 0 \right] \left( \sum_{i<j} a^4 p_i^2 p_j^2 + \sum_i (ap_i)^4 \right) + O(a^6)\end{aligned}$$

$\rho = \frac{3-\sqrt{3}}{2} \simeq 0.634$  (“magic value”) establishes  $2\rho^2 - 6\rho + 3 = 0$  [arXiv:1701.00726].

# Dispersion relations of 4D fermion actions

## • Dispersion relation of naive fermion

$$D_{\text{nai}} = \sum_{\mu} \gamma_{\mu} \nabla_{\mu} + m = i \sum_{\mu} \gamma_{\mu} \bar{p}_{\mu} + m$$

$$G_{\text{nai}} = \frac{-i \sum_{\mu} \gamma_{\mu} \bar{p}_{\mu} + m}{(i \sum_{\rho} \gamma_{\rho} \bar{p}_{\rho} + m)(-i \sum_{\sigma} \gamma_{\sigma} \bar{p}_{\sigma} + m)} = \frac{-i \sum_{\mu} \gamma_{\mu} \bar{p}_{\mu} + m}{\bar{p}^2 + m^2}$$

$$aE = \sqrt{a \sinh\left(\sum_i \sin^2(ap_i) + (am)^2\right)}$$

## • Dispersion relation of Wilson fermion

At  $r = 1$  the DR for Wilson fermion simplifies to

$$2 \cosh(aE) \left[ d + am - \sum_i \cos(ap_i) \right] = 1 + \sum_i \sin^2(ap_i) + \left[ d + am - \sum_i \cos(ap_i) \right]^2$$

which one solves for  $aE > 0$  by means of  $\text{acosh}(x) = \ln(x + \sqrt{x^2 - 1})$  for  $x > 1$ .

- Dispersion relation of KW fermion

$$G_{\text{KW}} = \frac{-i \sum_{\mu} \gamma_{\mu} \bar{p}_{\mu} - i \frac{ar}{2} \gamma_d \sum_{i=1}^{d-1} \hat{p}_i^2 + m}{\sum_{i=1}^{d-1} \bar{p}_i^2 + (\bar{p}_d + \frac{ar}{2} \sum_{i=1}^{d-1} \hat{p}_i^2)^2 + m^2}$$

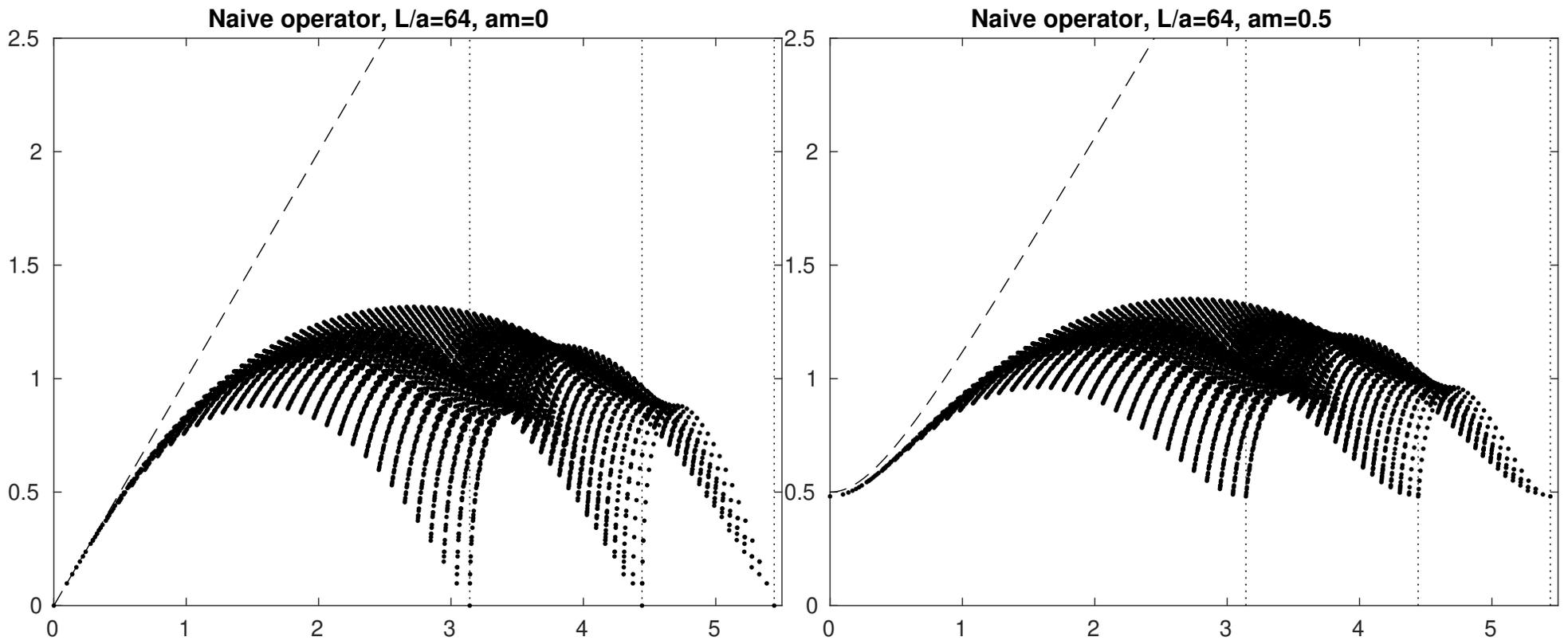
$$\sinh(aE) = ir \sum_{i=1}^{d-1} \{1 - \cos(ap_i)\} \pm \sqrt{\sum_{i=1}^{d-1} \sin^2(ap_i) + (am)^2}$$

