

Implications of Taste Structure of Minimally Doubled Fermions

Johannes H. Weber

(HU Berlin)
RTG 2575

Novel lattice fermions and their
suitability for HPC & Perturbation Theory

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This talk follows:

Pos LAT13 122 (2013)
Pos LAT14 071 (2014)
PhD thesis
Pos LAT16 250 (2016)

arXiv: 1312 . 0488
arXiv: 1601 . 06669
arXiv: 1706 . 07104
arXiv: 1611 . 08388

Naïve fermions : setting the stage and terminology

Naïve discretization on a hypercubic lattice with spacing a

$$\mathcal{D}(x, y) = \sum_{n=1}^d \gamma_n \frac{1}{a} (\delta_{x+a\hat{e}_n, y} - \delta_{x-a\hat{e}_n, y})$$

$$\left. \begin{array}{l} \partial_n^f \psi(x) = \frac{\psi(x+a\hat{e}_n) - \psi(x)}{a} \\ \partial_n^b \psi(x) = \frac{\psi(x) - \psi(x-a\hat{e}_n)}{a} \end{array} \right\} \text{turned into one another by reflections or charge conjugation}$$

$$\partial_n^s \psi(x) = \frac{1}{2} (\partial_n^f + \partial_n^b) \rightarrow \text{symmetries are ok}$$

Symmetries (remnant from the continuum theory):

W_d (remnant of $O(d)$)

C, P, T (Euclidean reflections R_ν : $\psi(x_\nu, x_i) \rightarrow \bar{\psi}(x_\nu, x_i)$; $P = \prod_{\nu=1}^{d-1} R_\nu$, $T = R_d$)

chiral symmetry $\{\gamma_5, \mathcal{D}\} = 0$

γ_5 -hermiticity : $\gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}^\dagger$

$$\text{In momentum space } \mathcal{D}(p) = \sum_{n=1}^d \frac{1}{a} \gamma_n \sin(ap_n) \equiv \sum_{n=1}^d i \gamma_n \bar{p}_n \quad \text{with } a\bar{p}_n = \sin(ap_n)$$

Vanishes for $ap_n = 0, \pi$. $\mathcal{D}(p)\psi(p) = 0$ has 2^d solutions $\Rightarrow 2^d$ tastes!

$2^{d/2}$ spin components per taste $\Rightarrow 2^{3/2 d}$ fermionic do.f. vs

$2^{d/2}$ spin components per site \Rightarrow least 2^d sites needed to encode all spin-taste components

two spurious internal $U(4)$ symmetries (one of which is broken by interactions)

- spinor rotations with momentum shifts: $\tau_\nu \equiv \tau_{\nu, x} = i \gamma_2 \gamma_5 (-1)^{x_\nu/a}$ ($\psi \rightarrow \tau \psi$, $\bar{\psi} \rightarrow \bar{\psi} \tau^\dagger$, swapping two sets of $2^{d/2-1}$ poles against each other)
- translations within a hypercube: (\rightarrow staggered shift symmetry)

Karsten-Wilczek fermions

L. H. Karsten
F. Wilczek

PLB 104 (1981)
PRL 59 (1987)

$$D_{KW}(x,y) = D(x,y) - \frac{ar}{2} i g_d \sum_{n=1}^{d-1} \Delta_n(x,y)$$

Wilczek parameter

In momentum space:

$$\begin{aligned} D_{KW}(p) &= D(p) + r/a i p_d \sum_{n=1}^{d-1} (1 - \cos(\alpha_{pn})) \\ &= i \sum_{n=1}^d \gamma_n \bar{p}_n + i \frac{ar}{2} g_d \sum_{n=1}^{d-1} \hat{p}_n^2 \\ \hat{p}_n &= \frac{a}{2} \sin\left(\frac{\alpha_{pn}}{2}\right) \end{aligned}$$

New term spreads the spectrum of $D(p)$ along the imaginary axis

$$|\lambda_{KW}| \leq \begin{cases} \sqrt{\frac{d+2(d-1)r^2}{1-(d-1)r^2}} & r < \frac{1}{2} \\ 1+2(d-1)r & r \geq \frac{1}{2} \end{cases} \quad \begin{matrix} d=4 \\ |1/3| \\ |1/2| \end{matrix}$$

Survivors at $\alpha_{pd} = 0, \pi$ & spatial $\alpha_{pn} = 0$
independent of r (only two if $r > 1/2$)

W_3 (exchange of spatial axes)

