Analytic aspects of Karsten-Wilczek and Boriçi-Creutz fermions

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partly in collaboration with **Michael Creutz**, **Johannes Weber** and **Hartmut Wittig**

Graphene

Graphene: a *"cousin"* of **diamond** and **graphite**

Nanoscopic material, ultra-thin sheet of matter – a form of the element carbon that is just a single atom thick

Is a single layer of graphite consisting of a 2-dimensional hexagonal lattice of carbon atoms

Graphite (*pencil, 1564*): essentially a jumbled mass of tiny scraps of graphene

Graphene

Writing with a pencil on paper actually produces graphene stacks

. . . somewhere among them, there could be individual graphene layers

Graphene was identified as a theoretical possibility as early as 1947 *(Wallace)*

However, for many years it was thought that it couldn't exist in nature – no one expected graphene to exist in the free state

Graphene is presumably produced every time someone writes with a pencil

however, no experimental tools existed to search for macroscopic one-atom-thick flakes among the pencil debris

Only in October 2004 the existence of graphene as a real separate material was first demonstrated *(University of Manchester, UK)*

Andre Geim and **Konstantin Novoselov, Physics Nobel Prize 2010**

In graphene, electrons behave **as if** they were **relativistic massless** particles

$\Rightarrow$ ultra-high mobilities exhibited by graphene devices

$\Rightarrow$ a variety of unique, and potentially very useful, characteristics

Graphene

Its unique electrical characteristics could make graphene the successor to silicon in a whole new generation of microchips

$\Rightarrow$ further development of ever-smaller, ever-faster silicon chips

Because of its single-atom thickness, pure graphene is transparent, and can be used to make transparent electrodes for light-based applications such as LEDs or improved solar cells

Graphene could also substitute for copper to make the electrical connections between computer chips and other electronic devices, providing much lower resistance and thus generating less heat

It has also potential uses in quantum-based electronic devices that could enable a new generation of computation and processing

This field is really in its infancy

There isn't any other material like graphene

Its strength is 200 times that of steel

The mobility of electrons in graphene is by far the highest of any known material

Graphene

Striking: it contains 2 massless Dirac particles
(Hou, Chamon and Mudry, 2006; Jackiw and Pi, 2007)

Creutz's original motivation: the low energy electronic excitations are described by the massless relativistic Dirac equation

The solution to a theory of fermions hopping on a hexagonal lattice displays two Dirac cones

The massless structure is robust , thanks to the topological stability, related to chirality: map of circles onto circles

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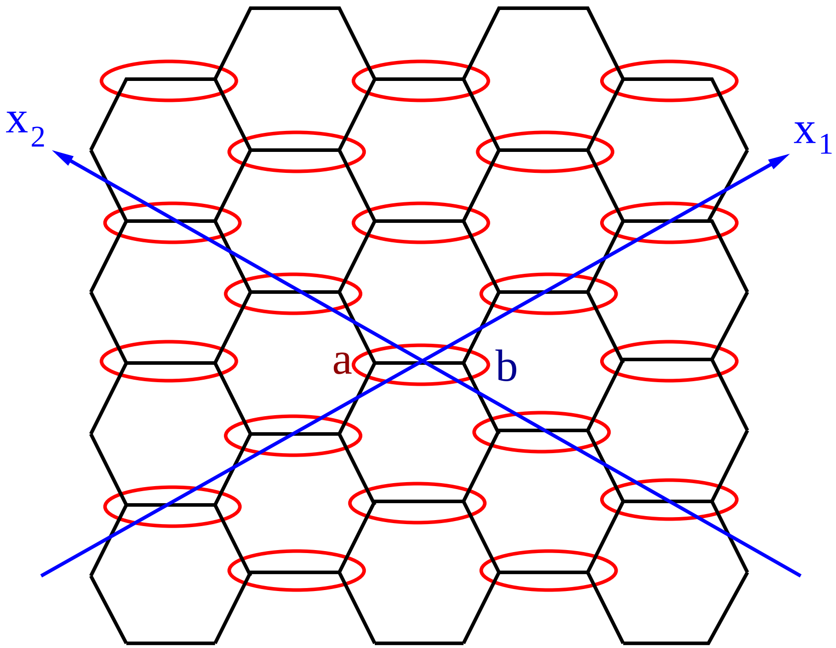
These electrons mimic Dirac fermions, but don't move at the speed of light – they actually move in graphene with a speed

$$\frac{v_F}{c} \approx \frac{1}{300}$$

(comparable to that in half-filled metals)

Graphene



Michael Creutz, JHEP 0804:017, 2008

Orient one third of the bonds horizontal, one third sloping up at 60 degrees, and one third sloping down

Clever choice of coordinates:

organize the graphene structure into two-atom “sites” involving “collapsed” horizontal bonds *(as enclosed in ellipses)*

use a non-orthogonal coordinate system with axes sloping up and down at 30 degrees intersecting the corresponding sites

Graphene

The Hamiltonian contains only nearest-neighbor hoppings between a and b type sites:

$$H = K\sum_{x_1,x_2}\Big(a^\dagger_{x_1,x_2}b_{x_1,x_2} + b^\dagger_{x_1,x_2}a_{x_1,x_2} + a^\dagger_{x_1+1,x_2}b_{x_1,x_2} + b^\dagger_{x_1-1,x_2}a_{x_1,x_2} + a^\dagger_{x_1,x_2-1}b_{x_1,x_2} + b^\dagger_{x_1,x_2+1}a_{x_1,x_2}\Big)$$

K is the “hopping” parameter and sets the energy scale

In momentum space

$$H = K\left[\tilde{a}^\dagger_{p_1,p_2}\tilde{b}_{p_1,p_2}\left(1+e^{-ip_1}+e^{ip_2}\right) + \tilde{b}^\dagger_{p_1,p_2}\tilde{a}_{p_1,p_2}\left(1+e^{ip_1}+e^{-ip_2}\right)\right]$$

can be represented by a matrix $K\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$, where $z = 1+e^{-ip_1}+e^{ip_2}$

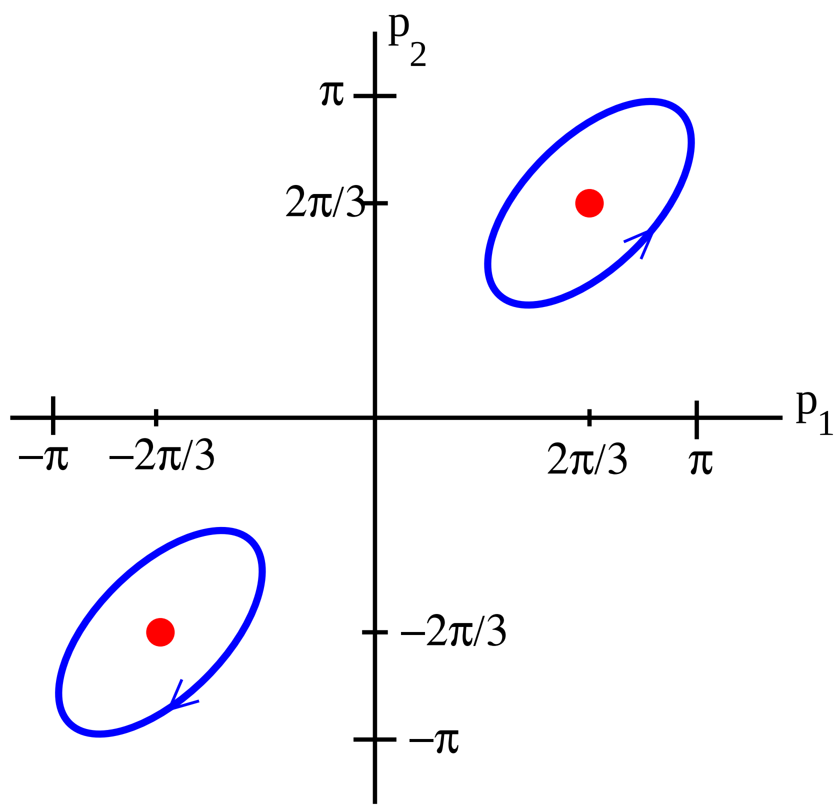
Eigenvalues of the energy : $\pm K|z|$

The energy vanishes when $|z|$ does

$\Rightarrow$ only 2 zeros: $p_1 = p_2 = \pm 2\pi/3$

Consider contours of constant $|z|$ around the zeros

Graphene



Contours of constant energy wrapping around one of the zero points

Traversing a contour, the phase of z wraps nontrivially around the unit circle

Then, when one collapses a contour and shrinks it to a point, the energy at this Dirac point $|z| = 0$ must vanish

When one fully goes around the contour, the spinor wave function acquires a minus sign

Graphene

This is the behavior of a half integer spin system

$\rightarrow$ **the fermion spin has emerged**

spin in two space dimensions is different than in three – there are no helicity states, but rotations about an axis orthogonal to the spatial plane

one might think of the two cones as representing spin up and spin down in the direction orthogonal to the spatial plane

We can see again the close ties between the doubling issues and topology

We have here another instance of the Nielsen-Ninomiya no-go theorem that applies to all lattice actions including mass terms

Since the Brillouin zone is periodic, any contour expanded to the boundaries of this zone cannot wrap z non-trivially

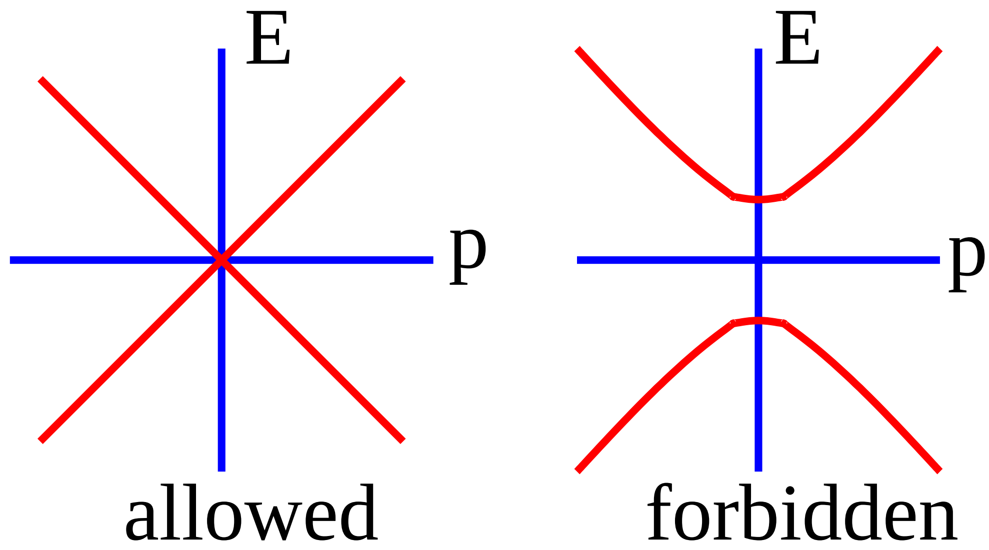
So, given any Dirac cone, there must exist another about which the topology unwraps

$\Rightarrow$ an **even number** of Dirac cones

Two Dirac cones is the minimum possible without breaking the symmetries

The chiral properties of the two cones must be opposite

Graphene



This mechanism prevents a band gap from opening in the spectrum

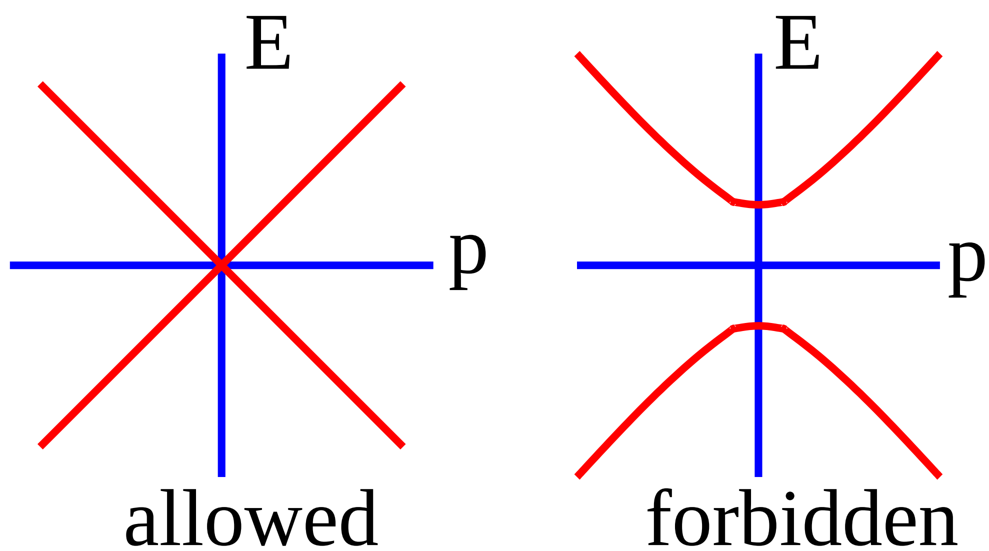
$\Rightarrow$ linear dispersion relation

$\Rightarrow$ graphite is black and a conductor

$\Rightarrow$ Dirac equation

Topological stability of the massless structure

Graphene



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Topological stability of the massless structure

Creutz then constructed an action with similar properties in four dimensions
(in the same paper, JHEP 0804:017, 2008)

Afterwards (2008): further developed by Boriçi , and then again by Creutz

$\rightarrow$ **Boriçi-Creutz** fermions

Minimally doubled fermions

In addition to spin, this model has an emergent chiral symmetry

$b \to -b$ changes the sign of H, because all hoppings couple a and b sites

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ anticommutes with the Hamiltonian $H(p_1, p_2) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$

$\to$ **in four dimensions it would correspond to** γ_5

Four-dimensional extension of the graphene (*Creutz*):

complex numbers $\to$ **quaternions**

Look for an analogous form $H(p_\mu) = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$ in four dimensions

$H(p_\mu)$ is now a 4x4 matrix

$z = z(p_1, p_2, p_3, p_4)$ are 2x2 matrices in a quaternionic space:

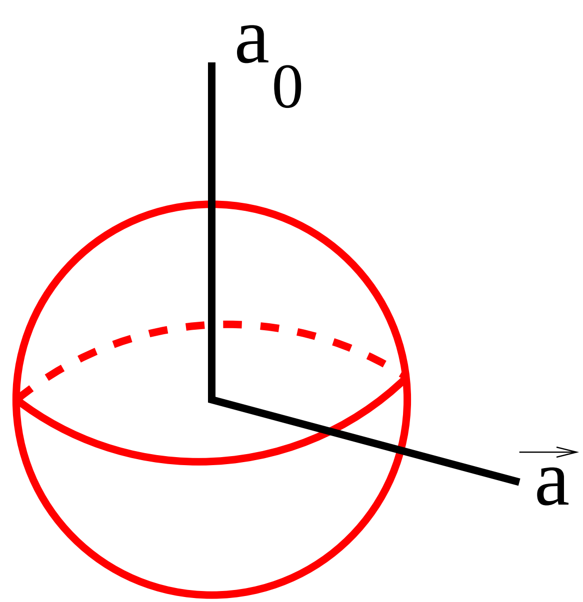
$$z = a_0 + i\,\vec{a}\,\vec{\sigma}, \qquad \text{with } |z|^2 = \sum_\mu a_\mu^2$$

where a_μ is a real 4-vector

Minimally doubled fermions

Eigenvalues of the energy: still $\pm K|z|$

Generalize topology to mapping 3-spheres onto 3-spheres



Constant energy surfaces must involve non-trivial mappings in the quaternionic space near the zeros ($= a_\mu$ *vanishing as a 4-vector*)

$\Rightarrow$ **topological stability of the massless structure**

Minimally doubled fermions

Gamma matrices:

$$\vec{\gamma} = \sigma_1 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$\gamma_4 = -\sigma_2 \otimes 1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\gamma_5 = \sigma_3 \otimes 1 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The lattice implementation of $\gamma_5 D = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$ is not unique – we only need a $z(p)$ with two zeros

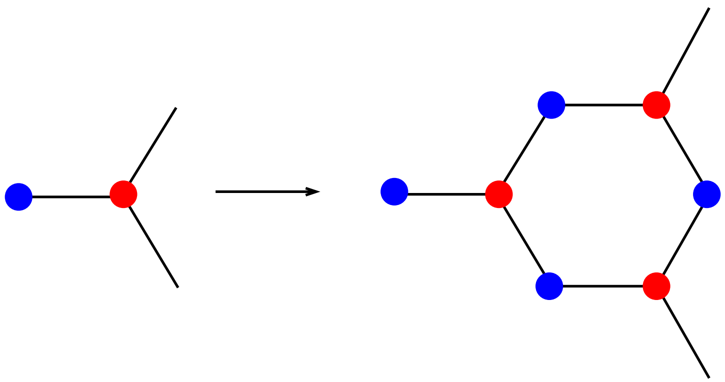
Creutz's proposal:

$$\begin{aligned} z \quad = \quad & B\,(\,4\,C - \cos p_1 + \cos p_2 - \cos p_3 - \cos p_4\,) \\ & + i\sigma_x\,(\sin p_1 + \sin p_2 - \sin p_3 - \sin p_4) \\ & + i\sigma_y\,(\sin p_1 - \sin p_2 - \sin p_3 + \sin p_4) \\ & + i\sigma_z\,(\sin p_1 - \sin p_2 + \sin p_3 - \sin p_4) \end{aligned}$$

B and C control anisotropic distortions

Minimally doubled fermions

Graphene (2 d): one bond splits into two



and iterate

smallest loops are hexagons

Minimally doubled fermions

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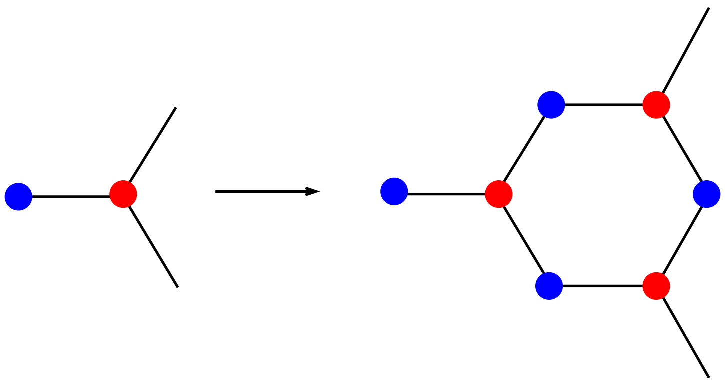

and iterate

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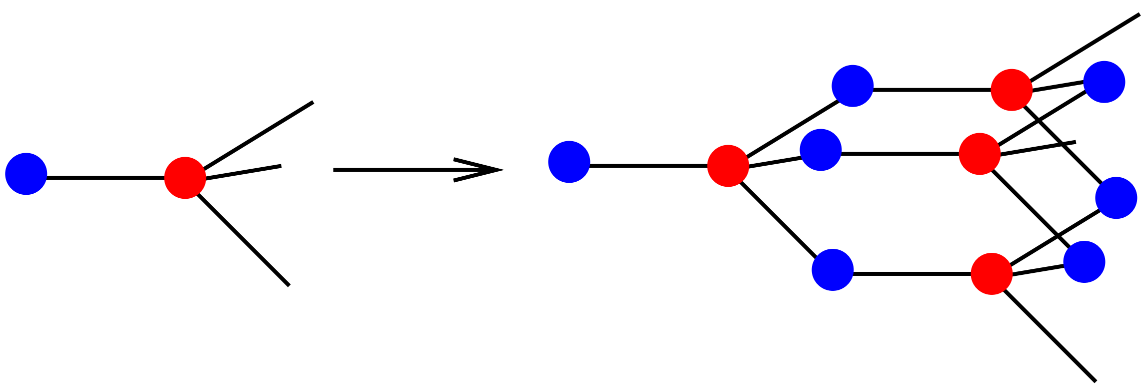
Diamond (3 d): one bond splits into three

