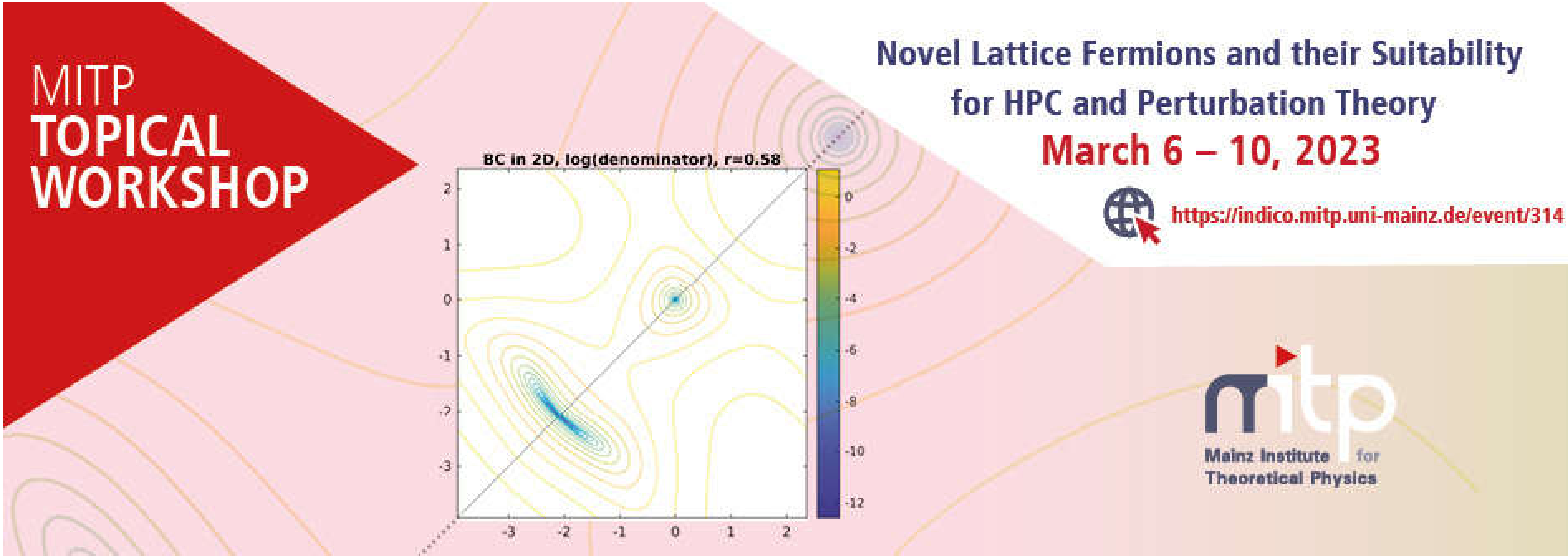

Restoration of hypercubic symmetry with Boriçi - Creutz fermions

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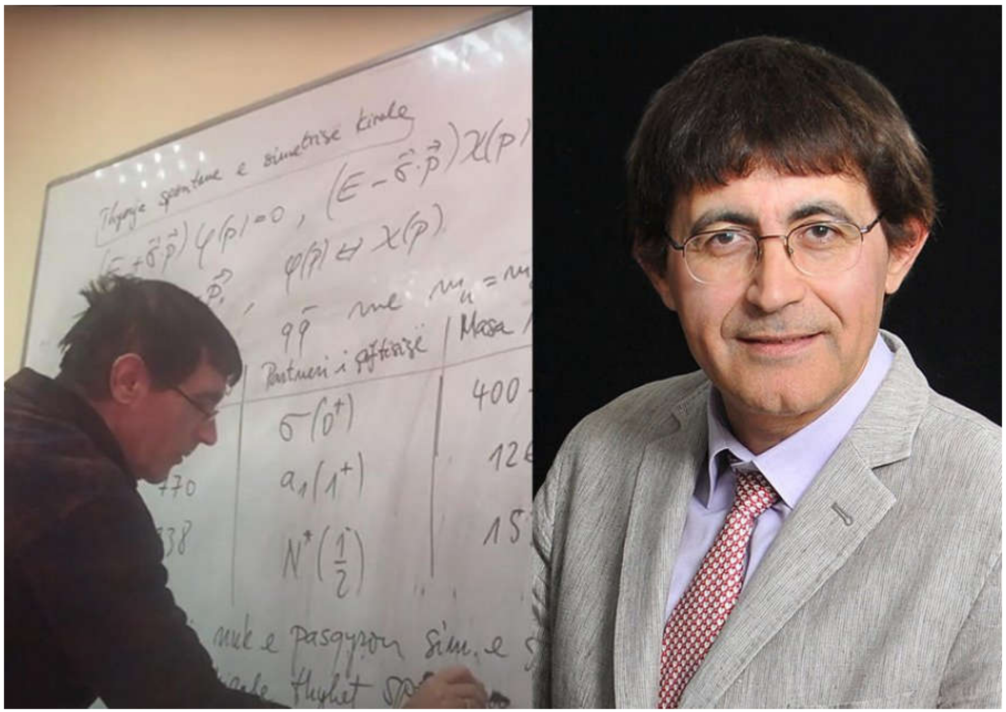
In memoriam Artan BORIÇI
(19.04.1965 - 25.03.2021)