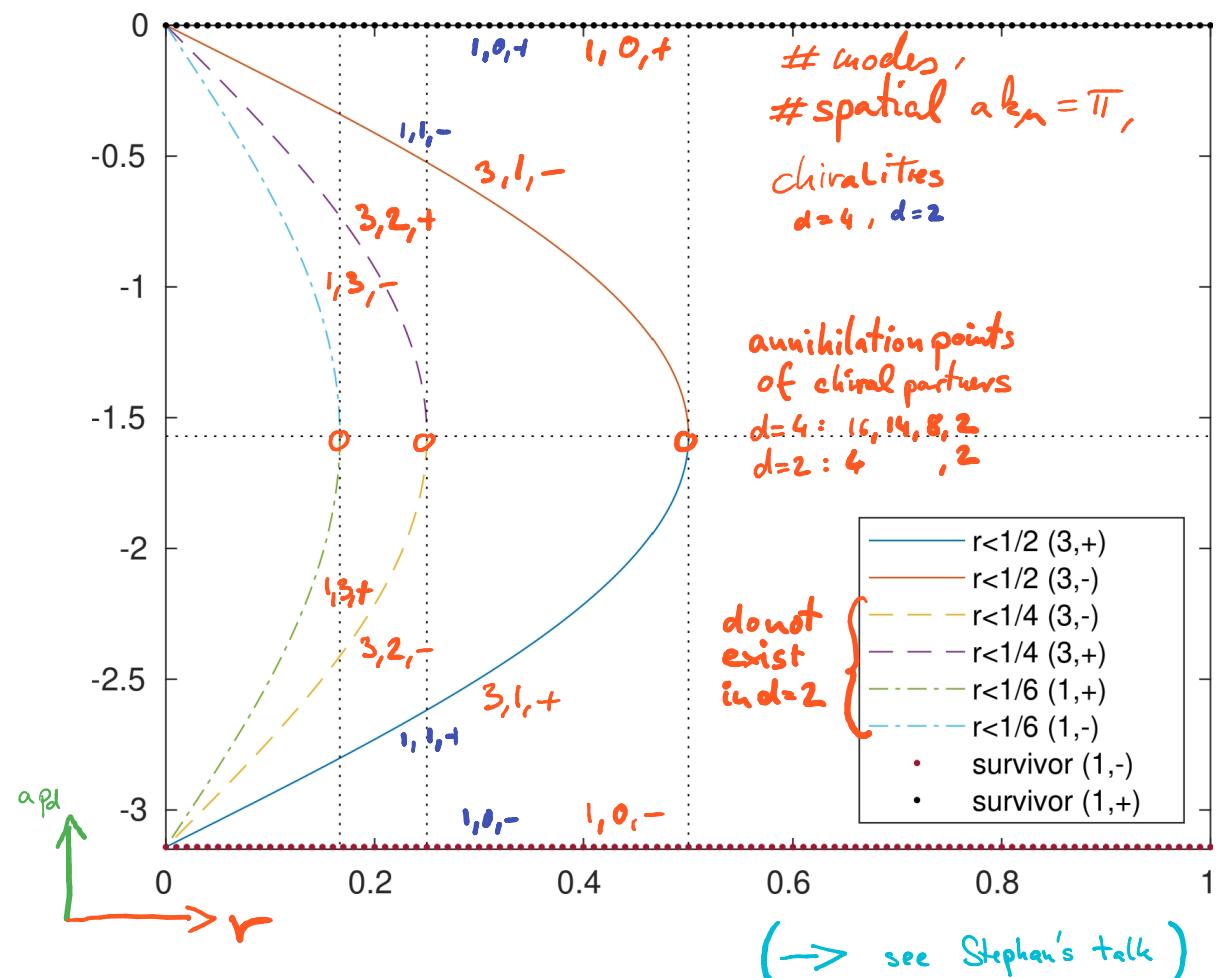
PCT

chiral: $\gamma_5 D_{KW} + D_{KW} \gamma_5 = 0$

one internal non-singlet C or T symmetry

$$\hookrightarrow \tau_d \equiv \tau_{d,x} = i \int_d \gamma_5 (-1)^{x \cdot d \alpha}$$

Species chain of Karsten-Wilczek fermions.
(Dürr, Weber PRD 102 (2020))



mirror fermion symmetry (M. Pernici PLB 346 (1995))

Karsten-Wilczek fermions in perturbation theory: part 1

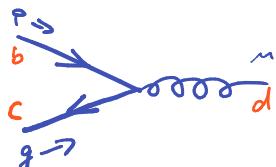
$$D_{K\omega}(x,y) = D(x,y) - \alpha \frac{r}{2} i \gamma_d \sum_{n=1}^{d-1} \Delta_n(x,y) + m_0 \delta_{xy} \quad \text{with minimal coupling to a gauge field}$$

propagator:

$$D_{K\omega}^{-1}(p) = \frac{-i \left(\sum_{n=1}^d \gamma_n \bar{P}_n + \alpha \frac{r}{2} \gamma_d \sum_{n=1}^{d-1} \hat{P}_n^2 \right) + m_0}{\sum_{n=1}^{d-1} \bar{P}_n^2 + \underbrace{\left(\bar{P}_d + \alpha \frac{r}{2} \sum_{n=1}^{d-1} \hat{P}_n^2 \right)^2 + m_0^2} \delta^{ab}}$$

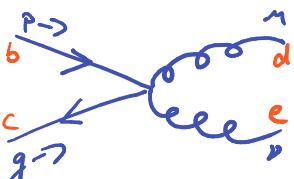
denominator is NOT EVEN in p_d or r

ggg vertex:



$$V_1^\mu(p,q) = -i \frac{g_0}{2} \left[\gamma_\mu \cos\left(\alpha \frac{p_a + q_m}{2}\right) + r \gamma_d (1 - \delta_{ad}) \sin\left(\alpha \frac{p_a + q_m}{2}\right) \right] (T^d)^{bc}$$

ggg² vertex:



$$V_2^{\mu\nu}(p,q) = +i \alpha \frac{g_0^2}{4} \delta^{\mu\nu} \left\{ \gamma_\mu \sin\left(\alpha \frac{p_a + q_m}{2}\right) - r \gamma_d (1 - \delta_{ad}) \cos\left(\alpha \frac{p_a + q_m}{2}\right) \right\} \{ T^d, T^e \}^{bc}$$