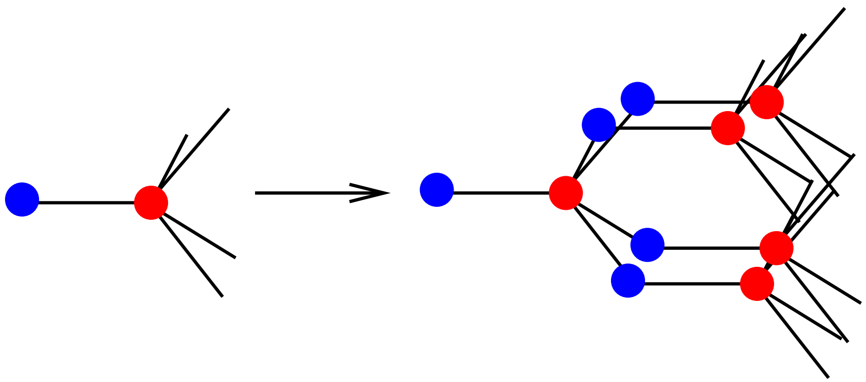


and iterate

smallest loops are cyclohexane “chairs”

Minimally doubled fermions

Graphene (4 d): one bond splits into four



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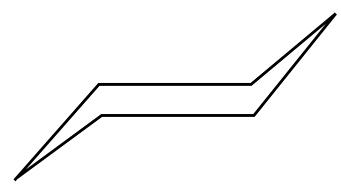

Minimally doubled fermions

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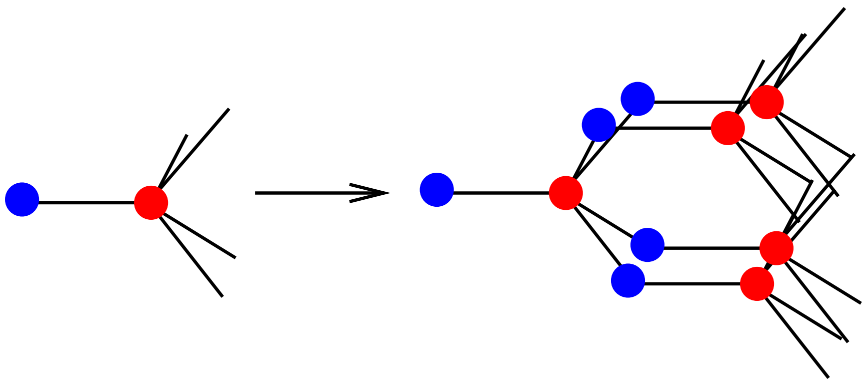

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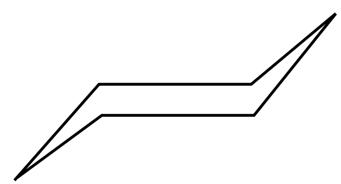
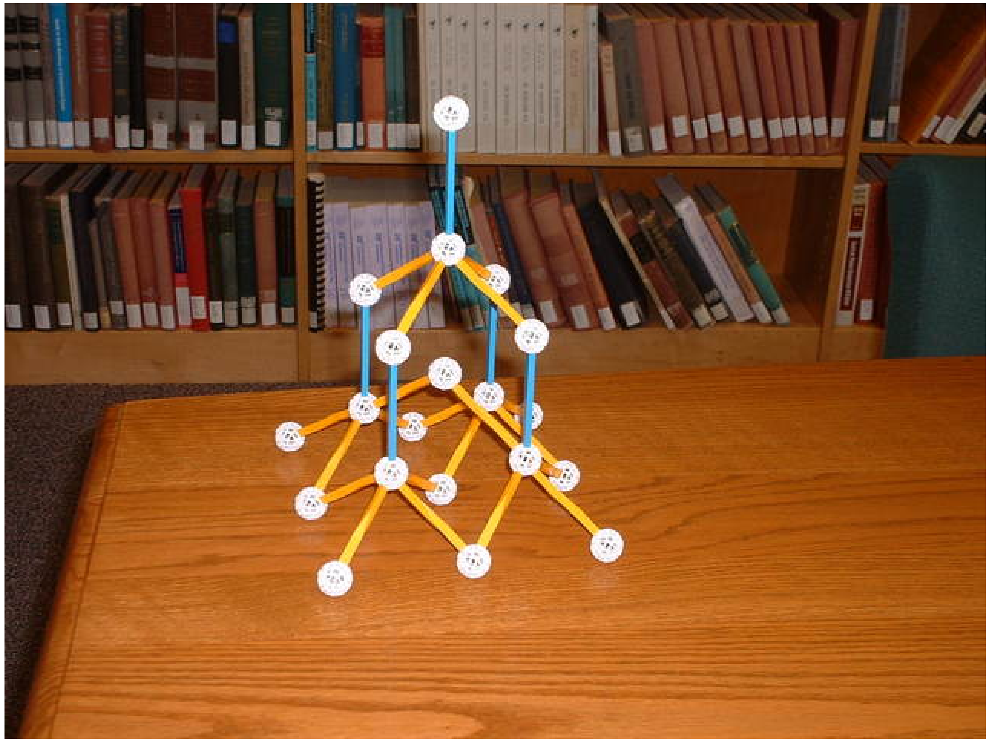


(thanks to Mike Creutz for providing many of these pictures)

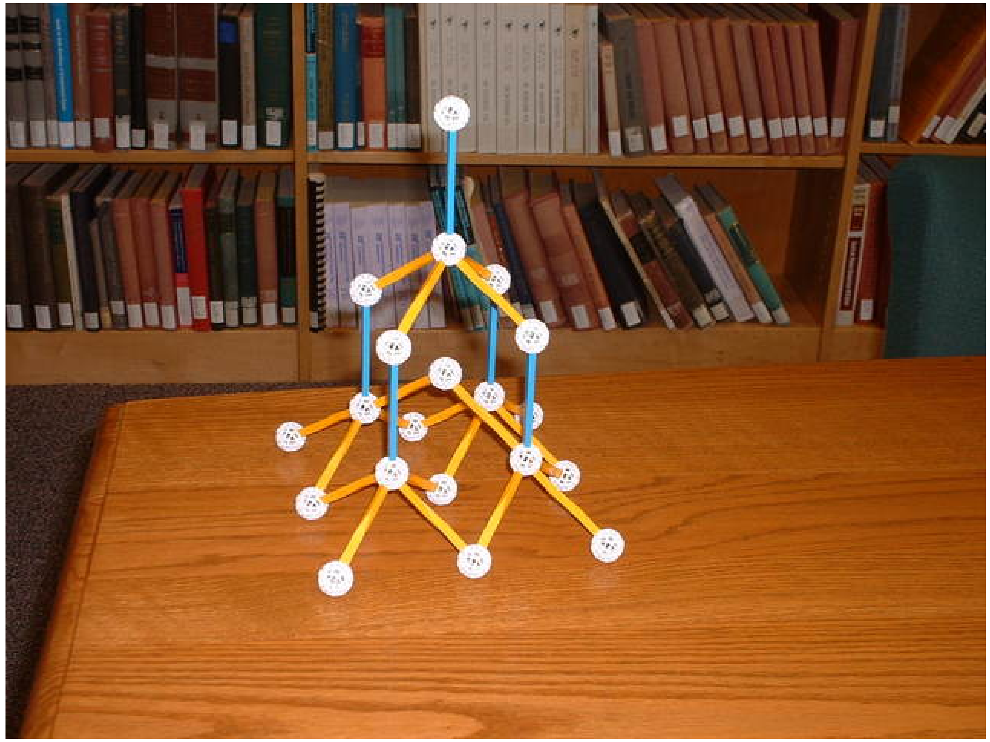
Minimally doubled fermions

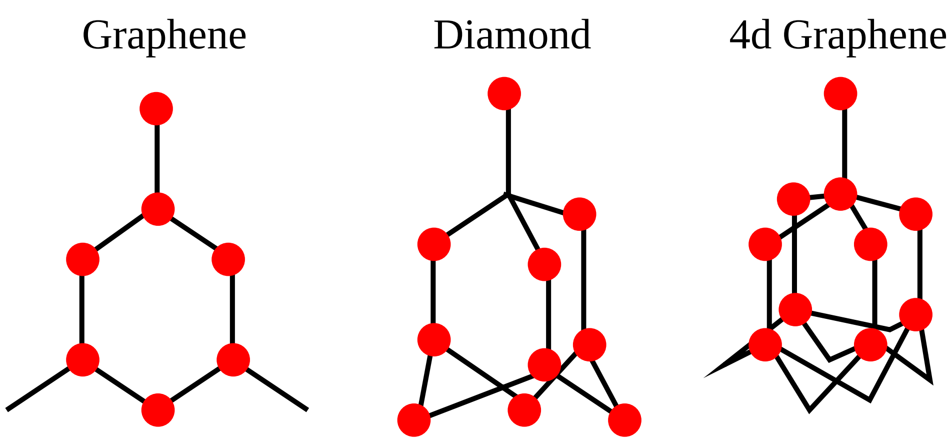
4d graphene:



Minimally doubled fermions

4d graphene:





Minimally doubled fermions

Boriçi : General family of (massless) actions on non-orthogonal lattices

$$D(p) = iB\gamma_4 \left(4C - \sum_{\mu} \cos p_\mu\right) + i \sum_{k=1}^{3} \gamma_k s_k(p)$$

where

$$\begin{aligned} s_1(p) &= \sin p_1 + \sin p_2 - \sin p_3 - \sin p_4 \\ s_2(p) &= \sin p_1 - \sin p_2 - \sin p_3 + \sin p_4 \\ s_3(p) &= \sin p_1 - \sin p_2 + \sin p_3 - \sin p_4 \end{aligned}$$

All these actions have two zeros, at $(\tilde{p}, \tilde{p}, \tilde{p}, \tilde{p})$ and $(-\tilde{p}, -\tilde{p}, -\tilde{p}, -\tilde{p})$, with

$$C = \cos \tilde{p}$$

Now we go on orthogonal lattices, where $B \sin \tilde{p} = C$

When we then put $B = 1$, after some translations of the momenta and rescalings we obtain the Boriçi-Creutz action

The Boriçi-Creutz action can be also constructed directly as a linear combination of two naive fermion formulations (*Creutz*)

Boriçi-Creutz fermions

Boriçi and Creutz: fermionic action with the free Dirac operator (in momentum space)

$$\boxed{D(p) = i\sum_\mu (\gamma_\mu \sin p_\mu + \gamma'_\mu \cos p_\mu) - 2i\Gamma + m_0}$$

where

$$\boxed{\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)} \qquad (\Gamma^2 = 1)$$

and

$$\gamma'_\mu = \Gamma\gamma_\mu\Gamma = \Gamma - \gamma_\mu$$

Useful relations:

$$\sum_\mu \gamma_\mu = \sum_\mu \gamma'_\mu = 2\Gamma, \quad \{\Gamma, \gamma_\mu\} = 1, \quad \{\Gamma, \gamma'_\mu\} = 1$$

The action vanishes at $p_1 = (0,0,0,0)$ and $p_2 = (\pi/2, \pi/2, \pi/2, \pi/2)$

$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$ selects a special direction $\rightarrow$ hypercubic breaking

A linear combination of two (physically equivalent) naive fermions , corresponding to the first two terms in the action

Boriçi-Creutz fermions

Consider the massless case:

$$D(p) = i \sum_{\mu} \gamma_\mu \sin p_\mu + i \sum_{\mu} \gamma'_\mu \cos p_\mu - 2i\Gamma$$

Boriçi-Creutz fermions

Consider the massless case:

$$D(p) = \underbrace{i \sum_\mu \gamma_\mu \sin p_\mu}_{\text{16 doublers}} + i \sum_\mu \gamma'_\mu \cos p_\mu - 2i\Gamma$$

The first term, as well-known, has 16 zeros in the first Brillouin zone, that is when any component of the momentum is 0 or π

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These are at maximal distance from the zeros of the first term

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No!

The massless Boriçi-Creutz action has only the two zeros $p_1 = (0, 0, 0, 0)$ (from the first term) and $p_2 = (\pi/2, \pi/2, \pi/2, \pi/2)$ (from the second term)

Boriçi-Creutz fermions

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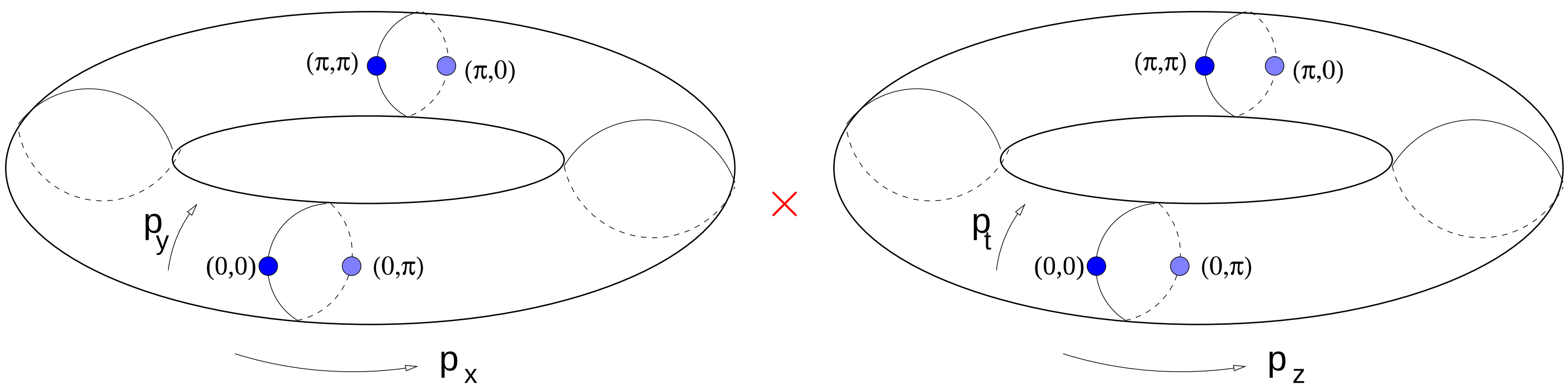
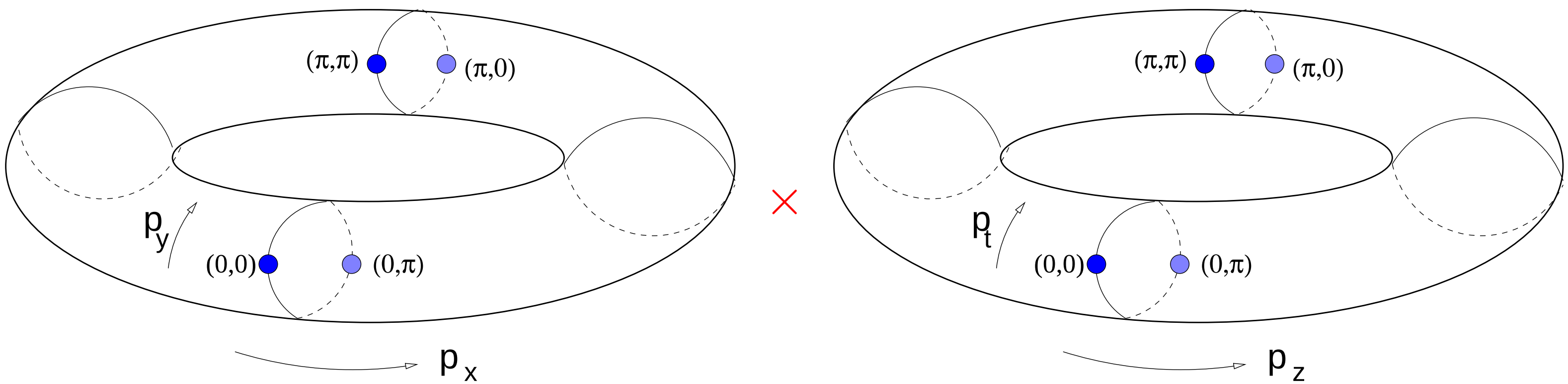
How does this happen?