- Dispersion relation of BC fermion

$$G_{\text{BC}} = \frac{-i \sum_{\mu} \gamma_{\mu} \bar{p}_{\mu} - i \frac{ar}{2} \sum_{\mu} \gamma'_{\mu} \hat{p}_{\mu}^2 + m}{\sum_{\lambda} \bar{p}_{\lambda}^2 - ar \sum_{\lambda} \bar{p}_{\lambda} \hat{p}_{\lambda}^2 + \frac{a^2 r^2}{4} \sum_{\lambda} \hat{p}_{\lambda}^4 + \frac{2ar}{d} \sum_{\rho, \sigma} \bar{p}_{\rho} \hat{p}_{\sigma}^2 + m^2}$$

$$\begin{aligned} 0 &= \sum_i \left[ \sin(ap_i) - r \{1 - \cos(ap_i)\} \right]^2 + \left[ i \sinh(aE) - r \{1 - \cosh(aE)\} \right]^2 \\ &+ \frac{4r}{d} \sum_{i,j} \sin(ap_i) \{1 - \cos(ap_j)\} + \frac{4ir}{d} \sinh(aE) \sum_j \{1 - \cos(ap_j)\} \\ &+ \frac{4r}{d} \sum_i \sin(ap_i) \{1 - \cosh(aE)\} + \frac{4ir}{d} \sinh(aE) \{1 - \cosh(aE)\} + (am)^2 \end{aligned}$$

