

Lattice perturbation theory with Brillouin Fermions and the calculation of c_{SW} at one-loop order

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BC in 2D, leg(denominator), $c=0.58$

Novel Lattice Fermions and their Suitability
for HPC and Perturbation Theory
March 6 – 10, 2023

<https://indico.mitp.uni-mainz.de/event/314>

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Why lattice perturbation theory?

- Explore qualitative features
- Matching to the continuum
- Calculate important quantities in the large β /small g_0 regime
i.e. m_{crit} , c_{SW} , ...

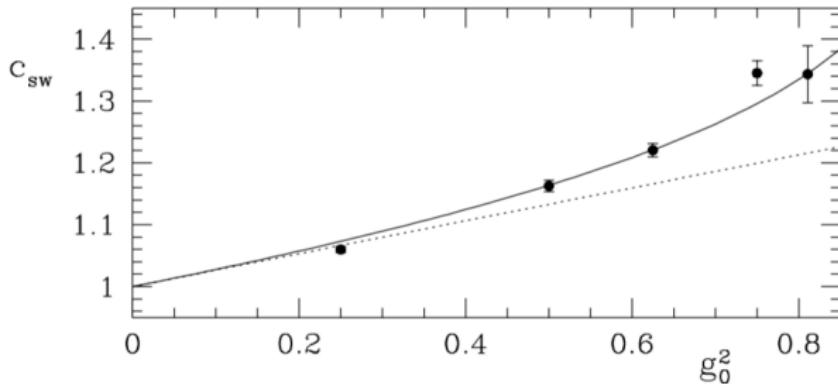


Figure: Non-perturbative determination of c_{SW} (data points), one-loop value (dashed line) and rational fit (solid line). Graphic from Lüscher et al. 1996 [hep-lat/9609035]

- Principle of Lattice perturbation theory:

Relate link variables $U_\mu(x) \in \text{SU}(N_c)$ to gluon fields $A_\mu(x) \in \mathfrak{su}(N_c)$

$$U_\mu(x) = e^{ig_0 A_\mu(x)}$$

$$A_\mu(x) = \sum_{a=1}^{N_c^2 - 1} T^a A_\mu^a(x)$$

→ Insert into action and expand to needed order in g_0 .

→ Feynman Rules

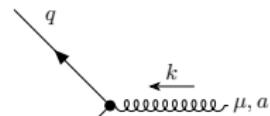
- At 0th order → Propagators
e.g. Gluon propagator (plaquette action)

$$G_{\mu\nu}(k) = \frac{\delta_{\mu\nu}}{\frac{4}{a^2} \sum_{\mu} \sin^2\left(\frac{a}{2} k_{\mu}\right)} = \frac{\delta_{\mu\nu}}{\hat{k}^2}, \quad \hat{k}_{\mu} = \frac{2}{a} \sin\left(\frac{a}{2} k_{\mu}\right)$$

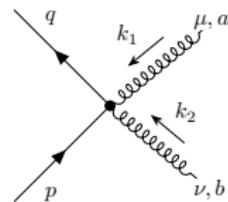
Fermion propagator (Wilson action)

$$S(k) = a \frac{-i \sum_{\lambda} \gamma_{\lambda} \sin(ak_{\lambda}) + 2r \sum_{\rho} \sin^2\left(\frac{a}{2} k_{\rho}\right)}{\sum_{\lambda} \sin^2(ak_{\lambda}) + 4r^2 \left(\sum_{\rho} \sin^2\left(\frac{a}{2} k_{\rho}\right)\right)^2}$$

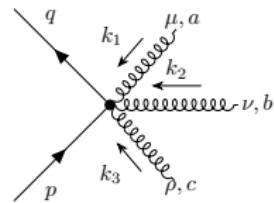
- At order $g_0^n \rightarrow$ quark-quark-n-gluon-vertex.



$$\sim g_0 T^a \rightarrow i g_0 \gamma_\mu T^a$$

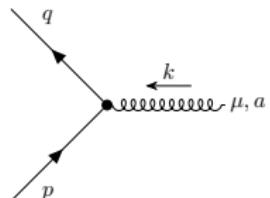


$$\sim a \cdot g_0^2 T^a T^b \rightarrow 0$$

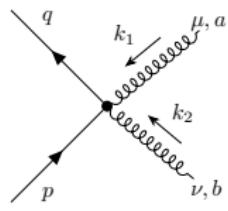


$$\sim a^2 \cdot g_0^3 T^a T^b T^c \rightarrow 0$$

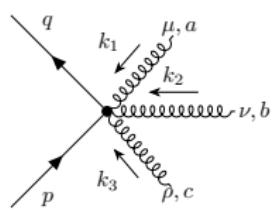
- Example: Wilson fermion action



$$-g_0 T^a \left(i\gamma_\mu \cos \left(\frac{a}{2}(p_\mu + q_\mu) \right) + r \sin \left(\frac{a}{2}(p_\mu + q_\mu) \right) \right)$$



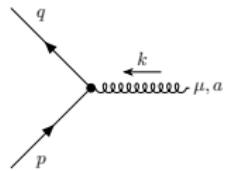
$$\frac{a}{2} g_0^2 \delta_{\mu\nu} T^a T^b \left(i\gamma_\mu \sin \left(\frac{a}{2}(p_\mu + q_\mu) \right) - r \cos \left(\frac{a}{2}(p_\mu + q_\mu) \right) \right)$$