Boriçi-Creutz fermions



The 16 doublers of the first naive fermion action, representing momentum space as a product of toroids

Boriçi-Creutz fermions



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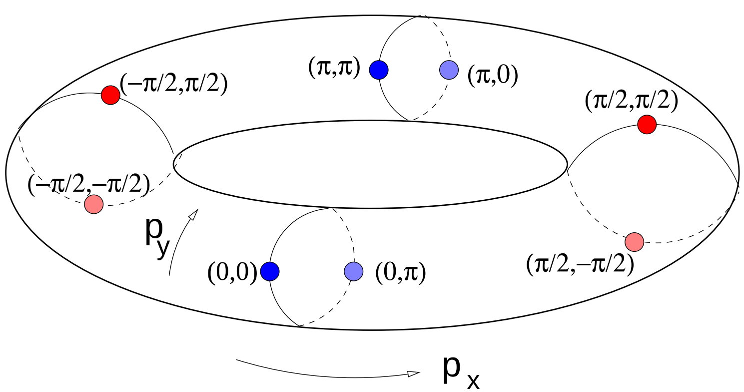

The 16 doublers of the second naive action are located at $p_\mu = \pm\pi/2$, furthest from the ones of the first naive action

(Michael Creutz, PoS LATTICE2008:080, 2008)

Boriçi-Creutz fermions

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Boriçi-Creutz fermions

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Boriçi-Creutz fermions

$$D(p) = \underbrace{i\sum_\mu \gamma_\mu \sin p_\mu}_{\substack{at\ p_2=(\pi/2,\pi/2,\pi/2,\pi/2)\\ = i\sum_\mu \gamma_\mu = 2i\Gamma}} + \underbrace{i\sum_\mu \gamma'_\mu \cos p_\mu}_{\substack{at\ p_1=(0,0,0,0)\\ = i\sum_\mu \gamma'_\mu = 2i\Gamma}} \quad \underbrace{-2i\Gamma}_{\substack{restores\ p_1\ and\ p_2\ as\ zeros\\ of\ the\ total\ action}}$$

Since at $p_2 = (\pi/2, \pi/2, \pi/2, \pi/2)$ one has $i\sum_\mu \gamma_\mu \sin p_\mu = i\sum_\mu \gamma_\mu = 2i\Gamma$, and (complementarily) at $p_1 = (0,0,0,0)$ one has $i\sum_\mu \gamma'_\mu \cos p_\mu = i\sum_\mu \gamma'_\mu = 2i\Gamma$, the addition of a third term in the action, $-2i\Gamma$, is required in order for these two values of p to remain zeros (when $m_0 = 0$) also of the combination of the two naive fermion actions

All the other 30 doublers were already lifted when one put the first and second term together, and the third term does not change this

$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)$ selects a special direction $\rightarrow$ hypercubic breaking

Note: this Dirac operator is purely anti-hermitian

Karsten-Wilczek fermions

Already in the Eighties: Karsten (1981), on a suggestion of Nielsen , and then Wilczek (1987), proposed particular actions for minimally doubled fermions

Unitary equivalent to each other, after phase redefinitions

Wilczek [PRL 59, 2397 (1987)] proposed a special choice of the function $P_\mu(p)$ which minimizes the numbers of doublers

The free Karsten-Wilczek Dirac operator

$$\boxed{D(p) = i \sum_{\mu=1}^{4} \gamma_\mu \sin p_\mu + i\gamma_4 \sum_{k=1}^{\mathbf{3}} (1 - \cos p_k)}$$

has zeros at $p_1 = (0,0,0,0)$ and $p_2 = (0,0,0,\pi)$

Drawback: it destroys the equivalence of the four directions under discrete permutations

$\rightarrow$ breaking of the hypercubic symmetry

Counterterms

The actions of minimally doubled fermions have two zeros

$\Rightarrow$ there is always a special direction in euclidean space
(the line that connects these two zeros)

Thus, these actions cannot maintain a full hypercubic symmetry

They are symmetric only under the **subgroup** of the hypercubic group which preserves (up to a sign) a **fixed direction**

For the Boriçi-Creutz action this is a major hypercube diagonal, while for other minimally doubled formulations it may not be a diagonal – for example for the Karsten-Wilczek action is the x_4 axis

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For the Boriçi-Creutz action this is a major hypercube diagonal, while for other minimally doubled formulations it may not be a diagonal – for example for the Karsten-Wilczek action is the x_4 axis

Each of these two bare actions does not contain all possible operators allowed by these respective symmetries (broken hypercubic group)

Radiative corrections generate new operators which are not present in the original bare actions

Counterterms are then necessary in order to have a **consistent renormalized theory**

This consistency requirement will uniquely determine their coefficients

Counterterms

Our task: add to the bare actions all possible counterterms allowed by the remaining symmetries (after hypercubic symmetry has been broken)

They are lattice artefacts peculiar to minimally doubled fermions

We consider operators of dimension four or lower, and we write them first in a continuum form

Afterwards, we look for convenient discretizations of these counterterms

In the following we will consider the massless case $m_0 = 0$

Chiral symmetry strongly restricts the number of possible counterterms

Since they have to anticommute with γ_5, we look only for operators in which a γ_μ matrix (or a sum of them) can be present – but not other matrices like 1, γ_5, $\gamma_\mu\gamma_5$ and $\sigma_{\mu\nu}$

For Boriçi-Creutz fermions, operators are allowed where summations over just single indices are present (in addition to the standard Einstein summation over two indices)

Then objects like $\sum_\mu \gamma_\mu = 2\Gamma$ appear

Counterterms

Three counterterms required for massless Boriçi-Creutz fermions
(S. C., M. Creutz, J. Weber & H. Wittig (2010))

Here operators are allowed with summations over single indices – then objects like $\sum_\mu \gamma_\mu = 2\Gamma$ appear

Dimension-4 fermionic counterterm: $\boxed{c_4(g_0)\,\overline{\psi}\,\Gamma\sum_\mu D_\mu\psi}$

Dimension-3 fermionic counterterm: $\boxed{\dfrac{ic_3(g_0)}{a}\,\overline{\psi}(x)\,\Gamma\,\psi(x)}$

There are counterterms also for the pure gauge part

Although at the bare level the breaking of hypercubic symmetry is a feature of the fermionic actions only, in the renormalized theory it propagates (*via the interactions between quarks and gluons*) also to the pure gauge sector

Purely gluonic counterterm for the Boriçi-Creutz action:

$$\boxed{c_P(g_0)\sum_{\lambda\rho\tau}\mathrm{tr}\,F_{\lambda\rho}(x)\,F_{\rho\tau}(x)}$$

Counterterms

Three counterterms required for massless Karsten-Wilczek fermions
(S. C., M. Creutz, J. Weber & H. Wittig (2010))

Here objects appear in which any index can be constrained to be equal to 4

Dimension-4 fermionic counterterm: $\boxed{d_4(g_0)\,\overline{\psi}\,\gamma_4 D_4\,\psi}$

Dimension-3 fermionic counterterm: $\boxed{\frac{id_3(g_0)}{a}\,\overline{\psi}(x)\,\gamma_4\,\psi(x)}$

It is not hard to imagine that in the case of Karsten-Wilczek fermions the temporal plaquettes will be renormalized differently from the other plaquettes

Indeed, the gluonic counterterm should compensate the asymmetry between these two kinds of plaquettes:

$$\boxed{d_P(g_0)\,\sum_{\rho\lambda} \mathrm{tr}\, F_{\rho\lambda}(x)\, F_{\rho\lambda}(x)\,\delta_{\rho 4}}$$

This is the only purely gluonic counterterm needed for this action, since introducing also a $\delta_{\lambda 4}$ in the above expression will produce a vanishing object

Counterterms

We can determine all these coefficients by requiring that the renormalized 1-loop propagators assume their standard forms

Perturbative calculation: S. C., M. Creutz, J. Weber & H. Wittig (2010)

Boriçi-Creutz fermions:

$$c_3(g_0) = 29.54170 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

$$c_4(g_0) = 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

$$c_P(g_0) = -0.9094 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)$$

where $\mathrm{Tr}\,(\mathrm{t^a t^b}) = \mathrm{C_2}\,\delta^{\mathrm{ab}}$

Karsten-Wilczek fermions:

$$d_3(g_0) = -29.53230 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

$$d_4(g_0) = -0.12554 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4)$$

$$d_P(g_0) = -12.69766 \cdot \frac{g_0^2}{16\pi^2} C_2 + O(g_0^4)$$

Counterterms

It is interesting to see how this works in the vacuum polarization

For Boriçi-Creutz fermions, without the purely gluonic counterterm :

$$\Pi^{(f)}_{\mu\nu}(p) = \Big(p_\mu p_\nu - \delta_{\mu\nu}p^2\Big)\left[\frac{g_0^2}{16\pi^2}C_2\Big(-\frac{8}{3}\log p^2a^2 + 23.6793\Big)\right] - \left((p_\mu + p_\nu)\sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu}\Big(\sum_\lambda p_\lambda\Big)^2\right)\frac{g_0^2}{16\pi^2}C_2 \cdot 0.9094$$

For Karsten-Wilczek fermions, without the purely gluonic counterterm :

$$\Pi^{(f)}_{\mu\nu}(p) = \Big(p_\mu p_\nu - \delta_{\mu\nu}p^2\Big)\left[\frac{g_0^2}{16\pi^2}C_2\Big(-\frac{8}{3}\log p^2a^2 + 19.99468\Big)\right] - \Big(p_\mu p_\nu\,(\delta_{\mu4} + \delta_{\nu4}) - \delta_{\mu\nu}\,(p^2\,\delta_{\mu4}\delta_{\nu4} + p_4^2)\Big)\frac{g_0^2}{16\pi^2}C_2 \cdot 12.69766$$

New terms appear, compared with a “normal” case like Wilson fermions

Although each of these actions breaks hypercubic symmetry in its appropriate and peculiar way, these new terms still satisfy the Ward identity $p^\mu \Pi^{(f)}_{\mu\nu}(p) = 0$

The cancellation of the hypercubic breaking terms of the vacuum polarization determines the coefficients of the gluonic counterterm

Counterterms

All counterterms remain of the same form at all orders of perturbation theory

Only the values of their coefficients depend on the number of loops

Exactly the same counterterms appear at the nonperturbative level, and they are required for a consistent simulation of these fermions

Counterterms not only provide additional Feynman rules for the calculation of loop amplitudes

They can also modify Ward identities – in particular, they contribute additional terms to the expressions of the conserved currents

Quark propagator and vertices

Inverting the Boriçi-Creutz action we obtain the fermion propagator $S(p)$ as

$$a\,\frac{-i\sum_\mu \gamma_\mu(\sin ap_\mu - \cos ap_\mu) - i\Gamma\left(\sum_\mu \cos ap_\mu - 2\right) + am_0}{\sum_\mu(\sin ap_\mu \sum_\nu \cos ap_\nu - 2\sin ap_\mu(\cos ap_\mu + 1) - 2\cos ap_\mu) + 8 + (am_0)^2}$$

The denominator of this propagator cannot be cast (*as instead is conveniently done for many standard actions*) in a form which possesses a definite behavior under parity transformation of each single coordinate ($p_i \to -p_i$)

By using $\{\gamma_\mu, \gamma_\nu\} = \{\gamma'_\mu, \gamma'_\nu\} = 2\delta_{\mu\nu}$ and $\{\gamma_\mu, \gamma'_\nu\} = 1 - 2\delta_{\mu\nu}$, the above quark propagator can also be written in the more convenient form

$$S(p) = a\,\frac{-i\sum_\mu \left[\gamma_\mu \sin ap_\mu - 2\,\gamma'_\mu\, \sin^2 ap_\mu/2\right] + am_0}{4\sum_\mu \left[\sin^2 ap_\mu/2 + \sin ap_\mu\left(\sin^2 ap_\mu/2 - \frac{1}{2}\sum_\nu \sin^2 ap_\nu/2\right)\right] + (am_0)^2}$$

where the limit of small p (continuum limit) is more transparent

The second pole at $ap = (\pi/2, \pi/2, \pi/2, \pi/2)$ describes (as expected) a particle of opposite chirality to the one at $ap = (0, 0, 0, 0)$

Quark propagator and vertices

Quark propagator for Karsten-Wilczek fermions (2nd pole at $ap = (0,0,0,\pi)$):

$$S(p) = a\,\frac{-i\sum_{\mu=1}^{4}\gamma_\mu \sin ap_\mu - 2i\,\gamma_4 \sum_{k=1}^{3}\sin^2\frac{ap_k}{2} + am_0}{\sum_{\mu=1}^{4}\sin^2 ap_\mu + 4\sin ap_4 \sum_{k=1}^{3}\sin^2\frac{ap_k}{2} + 4\left(\sum_{k=1}^{3}\sin^2\frac{ap_k}{2}\right)^2 + (am_0)^2}$$

Quark-quark-gluon and quark-quark-gluon-gluon vertices (Boriçi-Creutz):

$$\begin{aligned} V_1(p_1,p_2) &= -ig_0\left(\gamma_\mu \cos\frac{a(p_1+p_2)_\mu}{2} - \gamma'_\mu \sin\frac{a(p_1+p_2)_\mu}{2}\right) \\ V_2(p_1,p_2) &= \frac{1}{2}iag_0^2\left(\gamma_\mu \sin\frac{a(p_1+p_2)_\mu}{2} + \gamma'_\mu \cos\frac{a(p_1+p_2)_\mu}{2}\right) \end{aligned}$$

Quark-quark-gluon and quark-quark-gluon-gluon vertices (Karsten-Wilczek):

$$\begin{aligned} V_1(p_1,p_2) &= -ig_0\left(\gamma_\mu \cos\frac{a(p_1+p_2)_\mu}{2} + \gamma_4\,(1-\delta_{\mu 4})\,\sin\frac{a(p_1+p_2)_\mu}{2}\right) \\ V_2(p_1,p_2) &= \frac{1}{2}iag_0^2\left(\gamma_\mu \sin\frac{a(p_1+p_2)_\mu}{2} - \gamma_4\,(1-\delta_{\mu 4})\,\cos\frac{a(p_1+p_2)_\mu}{2}\right) \end{aligned}$$

(p_1 and p_2, momenta in and out of the vertex)

Self-energy

The tadpole of the self-energy can be easily computed from the vertex $V_2(p,p)$

The relevant expression for Boriçi-Creutz fermions is, in a general covariant gauge $\partial_\mu A_\mu = 0$,

$$\frac{1}{a^2}\cdot\frac{Z_0}{2}\left(1-\frac{1}{4}(1-\alpha)\right)\cdot iag_0^2 C_F\sum_\mu\Big(\gamma_\mu a p_\mu + (\Gamma-\gamma_\mu)(1+O(a^2))\Big)$$

which is equal to

$$g_0^2 C_F\,\frac{Z_0}{2}\left(1-\frac{1}{4}(1-\alpha)\right)\left(i\not{p}+\frac{i}{a}\sum_\mu(\Gamma-\gamma_\mu)\right)+O(a)$$

where

$$Z_0 = \int\frac{dp}{(2\pi)^4}\frac{1}{\widehat{p}^2} = 0.1549333....... = 24.466100\,\frac{1}{16\pi^2}$$

Terms of $O(a)$ and higher are not important here

Since $\sum_\mu \gamma_\mu = 2\Gamma$, the result of the one-loop tadpole is

$$g_0^2 C_F\frac{Z_0}{2}\left(1-\frac{1}{4}(1-\alpha)\right)\left(i\not{p}+\frac{2i\Gamma}{a}\right)$$

Self-energy

The $i\not{p}$ term is the same as for Wilson fermions, while the other term (as already noted by *Bedaque, Buchoff, Tiburzi and Walker-Loud* in 2008) would imply a power-divergent $1/a$ mixing with the dimension-3 operator $\overline{\psi}\,\Gamma\,\psi$ ...

... if not compensated by an analogous term coming from the other diagram of the self-energy, the sunset diagram

In our work we have shown that there is no such compensation

The result of the sunset diagram is

$$i\not{p}\cdot\frac{g_0^2}{16\pi^2}\,C_F\,\Bigg[\log a^2p^2-5.42642+(1-\alpha)\Big(-\log a^2p^2+7.850272\Big)\Bigg]$$

$$+m_0\cdot\frac{g_0^2}{16\pi^2}\,C_F\,\Bigg[4\,\log a^2p^2-29.48729+(1-\alpha)\Big(-\log a^2p^2+5.792010\Big)\Bigg]$$

$$+1.52766\cdot\frac{g_0^2}{16\pi^2}\,C_F\cdot i\,\Gamma\sum_\mu p_\mu$$

$$+(5.07558+6.11653\,(1-\alpha))\cdot\frac{g_0^2}{16\pi^2}\,C_F\cdot i\,\frac{\Gamma}{a}$$

Self-energy

Note that gauge invariance forces the terms proportional to $1-\alpha$ to be the same as (for example) Wilson or overlap fermions

This is an important check of the correctness of our calculations

The total self-energy (without counterterms) of a Boriçi-Creutz fermion is then given at this order by

$$\Sigma(p,m_0) = i\not{p}\,\Sigma_1(p) + m_0\,\Sigma_2(p) + c_1(g_0)\cdot i\,\Gamma\sum_\mu p_\mu + c_2(g_0)\cdot i\,\frac{\Gamma}{a}$$

with

$$\Sigma_1(p) = 1+\frac{g_0^2}{16\pi^2}\,C_F\left[\log a^2p^2+6.80663+(1-\alpha)\Big(-\log a^2p^2+4.792010\Big)\right]+O(g_0^4)$$

$$\Sigma_2(p) = 1+\frac{g_0^2}{16\pi^2}\,C_F\left[4\,\log a^2p^2-29.48729+(1-\alpha)\Big(-\log a^2p^2+5.792010\Big)\right]+O(g_0^4)$$

$$c_1(g_0) = 1.52766\cdot\frac{g_0^2}{16\pi^2}\,C_F+O(g_0^4)$$

$$c_2(g_0) = 29.54170\cdot\frac{g_0^2}{16\pi^2}\,C_F+O(g_0^4)$$

Self-energy

As expected, the two terms Γ/a coming from the tadpole and the half-circle diagrams do not cancel – in fact, they have the same sign

Notice that the parts proportional to $1-\alpha$ instead exactly cancel, as required by gauge invariance

The full inverse propagator at one loop can be written (*without counterterms*) as

$$\Sigma^{-1}(p,m_0) = \Big(1-\Sigma_1\Big)\cdot\Big\{i\not{p}+m_0\Big(1-\Sigma_2+\Sigma_1\Big)-\frac{ic_1}{2}\sum_\mu\gamma_\mu\sum_\nu p_\nu-\frac{ic_2}{a}\Gamma\Big\}$$

We can only cast the renormalized propagator in the standard form

$$\Sigma(p,m_0) = \frac{Z_2}{i\not{p}+Z_m\,m_0}$$

with the wave-function and quark mass renormalization given by

$$Z_2 = \Big(1-\Sigma_1\Big)^{-1}, \qquad Z_m = 1-\Big(\Sigma_2-\Sigma_1\Big)$$

if we cancel the Lorentz non-invariant factors (c_1 and c_2) by using the counterterms

Self-energy

The term proportional to c_1 can be eliminated by using the counterterm of the form $\overline{\psi} \sum_\mu \gamma_\mu \sum_\nu D_\nu \psi$ (permitted by the symmetries of the theory)

The term proportional to c_2 can be eliminated using the counterterm

$$\frac{1}{a} \overline{\psi} \Gamma \psi$$

which is already present in the action:

$$S(x) = \cdots + a^4 \sum_x \overline{\psi}(x) \left(m_0 - \frac{2i\Gamma}{a} \right) \psi(x)$$

For Boriçi-Creutz fermions we then determine at one loop

$$\begin{aligned} c_3(g_0) &= 29.54170 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4) \\ c_4(g_0) &= 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F + O(g_0^4) \end{aligned}$$

Self-energy

For Karsten-Wilczek fermions the result of the tadpole is

$$g_0^2 C_F \, \frac{Z_0}{2} \left(1 - \frac{1}{4}(1-\alpha)\right) \left(i\not{p} - \frac{3i\gamma_4}{a}\right)$$

The complete self-energy (without counterterms) comes out as

$$\Sigma(p, m_0) = i\not{p}\,\Sigma_1(p) + m_0\,\Sigma_2(p) + d_1(g_0)\cdot i\,\gamma_4 p_4 + d_2(g_0)\cdot i\,\frac{\gamma_4}{a}$$

where

$$\Sigma_1(p) = \frac{g_0^2}{16\pi^2}\,C_F \left[\log a^2 p^2 + 9.24089 + (1-\alpha)\Big(-\log a^2 p^2 + 4.792010\Big)\right]$$

$$\Sigma_2(p) = \frac{g_0^2}{16\pi^2}\,C_F \left[4\,\log a^2 p^2 - 24.36875 + (1-\alpha)\Big(-\log a^2 p^2 + 5.792010\Big)\right]$$

$$d_1(g_0) = -0.12554 \cdot \frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

$$d_2(g_0) = -29.53230 \cdot \frac{g_0^2}{16\pi^2}\,C_F + O(g_0^4)$$

Self-energy

The full inverse propagator at one loop can be written, without counterterms, as

$$\Sigma^{-1}(p, m_0) = \Big(1 - \Sigma_1\Big) \cdot \Big(i\not{p} + m_0 \Big(1 - \Sigma_2 + \Sigma_1\Big) - id_1\, \gamma_4 p_4 - \frac{id_2}{a}\, \gamma_4\Big)$$