- **Dispersion relation of naive fermion**

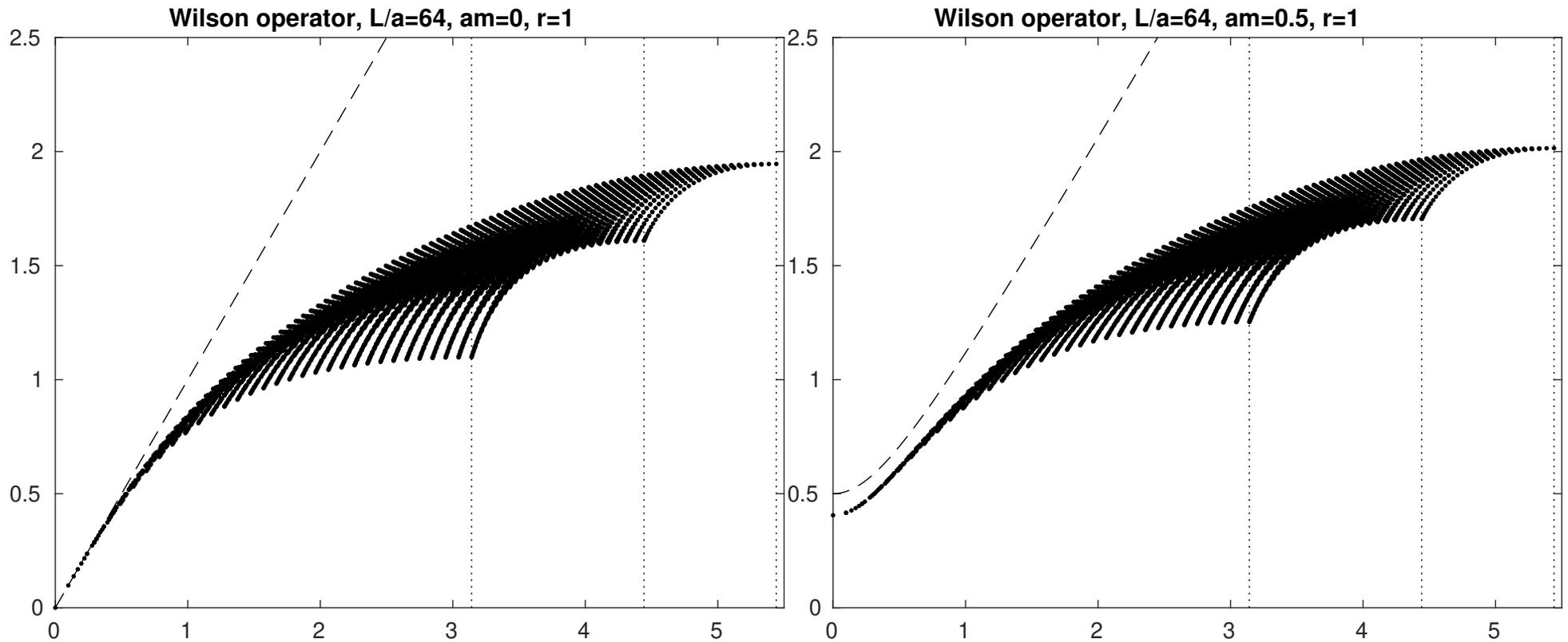


Momentum configurations with  $|\vec{p}| = 0, \pi, \sqrt{2}\pi, \sqrt{3}\pi, 2\pi$  realize 1,4,6,4,1 species.

Useful feature for heavy-quark physics: cut-off effects at  $|a\vec{p}| = 0$  are quadratic:

$$aE = am \left\{ 1 - \frac{1}{6}(am)^2 + \frac{3}{40}(am)^4 + O((am)^6) \right\}$$

- **Dispersion relation of Wilson fermion**

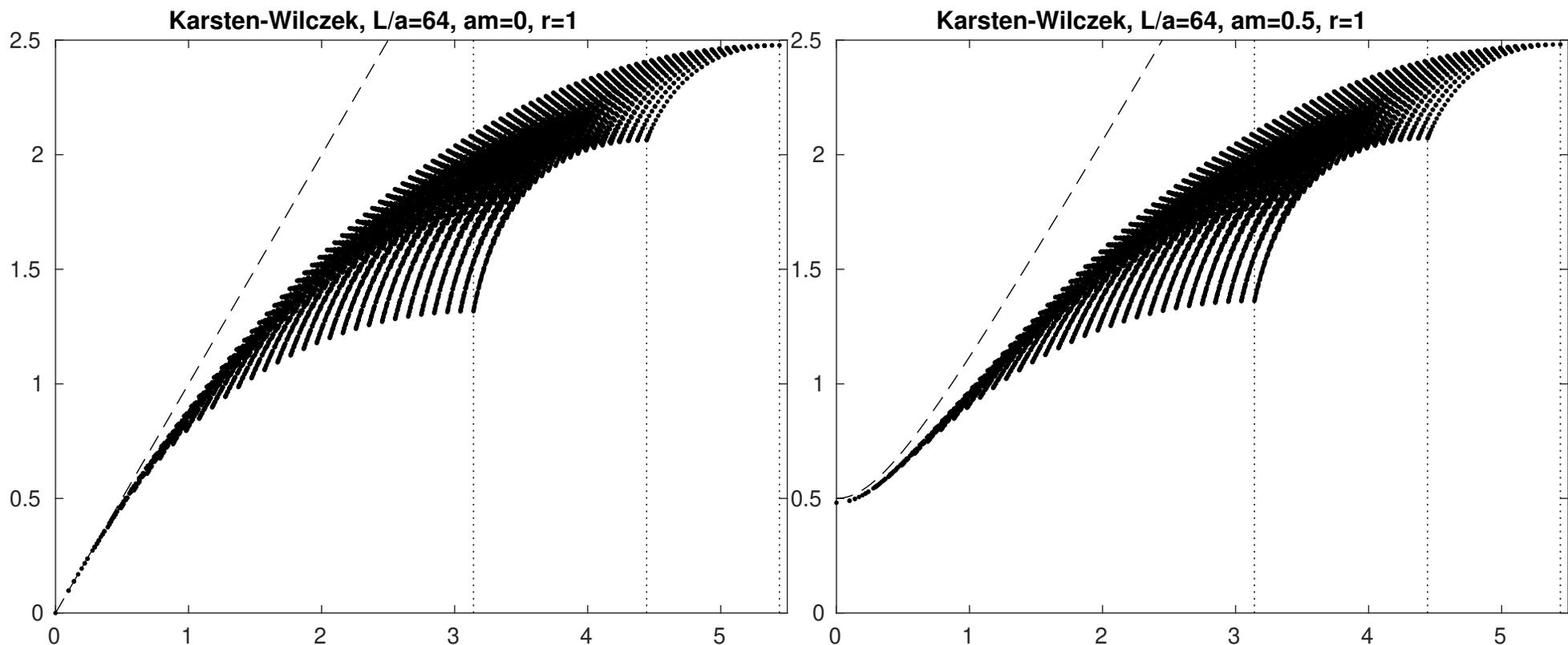


Inconvenient feature for heavy-quark physics: cut-off effects at  $|a\vec{p}| = 0$  are linear:

$$aE = am \left\{ 1 - \frac{r}{2}am + \frac{3r^2 - 1}{6}(am)^2 - \frac{[5r^2 - 3]r}{8}(am)^3 + O((am)^4) \right\}$$

Non-zero momenta up to  $|a\vec{p}| = O(1)$  seem affected by common mismatch in  $am$ .