(Like vertices for Wilson fermions with $1 \rightarrow i \gamma_d (1 - \delta_{ad})$) (\rightarrow see Stefano's talk)

Karsten-Wilczek fermions in perturbation theory : part 2

(Capitani, Creutz,
Weber, Wittig
JHEP09 (2010))

Using standard gluon propagator (plaquette action) to compute self-energy

$$\begin{aligned} \Sigma(p, m_0; g_0^2, r, a) &= \left[\text{Diagram 1} + \text{Diagram 2} \right]_{\text{ext. prop. terms}} \\ &= \left[i g \cdot p \sum_1 (g_0^2, r^2, (ap)^2) + m_0 \sum_2 (g_0^2, r^2, (ap)^2) + i \int_a^r f d \cdot d_3 (g_0^2, r^2) + i \int_a^r p d \cdot d_4 (g_0^2, r^2) + O(a, m_0^2) \right] \delta^{ab} \end{aligned}$$

present in fermion action at tree-level

$$\Rightarrow \text{for } r=1: \sum_1 = G_F (L + 9.24089) \quad \sum_2 = G_F (4L - 24.36875)$$

$$d_3 = G_F (-29.53228) \quad d_4 = G_F (-0.12554)$$

$$G_F = \frac{g_0^2 C_F}{16\pi^2}$$

$$L = \log (ap)^2$$

(also: Local currents & both Noether currents at 1-loop level)

and fermionic contribution to the vacuum polarization

$$\begin{aligned} \Pi_{\mu\nu}(p, m_0; g_0^2, r, a) &= \left[\text{Diagram 3} + \text{Diagram 4} \right]_{\text{ext. prop. terms}} \\ &= \left[(p_\mu p_\nu - \delta_{\mu\nu} p^2) \Pi_F (g_0^2, r^2, (ap)^2) + \underbrace{(p_\mu p_\nu (\delta_{ad} - \delta_{bd}) - \delta_{\mu\nu} (p^2 \delta_{ad} \delta_{bd} + p_d^2))}_{\equiv A_{\mu\nu}^{kw}} d_{4p} (g_0^2, r^2) + O(a, m_0^2) \right] \delta^{ab} \end{aligned}$$

present in fermion action at tree-level,
NOT in gauge action

$$\Rightarrow \text{for } r=1: \Pi_F = G_A \left(-\frac{8}{3} L + 19.19468 \right) \quad d_{4p} = G_A (-12.69766)$$

Ward identities ok: $p_\mu A_{\mu\nu}^{kw} = 0.$

$$G_A = \frac{g_0^2 C_A}{16\pi^2}$$

(→ see Stefano's talk)

Due to broken time reflection symmetry, odd powers of temporal indices cannot be ruled out in terms of naive arguments; e.g. $a D_\mu F_{\nu\rho} F_{\rho\lambda}$ is not obviously forbidden...

Karsten-Wilczek fermions: taste structure part 1

$$D_{KW}(p) = D(p) + r \sum_{n=1}^{d-1} \left(\frac{i}{a} \gamma_d - \frac{i}{a} \gamma_d \cos(a p_n) \right) + m_0 \mathbb{1} + m_3 M(p) \xrightarrow{\text{to be defined later}}$$

Operator Transformation	$D = \sum_{n=1}^d i \gamma_n \bar{p}_n$	$K = +\frac{i}{a} (d-1) \gamma_1$	$W = -\frac{i}{a} \gamma_d \sum_{n=1}^{d-1} \cos(a p_n)$	1	M
C	+D	-K	-W	+1	+M
T	+D	-K	-W	+1	+M
$\overline{C_d}$	+D	-K	-W	+1	-M
γ_5	-D	-K	-W	+1	+M

$su(2)$ - representation involving T_d such that all operators have well-defined transformations?

$$\tau_5 \equiv \gamma_5 \epsilon = \gamma_5 \underbrace{(-1)^{\sum_{n=1}^d x_n}}_{\text{staggered } \epsilon_x} \quad \text{and} \quad \tau_{d5} \equiv \gamma_d (-1)^{\sum_{n=1}^{d-1} x_n/a} \Rightarrow [\tau_{d5}, \tau_d] = 2i \tau_5 \quad \text{and cyclic}$$

for later convenience: $\bar{x} = \sum_{n=1}^d x_n/a$

τ_5	+D	-K	+W	+1	+M
τ_{d5}	+D	+K	-W	+1	-M

Karsten-Wilczek fermions: taste structure part 2

$$\bar{\tau}_{d5} \equiv \gamma_d (-1)^{\sum_{\mu=1}^{d-1} x_\mu / a}, \quad \bar{\tau}_d = i \gamma_d \gamma_5 (-1)^{x_d / a}, \quad \bar{\tau}_5 \equiv \gamma_5 \varepsilon = \gamma_5 (-1)^{\sum_{\mu=1}^d x_\mu / a}$$

chiral rep: $\gamma_\mu = \begin{pmatrix} 0 & -i\sigma_n \\ i\sigma_n & 0 \end{pmatrix} = \sigma_2 \otimes \sigma_n \quad \mu = 1, \dots, d-1; \quad \gamma_d = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \otimes 1 \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \otimes 1$

$$\bar{\tau}_{d5} \propto \sigma_1 \otimes 1, \quad \bar{\tau}_d \propto \sigma_2 \otimes 1, \quad \bar{\tau}_5 \propto \sigma_3 \otimes 1$$

\Rightarrow natural interpretation as taste su(2) algebra
Weyl spinors have well-defined taste interpretation

Remark: su(2) representation ceases to work in the continuum limit due to phase factors!