Similarly to before, by adding to the Karsten-Wilczek action counterterms of the form

$$\overline{\psi}\, \gamma_4\, D_4\, \psi, \qquad \frac{1}{a}\, \overline{\psi}\, \gamma_4\, \psi$$

the contributions which are not Lorentz invariant can be eliminated, and the renormalized propagator can be written in the standard form

$$\Sigma(p, m_0) = \frac{Z_2}{i\not{p} + Z_m\, m_0}$$

Then, at one loop

$$\begin{aligned} d_3(g_0) &= -29.53230 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4) \\ d_4(g_0) &= -0.12554 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4) \end{aligned}$$

Renormalization of the mass

Chiral symmetry protects the quark mass m_0 from an additive renormalization

The relation between the bare and renormalized quark masses, m_0 and $m_{\rm R}$, is then

$$m_{\rm R} = Z_m \, m_0$$

The full expression for the renormalization factors of the scalar and pseudo-scalar densities at one loop is

$$Z_S = Z_P = 1 - \Big(\Lambda_S + \Sigma_1 \Big)$$

where Λ_S is the result for the one-loop vertex diagram of the scalar density

Λ_S is exactly equal to the $O(g_0^2)$-contribution to the quark self-energy Σ_2, but comes with an opposite sign

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Λ_S is exactly equal to the $O(g_0^2)$-contribution to the quark self-energy Σ_2, but comes with an opposite sign

Then, the renormalization factors Z_S and Z_P satisfy

$$1/Z_m = Z_S = Z_P$$

The last equality is a consequence of chiral symmetry

The renormalization of the quark mass for minimally doubled fermions has the same form as (for instance) overlap fermions

Conserved vector and axial currents

Z_V and Z_A (of the local currents) are not equal to one

The local vector and axial currents are not conserved

We need to consider the chiral Ward identities in order to work with currents which are protected from renormalization

We have constructed the conserved vector and axial currents, and verified that at one loop their renormalization constants are equal to one

We act on the Boriçi-Creutz action in position space

$$S = a^4 \sum_x \left[\frac{1}{2a} \sum_\mu \left[\overline{\psi}(x)\,(\gamma_\mu + i\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) \right.\right.$$
$$\left.\left. -\overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\gamma'_\mu)\,U^\dagger_\mu(x)\,\psi(x)\right] + \overline{\psi}(x)\left(m_0 - \frac{2i\Gamma}{a}\right)\psi(x) \right]$$

with the vector transformation

$$\delta_V \psi = i\alpha\,\psi, \quad \delta_V \overline{\psi} = -i\alpha\,\overline{\psi}$$

or the axial transformation

$$\delta_A \psi = i\alpha\,\gamma_5 \psi, \quad \delta_A \overline{\psi} = i\alpha\,\overline{\psi}\gamma_5$$

Conserved vector and axial currents

Take the Ward identity

$$\left\langle \frac{\delta O(x_1 \cdots x_n)}{\delta \alpha(x)} \right\rangle = \left\langle O(x_1 \cdots x_n) \, \frac{\delta S}{\delta \alpha(x)} \right\rangle$$

For an axial transformation we have

$$i \left\langle \frac{\delta S}{\delta \alpha(x)} \right\rangle = \nabla_x^{\mu} \left\langle O(x_1 \cdots x_n) \, A_\mu(x) \right\rangle$$

(and similarly for a vector transformation)

For on-shell matrix elements, $O(x_1 \cdots x_n)$ is a product of the operators which generate the required initial and final states from the vacuum

Applying the axial transformation $\delta_A \psi$, we look for a current $A_\mu^{cons}(x)$ which satisfies

$$i \, \frac{\delta S}{\delta \alpha(x)} = \nabla^{\star} \, A_\mu^{cons}(x) = A_\mu^{cons}(x) - A_\mu^{cons}(x - a\widehat{\mu})$$

If the axial transformation is a symmetry of the action, then this current $A_\mu^{cons}(x)$ is conserved

(and similarly for the vector current)

Conserved vector and axial currents

Using translational invariance, the vector transformation gives

$$\delta S = \frac{ia^3}{2} \sum_x \alpha(x) \sum_\mu \Big[\overline{\psi}(x - a\widehat{\mu})\,(\gamma_\mu + i\gamma'_\mu)\,U_\mu(x - a\widehat{\mu})\,\psi(x) \\ -\overline{\psi}(x + a\widehat{\mu})\,(\gamma_\mu - i\gamma'_\mu)\,U^\dagger_\mu(x)\,\psi(x) \\ -\overline{\psi}(x)\,(\gamma_\mu + i\gamma'_\mu)\,U_\mu(x)\,\psi(x + a\widehat{\mu}) \\ +\overline{\psi}(x)\,(\gamma_\mu - i\gamma'_\mu)\,U^\dagger_\mu(x - a\widehat{\mu})\,\psi(x - a\widehat{\mu}) \Big]$$

The corresponding expression for the axial transformation is (for $m_0 = 0$)

$$\delta S = \frac{ia^3}{2} \sum_x \alpha(x) \sum_\mu \Big[\overline{\psi}(x - a\widehat{\mu})\,(\gamma_\mu + i\gamma'_\mu)\,\gamma_5\,U_\mu(x - a\widehat{\mu})\,\psi(x) \\ -\overline{\psi}(x + a\widehat{\mu})\,(\gamma_\mu - i\gamma'_\mu)\,\gamma_5\,U^\dagger_\mu(x)\,\psi(x) \\ -\overline{\psi}(x)\,(\gamma_\mu + i\gamma'_\mu)\,\gamma_5\,U_\mu(x)\,\psi(x + a\widehat{\mu}) \\ +\overline{\psi}(x)\,(\gamma_\mu - i\gamma'_\mu)\,\gamma_5\,U^\dagger_\mu(x - a\widehat{\mu})\,\psi(x - a\widehat{\mu}) \Big]$$

Axial symmetry only works for $m_0 = 0$: $\quad \overline{\psi}(x)\psi(x) \to 2i\alpha(x)\,\overline{\psi}(x)\gamma_5\psi(x)$

Conserved vector and axial currents

We then obtain the conserved vector current for Boriçi-Creutz fermions:

$$V_\mu^{cons}(x) = \frac{1}{2}\Big[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,U^\dagger_\mu(x)\,\psi(x)\Big]$$

The axial current (conserved in the case $m_0 = 0$) is

$$A_\mu^{cons}(x) = \frac{1}{2}\Big[\overline{\psi}(x)\,(\gamma_\mu + i\,\gamma'_\mu)\,\gamma_5\,U_\mu(x)\,\psi(x+a\widehat{\mu}) + \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu - i\,\gamma'_\mu)\,\gamma_5\,U^\dagger_\mu(x)\,\psi(x)\Big]$$

We can only obtain isospin-singlet currents, since the action describes a degenerate doublet of fermions

We have then computed the renormalization of these point-split currents

We give here the results for the individual diagrams of the conserved vector current

For the conserved axial current the numbers are the same, and one just needs to replace γ_μ with $\gamma_\mu\gamma_5$, and Γ with $\Gamma\gamma_5$

Conserved vector and axial currents

The vertex diagram gives the result

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[-\log a^2p^2 + 0.61800 + (1-\alpha)\Big(\log a^2p^2 - 1.73375 \Big) \right] + c_1^{vtx}(g_0)\, \Gamma$$

with the coefficient of the mixing given by

$$c_1^{vtx}(g_0) \;=\; -0.43749 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

The result of the sails is

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[4.80841 - 6.11653\,(1-\alpha) \right] + c_1^{sls}(g_0)\, \Gamma$$

with the coefficient of the mixing given by

$$c_1^{sls}(g_0) \;=\; -1.09017 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

Conserved vector and axial currents

Finally, the operator tadpole gives the same result as for Wilson fermions:

$$-g_0^2\, C_F\, \gamma_\mu\, \frac{Z_0}{2} \left(1 - \frac{1}{4}(1-\alpha)\right)$$

The sum of all these diagrams is

$$\frac{g_0^2}{16\pi^2}\, C_F\, \gamma_\mu \left[-\log a^2p^2 - 6.80664 + (1-\alpha)\left(\log a^2p^2 - 4.79202 \right) \right] \quad + c_1^{cv}(g_0)\, \Gamma$$

with the coefficient of the mixing given by

$$c_1^{cv}(g_0) = -1.52766 \cdot \frac{g_0^2}{16\pi^2}\, C_F + O(g_0^4)$$

The term proportional to γ_μ exactly compensates the contribution of $\Sigma_1(p)$ from the quark self-energy (wave-function renormalization)

Conserved vector and axial currents

Finally, the operator tadpole gives the same result as for Wilson fermions:

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The term proportional to γ_μ exactly compensates the contribution of $\Sigma_1(p)$ from the quark self-energy (wave-function renormalization)

But what about the mixing term, proportional to Γ ?

We should take into account the counterterms . . .

Conserved vector and axial currents

The counterterm $\overline{\psi}(x)\,\frac{i\Gamma}{a}\,\psi(x)$ does **<u>not</u>** modify these Ward identities

On the contrary, the counterterm

$$\frac{c_4(g_0)}{4}\sum_\mu\sum_\nu\Big(\overline{\psi}(x)\,\gamma_\nu\,U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\,\gamma_\nu\,U_\mu^\dagger(x)\,\psi(x)\Big)$$

generates new terms in the Ward identities and then in the conserved currents

The additional term in the conserved vector current so generated reads

$$\frac{c_4(g_0)}{4}\left[\overline{\psi}(x)\Big(\sum_\nu\gamma_\nu\Big)\,U_\mu(x)\,\psi(x+a\widehat{\mu})+\overline{\psi}(x+a\widehat{\mu})\Big(\sum_\nu\gamma_\nu\Big)\,U_\mu^\dagger(x)\,\psi(x)\right]$$

Its 1-loop contribution is easy to compute (c_4 *is already of order* g_0^2 !): <u>$c_4(g_0)\,\Gamma$</u>

The value of c_4 is known from the self-energy $\Rightarrow\quad c_4(g_0)\,\Gamma\;=\;-c_1^{cv}(g_0)\,\Gamma$

Only this value of c_4 exactly cancels the Γ mixing term present in the 1-loop conserved current without counterterms

Thus, we obtain that the renormalization constant of these point-split currents is one – which confirms that they are conserved currents

Everything is consistent. . .

Conserved vector and axial currents

Let us now consider the Karsten-Wilczek action in position space:

$$
\begin{aligned}
S \; = \; & a^4 \sum_x \left[\frac{1}{2a} \sum_{\mu=1}^{4} \left[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu(x)\, \psi(x + a\widehat{\mu}) \right.\right. \\
& \left.\left. - \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu^\dagger(x)\, \psi(x) \right] + \overline{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \right]
\end{aligned}
$$

Conserved vector and axial currents

Let us now consider the Karsten-Wilczek action in position space:

$$
\begin{aligned}
S \quad = \quad & a^4 \sum_x \left[\frac{1}{2a} \sum_{\mu=1}^{4} \left[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu(x)\, \psi(x + a\widehat{\mu}) \right.\right. \\
& \left.\left. - \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) U_\mu^\dagger(x)\, \psi(x) \right] + \overline{\psi}(x) \left(m_0 + \frac{3i\gamma_4}{a} \right) \psi(x) \right]
\end{aligned}
$$

After adding the counterterms, application of the chiral Ward identities gives for the conserved axial current of Karsten-Wilczek fermions

$$
\begin{aligned}
A_\mu^{\mathrm{c}}(x) \quad = \quad & \frac{1}{2} \left(\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) \gamma_5\, U_\mu(x)\, \psi(x + a\widehat{\mu}) \right. \\
& \left. + \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(1 - \delta_{\mu 4}\right)\right) \gamma_5\, U_\mu^\dagger(x)\, \psi(x) \right) \\
& + \frac{d_4(g_0)}{2} \left(\overline{\psi}(x)\, \gamma_4\gamma_5\, U_4(x)\, \psi(x + a\widehat{4}) + \overline{\psi}(x + a\widehat{4})\, \gamma_4\gamma_5\, U_4^\dagger(x)\, \psi(x) \right)
\end{aligned}
$$

Once more, is a simple expression which involve only nearest-neighbour sites

We checked explicitly that its renormalization constant is one

Vacuum polarization

Our focus here: the radiative corrections to the gluon propagator due to fermion loops

Contributions to the vacuum polarization due to loops of gluons and ghosts: independent of the lattice fermionic action chosen (*at one loop*)

$\Rightarrow$ do not provide informations relevant for hypercubic breaking

Only the fermionic loops are able to generate hypercubic-breaking terms (as it in the end happens for both Karsten-Wilczek and Boriçi-Creutz fermions)

The fermionic contribution to the vacuum polarization for one flavor of Wilson fermions *(where neither breaking of hypercubic symmetry nor fermion doubling occur)* is

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right) \left[\frac{g_0^2}{16\pi^2} C_t \left(-\frac{4}{3} \log p^2 a^2 + 4.337002 \right) \right]$$

where $\mathrm{Tr}\,(\mathrm{t^a t^b}) = \mathrm{C_2}\,\delta^{\mathrm{ab}}$

We can see that this (gauge invariant) result satisfies the Ward identity $p^\mu \Pi^{(f)}_{\mu\nu}(p) = 0$, which expresses the conservation of the fermionic current

Vacuum polarization

However, for both Karsten-Wilczek and Boriçi-Creutz fermions the quark loops are able to generate hypercubic-breaking terms , and this is what indeed happens

It is thus evident that these hypercubic-breaking contributions must be eliminated, and this can be achieved by employing the gluonic counterterms

Indeed, the expressions for the gluonic counterterms in momentum space are structurally identical to the additional terms in the vacuum polarization

We can then eliminate these hypercubic-breaking terms and so determine the coefficients of the gluonic counterterms

Vacuum polarization

For Boriçi-Creutz fermions (without the purely gluonic counterterm) :

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right)\left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3}\log p^2 a^2 + 23.6793\right)\right]$$
$$-\left((p_\mu + p_\nu)\sum_\lambda p_\lambda - p^2 - \delta_{\mu\nu}\left(\sum_\lambda p_\lambda\right)^2\right)\frac{g_0^2}{16\pi^2} C_2 \cdot 0.9094$$

For Karsten-Wilczek fermions (without the purely gluonic counterterm) :

$$\Pi^{(f)}_{\mu\nu}(p) = \left(p_\mu p_\nu - \delta_{\mu\nu} p^2\right)\left[\frac{g_0^2}{16\pi^2} C_2 \left(-\frac{8}{3}\log p^2 a^2 + 19.99468\right)\right]$$
$$-\left(p_\mu p_\nu (\delta_{\mu 4} + \delta_{\nu 4}) - \delta_{\mu\nu}\left(p^2\, \delta_{\mu 4}\delta_{\nu 4} + p_4^2\right)\right)\frac{g_0^2}{16\pi^2} C_2 \cdot 12.69766$$

There are new terms, compared with a standard situation like Wilson fermions

Although each of these actions breaks hypercubic symmetry in its appropriate and peculiar way, these new terms still satisfy the Ward identity $p^\mu \Pi^{(f)}_{\mu\nu}(p) = 0$

Very important: there are no power divergences ($1/a^2$ or $1/a$) in our results for the vacuum polarization!

Vacuum polarization

In principle divergences like $1/a^2$ or $1/a$ could have arisen

We have checked that tadpole contributions, when nonzero, are always of equal magnitude and opposite sign with respect to the sunset diagram

It is interesting to note that the numbers for these diagrams are much larger than in the case of Wilson fermions, where the coefficient of $g_0^2 C_2/16\pi^2$ *for the tadpole is* -9.67590

For Karsten-Wilczek fermions this number turns out to be -36.31464 *for each spatial component and* 7.12931 *for the temporal component, and for Boriçi-Creutz fermions it is even larger,* -73.71980

We can understand on general grounds why such power-divergences cannot appear, from the fact that to construct hypercubic breaking terms one has to employ objects like Γ and $\sum_\mu p_\mu$ (for Boriçi-Creutz fermions) and γ_4 and p_4 (for Karsten-Wilczek fermions)

However, after the traces of the fermions loops are evaluated there are no Dirac structures left, and no momenta can appear at the $1/a^2$ level

Linear pieces in the momenta, which would be required in case of a $1/a$ power divergence, are instead prohibited by the symmetry of the diagrams

Vacuum polarization

The hyper-cubic-breaking terms in the vaccum polarizazion can be put for both actions in the same algebraic form :

$$p^2\{\gamma_\mu,\Gamma\}\{\gamma_\nu,\Gamma\}+\delta_{\mu\nu}\{\not p,\Gamma\}\{\not p,\Gamma\}-\frac{1}{2}\{\not p,\Gamma\}\Big(\{\gamma_\mu,\not p\}\{\gamma_\nu,\Gamma\}+\{\gamma_\nu,\not p\}\{\gamma_\mu,\Gamma\}\Big)$$

In the case of Karsten-Wilczek fermions we have the same expression but with Γ replaced by $\gamma_4/2$

This substitution is suggested by comparison of the standard relation of Boriçi-Creutz fermions

$$\Gamma=\frac{1}{4}\sum_\mu(\gamma_\mu+\gamma'_\mu)$$

with the formula

$$\gamma_4=\frac{1}{2}\sum_\mu(\gamma_\mu+\gamma'_\mu)$$

expressing the symmetries of the action (as can be seen expanding the propagator of the Karsten-Wilczek action around the second Fermi point)

Is there any deeper significance to this structural **equivalence** of the hyper-cubic-breaking structures in the vacuum polarizations?