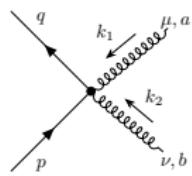
$$\frac{a}{6} g_0^3 \delta_{\mu\nu} \delta_{\mu\rho} T^a T^b T^c \left(i\gamma_\mu \cos \left(\frac{a}{2}(p_\mu + q_\mu) \right) + r \sin \left(\frac{a}{2}(p_\mu + q_\mu) \right) \right)$$

- Example: Clover improvement term

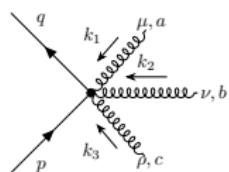
$$c_{\text{SW}} \cdot \sum_x \sum_{\mu < \nu} \bar{\psi}(x) \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu}(x) \psi(x) ,$$



$$ig_0 T^a \frac{1}{2} c_{\text{SW}}^{(0)} \sum_{\nu} \sigma_{\mu\nu} \cos \left(\frac{a}{2} (p_\mu - q_\mu) \right) \sin \left(a(p_\nu - q_\nu) \right)$$



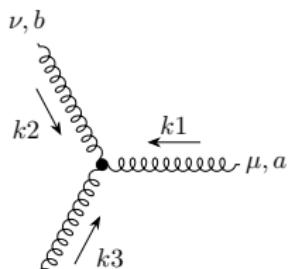
$$-g_0^2 f^{abc} T^c \frac{a}{8} c_{\text{SW}}^{(0)} \left(\delta_{\mu\nu} \sum_{\rho} \sigma_{\mu\rho} \dots + \sigma_{\mu\nu} \dots \right)$$



$$-ig_0^3 \frac{a^2}{2} c_{\text{SW}}^{(0)} T^a T^b T^c \left(\dots \right)$$

- Example: Lüscher-Weisz gauge action

$$S_g[U] = -\frac{2}{g_0^2} \sum_x \text{Re} \left[c_0 \sum_{\text{plaq}} \text{Tr} \left(\mathbf{1} - U^{\text{plaq}}(x) \right) + c_1 \sum_{\text{rect}} \text{Tr} \left(\mathbf{1} - U^{\text{rect}}(x) \right) \right]$$



$$-\frac{i g_0}{6} f^{abc} \left(c_0 V_{g^3 \mu \nu \rho}^{(0)}(k_1, k_2, k_3) + c_1 V_{g^3 \mu \nu \rho}^{(1)}(k_1, k_2, k_3) \right)$$

- Wilson operator with standard derivatives → 9-point stencil

$$D_W(x, y) = \sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{std}}(x, y) - \frac{r}{2} \Delta^{\text{std}}(x, y)$$

- Brillouin operator with 81-point stencil

$$D_B(x, y) = \sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{iso}}(x, y) - \frac{r}{2} \Delta^{\text{bri}}(x, y) .$$

- Brillouin operator with 81-point stencil

$$\begin{aligned} D_B(x, y) = & -r \frac{\lambda_0}{2} \delta(x, y) + \sum_{\mu=\pm 1}^{\pm 4} \left(\rho_1 \gamma_\mu - r \frac{\lambda_1}{2} \right) W_\mu(x) \delta(x + \hat{\mu}, y) \\ & + \sum_{\substack{\mu, \nu = \pm 1 \\ |\mu| \neq |\nu|}}^{\pm 4} \left(\rho_2 \gamma_\mu - r \frac{\lambda_2}{4} \right) W_{\mu\nu}(x) \delta(x + \hat{\mu} + \hat{\nu}, y) \\ & + \sum_{\substack{\mu, \nu, \rho = \pm 1 \\ |\mu| \neq |\nu| \neq |\rho|}}^{\pm 4} \left(\frac{\rho_3}{2} \gamma_\mu - r \frac{\lambda_3}{12} \right) W_{\mu\nu\rho}(x) \delta(x + \hat{\mu} + \hat{\nu} + \hat{\rho}, y) \\ & + \sum_{\substack{\mu, \nu, \rho, \sigma = \pm 1 \\ |\mu| \neq |\nu| \neq |\rho| \neq |\sigma|}}^{\pm 4} \left(\frac{\rho_4}{6} \gamma_\mu - r \frac{\lambda_4}{48} \right) W_{\mu\nu\rho\sigma}(x) \delta(x + \hat{\mu} + \hat{\nu} + \hat{\rho} + \hat{\sigma}, y) \end{aligned}$$