- In LQCD community known for the Boriçi - Creutz fermions
- More than 65 publications
- More than 50 international conferences & workshops
- Created the first group of LQCD in Albania
- PhD advisor of R.OSMANAJ & D. XHAKO
- Krylov subspace methods in lattice QCD
- Chiral fermions on the lattice

Fermions action discretisation:

- Doubling problem
- If no doublers, chiral symmetry is broken

Nielsen – Ninomiya No – go theorem *(Nielsen, Ninomiya, 1981):*

"*A local, real, free fermion lattice action, having chiral and translational invariance, necessarily has fermion doubling* "

Boriçi - Creutz fermions

*[Michael Creutz JHEP04(2008)017] & [Artan Boriçi Phys. Rev. D **78**, 074504 (2008)]*

- Minimally doubled fermions
  - Preserve exactly the chiral symmetry
  - Strictly local
  - 2 flavours (species of opposite chirality)

- Boriçi – Creutz fermionic action with the free Dirac operator (in the momentum space):

$$D(p) = \sum i\gamma_\mu \sin p_\mu + \sum i\gamma'_\mu \cos p_\mu - 2i\Gamma$$

Two zeros at (0,0,0,0) and $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\gamma'_\mu = \sum_\mu \gamma_\mu \Gamma \gamma_\mu$$

$$\Gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \; and \; \Gamma^2 = 1$$

$$\{\Gamma, \gamma_\mu\} = \{\Gamma, \gamma'_\mu\} = 1.$$

In the 4D space:

$$S_{BC} = \sum_n \left[\frac{1}{2}\sum_\mu \bar{\psi}_n \gamma_\mu (\psi_{n+\mu} - \psi_{n-\mu}) - \frac{ir}{2}\sum_\mu \bar{\psi}_n (\Gamma - \gamma_\mu)(2\psi_n - \psi_{n+\mu} - \psi_{n-\mu}) + m\bar{\psi}_n \psi_n\right]$$

Restoration of hypercubic symmetry of Boriçi – Creutz fermions

- There is always a special direction in euclidean space (given by the line that connects these two zeros: hypercubic diagonal)

- Thus, these actions cannot maintain a full hypercubic symmetry *(P. F. Bedaque et al, 2008)*.

- Hypercubic symmetry has to be restored $\longleftrightarrow$ counterterms needed to restore isotropy (Non – perturbative method) *(Perturbative calculations Capitani et al, 2010)*

  1. The dimension three counter term($c3$)
  2. 4 dimension counter d
  3. The third counterterm - Gauge field anisotropy due to feedback from **fermion loops**, needs dynamical fermions.

- For BC action we here only discuss the dimension three counter term($c3$) and for simplicity tune the coefficient of 4 dimension counter term to zero

$$D_{BC}(p) = \sum [i\gamma_\mu \sin p_\mu + i(\Gamma - \gamma_\mu) \cos p_\mu] + i(c_3 - 2)\Gamma$$

Restoration of hypercubic symmetry of Boriçi – Creutz fermions

In the 4D space, Boriçi – Creutz action, with the dimension three counteterm added is written as below:

$$\begin{aligned} S_{BC} &= \sum_n \Big[\frac{1}{2}\sum_\mu \bar{\psi}_n\gamma_\mu(\psi_{n+\mu}-\psi_{n-\mu}) \\ &\quad -\frac{ir}{2}\sum_\mu \bar{\psi}_n(\Gamma-\gamma_\mu)(2\psi_n-\psi_{n+\mu}-\psi_{n-\mu}) \\ &\quad +ic_3\bar{\psi}_n\Gamma\psi_n + m\bar{\psi}_n\psi_n\Big] \end{aligned}$$

How to find non perturbatively the proper counterterms?