Simulations

For Boriçi-Creutz fermions the renormalized action reads

$$
\begin{aligned}
S^f_{BC} \quad = \quad a^4 \sum_x \Bigg\{ \frac{1}{2a} \sum_{\mu=1}^{4} \Big[& \overline{\psi}(x)\,(\gamma_\mu + c_4(\beta)\,\Gamma + i\gamma'_\mu)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) \\
& -\overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu + c_4(\beta)\,\Gamma - i\gamma'_\mu)\,U^\dagger_\mu(x)\,\psi(x)\Big] \\
& +\overline{\psi}(x)\left(m_0 + \widetilde{c}_3(\beta)\,\frac{i\,\Gamma}{a}\right)\psi(x) \\
& +\beta \sum_{\mu<\nu}\left(1 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,P_{\mu\nu}\right) + c_P(\beta)\sum_{\mu\nu\rho}\mathrm{tr}\,F^{lat}_{\mu\rho}(x)\,F^{lat}_{\rho\nu}(x)\Bigg\}
\end{aligned}
$$

We have redefined the coefficient of the dimension-3 counterterm, using
$\widetilde{c}_3(\beta) = -2 + c_3(\beta)$ (which does not vanish at tree level)

F^{lat} is a lattice discretization of the field-strength tensor

Simulations

The renormalized action for Karsten-Wilczek fermions reads

$$S^f_{KW} = a^4 \sum_x \Bigg\{ \frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x)\,(\gamma_\mu (1 + d_4(\beta)\,\delta_{\mu 4}) - i\gamma_4\,(1-\delta_{\mu 4}))\,U_\mu(x)\,\psi(x+a\widehat{\mu})$$

$$-\overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu (1 + d_4(\beta)\,\delta_{\mu 4}) + i\gamma_4\,(1-\delta_{\mu 4}))\,U^\dagger_\mu(x)\,\psi(x) \Big]$$

$$+\overline{\psi}(x)\left(m_0 + \widetilde{d}_3(\beta)\,\frac{i\,\gamma_4}{a}\right)\psi(x)$$

$$+\beta \sum_{\mu<\nu} \left(1 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{tr}\,P_{\mu\nu}\right)\Big(1 + d_P(\beta)\,\delta_{\mu 4}\Big) \Bigg\}$$

where $\widetilde{d}_3(\beta) = 3 + d_3(\beta)$ has a non-zero value at tree level

In perturbation theory the coefficients of the counterterms have the expansions

$$\begin{aligned}
\widetilde{c}_3(g_0) &= -2 + c_3^{(1)} g_0^2 + c_3^{(2)} g_0^4 + \ldots; & \widetilde{d}_3(g_0) &= 3 + d_3^{(1)} g_0^2 + d_3^{(2)} g_0^4 + \ldots \\
c_4(g_0) &= c_4^{(1)} g_0^2 + c_4^{(2)} g_0^4 + \ldots; & d_4(g_0) &= d_4^{(1)} g_0^2 + d_4^{(2)} g_0^4 + \ldots \\
c_P(g_0) &= c_P^{(1)} g_0^2 + c_P^{(2)} g_0^4 + \ldots; & d_P(g_0) &= d_P^{(1)} g_0^2 + d_P^{(2)} g_0^4 + \ldots
\end{aligned}$$

Simulations

In perturbation theory the four-dimensional counterterm to the fermionic action is necessary for the proper construction of the conserved currents

Its coefficient, as determined from the one-loop self-energy, has exactly the right value for which the conserved currents remain <u>unrenormalized</u>

Another effect of radiative corrections is to move the poles of the quark propagator away from their tree-level positions

It is the task of the dimension-3 counterterm, for the appropriate value of the coefficient c_3 (or d_3), to bring the two poles back to their original locations

These shifts can introduce oscillations in some hadronic correlation functions *(similarly to staggered fermions)*

One possible way to determine c_3 (d_3): tune it in appropriately chosen correlation functions until these oscillations are removed

No sign problem for the Monte Carlo generation of configurations: the gauge action is real, and the eigenvalues of the Dirac operator come in complex conjugate pairs $\rightarrow$ fermion determinant always non-negative

Simulations

The purely gluonic counterterm for Boriçi-Creutz fermions introduces in the renormalized action operators of the kind $E \cdot B$, $E_1 E_2$, $B_2 B_3$ *(and similar)*

In a Lorentz invariant theory, instead, only the terms E^2 and B^2 are allowed

Fixing the coefficient c_P could then be done by measuring $\langle E \cdot B \rangle$, $\langle E_1 E_2 \rangle$, $\cdots$, and tuning c_P in such a way that one (or more) of these expectation values is restored to its proper value pertinent to a Lorentz invariant theory, i.e. zero

These effects could turn out to be rather small, given that in the tree-level action only the fermionic part breaks the hypercubic symmetry

It could also be that other derived quantities are more sensitive to this coefficient, and more suitable for its nonperturbative determination

In general one can look for Ward identities in which violations of the standard Lorentz invariant form, as functions of c_P, occur

For Karsten-Wilczek fermions the purely gluonic counterterm introduces an asymmetry between the plaquettes with a temporal index and the other ones

One could then fix d_P by computing a Wilson loop lying entirely in two spatial directions, and then equating its result to an ordinary Wilson loop which also extends in the time direction

Summary

- Boriçi-Creutz and Karsten-Wilczek fermions are described by a fully consistent renormalized quantum field theory
- Three counterterms need to be added to the bare actions
- All their coefficients can be calculated in perturbation theory – or nonperturbatively from Monte Carlo simulations
- After these subtractions are consistently taken into account, the power divergence in the self-energy is eliminated
- No other power divergences occur for all quantities that we calculated
- Scalar, pseudoscalar and tensor operators show no new mixings at all
- Local vector and axial currents mix with new operators which are not invariant under the hypercubic group
- The vacuum polarization does not present new divergences
- Conserved vector and axial currents can be defined, and they involve only nearest-neighbors sites
  - they do not have mixings, and their renormalization constant is one
  - one of the very few cases where one can define a simple conserved axial current (also ultralocal)

Towards better actions

It would be of substantial interest to find minimally doubled actions that (like the above two standard cases) have the correct continuum limit, but that require fewer counterterms, or even possibly none at all

We have made some investigations to explore these issues

Can we have minimally doubled fermions which require fewer than three counterterms?

. . . *maybe even just one?*

. . . *and maybe even none?*

We introduce here new nearest-neighbor minimally doubled actions which depend on 2 continuous parameters

For each counterterm, there exist curves in the parameter space on which its coefficient vanishes

$\Rightarrow$ renormalized actions with only 2 counterterms

Besides these generalized Karsten-Wilczek actions *(and moreover some also with next-to-nearest-neighbor interactions)*, we have also constructed generalized Boriçi-Creutz actions

Towards better actions

For all generalized Karsten-Wilczek actions that we introduce here, the 3 possible counterterms are the same of the standard Karsten-Wilczek fermions

This happens because both poles of the quark propagator still lie entirely on the temporal axis, and thus the temporal direction is always selected as the special one *(irrespective of the values of the parameters* α *and* λ *describing the actions)*

Furthermore, the spinorial structure of all these actions is also the same

Thus, P is a symmetry, and also CT (Bedaque et al. , 2008), but T and C separately are violated *(unless the actions are properly renormalized)*

The values of the coefficients of the counterterms for which one obtains a consistent renormalized theory depend on the particular choices of α and λ

We investigate what happens when one varies these parameters, and see if one can remove some of the counterterms

The values of the coefficients of the counterterms for which the hypercubic symmetry is restored are continuous real functions of α and λ

$\rightarrow$ in general there will be values of the these parameters for which some of these functions vanish

Towards better actions

One of the motivations for these investigations:

`for standard Boriçi-Creutz and Karsten-Wilczek fermions the two diagrams of the 1-loop quark self-energy (sunset and tadpole) always give contributions of opposite sign to the dimension-three counterterm (the one which scales as` $1/a$`)`

One could suspect that using a generalization of these actions an exact cancellation can occur for some values of the parameters α and λ, with the effect that this counterterm (or possibly in general other counterterms) can be removed from the game

This is indeed what happens!

We have found a few curves in the parameter space spanned by α and λ for which one of the counterterms can be removed

Then, the renormalized actions corresponding to these particular choices of the parameters require only 2 counterterms

Moreover, this means in quenched QCD there are many choices of α and λ for which only one counterterm remains (out of originally 2)

Nearest-neighbor minimally doubled actions

We study the class of (bare) nearest-neighbor fermionic actions

$$S^f(x;\alpha,\lambda) = a^4 \sum_x \Bigg[\frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x) \left(\gamma_\mu - i\gamma_4 \left(\lambda + \delta_{\mu 4}(\cot\alpha - \lambda)\right)\right) U_\mu(x)\, \psi(x + a\widehat{\mu})$$

$$- \overline{\psi}(x + a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(\lambda + \delta_{\mu 4}(\cot\alpha - \lambda)\right)\right) U_\mu^\dagger(x)\, \psi(x) \Big]$$

$$+ \overline{\psi}(x) \left(m_0 + \frac{i\gamma_4}{a} \left(3\lambda + \cot\alpha\right) \right) \psi(x) \Bigg]$$

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$$-\overline{\psi}(x+a\widehat{\mu}) \left(\gamma_\mu + i\gamma_4 \left(\lambda + \delta_{\mu 4}(\cot\alpha - \lambda)\right)\right) U_\mu^\dagger(x)\, \psi(x) \Big]$$

$$+\overline{\psi}(x) \left(m_0 + \frac{i\gamma_4}{a} \left(3\lambda + \cot\alpha\right) \right) \psi(x) \Bigg]$$

These Wilson-like minimally doubled fermions satisfy γ_5-hermiticity and have $\mu = 4$ as a special direction *(like for the standard Karsten-Wilczek action)*

They can also be expressed in the simple form

$$a^4 \sum_x \overline{\psi}(x) \left\{ \frac{1}{2} \sum_\mu \left[\gamma_\mu (\nabla_\mu + \nabla_\mu^\star) - ia\gamma_4 \left(\lambda + \delta_{\mu 4}(\cot\alpha - \lambda)\right) \nabla_\mu^\star \nabla_\mu \right] + m_0 \right\} \psi(x)$$

where the lattice discretizations of the covariant derivative are

$$\nabla_\mu \psi(x) = \frac{U_\mu(x)\psi(x+a\widehat{\mu}) - \psi(x)}{a}, \quad \nabla_\mu^\star \psi(x) = \frac{\psi(x) - U_\mu^\dagger(x-a\widehat{\mu})\psi(x-a\widehat{\mu})}{a}$$

Nearest-neighbor minimally doubled actions

In momentum space the Dirac operator of the above minimally doubled fermions reads, in the free case,

$$\mathcal{D}^f\,(p;\alpha,\lambda) = \frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu \sin ap_\mu + \frac{i\gamma_4}{a}\left[\lambda\sum_{k=1}^{3}(1-\cos ap_k) + \cot\alpha\,(1-\cos ap_4)\right] + m_0$$

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The two zeros, at $a\bar{p}_1 = (0,0,0,0)$ and $a\bar{p}_2 = (0,0,0,-2\alpha)$, describe two fermions of equal mass and opposite chirality

The range of α can be taken as $0 < \alpha < \pi$

For $\alpha = 0$ and $\alpha = \pi$ the action becomes singular ($\cot\alpha = \infty$)

Although for the quark propagators corresponding to α and $\pi - \alpha$ the distance between the poles is the same, the actions corresponding to these two choices of α are not equivalent (even for the same value of λ)

Varying λ does not change the location of any of the zeros – this parameter has only the task of decoupling the 14 other fermions from the naive fermionic action giving them a mass of order $1/a$

It must also be $\lambda > (1-\cos\alpha)/(2\sin\alpha)$ to avoid the appearance of other doublers

Nearest-neighbor minimally doubled actions

All the actions considered here have the correct leading behavior for small p *(irrespective of the values of α and λ)*

All these actions still contain only nearest-neighbor interactions, that is they are Wilson-like with hopping terms of only one unit of lattice spacing

For this reason they are rather cheap to simulate – they are a little more expensive than Wilson fermions because the spinor matrices are slightly more complicated

The computational effort will be about a few times the one required for Wilson fermions

For $\lambda = 1/\sin\alpha$ our actions can be cast, after a redefinition of p_4, into the actions written by Creutz in Fourier space in 2010, which in the free massless case read

$$\mathcal{D}^{\mathrm{Creutz}}(p;\alpha) = \frac{i}{a}\sum_{k=1}^{3}\gamma_k \sin ap_k + \frac{i\gamma_4}{a\sin\alpha}\left(\cos\alpha + 3 - \sum_{\mu=1}^{4}\cos ap_\mu\right)$$

Furthermore, when this choice of λ is taken, the standard Karsten-Wilczek action can be then obtained as a special case by setting $\alpha = \pi/2$

Nearest-neighbor minimally doubled actions

P is a symmetry, and also CT, but T and C separately are violated unless the action is properly renormalized – like for the standard Karsten-Wilczek action

Then, the counterterms that must be added to these generalized actions are the same needed for the standard Karsten-Wilczek action

In quenched QCD only 2 of them are needed

Nearest-neighbor minimally doubled actions

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Then, the counterterms that must be added to these generalized actions are the same needed for the standard Karsten-Wilczek action

In quenched QCD only 2 of them are needed

One can construct a conserved axial current for all these actions, which only involves nearest-neighbor sites:

$$A_\mu^{cons}(x;\alpha,\lambda) \;=\; \frac{1}{2}\Big(\overline{\psi}(x)\,(\gamma_\mu - i\gamma_4\,(\lambda + \delta_{\mu 4}(\cot\alpha - \lambda)))\,\gamma_5\,U_\mu(x)\,\psi(x+a\widehat{\mu})$$

$$+\overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu + i\gamma_4\,(\lambda + \delta_{\mu 4}(\cot\alpha - \lambda)))\,\gamma_5\,U_\mu^\dagger(x)\,\psi(x)\Big)$$

$$+\frac{d_4(g_0)}{2}\Big(\overline{\psi}(x)\,\gamma_4\gamma_5\,U_4(x)\,\psi(x+a\widehat{4}) + \overline{\psi}(x+a\widehat{4})\,\gamma_4\gamma_5\,U_4^\dagger(x)\,\psi(x)\Big)$$

This is particularly important, as not many fermionic formulations exist for which a conserved axial current exists and is of such a simple form

1-loop calculations

The values of the coefficients of the counterterms for which these actions are properly renormalized can be determined by computing in the cases of d_3 and d_4 the quark self-energy

For these specific values, the hypercubic-breaking factors in the radiative corrections disappear

In the case of d_{P} one enforces the restoration of the hypercubic symmetry on the renormalized vacuum polarization of the gluon

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In the case of d_{P} one enforces the restoration of the hypercubic symmetry on the renormalized vacuum polarization of the gluon

Due to the non-trivial form of the denominator of the quark propagator, it is not possible to provide results with an analytic dependence on α or λ

The search for the special values of these parameters which remove the hypercubic-breaking factors in the 1-loop quark self-energy and vacuum polarization must then be carried out numerically , through a sample of many values of α and λ

The tadpole of the self-energy however can be calculated analytically , and its result has a simple dependence on α and λ

1-loop calculations

In a general covariant gauge one obtains

$$T = \frac{1}{a^2}\cdot\frac{Z_0}{2}\left(1-\frac{1}{4}(1-\xi)\right)\cdot iag_0^2\,C_F\sum_{\mu=1}^{4}\Big(\gamma_\mu a p_\mu - \gamma_4\left(\lambda+\delta_{\mu 4}(\cot\alpha-\lambda)\right)\Big)$$

$$= g_0^2\,C_F\,\frac{Z_0}{2}\left(1-\frac{1}{4}(1-\xi)\right)\left(i\not{p}-\frac{i\gamma_4}{a}\,(3\lambda+\cot\alpha)\right)$$

The quantity Z_0 is an often-recurring lattice integral, defined as

$$Z_0 = \int_{-\pi/a}^{\pi/a}\frac{d^4p}{(2\pi)^4}\,\frac{1}{\widehat{p}^2} = 0.1549333\ldots = \frac{24.466100\ldots}{16\pi^2}, \qquad \widehat{p}^2 = \frac{4}{a^2}\sum_{\mu}\sin^2\left(\frac{ap_\mu}{2}\right)$$

The result for the $i\not{p}$ term is the same of Wilson fermions

The other term, which is linearly divergent as $1/a$, has a functional form already present in the bare minimally doubled action, where however its coefficient is a fixed number

In the renormalized action instead it becomes a counterterm, whose coefficient must be properly adjusted as a function of the gauge coupling

1-loop calculations

The $1/a$ term of the tadpole diverges not only when $a \to 0$ *(contributing so to the relevant counterterm d_3)*, but also when $\alpha \to 0$ – in the latter case with a behavior which goes like $1/\sin\alpha$ *(for fixed lattice spacing)*

It also diverges at the other end of the range, $\alpha \to \pi$, with a similar behavior

To carry out the calculations of the two other diagrams required for the tuning of the counterterms we have used a set of computer codes written in the algebraic manipulation language FORM – *extended to include the special features of the actions presented here*

Every curve of zeros separates, for its corresponding counterterm, the region where its coefficients is positive from the region where it is negative

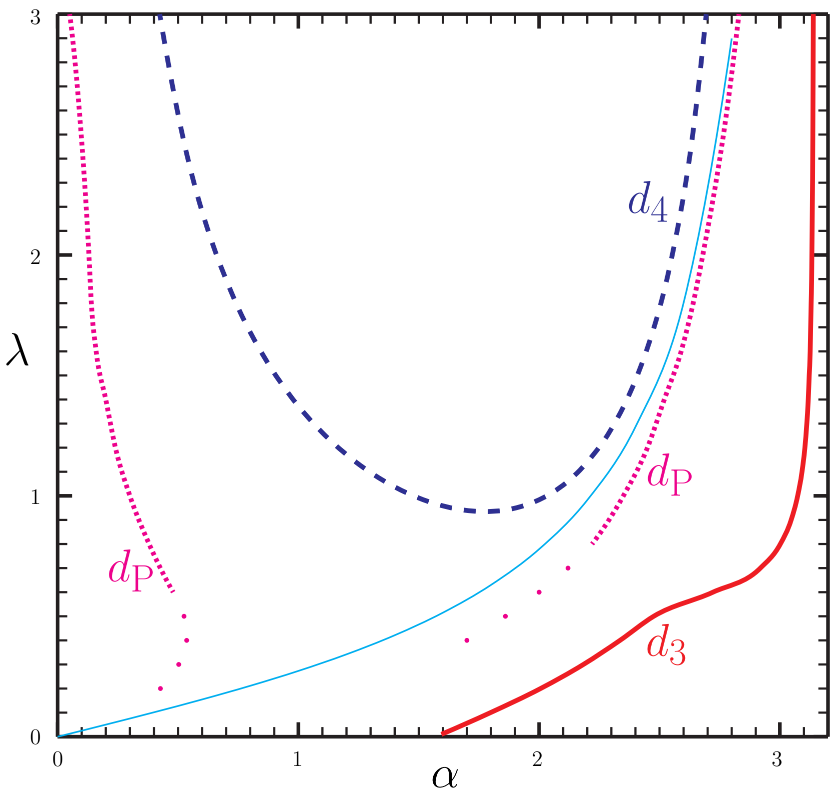
The dependence is <u>rather smooth</u>

One interpolates between values in the positive and negative regions, and so determines the exact values of α and λ for which this coefficient is indeed zero

Some of our results are summarized in the following figures

They show the curves for which each counterterm has a vanishing coefficient

1-loop calculations



Curves of zeros for the coefficients of the counterterms – interpolations of points obtained from 1-loop calculations

Our calculations show no intersections between these curves

The curve corresponding to a zero of d_4 is not symmetric with respect to the reflection $\alpha \to \pi/2 - \alpha$

The distance between the 2 poles of the quark propagator does not change when $\alpha \to \pi/2 - \alpha$, but these values of α correspond to different actions

The purpose here is not the computation of all zeros with a high precision, but rather to show that such curves of zeros exist and see what shape they have

1-loop calculations

The curve corresponding to the vanishing of the dimension-3 counterterm, d_3, has instead a domain which is restricted to $\alpha > \pi/2$

For $\alpha \to \pi/2$ (from above) along this curve, λ goes asymptotically to zero

For $\alpha \to \pi$ instead, λ grows very rapidly and tends to infinity

This is a behavior which is substantially different from the one of the dimension-4 counterterms

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This is a behavior which is substantially different from the one of the dimension-4 counterterms

The locations of the zeros of the coefficient of the gluonic counterterm could be determined only with an error of about ten per cent

This is due to the difficulty of evaluating the numerous integrals needed for this diagram, which arise from a quadratic Taylor expansion in the lattice spacing and are quite expensive to compute

Since the number of terms is some orders of magnitude larger than in the case of the quark self-energy, and moreover the vacuum polarization is divergent at both poles of the fermion propagator, the search for the zeros of $d_{\rm P}$ turns out to be much more expensive than for the fermionic counterterms d_3 and d_4, and the precision that can be achieved is much smaller