- Brillouin operator with 81-point stencil

$$\begin{aligned}
 D_B(x, y) = & -r \frac{\lambda_0}{2} \delta(x, y) + \sum_{\mu=\pm 1}^{\pm 4} \left(\rho_1 \gamma_\mu - r \frac{\lambda_1}{2} \right) W_\mu(x) \delta(x + \hat{\mu}, y) \\
 & + \sum_{\substack{\mu, \nu = \pm 1 \\ |\mu| \neq |\nu|}}^{\pm 4} \left(\rho_2 \gamma_\mu - r \frac{\lambda_2}{4} \right) W_{\mu\nu}(x) \delta(x + \hat{\mu} + \hat{\nu}, y) \\
 & + \sum_{\substack{\mu, \nu, \rho = \pm 1 \\ |\mu| \neq |\nu| \neq |\rho|}}^{\pm 4} \left(\frac{\rho_3}{2} \gamma_\mu - r \frac{\lambda_3}{12} \right) W_{\mu\nu\rho}(x) \delta(x + \hat{\mu} + \hat{\nu} + \hat{\rho}, y) \\
 & + \sum_{\substack{\mu, \nu, \rho, \sigma = \pm 1 \\ |\mu| \neq |\nu| \neq |\rho| \neq |\sigma|}}^{\pm 4} \left(\frac{\rho_4}{6} \gamma_\mu - r \frac{\lambda_4}{48} \right) W_{\mu\nu\rho\sigma}(x) \delta(x + \hat{\mu} + \hat{\nu} + \hat{\rho} + \hat{\sigma}, y)
 \end{aligned}$$

$$\begin{aligned}
 (\rho_1, \rho_2, \rho_3, \rho_4) &= \frac{1}{2}(1, 0, 0, 0), \quad (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (-8, 1, 0, 0, 0) \rightarrow \text{Wilson} \\
 (\rho_1, \rho_2, \rho_3, \rho_4) &= \frac{1}{432}(64, 16, 4, 1), \quad (\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{64}(-240, 8, 4, 2, 1) \rightarrow \text{Brillouin}
 \end{aligned}$$

The propagator of the free Brillouin fermion

$$S_B(k) = \left(\sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{iso}}(k) - \frac{r}{2} \Delta^{\text{bri}}(k) \right)^{-1} = \frac{- \sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{iso}}(k) - \frac{r}{2} \Delta^{\text{bri}}(k)}{\frac{r^2}{4} \Delta^{\text{bri}}(k)^2 - \left(\sum_{\mu} \nabla_{\mu}^{\text{iso}}(k)^2 \right)}$$

with the Fourier transforms of the “free” derivative $\nabla_{\mu}^{\text{iso}}$ and Laplace operator Δ^{bri}

$$\nabla_{\mu}^{\text{iso}}(k) = \frac{i}{27} \bar{s}(k_{\mu}) \prod_{\nu \neq \mu} (\bar{c}(k_{\nu}) + 2)$$

$$\Delta^{\text{bri}}(k) = 4 \left(c(k_1)^2 c(k_2)^2 c(k_3)^2 c(k_4)^2 - 1 \right)$$

with

$$s(k_{\mu}) = \sin(\frac{1}{2} k_{\mu})$$

$$c(k_{\mu}) = \cos(\frac{1}{2} k_{\mu})$$

$$\bar{s}(k_{\mu}) = \sin(k_{\mu})$$

$$\bar{c}(k_{\mu}) = \cos(k_{\mu})$$

$$s^2(k) = \sum_{\mu} s(k_{\mu})^2$$

$$\bar{s}^2(k) = \sum_{\mu} \bar{s}(k_{\mu})^2 .$$

- The propagator of the free Brillouin fermion

$$S_B(k) = \left(\sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{iso}}(k) - \frac{r}{2} \Delta^{\text{bri}}(k) \right)^{-1} = \frac{- \sum_{\mu} \gamma_{\mu} \nabla_{\mu}^{\text{iso}}(k) - \frac{r}{2} \Delta^{\text{bri}}(k)}{\frac{r^2}{4} \Delta^{\text{bri}}(k)^2 - \left(\sum_{\mu} \nabla_{\mu}^{\text{iso}}(k)^2 \right)}$$

with the Fourier transforms of the “free” derivatives

$$\nabla_{\mu}^{\text{iso}}(k) = \frac{i}{27} \bar{s}(k_{\mu}) \prod_{\nu \neq \mu} (\bar{c}(k_{\nu}) + 2)$$

$$\Delta^{\text{bri}}(k) = 4 \left(c(k_1)^2 c(k_2)^2 c(k_3)^2 c(k_4)^2 - 1 \right)$$

Thus

$$S_B(k) = \frac{-\frac{i}{27} \sum_{\mu} \left(\gamma_{\mu} \bar{s}(k_{\mu}) \prod_{\nu \neq \mu} (\bar{c}(k_{\nu}) + 2) \right) - 2r \left(c(k_1)^2 c(k_2)^2 c(k_3)^2 c(k_4)^2 - 1 \right)}{4r^2 \left(c(k_1)^2 c(k_2)^2 c(k_3)^2 c(k_4)^2 - 1 \right)^2 + \frac{1}{729} \sum_{\mu} \left(\bar{s}(k_{\mu})^2 \prod_{\nu \neq \mu} (\bar{c}(k_{\nu}) + 2)^2 \right)}$$