Metodology

- Evaluation of the broken hypercubic symmetry mass

$$\Delta(M_\pi^2) = (M_\pi^{(1,1,1,1})^2 - (M_\pi^{(1,0,0,0})^2 \qquad \Delta M = M_\pi^{diag} - M_\pi^{x_1}$$

  - Point splitting (*u(x),d(x)*)
  - Pion propogator calculation
  - Calculation of the pion mass from two different directions

- Non perturbative restaturation of broken hypercubic symmetry

Fully-tuned counterterm coefficients must minimise the degree of anisotropy which is observed at finite lattice spacing. Hence, the most straightforward strategy for non-perturbative tuning is a comparison of computations of correlation functions in different euclidean directions.

$\Downarrow$

- 1 –st Method : $\Delta(M_\pi^2) = (M_\pi^{(1,0,0,0)})^2 - (M_\pi^{(1,1,1,1)})^2 = 0$

$$aM_\pi^2 \underset{m_{q\to 0}}{\to} 0$$

- 2- nd Method: Exploring the phase – structure of QCD in T=0K

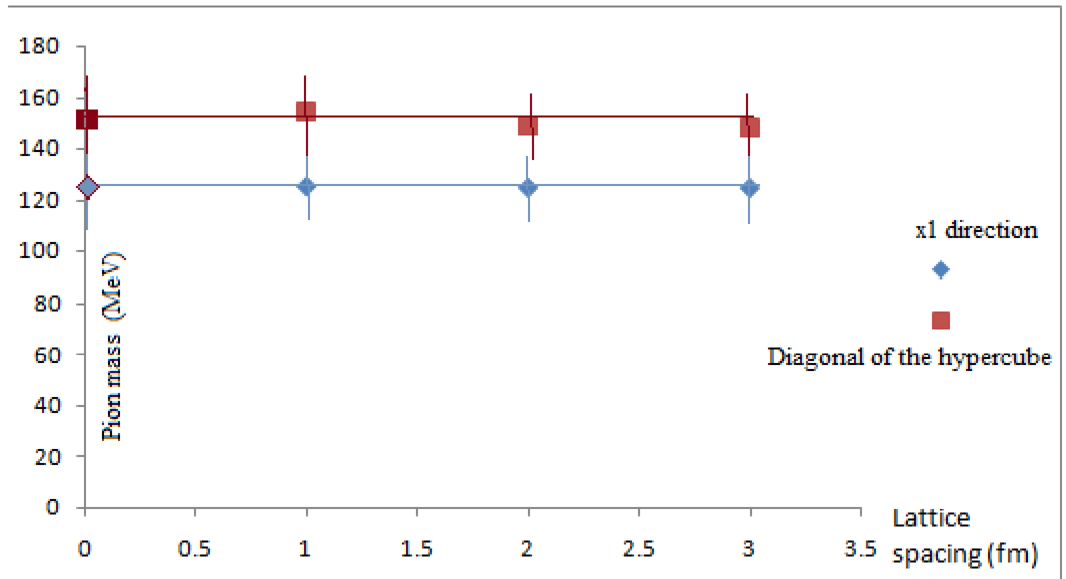
(*Banks- Casher relation and Lanczos quadrature)*

Naïve evaluation of the broken hypercubic symmetry mass

- Calculation for the pion mass from two different directions

- Lattices 16^4, 28^4 dhe 32^4
- Quenched approximation
- BC fermionic action
- Wilson gauge action
- 500 configurations

- Point source for quark propagators
- Five different quark masses
- BiCGstab inverter
- Point splitting method
- FermiQCD /QCDLAB



$$m_{\pi^+(diag)} = 147.54 \pm 5.1 MeV$$

$$m_{\pi^+(x_1)} = 124.59 \pm 5.2 MeV$$

Quark propagators are computed using the `quark_propagator` routine.