1-loop calculations

For the fermionic counterterms the zeros could instead be easily determined with a precision of about 10^{-4}

Of course one can always compute a few selected zeros in a small region of the space of parameters with very high precision

The complete mapping of the whole space of parameters requires however an extremely larger computational effort, so that only a much lower precision can be accomplished

At any rate the main purpose of the present investigations is not the exact computation of all zeros with a high precision

We want rather to show that such curves of zeros exist and see what shape they have

These curves could also be connected to some symmetries

There is no need here to compute these curves of zeros with high precision

Of course when one will eventually be able to construct a nonperturbatively renormalized action with just one or no counterterm, a determination with higher precision of the corresponding parameters will be desirable

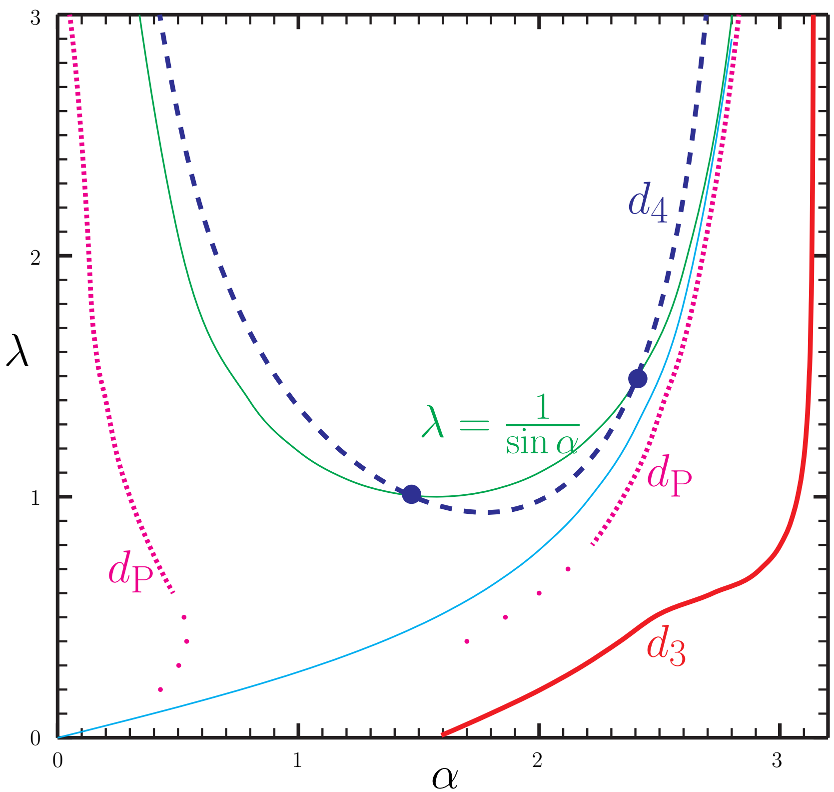
1-loop calculations

For the special $\alpha = \pi/2$ ($=$ standard Karsten-Wilczek action) there is no way for d_3 (the coefficient of the $O(1/a)$ counterterm) to become zero by varying the value of λ

The reason is that, although the tadpole and sunset diagrams have opposite sign and their absolute values decrease as λ is lowered, the sunset always remains in magnitude much smaller than the tadpole and so a cancellation can never take place

The calculations presented in this work show that, on the contrary, such cancellations can take place when $\alpha > \pi/2$

Indeed, for any $\pi/2 < \alpha < \pi$ there is always a value of λ for which a cancellation happens



It is interesting that there are two points for which the curve $\lambda = 1/\sin\alpha$ intersects the curve of zeros of d_4

Then, the action proposed by Creutz, which in general requires three counterterms, needs only two of them when either of the following two choices of α is made:

$$(\alpha, \lambda) = (1.47, 1.01)$$

or

$$(\alpha, \lambda) = (2.41, 1.49)$$

In both cases it is the fermionic counterterm of dimension 4 which is eliminated

1-loop calculations

Generalized Karsten-Wilczek fermions: the counterterms needed for a consistent 1-loop renormalized theory can be fewer than the 3 required for the standard massless Karsten-Wilczek and Boriçi-Creutz actions

There are many choices of α and λ for which a counterterm can be left out $\rightarrow$ the corresponding actions are cheaper and more convenient to simulate

If some of the curves of zeros had an intersection point , this would give a renormalized minimally doubled action which requires only one counterterm

Unfortunately the (*perturbative*) curves that we have obtained do not intersect, and so one remains always with at least two counterterms – *at least in perturbation theory and within the families of actions considered in this paper*

However one can still choose in some convenient way which counterterms to keep and which one to discard

In the quenched case, one has the possibility to construct an action with just one counterterm

In full QCD, one can choose to use only the 2 fermionic counterterms, and then there is no need to fine-tune and employ a gluonic operator of the FF form

Going nonperturbative

Will the qualitative pattern of the curves that we have found be reproduced also nonperturbatively?

The dependence of the coefficients of the counterterms on the parameters of the action appears to be rather smooth

Then it will probably be not too expensive to perform first a quick rough tuning of the parameters around the curves of zeros that we have found perturbatively

Afterwards one can compute with more precision the positions of these nonperturbative zeros, using a much finer tuning

It could be that the locations of these zeros do not differ too much from the perturbative results, and so one could take them as a good starting guess

It is also possible that nonperturbatively the vanishing of the counterterm of dimension 3 occurs in the region where there is minimal doubling

Since this is the only relevant counterterm, in this case only two marginal counterterms (of dimension 4, whose coefficient is likely to be small) would remain to be tuned in order to carry out consistent Monte Carlo simulations, leading to milder numerical cancellations

Going nonperturbative

It could happen that nonperturbatively an intersection point does exist

This would make possible to simulate renormalized minimally doubled actions with at most one counterterm

In the case in which the (nonperturbative) curves indeed intersect, the intersection points will be the most important numbers to find

Since there will likely be not many of them, it will not be overly expensive to determine them with high precision

Even when it is not possible to remove all counterterms, it is covenient to accomplish a reduction in the dimensionality of the parameter space of their coefficients – it makes their numerical determination easier

In particular, if there is only one counterterm left, it is much simpler to carry out the determination of its coefficient, because one has to deal with just a one-dimensional space instead of a multi-parameter one

Besides the removal of counterterms, it is always useful to have as many minimally doubled actions as possible and keep on trying to construct new ones – some particular actions could turn out to have better theoretical or practical properties, and be particularly advantageous for lattice simulations of chiral fermions

Still more actions?

The effective amount of important physical quantities such as the mass splittings within otherwise degenerate multiplets, could turn out to be rather small only for a few of these actions

By moving the distance between the two poles one could minimize in the continuum limit the effects coming from having only a $U(1)$ chiral symmetry

In general it is convenient to have minimally doubled actions where the distance between the two poles of the quark propagator can be varied

Special values of this distance could turn out to be particularly convenient for efficient numerical simulations of minimally doubled fermions

It is possible that still cleverer minimally doubled actions can be constructed – and maybe arrive at the optimal situation where a maximal reduction can be accomplished, that is <u>no counterterms at all are needed</u>

Then one will be able to obtain consistent physical results from simulations using just the bare tree-level actions – no tuning of counterterms needed

Simulations of minimally doubled actions without counterterms will be cheaper than when one needs to add counterterms to the bare actions – and than the already convenient standard Karsten-Wilczek fermions

Next-to-nearest-neighbor actions

We would like to have actions for which intersections between the curves of zeros exist, so that 2 or even more of the possible counterterms can then be removed

One can think of widening the pool by considering also couplings between **next-to-nearest-neighbor** lattice sites

In the quest for minimally doubled actions without counterterms, investigating such kind of actions could turn out at the end to be rewarding

We do not know in fact whether there could be theoretical impediments in principle to countertermless minimally doubled actions when one only considers nearest-neighbor interactions

It is conceivable that introducing interactions also at distance $2a$ or larger could allow actions with different kinds of properties

The hope is that at the end some of these actions will not require any counterterms to be properly renormalized

We find then useful to propose here a first example of a class of minimally doubled actions with next-to-nearest-neighbor interactions:

Next-to-nearest-neighbor actions

$$S^f_{ntn}(x;\alpha,\lambda,\lambda',\rho) = a^4 \sum_x \Bigg[\frac{1}{2a} \sum_{\mu=1}^{4} \Big[\overline{\psi}(x)\,(\gamma_\mu - i\gamma_4\, f^{(1)}_\mu)\, U_\mu(x)\, \psi(x+a\widehat{\mu})$$

$$- \overline{\psi}(x+a\widehat{\mu})\,(\gamma_\mu + i\gamma_4\, f^{(1)}_\mu)\, U^\dagger_\mu(x)\, \psi(x) \Big]$$

$$+ \frac{i}{4a} \sum_{\mu=1}^{4} f^{(2)}_\mu \cdot \Big[\overline{\psi}(x)\,\gamma_4\, U_\mu(x)\, U_\mu(x+a\widehat{\mu})\, \psi(x+2a\widehat{\mu})$$

$$+ \overline{\psi}(x+2a\widehat{\mu})\,\gamma_4\, U^\dagger_\mu(x+a\widehat{\mu})\, U^\dagger_\mu(x)\, \psi(x) \Big]$$

$$+ \overline{\psi}(x) \left(m_0 + \frac{i\gamma_4}{a}\, f^{(0)} \right) \psi(x) \Bigg]$$

where

$$f^{(0)}(\alpha,\lambda,\lambda',\rho) = 3\lambda + \frac{9}{2}\lambda' + \left(\rho + \frac{3}{4}\frac{1-\rho}{\sin^2\alpha} \right) \cot\alpha$$

$$f^{(1)}_\mu(\alpha,\lambda,\lambda',\rho) = \lambda + 2\lambda' + \delta_{\mu 4} \left(\left(\rho + \frac{1-\rho}{\sin^2\alpha} \right) \cot\alpha - \lambda - 2\lambda' \right)$$

$$f^{(2)}_\mu(\alpha,\lambda',\rho) = \lambda' + \delta_{\mu 4} \left(\frac{1-\rho}{2\sin^2\alpha} \cot\alpha - \lambda' \right)$$

are functions diagonal in spinor and color space

Next-to-nearest-neighbor actions

There are simple relations between these functions, and if one defines

$$f_\mu^{(h)}(\alpha, \lambda, \rho) = \lambda + \delta_{\mu 4}\left(\rho \cot \alpha - \lambda\right)$$

then knowing $f_\mu^{(1)}$ one can obtain

$$\begin{aligned} f_\mu^{(2)} &= \frac{1}{2}\left(f_\mu^{(1)} - f_\mu^{(h)}\right) \\ f^{(0)} &= \sum_{\mu=1}^{4}\left(\frac{3}{4}\, f_\mu^{(1)} + f_\mu^{(h)}\right) = \sum_{\mu=1}^{4}\left(f_\mu^{(h)} + \frac{3}{2}\, f_\mu^{(2)}\right) \end{aligned}$$

Next-to-nearest-neighbor actions

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The corresponding momentum-space actions are given in the free case by

$$\frac{i}{a}\sum_{\mu=1}^{4}\gamma_\mu \sin ap_\mu + \frac{i\gamma_4}{a}\Bigg\{\sum_{k=1}^{3}\Big(\lambda\,(1-\cos ap_k) + \lambda'\,(1-\cos ap_k)^2\Big)$$
$$+\cot\alpha\,\Big(\rho\,(1-\cos ap_4) + \frac{1-\rho}{2\sin^2\alpha}\,(1-\cos ap_4)^2\Big)\Bigg\} + m_0$$

For $\lambda' = 0$ & $\rho = 1$ one falls back to the case of the nearest-neighbor actions

Next-to-nearest-neighbor actions

These actions satisfy γ_5 -hermiticity, and the temporal direction is again the special one which is selected and which then breaks hypercubic symmetry

Same symmetries of the Karsten-Wilczek action: P is a symmetry but T and C separately are violated, unless the action is properly renormalized

So, the counterterms that must be added to these generalized actions are again the same needed for the standard Karsten-Wilczek action

The parameter α regulates the distance between the two zeros, which are at the same positions $a\bar{p}_1 = (0, 0, 0, 0)$ and $a\bar{p}_2 = (0, 0, 0, -2\alpha)$ as in the nearest-neighbor actions

That there are only two zeros is certain if $-3 \leq \rho \leq 1$ and $-\pi/2 < \alpha < \pi/2$

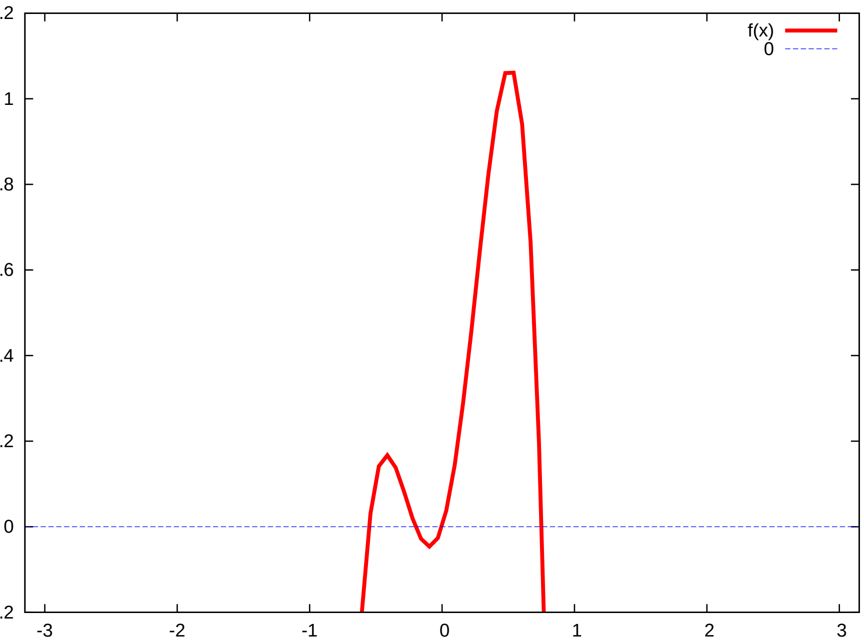
For choices of ρ outside of this range, additional zeros can in general appear, and one can still get minimally doubled actions but only for a restricted domain of α (whose extension depends on the value of ρ)

One must also take, to ensure that there are no more than two fermions,

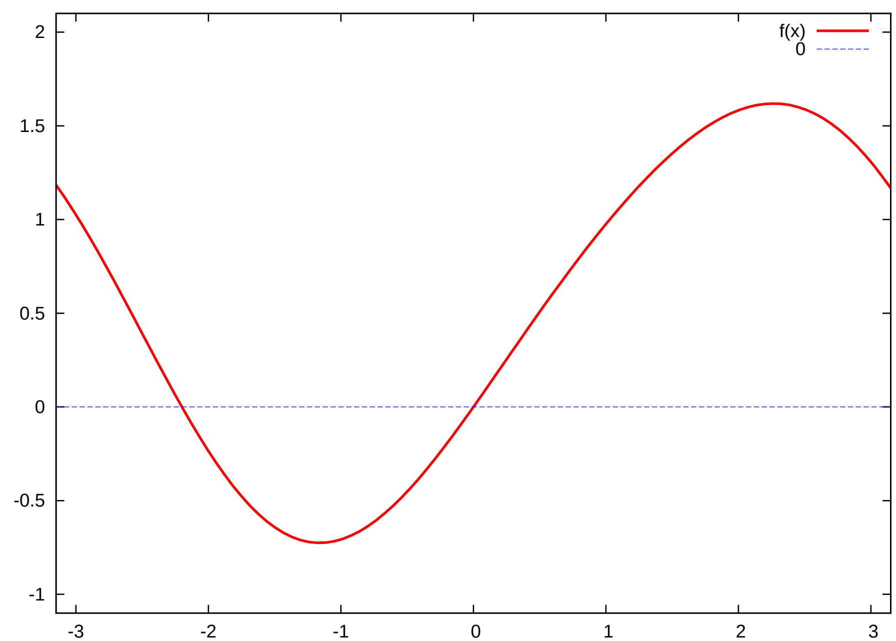
$$\lambda + 2\lambda' > -\min \{\sin x + \cot\alpha \,(\rho\,(1 - \cos x) + (1-\rho)\,(1-\cos x)^2/(2\sin^2\alpha))\}/2$$

Next-to-nearest-neighbor actions

Obtaining minimally doubled actions is not trivial: profile of the action *(proportional to* γ_4*)* vs. p_4 *(for* $\vec{p} = (0,0,0)$*)* in the case $(\alpha, \rho) = (0.1, 1.1)$



Everything's ok for $(\alpha, \rho) = (1.1, 0.4)$:



Next-to-nearest-neighbor actions

It is worth noting that the above actions in position space can also be written more concisely in the simple form

$$a^4 \sum_x \overline{\psi}(x) \left\{ \sum_\mu \left[\frac{1}{2}\, \gamma_\mu (\nabla_\mu + \nabla^\star_\mu) - i a \gamma_4 \left\{ \frac{1}{2}\, f^{(1)}_\mu\, \nabla^\star_\mu \nabla_\mu - f^{(2)}_\mu\, \widetilde{\nabla}^\star_\mu \widetilde{\nabla}_\mu \right\} \right] + m_0 \right\} \psi(x)$$

where in addition to the standard ∇_μ and $\nabla^\star_\mu$ one has also introduced another discretization for the lattice covariant derivative, extending this time over two lattice sites:

$$\widetilde{\nabla}_\mu\, \psi(x) = \frac{U_\mu(x)\, U_\mu(x + a\widehat{\mu})\, \psi(x + 2a\widehat{\mu}) - \psi(x)}{2a}$$

$$\widetilde{\nabla}^\star_\mu\, \psi(x) = \frac{\psi(x) - U^\dagger_\mu(x - a\widehat{\mu})\, U^\dagger_\mu(x - 2a\widehat{\mu})\, \psi(x - 2a\widehat{\mu})}{2a}$$

Note that in this concise notation it is apparent that there is no mass term left if one sets $m_0 = 0$

This was also true for the nearest-neighbor actions

Terms like $i\overline{\psi}(x)\gamma_4\psi(x)/a$ are in fact part of the various Laplacians

Next-to-nearest-neighbor actions

Our primary motivation for introducing these next-to-nearest-neighbor actions is that for special choices of the parameters one could hit on renormalized actions which do not require any counterterms

Since there are 4 parameters, and not just 2 as in the nearest-neighbor case, there should be many more “curves” on which the counterterms become zero and, above all, more chances for intersections among these curves

(Actually, the “curves” are likely to be 3-dimensional manifolds)

It could then happen that there are some values of the parameters for which one ends up with just one counterterm, or none at all

Of course to explore adequately this larger parameter space will be more expensive than for the nearest-neighbor actions

It is probably not too difficult to go one step further and construct minimally doubled fermions with hopping terms extending to 3 (or more) lattice spacings

This will enlarge even further the space in which to search for actions which do not require counterterms – although incrementing the range of the couplings renders such actions increasingly less convenient for simulations

Generalized Boriçi-Creutz actions

We have also generalized the **Boriçi-Creutz** action

The second zero α_μ can be moved to an arbitrary position

$$-\pi < \alpha_\mu < \pi \quad (\alpha_\mu \neq 0)$$

and the direction which breaks the hypercubic symmetry can also be arbitrarily chosen