Two tastes, $2^{d/2}$ spin components per taste or per site

\Rightarrow two sites are necessary to define both tastes

Symmetry demands that these are separated by one step in the d -direction

The naive operator $\mathcal{D}(p)$ is taste singlet

\Rightarrow Weyl spinors alternate in taste interpretation between even and odd sites in all directions

γ_5 is not invariant under this su(2) representation:

$$\{\bar{\tau}_{d5}, \gamma_5\} = 0, \quad \{\bar{\tau}_d, \gamma_5\} = 0, \quad \{\bar{\tau}_5, \gamma_5\} = 2\varepsilon \quad \Rightarrow \varepsilon \text{ is the taste singlet } \gamma_5!$$

Karsten-Wilczek fermions: taste structure part 3

Arbitrary spin taste operators can be constructed by combining Dirac matrices and hopping terms:

Invariance under $\begin{Bmatrix} \tau_d \\ \tau_s \\ \tau_{ds} \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \text{odd # of temporal hops} \\ \text{odd # of spatial hops} \\ \text{odd # of both types} \end{Bmatrix}$

KW spin-taste notation $\Gamma_{\mu\nu} = \gamma_\mu \otimes \xi_\nu \Rightarrow \begin{Bmatrix} \epsilon = \Gamma_{50}, \gamma_5 = \Gamma_{53}, \tau_5 = \Gamma_{03} \\ \text{GL}(4, \mathbb{C}) \text{ spin} \quad \text{su}(2) \text{ taste} \end{Bmatrix}$

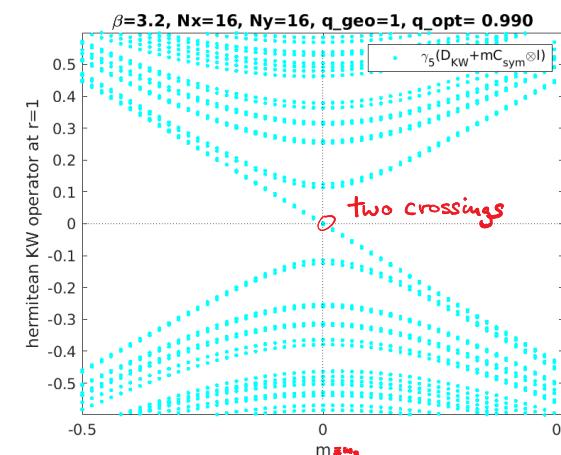
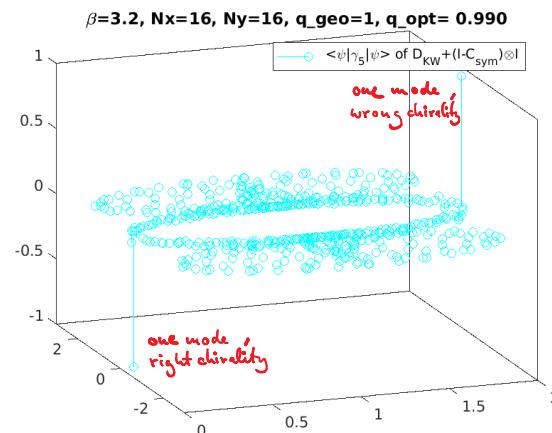
What is a suitable operator M ? e.g. $\prod_{n=1}^d \cos(\alpha p_n)$ or $\frac{1}{d-1} \sum_{n=1}^{d-1} \cos(\alpha p_n) \cos(\alpha p_d)$

(needs to be gauged, symmetrized, etc.; both are identical in $d=2$: $M = C_{sym} = \prod_{n=1}^d \cos(\alpha p_n)$ in $d=2$)

Plays the role of a taste non-singlet mass or an Adams-like chirality-splitter

Chiral eigenmodes
of $\gamma_5 M$

$$r=1, m_0=1, m_3=1$$



(Dürr, Weber PRD 105 (2022))

Eigenvalue flow
of hermitian op.

$$r=1$$

(→ see Stephan's talk)

Karsten-Wilczek fermions: the determinant

(Weber Pos LAT14 071 (2014))
 (Weber Pos LAT16 250 (2016))

$$\mathcal{D}_{Kw}[u; a; m_0, m_3, r, d_3, d_4] = \mathcal{D}_s[u_s] + (1+d_4) \mathcal{D}_t[u_t] + r \left(1 + \frac{d_3}{d-1}\right) K + r W[u_s] + m_0 I + m_3 M[U_s, \bar{U}_t]$$

$$\det(\mathcal{D}_{Kw}) \equiv \Delta(a; m_0, m_3, r, d_3, d_4) = \det(\mathcal{D}_{Kw} X^T X) = \det(X \mathcal{D}_{Kw} X^T) \text{ for arbitrary unitary operator } X$$

$$X \rightarrow i\gamma_5 \Rightarrow \Delta(\dots) = \Delta(a; m_0^2, m_3^2, m_0 m_3; r, d_3, d_4)$$

$$X \rightarrow T_d \Rightarrow \Delta(\dots) = \Delta(a; m_0^2, m_3^2, m_0 m_3 r, r^2, d_3, d_4)$$

$$X \rightarrow C \Rightarrow \Delta(\dots) = \Delta(a^2; m_0^2, m_3^2, r^2, d_3, d_4) \quad (\text{assuming gauge is invariant under } C)$$

\Rightarrow Residual taste-symmetry leads to automatic $\mathcal{O}(a)$ improvement of the determinant. (tasted charge conjugation symmetry)

Remark: could have been inferred from the fact that $(\det \mathcal{D})$ must be a taste singlet...
 \rightarrow same idea as for twisted-mass at maximal twist.