It calls the BiCGstab algorithm [8, 9] as a Dirac solver:

$$q = D^{-1} b \, ,$$

where D is the Wilson operator and b a point source at the origin of the lattice for each color and Dirac spin. Therefore, at each lattice site x the propagator q_x is a 12×12 matrix. The pion and rho propagators are defined as:

$$G_{\pi,x} = \mathrm{Tr}\; q_x q_x^* \, , \qquad G_{\rho,x} = \mathrm{Tr}\; \gamma_5 \gamma_k q_x \gamma_k \gamma_5 q_x^* \, ,$$

where the trace and Hermitian conjugation is performed in the tensor product space of color and spin. We sum over space-like lattice sites in order to get particle masses. For example, at long Euclidean time separation T the pion propagator is dominated by its ground state contribution:

$$G_{\pi,T} = \sum_{\vec{x}} G_{\pi,(T,\vec{x})} \simeq C e^{-m_\pi T} \, .$$

Since we simulate with periodic boundary conditions in all directions the propagator decays exponentially also with respect to reflected times, which are translated by the lattice size N_4:

$$G_{\pi,T} \simeq C \left[e^{-m_\pi T} + e^{-m_\pi (N_4 - T)} \right] \propto \cosh\left(T - \frac{N_4}{2} \right) \, .$$

$$M_{\text{eff}}(T) = -\log \frac{G_{\pi,T+1}}{G_{\pi,T}}$$

Point splitting method for BC fermions

Creutz M. (2010): Minimal doubling and point splitting (KW fermions)
Osmanaj, Boriçi (2015) Point splitting method for BC fermions

The fundamental field ψ can create either of the two species. For a quantity that creates only one of them, it is natural to combine fields on nearby sites in such a way as to cancel the other. In other words, one can point split the fields to separate the poles which occur at distinct "bare momenta." For the free theory, one construction that accomplishes this is to consider

$$M\begin{pmatrix} \psi(p) \\ \psi(p+\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix} \Leftrightarrow \begin{pmatrix} \frac{1}{4}\sum_{\mu}(1-\sin p_\mu) & 0 \\ 0 & \frac{\Gamma}{4}\sum_{\mu}(1-\cos p_\mu) \end{pmatrix}\begin{pmatrix} \psi(p) \\ \psi(p+\frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} u \\ d \end{pmatrix}$$

$$u(p) = \frac{1}{4}\sum_{\mu}(1-\sin p_\mu)\psi(p)$$

$$d(p) = \frac{\Gamma}{4}\sum_{\mu}(1-\cos p_\mu)\psi(p+\frac{\pi}{2})$$

Point splitting method for BC fermions

$$u_x = \frac{1}{4}\sum_p e^{ipx}\left(\sum_{\mu=1}^{4}(1+i\frac{e^{ip_\mu a}-e^{-ip_\mu a}}{2})\psi(p)\right)$$

$$u_x = \frac{1}{4}\left[4\psi(\mathrm{x})+\frac{i}{2}\sum_{\mu=1}^{4}(U_{x,\mu}\psi(\mathrm{x+a}\,\mu)-U^{\dagger}{}_{x-\mathrm{a}\,\mu,\mu}\psi(\mathrm{x-a}\,\mu)\right]=A\psi(x)$$

$$d_x = \frac{\Gamma}{4}\sum_p e^{ipx}\left(\sum_{\mu=1}^{4}(1-i\frac{e^{ip_\mu a}+e^{-ip_\mu a}}{2})\psi(p+\frac{\pi}{2})\right)$$

$$d_x = \frac{\Gamma}{4}\sum_{p'} e^{ip'x}e^{-i\frac{\pi}{2}x}\left(\sum_{\mu=1}^{4}(1-i\frac{e^{ip'_\mu a}-e^{-ip'_\mu a}}{2})\psi(p')\right)$$

$$d_x = \frac{\Gamma}{4}e^{-i\frac{\pi}{2}(\mathrm{x}_1+x_2+x_3+x_4)}\left[4\psi(\mathrm{x})+\frac{i}{2}\sum_{\mu=1}^{4}(U_{x,\mu}\psi(\mathrm{x+a}\,\mu)-U^{\dagger}{}_{x-\mathrm{a}\,\mu,\mu}\psi(\mathrm{x-a}\,\mu)\right]$$