- **Dispersion relation of KW fermion**

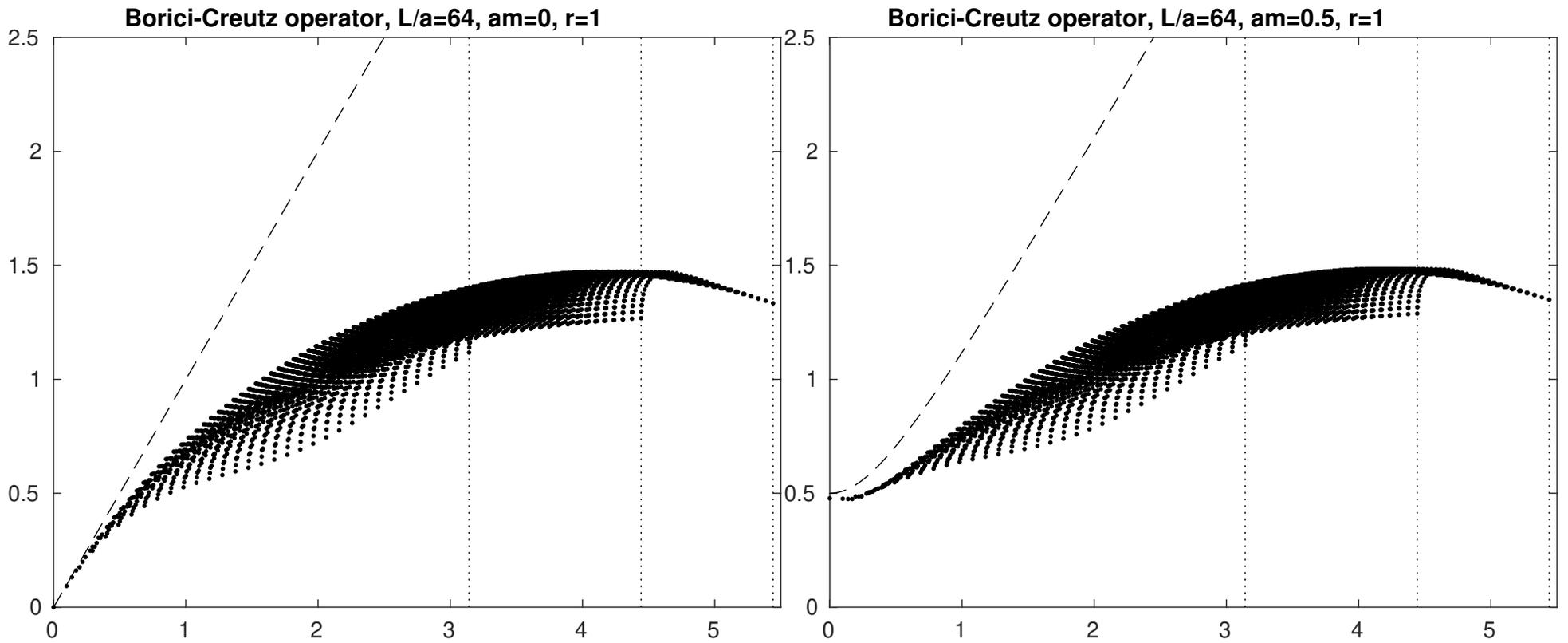


Feature for heavy-quark physics: cut-off effects at  $|a\vec{p}| = 0$  are quadratic:

$$aE = am \left\{ 1 - \frac{1}{6}(am)^2 + \frac{3}{40}(am)^4 + O((am)^6) \right\}$$

Non-zero momenta up to  $|a\vec{p}| = O(1)$  seem well represented [arXiv:2003.10803].

- **Dispersion relation of BC fermion**



Feature for heavy-quark physics: cut-off effects in real/imag part are linear/quadratic

$$aE = am \left\{ 1 + \frac{ir}{4}am - \frac{3r^2+16}{96}(am)^2 + \frac{i[r^3-3r]}{16}(am)^3 - \frac{805r^4-960r^2-768}{10240}(am)^4 \right\}$$

Questionable features for tiny momenta at  $am \simeq 0.5$  [arXiv:2003.10803].