The components of α_μ do not need to be equal, and they can even be all different from one another

The direction of hypercubic breaking can never exactly correspond to one of the p_μ axes

In this sense, “complementary” to the generalized Karsten-Wilczek actions

The action has still the correct continuum limit

Minimal doubling is guaranteed if the distance between the two zeros does not become too large

Generalized Boriçi-Creutz actions

Boriçi-Creutz fermions: special place among minimally doubled fermions

They have sparked off the revival of this class of ultralocal chiral formulations – and their particular construction has arisen from investigations of the properties of electrons in graphene

Boriçi-Creutz fermions are an instructive example of models based on spinless fermions hopping on a lattice, in which the low-energy excitations come out at the end to carry half-integer spin

This view of the **emergence of spin from spinless particles** has been discussed by **Creutz**:

"Emergent spin", arXiv:1308.3672, Ann. Phys. (Amsterdam) 342, 21 (2014)

How the spin arises is dictated by the topological properties of the action in momentum space, which

- protect from additive mass renormalization
- constrain the fermionic flavors to appear only in an **even number**

  $\rightarrow$ intriguing picture of the workings of the Nielsen-Ninomiya theorem

Generalized Boriçi-Creutz actions

Dirac operator:

$$D = \frac{1}{2}\left\{\sum_{\mu=1}^{4}\gamma_\mu(\nabla_\mu+\nabla_\mu^*) + ia\sum_{\mu=1}^{4}\Big(\gamma_\mu\cot\alpha_\mu+\gamma'_\mu\csc\alpha_\mu\Big)\,\nabla_\mu^*\nabla_\mu\right\} + m_0$$

where γ' is another set of Dirac matrices

After expanding the covariant derivatives this fermionic action reads

$$\begin{aligned} a^4\sum_x\Bigg\{\frac{1}{2a}\sum_{\mu=1}^{4}\Bigg[&\overline{\psi}(x)\Big(\gamma_\mu+i\Big(\gamma_\mu\cot\alpha_\mu+\gamma'_\mu\csc\alpha_\mu\Big)\Big)\,U_\mu(x)\,\psi(x+a\widehat{\mu}) \\ &-\overline{\psi}(x+a\widehat{\mu})\Big(\gamma_\mu-i\Big(\gamma_\mu\cot\alpha_\mu+\gamma'_\mu\csc\alpha_\mu\Big)\Big)\,U_\mu^\dagger(x)\,\psi(x)\Bigg] \\ &+\overline{\psi}(x)\Big(m_0-\frac{i}{a}\sum_\mu\Big(\gamma_\mu\cot\alpha_\mu+\gamma'_\mu\csc\alpha_\mu\Big)\Big)\,\psi(x)\Bigg\}\end{aligned}$$

Only **nearest-neighbor** interactions (like the Wilson action)

Since $\{\gamma'_\mu,\gamma_5\}=0$, it preserves a $U(1)$ chiral symmetry (for $m_0=0$), which protects from additive mass renormalization, and also satisfies γ_5-hermiticity

Construction of the generalized action

The standard Boriçi-Creutz action can be viewed as the outcome of an ingenious construction, devised by Creutz

It can be represented as a linear combination of two physically equivalent naive fermion actions – the second one having been given a momentum shift

We first try to use again two naive fermions

Make a translation in momentum space of the second naive fermion action:

$$D^{BC'}(p) = i\sum_\mu \Big(\gamma_\mu \sin p_\mu + \gamma'_\mu \sin(p_\mu + \pi - \alpha_\mu)\Big) - i\sum_\mu \gamma'_\mu \sin\alpha_\mu + m_0$$

Then the second zero of the whole action is now at $p_\mu = \alpha_\mu$

The Γ term has to be modified in order to achieve the desired minimal doubling

$\rightarrow$ now is $\Gamma = (1/2)\sum_\mu \gamma_\mu \sin\alpha_\mu = (1/2)\sum_\mu \gamma'_\mu \sin\alpha_\mu$

This action can be written also as

$$\begin{aligned} D^{BC'}(p) &= i\sum_\mu \Big(\gamma_\mu \sin p_\mu + \gamma'_\mu \big(\sin(\alpha_\mu - p_\mu) - \sin\alpha_\mu\big)\Big) + m_0 \\ &= i\sum_\mu \Big(\gamma_\mu \big(\sin p_\mu - \sin\alpha_\mu\big) + \gamma'_\mu \sin(\alpha_\mu - p_\mu)\Big) + m_0 \end{aligned}$$

Construction of the generalized action

A major problem with this action: wrong continuum limit

Indeed, its leading term for small p is

$$D^{BC'}(p) \simeq i\not{p} - i\sum_{\mu} \gamma'_\mu \, p_\mu \cos\alpha_\mu$$

One consequence: the basic vertex for the emission of a gluon by a quark current is not simply proportional to γ_μ, but still contains also γ'_μ terms, even in the continuum limit

Wrong continuum limit: because at the point $p_\mu = (0,0,0,0)$, where the coefficient of $i\gamma_\mu$ vanishes, the first derivative of the function expressing the coefficient of $i\gamma'_\mu$ does not vanish

One then has to find a way to overcome this limitation

In order to obtain that this derivative becomes zero, we have to modify the shape of the naive actions in momentum space

$\rightarrow$ make the substitution

$$\sin p_\mu \longrightarrow \sin p_\mu - \cot\alpha_\mu \, (1 - \cos p_\mu)$$

Construction of the generalized action

The mechanism of minimal doubling is now similar as before

Main difference: at $p_\mu = 0$, where the coefficient of γ_μ is zero, that of γ'_μ has a maximum, and thus its first derivative is zero

At $p_\mu = \alpha_\mu$ the roles of γ_μ and γ'_μ are just reversed

The minimally doubled action coming out of this choice of modified naive actions is then

$$\begin{aligned} D(p) &= i \sum_\mu \Big[\gamma_\mu \Big(\sin p_\mu - \cot \alpha_\mu \left(1 - \cos p_\mu\right) \Big) \\ &\qquad + \gamma'_\mu \Big(\sin(p_\mu + \alpha_\mu) - \cot \alpha_\mu \left(1 - \cos(p_\mu + \alpha_\mu)\right) \Big) \Big] - in\Gamma + m_0 \\ &= i \sum_\mu \frac{1}{\sin \alpha_\mu} \Big[\gamma_\mu \Big(\cos(p_\mu - \alpha_\mu) - \cos \alpha_\mu \Big) \\ &\qquad + \gamma'_\mu \Big(\cos p_\mu - \cos \alpha_\mu \Big) \Big] - in\Gamma + m_0 \end{aligned}$$

$\rightarrow$ the continuum limit is now the correct one!

The new Gammas

A new definition of Γ must be now used!

The generalization of Γ that, when combined with the sum of the modified naive actions, builds a minimally doubled action is

$$\Gamma = \frac{1}{n} \sum_\mu \frac{1-\cos\alpha_\mu}{\sin\alpha_\mu}\, \gamma_\mu = \frac{1}{n} \sum_\mu \frac{1-\cos\alpha_\mu}{\sin\alpha_\mu}\, \gamma'_\mu \; ; \quad n = \sqrt{\sum_\mu \frac{(1-\cos\alpha_\mu)^2}{\sin^2\alpha_\mu}}$$

With this definition the action has always at least two zeros, located at the origin and at α_μ *(if the components of α_μ become large other zeros can appear)*

The matrix Γ encodes the generic direction of hypercubic breaking that is now possible to choose

One can also write it as $\Gamma = \frac{1}{n} \sum_\mu \gamma_\mu \, \tan(\alpha_\mu/2)$

So, there is a one-to-one correspondence between Γ and the direction of hypercubic breaking

$\Gamma^2 = 1$, and so this matrix is also unitary

The new Gammas

We can then observe that also the two modified naive actions out of which the action is built are physically equivalent

Indeed if we take the unitary transformations

$$
\begin{aligned}
\psi(x) &\rightarrow e^{-i\alpha_\mu x_\mu}\,\Gamma\,\psi(x)\\
\overline{\psi}(x) &\rightarrow e^{i\alpha_\mu x_\mu}\,\overline{\psi}(x)\,\Gamma
\end{aligned}
$$

in momentum space, the corresponding effect is given by the substitutions $\sin(p_\mu) \rightarrow \sin(p_\mu + \alpha_\mu)$ and $\cos(p_\mu) \rightarrow \cos(p_\mu + \alpha_\mu)$

Thus, under this unitary transformation the first modified naive action goes exactly into the second one

An important consequence of this equivalence is that the relation $\gamma'_\mu = \Gamma\gamma_\mu\Gamma$ of the standard Boriçi-Creutz action is still valid, even though now the explicit expressions of the γ'_μ matrices depend on the choice of α_μ

This equivalence can be seen from the fact that the unitary transformation brings the first zero onto the second one , and so

$$
\overline{\psi}\gamma_\mu\psi \;\rightarrow\; \overline{\psi}\Gamma\gamma_\mu\Gamma\psi = \overline{\psi}\gamma'_\mu\psi
$$

The new Gammas

Moreover, from this relation (and together with $\Gamma^2 = 1$) the equivalence of the two previous definitions of Γ can be verified, as well as that

$$\{\gamma'_\mu, \gamma'_\nu\} = \{\Gamma\gamma_\mu\Gamma, \Gamma\gamma_\nu\Gamma\} = \Gamma\{\gamma_\mu, \gamma_\nu\}\Gamma = 2\delta_{\mu\nu}$$

which shows that the matrices γ'_μ are a fully legitimate set of Dirac matrices

They are a linear combination of the γ_μ, which can be expressed as $\gamma'_\mu = \sum_\nu a_{\mu\nu}\gamma_\nu$, where a is an orthogonal matrix (Borici, 2007)

The specific values of the entries of γ'_μ depend on the actual location of the second zero

The new Gammas

Moreover, from this relation (and together with $\Gamma^2 = 1$) the equivalence of the two previous definitions of Γ can be verified, as well as that

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The specific values of the entries of γ'_μ depend on the actual location of the second zero

Another useful relation for the γ'_μ matrices is

$$\gamma'_\mu = \{\Gamma, \gamma_\mu\}\Gamma - \gamma_\mu = \frac{2}{n}\,\frac{1 - \cos\alpha_\mu}{\sin\alpha_\mu}\,\Gamma - \gamma_\mu$$

This relation was $\gamma'_\mu = \Gamma - \gamma_\mu$ for the standard Boriçi-Creutz action

The relation $\gamma'_\mu = \Gamma\gamma_\mu\Gamma$ remains instead unmodified also in the generalized Boriçi-Creutz action, and so it looks as though it could be the more fundamental of the two main ways of expressing γ'_μ in terms of γ_μ

The new Gammas

So, the relation $\gamma'_\mu = \Gamma\gamma_\mu\Gamma$ of the standard Boriçi-Creutz action is still valid – even though now the explicit expressions of the γ'_μ matrices depend on the choice of α_μ

It is also easy to see from $\gamma'_\mu = \Gamma\gamma_\mu\Gamma$ that $\{\gamma'_\mu, \gamma_5\} = 0$

Then chiral symmetry and the γ_5-hermiticity of the action immediately follow

It must be $\alpha_\mu \neq 0$ and $\alpha_\mu \neq \pi$, otherwise the two modified naive actions collapse onto each other, or their sum is identically zero, and so the construction of the action obviously degenerates

Regions of minimal doubling

What we have generalized here is the standard Boriçi-Creutz action whose second zero is conventionally taken at $(\pi/2, \pi/2, \pi/2, \pi/2)$, and hence its direction of hypercubic breaking is the positive major diagonal

However, from the second (modified) naive action one could choose any of its other 15 zeros out of $(\pm\pi/2, \pm\pi/2, \pm\pi/2, \pm\pi/2)$ to survive at the end

If for instance one picks $(\pi/2, -\pi/2, \pi/2, \pi/2)$, the new direction of hypercubic breaking is a different major hypercubic diagonal, and $\Gamma = \frac{1}{2}(\gamma_1 - \gamma_2 + \gamma_3 + \gamma_4)$

Each of these 16 possible choices corresponds to a four-dimensional orthant

The generalized action that we have derived is instead valid for all the sixteen orthants combined (except for the p_μ axes)

Also the expression for Γ already covers this general case, and for example if $-\pi < \alpha_2 < 0$ then the coefficient of γ_2 becomes automatically negative

Not all possible choices of α_μ preserve minimal doubling – additional zeros can appear if some components of α_μ become too large

It can however be proven that for a large region of choices of α_μ there are indeed only two flavors

Regions of minimal doubling

Any zero of the action has to satisfy the trace equations

$$\sum_\mu \frac{\cos\left(p_\mu - \alpha_\mu/2\right)}{\cos\left(\alpha_\mu/2\right)} = 4, \qquad \frac{\sin\left(p_\mu - \alpha_\mu/2\right)}{\sin\left(\alpha_\mu/2\right)} = \frac{\sin\left(p_\nu - \alpha_\nu/2\right)}{\sin\left(\alpha_\nu/2\right)}$$

They come out from imposing respectively $\mathrm{Tr}\,\Gamma\, D(p) = 0$ and $\mathrm{Tr}\,(\gamma_\mu \sin\alpha_\mu/(1-\cos\alpha_\mu) - \gamma_\nu \sin\alpha_\nu/(1-\cos\alpha_\mu))\, D(p) = 0$

With the help of these trace equations one can always check, by direct inspection, whether or not a given p_μ is a zero for a given choice of α_μ

Two important properties of the zeros can be inferred from the trace equations:

- symmetric under permutations of the coordinates
- symmetric under reflections of any of the coordinates axes

Then, each orthant can be studied separately, since the distribution patterns of the zeros is the same and every orthant, and only changes of signs have to be taken into account

We can then restrict our considerations to α_μ's which have only positive components – that is to the first orthant

On a major hypercubic diagonal

When α_μ lies on the positive major diagonal, $\alpha_\mu = (\alpha, \alpha, \alpha, \alpha)$, the trace equations for the zeros become much simpler

One can solve them analytically along the entire length of the diagonal

When $\alpha < 2\pi/3$ there cannot be additional zeros, and thus minimal doubling is preserved

When $\alpha \geq 2\pi/3$ additional doublers do appear:

$$p_\mu = \Big(\alpha/2 + \eta_+, \alpha/2 + \eta_+, \alpha/2 + \eta_+, \alpha/2 + \eta_-\Big), \quad \eta_\pm = \arccos\ (\pm 2 \cos \alpha/2)$$

For $\alpha_\mu = (2\pi/3, 2\pi/3, 2\pi/3, 2\pi/3)$, $p_\mu = (\pi/3, \pi/3, \pi/3, -2\pi/3)$ (and its nontrivial permutations) are the additional zeros

In the general case where α_μ is not on a major hypercubic diagonal, it is difficult to obtain exact solutions to the trace equations, however one can still obtain a lot of information

(These and the following ones are still tree-level considerations – the actual surfaces of demarcation between the regions of minimal doubling and those that contain additional doublers may be slightly different after all interactions have been taken into account)

Regions of minimal doubling

In the general case where α_μ is outside a major hypercubic diagonal, minimal doubling can be guaranteed if the components of α_μ do not become too large

A uniform bound for all components is given by $\cos\left(\alpha_\mu/2\right) \geq \frac{3}{5}$, which corresponds to $\alpha_\mu \leq 0.590334\,\pi \sim 106.26^o$

Then no other zeros can appear in the action beyond the “standard” two

One can also see, by direct inspection of the trace equations, that for the actions defined by

$$\cos\left(\alpha_\mu/2\right) = \left(\frac{3-3\delta}{5-4\delta}, \frac{3-3\delta}{5-4\delta}, \frac{3-3\delta}{5-4\delta}, 1-\delta\right)$$

there are extra zeros given by

$$\cos\left(p_\mu - \alpha_\mu/2\right) = (1,1,1,-1)$$

If one takes δ to be very small *(it has to be $\delta > 0$)*, the existence of these zeros shows that it is not possible to further improve the above uniform bound

If for all components $\cos\left(\alpha_\mu/2\right) \leq \frac{1}{2}$, then minimal doubling is lost, that is extra zeros always appear

This corresponds to $\alpha_\mu \geq 2\pi/3$

Counterterms

Standard Boriçi-Creutz fermions: appearance of sums involving only one Lorentz index, $\sum_\mu f_\mu$, which mirrors $2\Gamma = \sum_\mu \gamma_\mu$

Generalized Boriçi-Creutz fermions: the sums over only one Lorentz index must be of the form $\sum_\mu f_\mu (1 - \cos\alpha_\mu)/\sin\alpha_\mu$, which mirrors the generalized Γ

The fermionic counterterms should look formally like the ones of standard Boriçi-Creutz fermions

$$\overline{\psi}\,\Gamma \sum_\mu D_\mu \psi, \qquad \frac{1}{a}\,\overline{\psi}(x)\,\Gamma\,\psi(x)$$

where the explicit expressions now depend on the actual choice of α_μ

The gluonic counterterm also contains information about the special direction:

$$\sum_{\mu\nu\rho} \frac{1-\cos\alpha_\mu}{\sin\alpha_\mu}\,\frac{1-\cos\alpha_\nu}{\sin\alpha_\nu}\,\mathrm{Tr}\,F_{\mu\rho}(x)\,F_{\rho\nu}(x)$$

Many choices of α_μ are likely to have a reduced number of counterterms (as it has occurred in the case of generalized Karsten-Wilczek fermions)

No counterterms at all for some special value of α_μ?

This could be helped by the fact that α_μ can provide 4 independent parameters

Summary

New minimally doubled (families of) actions:

- generalized Karsten-Wilczek fermions
  - nearest-neighbors, *2 parameters*
  - next-to-nearest-neighbors, *4 parameters*
- generalized Boriçi-Creutz fermions, *4 parameters*

Generalized Karsten-Wilczek, nearest-neighbors:

- For special values of the parameters, counterterms can be eliminated
- The counterterm of dimension 3 (the only relevant one) can be always eliminated – *. . . but at lowest order in perturbation theory this does not happen in the domain of minimal doubling . . .*
- Are there intersection between the curves of zero?
- Wait for nonperturbative studies

Can we find a minimally doubled action with no counterterms?

This work can also be considered as an inspiration to undertake further searches for new minimally doubled actions which possess a reduced number of counterterms – and possibly (in the best of cases) none at all

BACKUP SLIDES

Chiral fermions on the lattice

Simplest discretization of the Dirac action: **naive** fermions

$$\partial_\mu \psi(x) \;\longrightarrow\; \frac{\psi(x+a\widehat{\mu})-\psi(x-a\widehat{\mu})}{2a}$$

Massless propagator of these naive fermions: $a\,\dfrac{-i\sum_\mu \gamma_\mu \sin ap_\mu}{\sum_\mu \sin^2 ap_\mu}$

This propagator has a pole at $ap=(0,0,0,0)$, as expected

But: $\sin ap_\mu$ vanishes whenever any component p_μ is either 0 or π/a

So, there are many other poles, at $ap=(\pi,0,0,0)$, $(0,\pi,0,0)$, ..., $(\pi,\pi,0,0)$, ..., (π,π,π,π) ($=$ the corners of the first Brillouin zone)

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Each pole of the propagator corresponds to a massless fermion in the theory

These Dirac particles are pair produced as soon as interactions are switched on – they appear in internal loops and contribute to intermediate processes

$\Rightarrow$ $2^4 = 16$ particles are propagating on our lattice

Although they are a lattice artifact, one must then take into account all these 16 fermions in lattice computations

Chiral fermions on the lattice

Is there a way out of this?