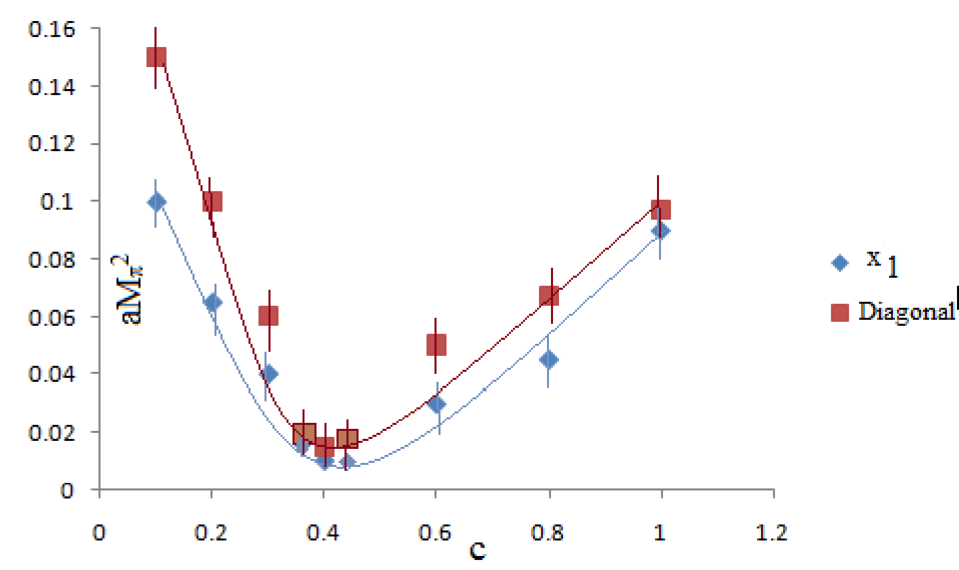
$$d_x = \Gamma e^{-i\frac{\pi}{2}(\mathrm{x}_1+x_2+x_3+x_4)}u_x$$

The pion operator admits the expression of the neutral pion operator:

$$\pi_0(x)=\overline{\psi(x)}\gamma_5\tau_3\psi(x)=\overline{\mathrm{u}(x)}\gamma_5\,\mathrm{u}(x)-\overline{\mathrm{d}(x)}\gamma_5\,\mathrm{d}(x)$$

$$\pi_0(x)=\overline{\mathrm{u}(x)}\gamma_5\,\mathrm{u}(x)-\overline{\mathrm{u}(x)}e^{i\frac{\pi}{2}(\mathrm{x}_1+x_2+x_3+x_4)}\Gamma\gamma_5\Gamma e^{-i\frac{\pi}{2}(\mathrm{x}_1+x_2+x_3+x_4)}u_x=2\overline{\mathrm{u}(x)}\gamma_5\,\mathrm{u}(x)$$

1 – st Attempt to restoration:



- Lattice 32^4
- Quenched approximation
- BC fermionic action
- Wilson gauge action
- 300 configurations

- Point source for quark propagators
- Quark mass = 0.02
- BiCGstab inverter
- Point splitting method
- FermiQCD /QCDLAB

$\beta = 6$, L=32, am_q= 0.02

2 – nd Attempt to restoration:

- When hypercubic symmetry is broken, as in our case, the total momentum of the couple quark antiquark with opposite chirality is different from zero , so is needed energy for their creation from the QCD action.

The lack of hyper cubic symmetry , doesn’t allow the creation of these couples and the chiral condensate, so the chiral symmetry is exact.

In order to have a chiral condensate different form zero, the Lorentz symmetry should be present .

By adding the proper counteterm , in order to explore and observe the phase structure of QCD in T = 0K, we can partially restore the broken hypercubic symmetry.

2 – nd Attempt to restoration:

- Banks - Casher relation *(T. Banks, A, Casher, 1980)*, provides a link between chiral condensate and the spectral density .

$$\lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda,m)=\frac{\Sigma}{\pi} \qquad \Sigma=-\lim_{m\to 0}\lim_{V\to\infty}\langle\bar{\psi}\psi\rangle$$

(If chiral symmetry is spontaneously broken by a non – zero value of the condensate the density of the quark modes in infinite volume does not vanish at the origin. A non – zero density conversely implies that the symmetry is broken)

- Instead of the spectral density, the avarage number *υ (M,m)* of eigenmodes of the Dirac operator with eigenvalues $\alpha \le M^2$ turns out to be a more convenient quantity to consider. Since

$$\nu(\Lambda)=V\int_{-\Lambda}^{\Lambda}d\lambda\rho(\lambda,m)$$

the mode number ultimately carries the same information as the spectral density. *(L. Giusti, M. Luscher, 2009)*

- Count modes using Gauss – Lanczos quadrature:

$$\Sigma_{eff}=\frac{\pi}{2}\frac{\nu(\Lambda)}{\Lambda V} \qquad \Lambda=\sqrt{M^2-m^2}$$

Algorithm

Gauss – Lanczos algorithm

Calculate α_i *dhe* β_i *using Lanczos algorithm for Ax=b*

Define $(T_n)_{i,i} = \alpha_i, \quad (T_n)_{i+1,i} = (T_n)_{i,i+1} = \beta_i, \quad (T_n)_{i,j} = 0$