- The qqg-vertex

$$\begin{aligned}
 V_{1\mu}^a(p, q) = & -g_0 T^a \left[r\lambda_1 s(p_\mu + q_\mu) + 2i\rho_1 c(p_\mu + q_\mu)\gamma_\mu \right. \\
 & + \sum_{\substack{\nu=1 \\ \nu \neq \mu}}^4 \left\{ r\lambda_2 K_{\mu\nu}^{(sc)}(p, q) + 2i\rho_2 (K_{\mu\nu}^{(cc)}(p, q)\gamma_\mu - K_{\mu\nu}^{(ss)}(p, q)\gamma_\nu) \right\} \\
 & + \frac{1}{3} \sum_{\substack{\nu, \rho=1 \\ \neq (\nu, \rho; \mu)}}^4 \left\{ r\lambda_3 K_{\mu\nu\rho}^{(ssc)}(p, q) + 2i\rho_3 (K_{\mu\nu\rho}^{(ccc)}(p, q)\gamma_\mu - 2K_{\mu\nu\rho}^{(ssc)}(p, q)\gamma_\nu) \right\} \\
 & \left. + \frac{1}{9} \sum_{\substack{\nu, \rho, \sigma=1 \\ \neq (\nu, \rho, \sigma; \mu)}}^4 \left\{ r\lambda_4 K_{\mu\nu\rho\sigma}^{(scsc)}(p, q) + 2i\rho_4 (K_{\mu\nu\rho\sigma}^{(cccc)}(p, q)\gamma_\mu - 3K_{\mu\nu\rho\sigma}^{(sssc)}(p, q)\gamma_\nu) \right\} \right]
 \end{aligned}$$

with $K_{\mu\nu}^{(sc)}(p, q) = s(p_\mu + q_\mu)[\bar{c}(p_\nu) + \bar{c}(q_\nu)]$ etc.

Perturbation Theory with Brillouin Fermions

- The qqgg- and qqggg-vertices

$$\begin{aligned}
V_{\mu\nu}^{ab}(p, q, k_1, k_2) &= a g_0^2 T^a T^b \left(-\frac{1}{2} r \lambda_1 \delta_{\mu\nu} c(p_\mu + q_\mu) + i \rho_1 \delta_{\mu\nu} s(p_\mu + q_\mu) \gamma_\mu \right. \\
&+ r \lambda_2 \left((1 - \delta_{\mu\nu}) L_{\mu\nu}^{(ss)}(p, q, k_2) - \frac{1}{2} \delta_{\mu\nu} \sum_{\substack{\alpha=1 \\ \alpha \neq \mu}}^4 K_{\mu\alpha}^{(cc)}(p, q) \right) \\
&+ r \lambda_3 \left(\frac{2}{3} (1 - \delta_{\mu\nu}) \sum_{\substack{\rho=1 \\ \neq (\mu, \nu)}}^4 L_{\mu\nu\rho}^{(ssc)}(p, q, k_2) - \frac{1}{6} \delta_{\mu\nu} \sum_{\substack{\alpha, \rho=1 \\ \neq (\alpha, \mu)}}^4 K_{\mu\rho\alpha}^{(ccc)}(p, q) \right) \\
&+ r \lambda_4 \left(\frac{1}{6} (1 - \delta_{\mu\nu}) \sum_{\substack{\rho, \sigma=1 \\ \neq (\rho, \mu, \sigma)}}^4 L_{\mu\nu\rho\sigma}^{(sscc)}(p, q, k_2) - \frac{1}{18} \delta_{\mu\nu} \sum_{\substack{\alpha, \rho, \sigma=1 \\ \neq (\alpha, \rho, \sigma, \mu)}}^4 K_{\mu\rho\alpha\sigma}^{(cccc)}(p, q) \right) \\
&+ i \rho_2 \left(2(1 - \delta_{\mu\nu}) \left[L_{\mu\nu}^{(cs)}(p, q, k_2) \gamma_\mu + L_{\mu\nu}^{(sc)}(p, q, k_2) \gamma_\nu \right] \right. \\
&\quad \left. + \delta_{\mu\nu} \sum_{\substack{\alpha=1 \\ \alpha \neq \mu}}^4 \left[K_{\mu\alpha}^{(sc)}(p, q) \gamma_\mu + K_{\mu\alpha}^{(cs)}(p, q) \right] \right) \\
&+ i \rho_3 \left(\frac{4}{3} (1 - \delta_{\mu\nu}) \sum_{\substack{\rho=1 \\ \neq (\rho, \mu, \nu)}}^4 \left[L_{\mu\nu\rho}^{(csc)}(p, q, k_2) \gamma_\mu + L_{\mu\nu\rho}^{(acc)}(p, q, k_2) \gamma_\nu - L_{\mu\nu\rho}^{(sss)}(p, q, k_2) \gamma_\rho \right] \right. \\
&\quad \left. + \frac{1}{3} \delta_{\mu\nu} \sum_{\substack{\alpha, \rho=1 \\ \neq (\alpha, \rho, \mu, \nu)}}^4 \left[K_{\mu\alpha\rho}^{(sc)}(p, q) \gamma_\mu + 2 K_{\mu\alpha\rho}^{(ccs)}(p, q) \gamma_\rho \right] \right) \\
&+ i \rho_4 \left(\frac{1}{3} (1 - \delta_{\mu\nu}) \sum_{\substack{\rho, \sigma=1 \\ \neq (\rho, \sigma, \mu, \nu)}}^4 \left[L_{\mu\nu\rho\sigma}^{(csc)}(p, q, k_2) \gamma_\mu + L_{\mu\nu\rho\sigma}^{(scce)}(p, q, k_2) \gamma_\nu - 2 L_{\mu\nu\rho\sigma}^{(sscc)}(p, q, k_2) \gamma_\rho \right] \right. \\
&\quad \left. + \frac{1}{9} \delta_{\mu\nu} \sum_{\substack{\alpha, \rho, \sigma=1 \\ \neq (\alpha, \rho, \sigma, \mu, \nu)}}^4 \left[K_{\mu\alpha\rho\sigma}^{(scce)}(p, q) \gamma_\mu + 3 K_{\mu\alpha\rho\sigma}^{(ccce)}(p, q) \gamma_\rho \right] \right)
\end{aligned}$$