\Rightarrow The Dirac operator \mathcal{D}_{Kw} has $\mathcal{O}(a)$ effects accompanied by odd powers of r . As gauge configurations are functions of r^2 only, one may average observables with $\pm r$ on the same gauge configurations to eliminate $\mathcal{O}(a)$ effects in any observables without additional tuning.

Karsten-Wilczek fermions: propagators and correlators part 1

(Weber Pos LAT14 071 (2014)
 Weber Pos LAT16 250 (2016))

Suppose an operator $D_{\pm s} = E \pm s O$ and a broken symmetry X , such that $X D_{\pm s} X^T = D_{\mp s} = E \mp s O$

Then the fermion propagator $S_{\pm s}$ satisfying $S_{\pm s} D_{\pm s} = \delta$

$$\text{is } S_{\pm s} = [(E_{-s} O)(E_{+s} O)]^{-1}(E^{\mp s} O) = [E^2 + s [E, O] - s^2 O^2]^{-1}(E^{\mp s} O)$$

$$\text{or } S_{\pm s} = S_E^{\mp s} S_O \quad \text{with} \quad S_E = [E - s^2 O - E^{-1} O]^{-1}, \quad S_O = -S_E s O E^{-1}$$

Construct a meson correlation function such as $C_{pp}(v, w) = \langle \bar{q}_v \Gamma q_v \bar{q}_w \Gamma q_w \rangle$, where $X \Gamma X^T = \tilde{\Gamma} \Gamma$, $\tilde{\Gamma} = e^{i\varphi}$

$$\begin{aligned} C_{pp}(v, w) &= \langle (\Gamma \bar{q}_v \bar{q}_v) (\Gamma \bar{q}_w \bar{q}_w) - (\Gamma \bar{q}_v \bar{q}_w \Gamma \bar{q}_w \bar{q}_v) \rangle = \langle (\Gamma S_{+s}(v, v)) (\Gamma S_{+s}(w, w)) - (\Gamma S_{+s}(v, w) \Gamma S_{+s}(w, v)) \rangle \\ &= \underbrace{|\tilde{\Gamma}|^2}_{=1} \langle (\Gamma S_{-s}(v, v)) (\Gamma S_{-s}(w, w)) - (\Gamma S_{-s}(v, w) \Gamma S_{-s}(w, v)) \rangle \end{aligned}$$

\Rightarrow Observables that are even under C, T also cannot have odd powers in a .

This applies, e.g., to the correlator of the Goldstone pion $C_{ss}(x, y) = \langle \bar{\psi}(x) f_5 \psi(x) \bar{\psi}(y) f_5 \psi(y) \rangle$, which is invariant under charge conjugation. Due to that C -invariance, T or T -invariance is granted.

Absence of naively expected $O(a)$ effects in the pion correlator led to confusion... (K. Cichy, et al. NPB 800 (2008))

Karsten-Wilczek fermions: propagators and correlators part 2 (Weber PoS LATTICE 250 (2016))

$$S_{\pm} = S_E \pm S_O \quad \text{with} \quad S_E = [E - \sigma E^{-1} \sigma]^{-1}, \quad S_O = -S_E \sigma E^{-1}$$

Choose $X = \tau_d$: $E = D_s + D_\ell + m_0 \mathbf{1}$, $\sigma = v(k + w) + m_0 M$

There is another connected contribution, which is a mixed product of S_{\pm} and has a different interpolating operator structure.

$$\left\langle \overbrace{\tau_d^+ \tau_d^-}^{\tau_d^+ \tau_d^-} S_{-}(v,w) \overbrace{\tau_d^+ \tau_d^-}^{\tau_d^+ \tau_d^-} S_{+}(w,v) \right\rangle = \left\langle (\tau_d^+ S_{-} \tau_d^- \tau_d^+ S_{+}(w,v)) \right\rangle$$

For naive or KS fermions \Rightarrow time-alternating parity partner contribution,

e.g. $C(t) = \sum_i A_i e^{-E_i t} - (-1)^{t/\epsilon} \sum_j \tilde{A}_j e^{-\tilde{E}_j t}$, where two towers of states differ in parity.

In the free theory, these parity partners are degenerate at finite lattice spacing for naive or KS fermions that have no σ term.

This is NOT TRUE for KW fermions. Compare two pseudoscalars:

$$\left\langle \gamma_5 S_{+}(v,w) \gamma_5 S_{+}(w,v) \right\rangle = \left(\left\langle (\gamma_5 S_E(v,w) \gamma_5 S_E(w,v)) \right\rangle + \left\langle (\gamma_5 S_O(v,w) \gamma_5 S_O(w,v)) \right\rangle \right)$$

$$\begin{aligned} \left\langle \overbrace{\tau_d^+ \tau_d^-}^{\tau_d^+ \tau_d^-} S_{+}(v,w) \overbrace{\tau_d^+ \tau_d^-}^{\tau_d^+ \tau_d^-} S_{+}(w,v) \right\rangle &= -(-1)^{(v-w)/\epsilon} \left\langle (\gamma_5 S_{-}(v,w) \gamma_5 S_{+}(w,v)) \right\rangle \\ &= -(-1)^{(v-w)/\epsilon} \left(\left\langle (\gamma_5 S_E(v,w) \gamma_5 S_E(w,v)) \right\rangle - \left\langle (\gamma_5 S_O(v,w) \gamma_5 S_O(w,v)) \right\rangle \right) \end{aligned}$$