Chiral fermions on the lattice

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Naive fermions:

$$D(p) = \frac{i}{a} \sum_{\mu} \gamma_\mu \sin ap_\mu$$

Chiral fermions on the lattice

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Naive fermions:

$$D(p) = \frac{i}{a}\sum_\mu \gamma_\mu \sin ap_\mu$$

Wilson fermions:

$$D(p) = \frac{i}{a}\sum_\mu \gamma_\mu \sin ap_\mu \; + \frac{2r}{a}\sum_\mu \sin^2\frac{ap_\mu}{2}$$

Chiral fermions on the lattice

Is there a way out of this?

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$$D(p) = \frac{i}{a} \sum_{\mu} \gamma_\mu \sin a p_\mu$$

Wilson fermions:

$$D(p) = \frac{i}{a} \sum_{\mu} \gamma_\mu \sin a p_\mu \underbrace{+ \frac{2r}{a} \sum_{\mu} \sin^2 \frac{a p_\mu}{2}}_{\textbf{Wilson term}, \quad \sim \mathbf{O(a)}}$$

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Wilson term : "lifts" the mass of 15 of the 16 doublers to $O(1/a)$, and they disappear from the dynamics

$\Rightarrow$ Wilson fermions contain only one flavor of quarks

However: the Wilson term breaks chiral symmetry *(it's a mass term. . .)*

Lattice simulations of massless QCD with Wilson fermions do not preserve chiral symmetry $\rightarrow$ tuning of masses . . . *(critical mass)*

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Other *(simple)* solution which preserves chiral symmetry: staggered fermions

. . . but four doublers remain – and one gets a complicated intertwining of spinor indices and spacetime

Chiral fermions on the lattice

On the lattice:

it is impossible to eliminate the doublers in any fermion action without at the same time breaking chiral symmetry or some important property of field theory

This is a special case of a very important no-go theorem, established by Nielsen and Ninomiya many years ago

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No-go theorem: it is impossible to construct a lattice fermion formulation without fermion doubling and with an explicit continuous chiral symmetry – unless one gives up some other fundamental properties, like locality, unitarity, . . .

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No-go theorem: it is impossible to construct a lattice fermion formulation without fermion doubling and with an explicit continuous chiral symmetry – unless one gives up some other fundamental properties, like locality, unitarity, . . .

This statement only applies to the "standard" chiral symmetry, which acts on the spinor fields according to the transformations

$$\begin{aligned} \psi &\to \psi + \epsilon\gamma_5\psi \\ \overline{\psi} &\to \overline{\psi} + \epsilon\overline{\psi}\gamma_5 \end{aligned}$$

One of the major theoretical advances in this field (1998): there are other transformation laws that can define a lattice chiral symmetry – and which do not necessarily imply fermion doubling

$\Rightarrow$ *Ginsparg-Wilson fermions*

Chiral fermions on the lattice

NO-GO THEOREM OF Nielsen & Ninomiya (1981)

Any massless Dirac operator $D = \gamma_\mu D_\mu \equiv D(x-y)$ in a lattice fermionic action cannot satisfy the following properties at the same time:

- $D(x)$ is local (in the sense that is bounded by $Ce^{-\gamma|x|}$)

  i.e. D couples only fields $\overline{\psi}(x)$, $\psi(y)$ with $(x-y) = O(a)$

  (avoids interactions over macroscopic distances)

- its Fourier transform has the right continuum behavior for small p:
  $\widetilde{D}(p) = i\gamma_\mu p_\mu + O(ap^2)$

- $\widetilde{D}(p)$ is invertible for any $p \neq 0$

  $\Rightarrow$ avoidance of additional poles

  $\Rightarrow$ there are no massless doublers

- $\gamma_5 D + D\gamma_5 = 0$: it is invariant under chiral transformations
  (a realization of the chiral symmetry)

This is always true – there are no exceptions

Chiral fermions on the lattice

These 4 conditions cannot be fulfilled at the same time, by whatever lattice formulation

Therefore, for any lattice action that one can think of, at least one of these conditions has to fail

$\Rightarrow$ either fermion doubling, or explicit chiral symmetry breaking, or ...

All this can be seen already at the level of FREE fermions ($U_\mu = 1$)

Examples:

- `Naive fermions`: 16-fold degeneracy
- `Wilson fermions`: degeneracy completely removed, but they break chiral symmetry
- `staggered fermions`: 4-fold degeneracy; entanglement of flavor, spin and spacetime

  only a $U(1) \otimes U(1)$ subgroup of the full $SU(N_f) \otimes SU(N_f)$ chiral group remains unbroken; the doublers are removed only partially, and taken as different flavors (*tastes*)
- `SLAC fermions`: non-local

Chiral fermions on the lattice

We can intuitively understand why all this happens from general arguments regarding the free fermion propagator on the lattice, and the energy-momentum relation in the Brillouin zone

Minimal requirements: periodicity, continuum-like dispersion relation around $p = 0$, and (desirable) continuity

The general form of a propagator on the lattice for a massless chiral fermion ($=$ anticommutes with γ_5) is

$$\frac{1}{i \sum_\mu \gamma_\mu P_\mu(p)}$$

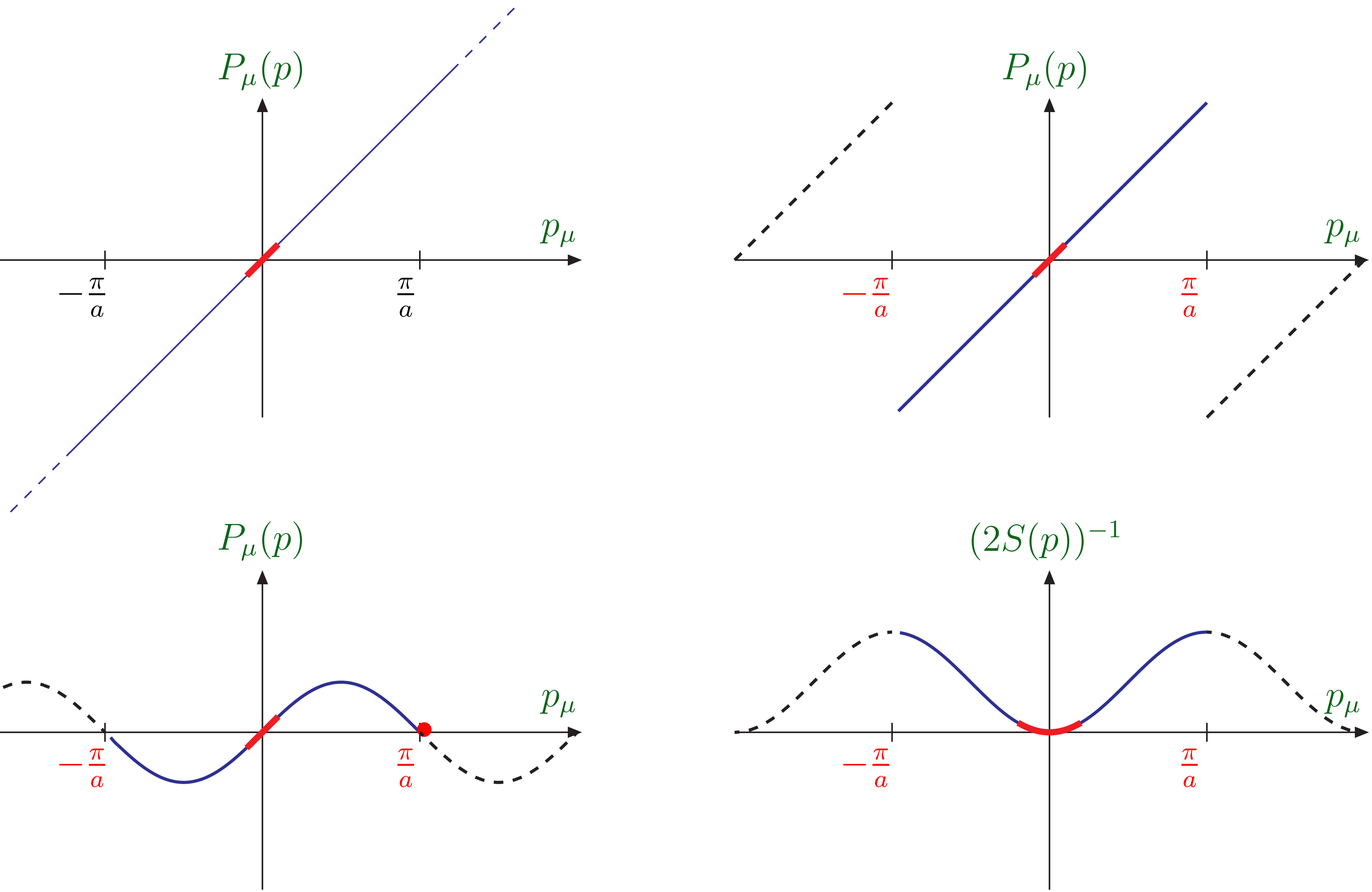
For naive fermions: $P_\mu(p) = \frac{1}{a} \sin a p_\mu$

Let us assume at first that $P_\mu(p)$ is a continuous function

Looking at a given coordinate μ: there is always a first order zero at $p_\mu = 0$, and because of periodicity and continuity there must be another zero somewhere else in the first Brillouin zone

This other crossing is a doubler – and must have a derivative of opposite sign, which means opposite chirality

Chiral fermions on the lattice



Chiral fermions on the lattice

It is unavoidable to have these extra particles in the theory

In four dimensions: $2^4 = 16$ doublers

There is an equal number of left-handed and right-handed fermions *(negative chirality: when an odd number of components has a zero different from* $p_\mu = 0$*)*

This argument is independent of the particular shape of the function $P_\mu(p)$, as long as this is continuous

The only possibility to avoid a second crossing: $P_\mu(p)$ must be a discontinuous function

Most famous example of this: the SLAC propagator [Drell, Weinstein and Yankielowicz, 1976], for which $P_\mu(p) = p_\mu$ throughout the whole Brillouin zone

However, this choice implies a nonlocality in the lattice action – it corresponds to a nonlocal lattice derivative:

$$\partial_\mu = \text{ infinite series in } (\nabla_\mu + \nabla^\star_\mu)^n$$

$\Rightarrow$ many problems: the very existence of the continuum limit is in doubt

(∂_μ: continuum derivative; ∇_μ, $\nabla^\star_\mu$: lattice finite differences)

Chiral fermions on the lattice

The fermion doubling occurs because the Dirac equation is of first order

For a scalar particle things are different, because it is described by a second-order equation

$\Rightarrow$ the linear crossings at $p = 0$ become now second-order zeros

Then, the function $P_\mu(p)$ does not need to have another zero , because at the origin behaves as $O(p^2)$, and thus does not need to become negative!

$\Rightarrow$ no further crossings $\Rightarrow$ no doublers

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How do Wilson fermions manage to avoid the necessity of doublers?

The form of the propagator is fundamentally different:

$$\frac{1}{i\sum_\mu \gamma_\mu P_\mu(p) + Q(p)} \qquad \left(P_\mu(p) = \frac{1}{a}\sin ap_\mu ;\;\; Q(p) = \frac{2r}{a}\sum_\mu \sin^2\frac{ap_\mu}{2} \right)$$

and at π/a the denominator, instead of being zero, is proportional to r/a

The price is that the additional term (a mass term) breaks chiral symmetry

Chiral fermions on the lattice

The issues with chiral symmetry are an unpleasant drawback of the lattice

…but the appearance of doublers is a necessity , and we can understand why by looking at the axial anomaly

Some symmetries of the classical action might not survive quantization

In the continuum the process of regularization destroys chiral symmetry – a mass scale appears in the renormalized theory

After the removal of the cutoff, it may happen that not all the unphysical degrees of freedom actually decouple

Then, we are left with an imperfect decoupling of the unphysical degrees of freedom needed to regularize the theory

When this occurs, not all the symmetries of the formal continuum action can be recovered

$\Rightarrow$ quantum anomalies appear

So, even in theories that are chirally symmetric classically, the axial current may acquire an anomalous divergence through quantum effects (Adler, Bell & Jackiw, 1969)

Chiral fermions on the lattice

The lattice regularization can in general preserve chiral invariance at every step of the transition from the classical to the quantum theory

Naive lattice fermions: a regularization of Dirac fermions that does not break chiral symmetry , for any finite value of a

Then there is no chiral anomaly

In this case, extra particles (the doublers) must necessarily appear on the lattice, with the task of canceling the "continuum" axial anomaly

The number of fermion species must always be even , so that the anomaly can cancel between pairs of them *(like it happens in staggered fermions)*

When one tries to remove the doublers from the game, the anomaly has to come back again – and then chiral symmetry must be violated

This is what happens with Wilson fermions: when one removes the doubler, the continuum anomaly is not canceled anymore – and the so recovered axial anomaly corresponds to a regularization which has to break chiral symmetry

Chiral fermions on the lattice

Contrary to what one would naively expect from the Nielsen-Ninomiya theorem, it is still possible to construct a Dirac operator which satisfies the first three conditions and it is also chirally invariant

Solution to this apparent paradox : the corresponding chiral symmetry is not the one associated with a Dirac operator which anticommutes with γ_5

The fourth condition of the theorem is instead replaced by the Ginsparg-Wilson relation: $\gamma_5 D + D\gamma_5$ is not zero, but proportional to $aD\gamma_5 D$

Thus, the actual lattice chiral symmetry is not what one would naively expect

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Lüscher [1998] has shown that Ginsparg-Wilson fermions are invariant under an exact global chiral symmetry at any finite lattice spacing, of the form

$$\begin{aligned} \psi &\to \psi + \epsilon\gamma_5\Big(1 - aD\Big)\psi \\ \overline{\psi} &\to \overline{\psi} + \epsilon\overline{\psi}\gamma_5 \end{aligned}$$

It is a sort of *"escape"* from the Nielsen-Ninomiya theorem

The Nielsen-Ninomiya theorem is still valid, but – in spite of this – one can still construct a formulation of chiral fermions with no doublers

Chiral fermions on the lattice

When the condition that the Dirac operator anticommutes with γ_5 is released (at $a \neq 0$), the lattice quark propagator is not restricted to be of the form

$$\frac{1}{i\sum_\mu \gamma_\mu P_\mu(p)}$$

Then, the previous considerations about the presence of the doublers that we derived from it are not valid anymore

Non-trivial solutions of the Ginsparg-Wilson relation (*1982* $\rightarrow$ 1997) were found:

- domain-wall fermions (Kaplan, Shamir & Furman, 1992/93)
- overlap fermions (Neuberger & Narayanan, 1992 $\rightarrow$ Neuberger, 1998)
- fixed-point fermions *[perfect actions]* (Hasenfratz & Niedermayer, 1993)

The divergence of the axial symmetry of Ginsparg-Wilson fermions has now the well-known anomaly – and one can simulate a single chiral fermion

But these actions are not ultralocal, and extremely costly – Ginsparg-Wilson fermions are much more complicated and computationally expensive than Wilson or staggered fermions

. . . and not everything seems to be really ok . . .

Chiral fermions on the lattice

The group of Lüscher's lattice chiral symmetry is not the same as the continuum one

Mandula (2009) : this chiral group has an infinite number of generators – indeed, there are an infinite number of lattice axial transformations corresponding to each continuum transformation:

$$\psi \to \psi + \epsilon\gamma_5\Big(1 - aD\Big)\psi \qquad \psi \to \psi + \epsilon\gamma_5\Big(1 - \frac{a}{2}D\Big)\psi \qquad \psi \to \psi + \epsilon\gamma_5\psi$$

$$\overline{\psi} \to \overline{\psi} + \epsilon\overline{\psi}\gamma_5 \qquad \overline{\psi} \to \overline{\psi} + \epsilon\overline{\psi}\Big(1 - \frac{a}{2}D\Big)\gamma_5 \qquad \overline{\psi} \to \overline{\psi} + \epsilon\overline{\psi}\Big(1 - aD\Big)\gamma_5$$

Infinite-parameter symmetry groups are often a sign of disease in a theory

Quite bad: many different axial transformations correspond to the same conserved Noether current

In the canonical formulation there is a one-to-one correspondence between them: *the generators of symmetry transformations are the space integrals of the time components of their conserved currents*

So, the Euclidean path integral does not automatically correspond to a canonical quantum field theory – indeed, the antifermions are represented by variables that are not the conjugates of the fermion variables

Chiral fermions on the lattice

The fact is that in the path integral fermion and antifermion variables are independent, and not necessarily conjugate as in the canonical formalism

Usually there is no problem with this – but for Ginsparg-Wilson symmetry transformations, this prevents the construction of a Hamiltonian theory

The noncanonical elements of this lattice chiral symmetry violate reflection positivity, produce singularities, and impede continuation to Minkowski space

So, for overlap, domain-wall, and perfect-action chiral fermions it seems to be possible to only define a path integral in Euclidean space

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Lüscher: if the gauge fields are smooth enough *(admissibility condition,* $|F_{\mu\nu}(x)| < \epsilon$*,* $0 < \epsilon < \pi/3$*)*, the topological charge can be given a unique lattice definition, and a unique index theorem follows

Creutz: forcing the gauge fields to be so smooth, and non-analytic, brings a violation of reflection positivity, and makes the Hamiltonian non-hermitian

. . . but if one does not impose the smoothness condition, there are gauge fields for which the index of the overlap operator is not uniquely defined . . .

Also: Lüscher was not able to construct a chiral gauge theory for nonabelian groups

Minimally doubled fermions

Nielsen-Ninomiya theorem:

- using two fermion flavors one can maintain an exact chiral symmetry for any finite lattice spacing a, together with locality and unitarity

A chiral symmetry of the standard type *(not Ginsparg-Wilson)* – for a degenerate doublet of quarks

Minimally doubled fermions can still be kept ultralocal , like Wilson fermions

$\rightarrow$ *cheap for simulations*

no tuning of masses is required – chiral symmetry protects masses from additive renormalization

One can construct a conserved axial current, which has a simple expression, involving only nearest-neighbors sites

One of the very few lattice discretizations in which one can give a simple expression (and ultralocal) for a conserved axial current

`A convenient implementation of chiral symmetry at nonzero lattice spacing`

Minimally doubled fermions

Compared with staggered fermions:

- same kind of $U(1)$ chiral symmetry
- 2 flavors instead of 4

  $\Rightarrow$ no uncontrolled extrapolations to 2 physical light flavors
- no complicated intertwining of spin and flavor

Ideal for $N_f = 2$ simulations: no rooting needed !

Much cheaper and simpler than Ginsparg-Wilson fermions

Very convenient for vector-like theories like QCD

Might be practical for simulations of finite temperature QCD , where staggered fermions are extensively used

Two realizations of minimally doubled fermions:

- Boriçi-Creutz fermions
- Karsten-Wilczek fermions

The twisted-ordering method by Creutz and Misumi (2010) can also be useful for constructing other minimally doubled actions