$$\begin{aligned}
&L_{\mu\nu\rho}^{(ss)}(p, q, k_1, k_2) = a^2 g_0^2 T^a T^b \sum_{\alpha=1}^4 \left[\frac{1}{2} \delta_{\mu\nu} \delta_{\rho\alpha} (\lambda_1 (p_\alpha + q_\alpha) + 2 \lambda_2 (p_\alpha + q_\alpha) \gamma_\alpha) \right. \\
&\quad \left. + \lambda_3 \left(\frac{1}{2} (1 - \delta_{\mu\nu}) \delta_{\rho\alpha}^2 (p_\alpha + q_\alpha) k_2 + \frac{1}{2} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) k_2 \right) \right]
\end{aligned}$$

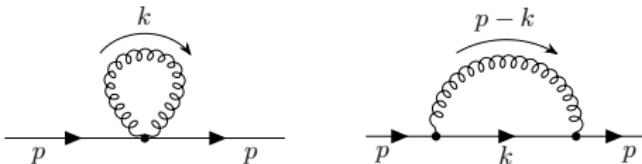
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$$\begin{aligned}
&+ \frac{1}{2} \delta_{\mu\nu} \delta_{\rho\alpha} \sum_{\beta=1}^4 K_{\mu\beta\alpha}^{(cc)}(p, q) \right) \\
&+ \lambda_3 \left(\frac{1}{2} (1 - \delta_{\mu\nu}) (1 - \delta_{\rho\alpha}) (1 - \delta_{\beta\gamma}) M_{\mu\beta\alpha\gamma}^{(cccc)}(p, q, k_1, k_2) \right. \\
&\quad \left. + \frac{1}{24} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 K_{\mu\beta\alpha}^{(cc)}(p, q, k_1, k_2) - \frac{1}{2} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 K_{\mu\beta\alpha}^{(cc)}(p, q, k_1, k_2) \right. \\
&\quad \left. + \frac{1}{12} \delta_{\mu\nu} \delta_{\rho\alpha} \sum_{\beta=1}^4 K_{\mu\beta\alpha}^{(cc)}(p, q, k_1, k_2) \right) \\
&+ \lambda_4 \left(\frac{1}{2} (1 - \delta_{\mu\nu}) (1 - \delta_{\rho\alpha}) (1 - \delta_{\beta\gamma}) \sum_{\delta=1}^4 K_{\mu\beta\alpha\delta}^{(cccc)}(p, q, k_1, k_2) \right. \\
&\quad \left. + \frac{1}{12} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 K_{\mu\beta\alpha}^{(cc)}(p, q, k_1, k_2) + \frac{1}{2} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 K_{\mu\beta\alpha}^{(cc)}(p, q, k_1, k_2) \right. \\
&\quad \left. + \frac{1}{12} \delta_{\mu\nu} \delta_{\rho\alpha} \sum_{\beta=1}^4 K_{\mu\beta\alpha}^{(cc)}(p, q, k_1, k_2) \right) \\
&+ i \rho_2 \left(\frac{1}{2} (1 - \delta_{\mu\nu}) \left[K_{\mu\nu}^{(cc)}(p, q, k_1, k_2) \gamma_\mu - K_{\mu\nu}^{(sc)}(p, q, k_1, k_2) \right] \right. \\
&\quad \left. + \delta_{\mu\nu} \left(\frac{1}{2} (1 - \delta_{\rho\alpha}) \left[K_{\mu\rho\alpha}^{(cc)}(p, q, k_1, k_2) \gamma_\mu - K_{\mu\rho\alpha}^{(sc)}(p, q, k_1, k_2) \right] \right. \right. \\
&\quad \left. \left. + \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \left[K_{\mu\rho\alpha}^{(cc)}(p, q, k_1, k_2) \gamma_\nu - K_{\mu\rho\alpha}^{(sc)}(p, q, k_1, k_2) \right] \right) \right) \\
&+ i \rho_3 \left(\frac{1}{2} (1 - \delta_{\mu\nu}) (1 - \delta_{\rho\alpha}) (1 - \delta_{\beta\gamma}) \right. \\
&\quad \left. + \left[M_{\mu\beta\alpha\gamma}^{(cccc)}(p, q, k_1, k_2) - M_{\mu\beta\alpha\gamma}^{(ccce)}(p, q, k_1, k_2) - M_{\mu\beta\alpha\gamma}^{(sscc)}(p, q, k_1, k_2) \right] \right. \\
&\quad \left. + \frac{1}{24} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\gamma}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\gamma}^{(ccce)}(p, q, k_1, k_2) \right. \right. \\
&\quad \left. \left. - K_{\mu\beta\alpha\gamma}^{(sscc)}(p, q, k_1, k_2) \right] \right. \\
&\quad \left. + \frac{1}{12} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\gamma}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\gamma}^{(ccce)}(p, q, k_1, k_2) \right. \right. \\
&\quad \left. \left. + K_{\mu\beta\alpha\gamma}^{(sscc)}(p, q, k_1, k_2) \right] \right. \\
&\quad \left. + \frac{1}{12} \delta_{\mu\nu} \delta_{\rho\alpha} \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\gamma}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\gamma}^{(ccce)}(p, q, k_1, k_2) \right. \right. \\
&\quad \left. \left. + K_{\mu\beta\alpha\gamma}^{(sscc)}(p, q, k_1, k_2) \right] \right) \\
&+ \frac{1}{3} \delta_{\mu\nu} \delta_{\rho\alpha} \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\gamma}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\gamma}^{(ccce)}(p, q, k_1, k_2) \right. \\
&\quad \left. + K_{\mu\beta\alpha\gamma}^{(sscc)}(p, q, k_1, k_2) \right]
\end{aligned}$$