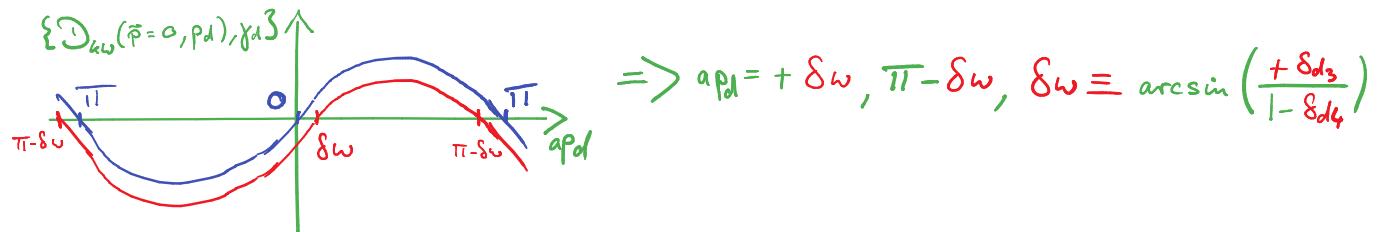
Karsten-Wilczek fermions: propagators and correlators part 3

Pole structure of the propagator of D_{Kw} at weak coupling, mis-tuned by δ_{d_4} and δ_{d_3} (assuming $m_0=0, m_3=0$):

$$D_{Kw}(p) D_{Kw}^+(p) = \sum_{n=1}^{d-1} \bar{p}_n^2 + ((1 - \delta_{d_4}) \bar{p}_d + \cancel{r} \sum_{n=1}^{d-1} \bar{p}_n^2 - \delta_{d_3}/a)^2 \stackrel{!}{=} 0 \quad \begin{array}{l} \text{minimal doubling} \\ \text{iff } \frac{2r - \delta_{d_3}}{1 - \delta_{d_4}} > 1 \end{array}$$

implies $a p_n = 0, \pi \quad \forall n = 1, \dots, d-1$ and $(1 - \delta_{d_4}) \sin(ap_d) + r \sum_{n=1}^{d-1} (1 - \cos(ap_n)) - \delta_{d_3} \stackrel{!}{=} 0 \Rightarrow ap_d = \arcsin\left(\frac{+\delta_{d_3}}{1 - \delta_{d_4}}\right)$

Mild mis-tuning $\delta_{d_3}, \delta_{d_4} \ll 1$:



Physical fermions at the poles : $\tilde{\zeta}_d = i \gamma_d \gamma_5 e^{\pm i(\pi - 2\delta w) x_d/a} = e^{\mp i 2\delta w x_d/a}$ $\tilde{\zeta}_d$ maps the pole at $+8w$ to the one at $\pi - 8w$
 $\tilde{\zeta}_d^+ = i \gamma_d \gamma_5 e^{\pm i(\pi - 2\delta w) x_d/a} = e^{\pm i 2\delta w x_d/a}$ $\tilde{\zeta}_d$ maps the pole at $\pi - 8w$ to the one at $+8w$

Tying the propagators together, the time alternating contribution $\langle (\overset{\uparrow}{\tau} S_{ir}(v,v) \overset{\uparrow}{\tau} S_{tr}(v,v)) \rangle = \langle (\overset{\uparrow}{\tau} X^+ S_{ir} X \overset{\uparrow}{\tau} S_{tr}(v,v)) \rangle$

becomes directly sensitive to the mis-tuning.

With $X = \tilde{\zeta}_d$: $C(t) = \sum_i A_i (e^{-E_i t} + e^{-E_i (aN_\tau - t)}) - \frac{1}{2} (e^{+i 2\delta w t/a} + e^{-i 2\delta w t/a}) (-1)^{t/a} \sum_j \tilde{A}_j \left(e^{-\tilde{E}_j t} + e^{-\tilde{E}_j (aN_\tau - t)} \right)$

swap pairing of poles

$$= \sum_i 2A_i e^{-E_i \frac{aN_\tau}{2}} \cosh(E_i (aN_\tau/2 - t)) - \underbrace{\cos(2\delta w t/a) (-1)^{t/a}}_{= \cos((\pi - 2\delta w) t/a)} \sum_j 2\tilde{A}_j e^{-\tilde{E}_j \frac{aN_\tau}{2}} \cosh(\tilde{E}_j (aN_\tau/2 - t))$$

Since $8w \approx \delta_{d_3}$, this shifted frequency can be put to practical use. If life gives you lemons ...