Calculate the eigenvalue and eigenvectors of the matrix Tn, where i=1…. n

Sort the eigenvalues and eigenvectors in the increasing order of eigenvalues

Define k as the maximal index s, the maximum index which correspond to the cut-off eigenvalue

Define θ_i *as the positive square root of the original eigenvalue*

Define z_i *as the first elements of the eigenvectors* v_i*,* $i = 1 \ldots n$

Define $\omega_i = z_i^2$

Calculate the mode number $\vartheta_k = \sum_i^k \omega_i$

Lanczos algorithm

% Let be $A = D^* D$ and $b = z_2$

% $\beta_0 = 0, \rho_1 = \|b\|_2^{-1}, q_0 = 0, q_1 = \rho_1 b$

for $i = 1 \ldots n$ ***do***

$v = Aq_i$

$\alpha_i = q_i^\dagger v$

$v := v - q_i\alpha_i - q_{i-1}\beta_{i-1}$

$\beta_i = \|v\|_2$

$$\rho_{i+1} = \frac{v}{\beta_i}$$

$$\rho_{i+1} = -\frac{(\rho_i\alpha_i + \rho_{i-1}\beta_{i-1})}{\beta_i}$$

if $\left|\frac{\rho_1}{\rho_{i+1}}\right| < tol$ ***then***

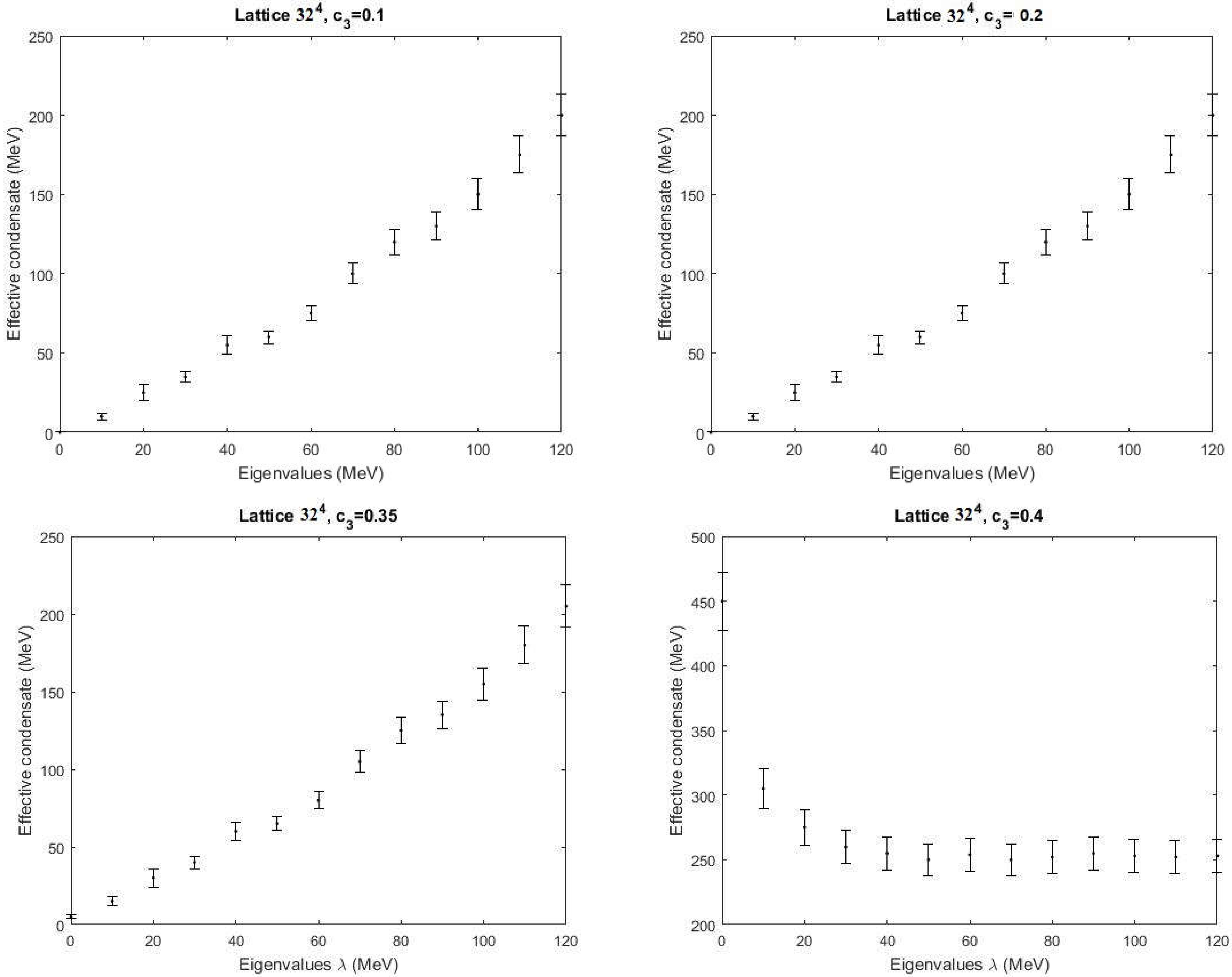
$n = i$

stop

end if

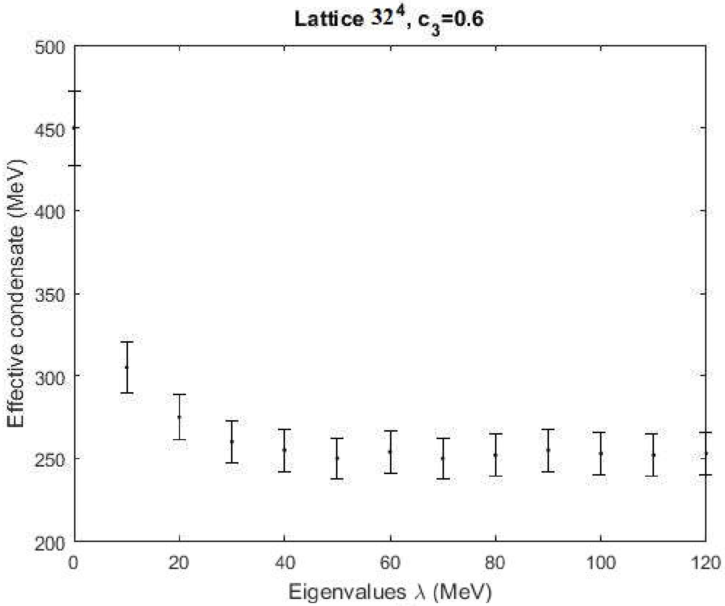
end for