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$$\begin{aligned}
&+ i \rho_4 \left(\frac{1}{2} (1 - \delta_{\mu\nu}) (1 - \delta_{\rho\alpha}) (1 - \delta_{\beta\gamma}) \sum_{\delta=1}^4 \left[K_{\mu\beta\alpha\delta}^{(cccc)}(p, q, k_1, k_2) \right. \right. \\
&\quad \left. \left. + M_{\mu\beta\alpha\delta}^{(ccce)}(p, q, k_1, k_2) + M_{\mu\beta\alpha\delta}^{(sscc)}(p, q, k_1, k_2) - M_{\mu\beta\alpha\delta}^{(sscc)}(p, q, k_1, k_2) \right] \right. \\
&\quad \left. + \frac{1}{24} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\delta}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\delta}^{(ccce)}(p, q, k_1, k_2) \right. \right. \\
&\quad \left. \left. - K_{\mu\beta\alpha\delta}^{(sscc)}(p, q, k_1, k_2) \right] \right. \\
&\quad \left. + \frac{1}{12} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\delta}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\delta}^{(ccce)}(p, q, k_1, k_2) \right. \right. \\
&\quad \left. \left. + K_{\mu\beta\alpha\delta}^{(sscc)}(p, q, k_1, k_2) \right] \right. \\
&\quad \left. + \frac{1}{12} \delta_{\mu\nu} \delta_{\rho\alpha} \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\delta}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\delta}^{(ccce)}(p, q, k_1, k_2) \right. \right. \\
&\quad \left. \left. + K_{\mu\beta\alpha\delta}^{(sscc)}(p, q, k_1, k_2) \right] \right) \\
&+ \frac{1}{9} \delta_{\mu\nu} (1 - \delta_{\rho\alpha}) \sum_{\beta=1}^4 \left[K_{\mu\beta\alpha\delta}^{(cccc)}(p, q, k_1, k_2) - K_{\mu\beta\alpha\delta}^{(ccce)}(p, q, k_1, k_2) \right. \\
&\quad \left. + K_{\mu\beta\alpha\delta}^{(sscc)}(p, q, k_1, k_2) \right]
\end{aligned}$$

- Tadpole and Sunset diagrams



- Expand to order $1/a$

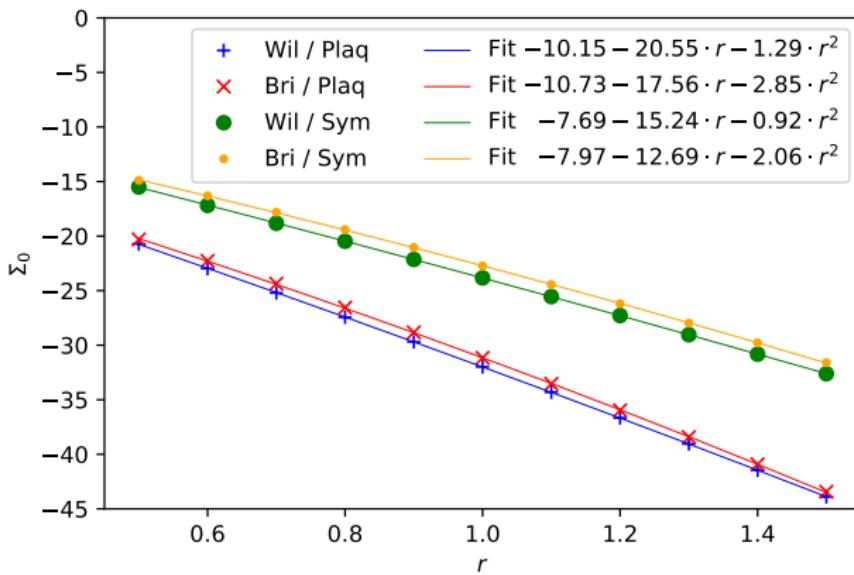
$$\Sigma = \frac{g_0^2 C_F}{16\pi^2} \left(\frac{\Sigma_0}{a} + \mathcal{O}(a^0, p) \right) = -m_{\text{crit}} + \mathcal{O}(a^0, p)$$