Karsten-Wilczek fermions: dropped KW fermions? (Kimura et al., JHEP 01 (2012))

$$\mathcal{D}_{KW}(p) = \cancel{\mathcal{D}}(p) + r \sum_{n=1}^{d-1} \left(\cancel{\frac{i}{a} \gamma_d} - \frac{i}{a} \gamma_d \cos(a p_n) \right) + m_0 \mathbf{1} + m_3 M(p)$$

operator transform	\mathcal{D}	$W = -\frac{i}{a} \gamma_d \sum_{n=1}^{d-1} \cos(a p_n)$	1	M
C	$+\cancel{\mathcal{D}}$	$-W$	+1	+M
T	$+\cancel{\mathcal{D}}$	$-W$	+1	+M
\overline{C}_d	$+\cancel{\mathcal{D}}$	$-W$	+1	-M
γ_5	$-\cancel{\mathcal{D}}$	$-W$	+1	+M
T_5	$+\cancel{\mathcal{D}}$	$+W$	+1	+M
\overline{T}_{d5}	$+\cancel{\mathcal{D}}$	$-W$	+1	-M

- a) has an exact taste symmetry under T_5 !
 b) loses minimal doubling: 4(d-1) tastes with poles

at $a p_n = 0, \pi$ with $N_\pi = 1, 2$, $a p_d = \pm S_{N_\pi}, \pi \mp S_{N_\pi}$

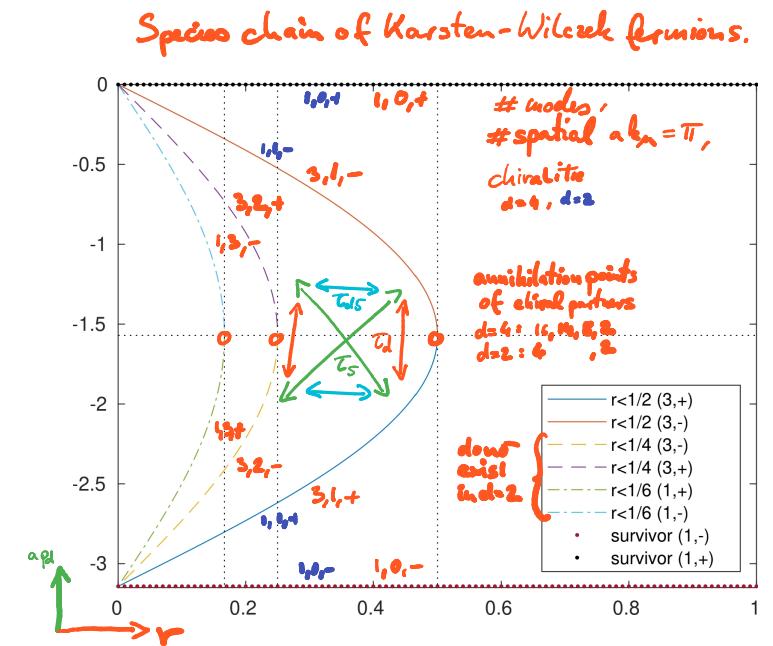
with $S_{N_\pi} = |(2N_\pi+1-d)| \arcsin r|$

chirality unaltered by XC of spatial axes

T_5 maps $(0, 0, \pi, \pi-s)$ into $(\pi, \pi, 0, \pi-s)$

while \overline{T}_d swap chiralities as $(0, 0, \pi, \pi-s)$ into $(0, 0, \pi, -s)$
 $(\pi, \pi, 0, \pi-s)$

- c) symmetries under $CT, T_d C, \overline{T}_d T$ persist. Equivalently $T_{d5} C, \overline{T}_{d5} T$.



- d) at $r = \pm 1$: $S_{N_\pi} = \pm \pi/2$
 taste pairs at $a p_d = S_{N_\pi}, \pi - S_{N_\pi}$ merge
 and lose well-defined chiralities!

=> Unclear, if dropped KW fermions are useful ...

Karsten-Wilczek Fermions: Symanzik effective theory part 1

Continuum building blocks: $\mathcal{D} = \sum_{n=1}^d \Gamma_n \Gamma_n \otimes 1$, $r \gamma_a \Gamma_d \otimes \sigma_1$, $m_0 1 \otimes 1$, $m_3 M = m_3 1 \otimes \sigma_3$, $r \gamma_a \Gamma_d \otimes \sigma_3$

Lattice spin-taste representation:

$\gamma_n \equiv \Gamma_n \otimes \sigma_2$	and	$\delta_{x \pm \hat{n}, y} = 1 \otimes \sigma_2$	$n = 1, \dots, d-1$
$\gamma_d \equiv \Gamma_d \otimes \sigma_1$	and	$\delta_{x \pm \hat{d}, y} = 1 \otimes \sigma_1$	

$$\text{with } D_n(x,y) = \frac{\delta_{x+\hat{n},y} - \delta_{x-\hat{n},y}}{2a} \text{ and } c_n(x,y) = \frac{\delta_{x+\hat{n},y} + \delta_{x-\hat{n},y}}{2}$$

\Rightarrow taste manipulation via $c_d \sim 1 \otimes \sigma_1$, $C = \frac{1}{d-1} \sum_{n=1}^{d-1} c_n \sim 1 \otimes \sigma_2$, $M = \frac{1}{2} \{C, c_d\} \sim 1 \otimes \sigma_3$

Lattice building blocks: $\mathcal{D} \sim \Gamma_n \otimes 1$, $\frac{a}{d-1} K = i \gamma_d \sim \Gamma_d \otimes \sigma_1$, $m_0 1 \sim 1 \otimes 1$, $m_3 M \sim 1 \times \sigma_3$

$$\frac{a}{d-1} W = i \gamma_d C \sim \Gamma_d \otimes \sigma_3 \quad (\text{arises naturally from interplay of } \mathcal{D} \text{ and } i \gamma_d)$$

Dimension 3:

O_o^F	$= \frac{r}{a} \bar{\psi} i \gamma_d \psi$	$= O_A^{F3}$	Note: at dimension 3
$[O_i^F]$	$= \frac{r}{a} \bar{\psi} i \gamma_d C \psi$	$= O_B^{F3}$	O_B^{F3} is indistinguishable,

starts to differ at dimension 5,
where it arises automatically.