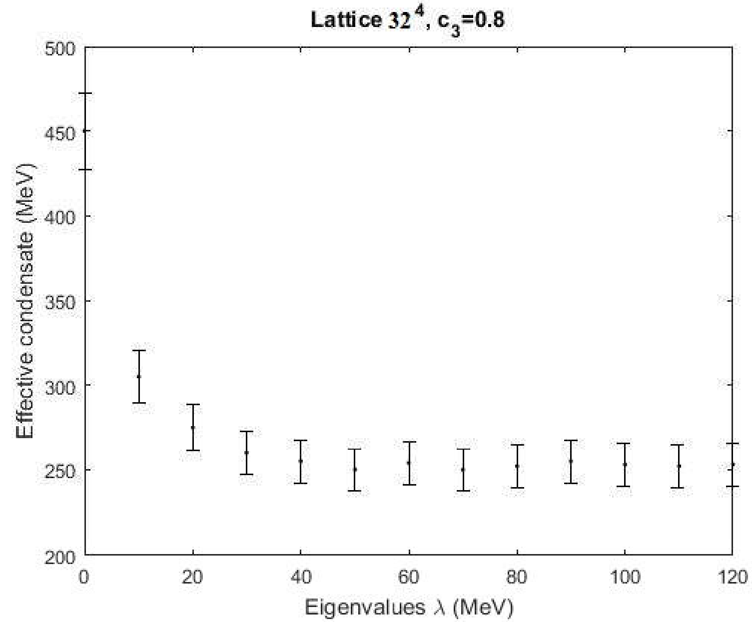
Results

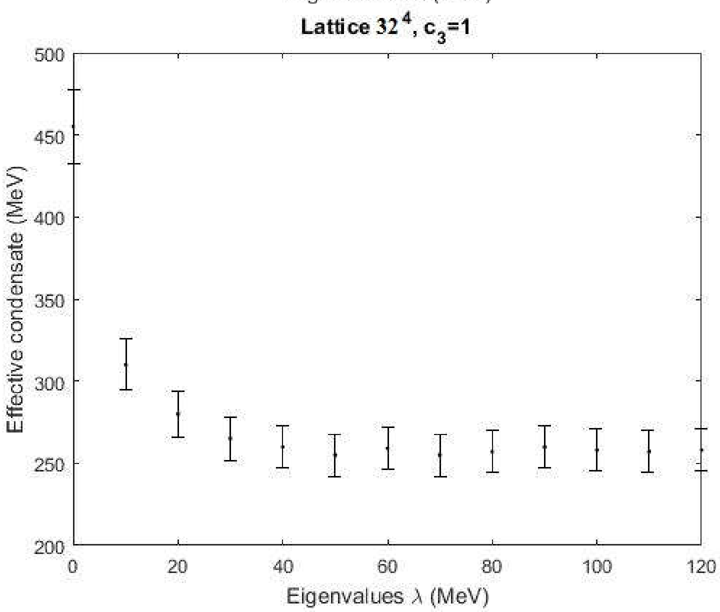


Lattice 32^4, c_3=0.1
Lattice 32^4, c_3= 0.2
Lattice 32^4, c_3=0.35
Lattice 32^4, c_3=0.4
Effective condensate (MeV)
Eigenvalues (MeV)
Eigenvalues λ (MeV)

Results



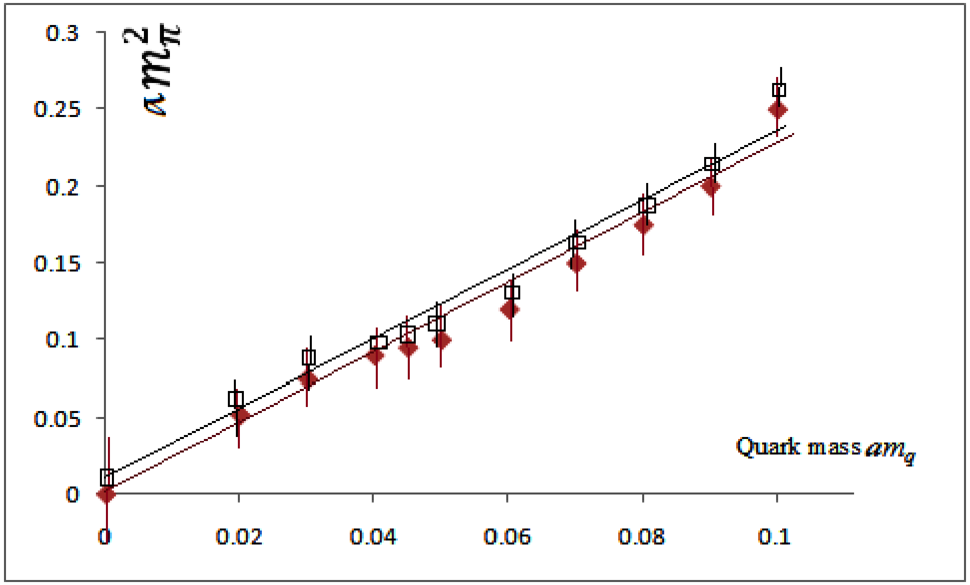




- Lattices 24^4 / 32^4
- Quenched approximation
- Wilson gauge action
- Boriçi – Creutz action
- Zero quark mass (BC fermions are chiral)
- Different values of counterterms c3

  (0.1, 0.2 0.35, 0.4, 0.6,0.8,1)
- Gauss – Lanczos algorithm

Preliminary results



Pion masses in two different directions, in the case of the modified BC action.

- Lattice 32^4
- Quenched approximation
- Modified BC fermionic action
- Wilson gauge action
- 300 configurations

- Point source for quark propagators
- Quark mass = 0.02
- BiCGstab inverter
- Point splitting method
- FermiQCD /QCDLAB

Summary

- Minimally doubled fermions preserve an exact chiral symmetry for a degenerate doublet of quarks, are strictly local, but have a preferred direction in euclidean spacetime → breaking of the hypercubic symmetry.

- The calculations of pion masses in two different directions show clearly the broken hypercubic symmetry.

- Chiral symmetry and spontanous chiral symmetry breaking is very important in QCD.

- The chiral condensate can be used as an order parameter for BC fermions, and help us to find the proper counterterms that restore partially the broken hypercubic symmetry.

- Using Lanczos quadrature and Banks – Casher relation we can explore the chiral symmetry breaking and find the counterterms for which we have spontanous chiral symmetry breaking

- The value of the counterterm, for which is realised the phase transition of QCD, give us a confirmation that we are working in the proper phase (the one with spontanous symmetry breaking).

- Both the methods proposed for tuning non perturbatively the counterterms, give us the same results, at least for the dimension 3 counterterm.

- Dimension 4 counteterm should be consider in order to have a full restoration.

Thank you!