- Evaluate corresponding integrals

$$g_0^2 C_F \Sigma_0^{(\text{tadpole})} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \sum_{\mu, \nu, a} \left[G_{\mu\nu}(k) V_{2\mu\nu}^{aa}(p, p, k, -k) \right]_{p=0}$$

$$g_0^2 C_F \Sigma_0^{(\text{sunset})} = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \sum_{\mu, \nu, a} \left[V_{1\mu}^a(p, k) G_{\mu\nu}(p - k) S(k) V_{1\nu}^a(k, p) \right]_{p=0}$$

Self energy of the Brillouin fermion



Perturbative determination of c_{SW}

- Perturbative expansion

$$c_{\text{SW}} = c_{\text{SW}}^{(0)} + g_0^2 c_{\text{SW}}^{(1)} + \mathcal{O}(g_0^4)$$

- Consider vertex function

$$\Lambda_\mu^a(p, q) = \sum_{L=0}^{\infty} g_0^{2L+1} \Lambda_\mu^{a(L)}(p, q)$$

- Tree level \rightarrow qqg-vertex

$$\begin{aligned} \Lambda_\mu^{a(0)}(p, q) &= V_{1\mu}^a(ap, aq) = -g_0 T^a \left(i\gamma_\mu (2\rho_1 + 12\rho_2 + 24\rho_3 + 16\rho_4) \right. \\ &\quad \left. + a \left[\frac{r}{2}(p_\mu + q_\mu)(\lambda_1 + 6\lambda_2 + 12\lambda_3 + 8\lambda_4) + \frac{i}{2}c_{\text{SW}}^{(0)} \sum_\nu \sigma_{\mu\nu}(p_\nu - q_\nu) \right] + \mathcal{O}(a^2) \right) \end{aligned}$$

- On-shell

$$\bar{u}(q)\Lambda_{\mu}^{a(0)}(p, q)u(p) = -g_0 T^a \bar{u}(q) \left(i\gamma_{\mu} (2\rho_1 + 12\rho_2 + 24\rho_3 + 16\rho_4) + \frac{a}{2} [r(\lambda_1 + 6\lambda_2 + 12\lambda_3 + 8\lambda_4) - c_{SW}^{(0)}] (p_{\mu} + q_{\mu}) \right) u(p) + \mathcal{O}(a^2)$$

- \Rightarrow Improvement condition

$$c_{SW}^{(0)} = r$$

As $(\lambda_1 + 6\lambda_2 + 12\lambda_3 + 8\lambda_4) = 1$ needs to be fulfilled for the action to have the correct continuum limit.

- One-loop vertex function

$$g_0^3 \Lambda_\mu^{a(1)} = -g_0^3 T^a \left(\gamma_\mu F_1 + a \not{q} \gamma_\mu F_2 + a \gamma_\mu \not{p} F_3 + a(p_\mu + q_\mu) G_1 + a(p_\mu - q_\mu) H_1 \right)$$

F_1, F_2 do not contribute on-shell, $H_1 = 0$ due to symmetry arguments.

(Aoki, Kuramashi 2003)

- On-shell

$$\begin{aligned} & g_0^3 \bar{u}(q) \left(\frac{a}{2} c_{\text{SW}}^{(1)}(p_\mu + q_\mu) T^a + \Lambda_\mu^{a(1)}(p, q) \right) u(p) \\ &= g_0^3 \bar{u}(q) \left(i \gamma_\mu F_1 + \frac{a}{2} (p_\mu + q_\mu) (c_{\text{SW}}^{(1)} - 2G_1) T^a \right) u(p) + \mathcal{O}(p^2, q^2) + \mathcal{O}(a^2) \end{aligned}$$

- \Rightarrow Improvement condition

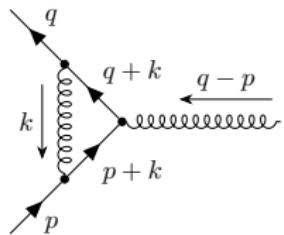
$$c_{\text{SW}}^{(1)} = 2G_1$$

- Extract G_1

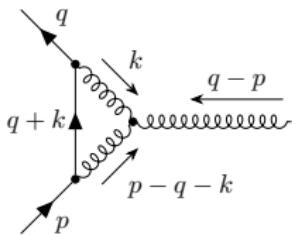
$$g_0^3 T^a G_1 = -\frac{1}{8} \text{Tr} \left[\left(\frac{\partial}{\partial p_\mu} + \frac{\partial}{\partial q_\mu} \right) \Lambda_\mu^{a(1)} - \left(\frac{\partial}{\partial p_\nu} - \frac{\partial}{\partial q_\nu} \right) \Lambda_\mu^{a(1)} \gamma_\nu \gamma_\mu \right]_{p,q=0}^{\mu \neq \nu}$$

- Six one-loop diagrams

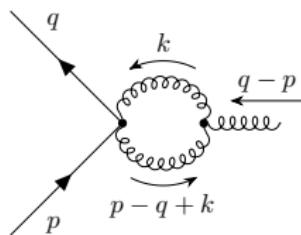
(a)



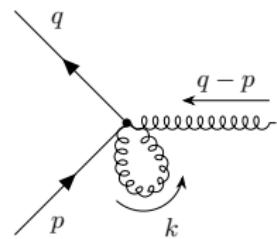
(b)



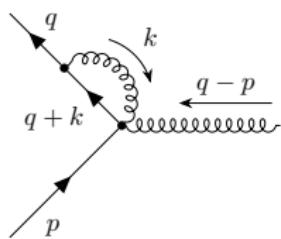
(c)



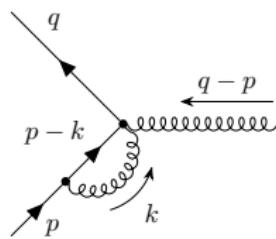
(d)



(e)



(f)



- Construct Feynman integrals corresponding to the diagrams
e.g. diagram (c):

$$\Lambda_{\mu}^{a(1)(c)} = 6 \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \sum_{\nu, \rho, \sigma, \tau} \sum_{b, c} V_{2\nu\rho}^{b,c}(p, q, k, q - p - k) G_{\nu\sigma}(k) G_{\rho\tau}(p - q + k) \\ \times V_{3g\mu\sigma\tau}^{abc}(q - p, -k, p - q + k)$$

- Insert Wilson/Brillouin fermion rules and Plaquette/Lüscher-Weisz gluon rules
- Extract G_1
- Add all diagrams
- Perform integration over k

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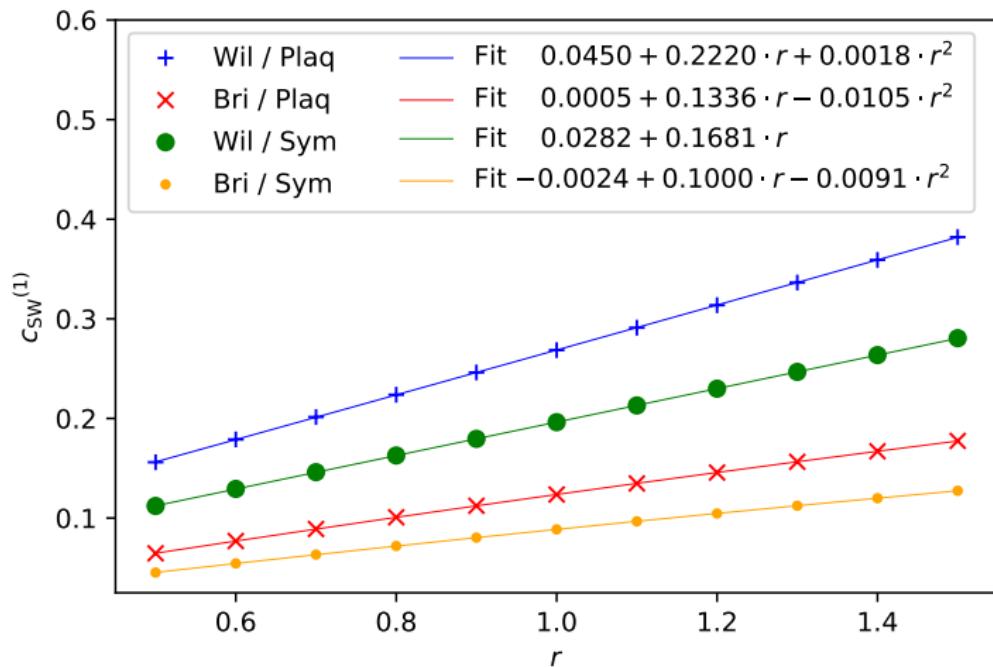
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Action	$c_{SW}^{(1)}$
Wilson/Plaq.	0.26858825(1)
Brillouin/Plaq.	0.12362580(1)
Wilson/Sym.	0.1962445(1)
Brillouin/Sym.	0.088601(1)

(for $N_c = 3$ and $r = 1$)

- In the Wilson case the numbers agree with
(Aoki,Kuramashi 2003), (Horsley et al. 2008)



- Calculate diagrams individually
- Logarithmic divergencies in all but one diagram
- Regulate by subtracting simple logarithmically divergent lattice integral with appropriate pre-factor

$$\mathcal{B}_2 = \int_{-\pi}^{\pi} \frac{d^4 k}{(2\pi)^4} \frac{1}{(\hat{k}^2)^2}, \quad \mathcal{B}_2(\mu) = \frac{1}{16\pi^2} \left(-\ln(\mu^2) + F_0 - \gamma_E \right) + \mathcal{O}(\mu^2)$$

Diag.	\mathcal{B}_2	Wilson/Plaq.	Brillouin/Plaq.	Wilson/Sym.	Brillouin/Sym.
(a)	-1/3	0.009852153(1)	0.0100402212(1)	0.01048401(1)	0.0108335(1)
(b)	-9/2	0.125895883(1)	0.098371668(1)	0.1285594(1)	0.102829(1)
(c)	+9/2	-0.124125079(1)	-0.100558858(1)	-0.1337781(1)	-0.1098254(1)
(d)	0	0.297394534(1)	0.142461144(1)	0.2354388(1)	0.1120815(1)
(e)	+1/6	-0.020214623(1)	-0.013344189(1)	-0.022229808(1)	-0.013659(1)
(f)	+1/6	-0.020214623(1)	-0.013344189(1)	-0.022229808(1)	-0.013659(1)
Sum	0	0.26858825(1)	0.12362580(1)	0.1962445(1)	0.088601(1)

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→ More detail in arXiv:2302.11261