Dimension 4:

O_o^F	$= m_0 \bar{\psi} \psi$	$= O_A^{F4}$
O_i^F	$= \bar{\psi} \mathcal{D} \psi$	$= O_B^{F4}$
O_2^F	$= r^2 \bar{\psi} i \gamma_d \mathcal{D} i \gamma_d \psi = r^2 \bar{\psi} (\mathcal{D} - 2 \mathcal{D}_d \gamma_d) \psi$	$= r^2 (O_B^{F4} - 2 O_c^{F4})$
O_3^F	$= m_3 \bar{\psi} M \psi$	$\therefore O_c^{F4} = \bar{\psi} \mathcal{D} i \gamma_d \psi$

O_o^G	$= \sum_{\mu < \nu=1}^d F_{\mu\nu}^2$	$= O_A^{G4}$	regular plaquette
O_i^G	$= r^2 \sum_{\mu=1}^{d-1} F_{\mu d}^2$	$= O_B^{G4}$	"electric" plaquette

Karsten-Wilczek Fermions: Symanzik effective theory part 2

(Weber PoS LAT14 071 (2014))

Lattice building blocks: $\mathcal{D} \sim P_m \otimes 1$, $\overset{\alpha}{K} = i\gamma_5 \not{d} \sim P_d \otimes \sigma_1$, $m_0 1 \sim 1 \otimes 1$, $m_3 M \sim 1 \times \sigma_3$

taste manipulation via $c_d \sim 1 \otimes \sigma_i$, $C = \frac{1}{d-1} \sum_{n=1}^{d-1} c_n \sim 1 \otimes \sigma_2$, $M = \frac{1}{2} \{C, c_d\} \sim 1 \otimes \sigma_3$

$$\text{Dimension 5: } O_0^{\text{FS}} = \alpha r m_0^2 \bar{\psi} i\gamma_5 \not{d} \psi$$

$$O_1^{\text{FS}} = \alpha r \bar{\psi} i\gamma_5 \sum_{n=1}^{d-1} \Delta_n \psi$$

$$O_2^{\text{FS}} = \alpha r \bar{\psi} i\gamma_5 \Delta_d \psi$$

$$O_3^{\text{FS}} = \alpha r \bar{\psi} i\gamma_5 \sum_{n=1}^{d-1} \gamma_n \beta_n \psi$$

$$O_4^{\text{FS}} = \alpha r m_0 m_3 \bar{\psi} M \not{d} \psi$$

$$O_5^{\text{FS}} = \alpha r m_3^2 \bar{\psi} i\gamma_5 \not{d} \psi$$

$$= O_1^{\text{FS}} = m_0^2 O_A^{\text{FS}}$$

$$= O_B^{\text{FS}} = \alpha^2 (d-1) (O_A^{\text{FS}} - O_B^{\text{FS}})$$

$$= O_C^{\text{FS}}$$

$$= O_D^{\text{FS}}$$

$$= O_E^{\text{FS}}$$

$$= O_F^{\text{FS}} = m_3^2 O_A^{\text{FS}}$$

Remarks on the constructions

odd powers of $i\gamma_5 \not{d}$ (one or three)

total even power of $\mathcal{D}, m_0, m_3 M$

$\Rightarrow 17$ operators to start with

6 are trivially zero

5 non-trivial relations (assuming $[100; 300] \neq 0$)

\Rightarrow change remaining 6 to preferred basis

2 trivial mass squared rescalings of O_A^{FS}

even powers of $i\gamma_5 \not{d}$ forbidden by chiral symmetry

The symmetry-breaking pattern of the 1-hop operator O_{C1}^{FS} is wrong!

Instead one must use the 2-hop operator O_{C2}^{FS} with $\tilde{\Delta}_d = \frac{\delta_{x+2\hat{d}, y} + \delta_{x-2\hat{d}, y}}{8} \sim \frac{\cos(2\pi p d)}{4}$.

no gauge field operators due to

- unbroken chiral symmetry
- broken taste symmetry
- unbroken taste reflection & charge conjugation symmetries

Operator Transform	h			
	1 hop O_{C1}^{FS}	2 hops $O_C^{\text{FS}} \equiv O_{C2}^{\text{FS}}$	O_D^{FS}	O_E^{FS}
Φ	$+ O_{C1}^{\text{FS}}$	$+ O_{C2}^{\text{FS}}$	$+ O_D^{\text{FS}}$	$+ O_E^{\text{FS}}$
C	$- O_{C1}^{\text{FS}}$	$- O_{C2}^{\text{FS}}$	$- O_D^{\text{FS}}$	$- O_E^{\text{FS}}$
T	$- O_{C1}^{\text{FS}}$	$- O_{C2}^{\text{FS}}$	$- O_D^{\text{FS}}$	$- O_E^{\text{FS}}$
\overline{c}_d	$+ O_{C1}^{\text{FS}}$	$- O_{C2}^{\text{FS}}$	$- O_D^{\text{FS}}$	$+ O_E^{\text{FS}}$
γ_5	$- O_{C1}^{\text{FS}}$	$- O_{C2}^{\text{FS}}$	$- O_D^{\text{FS}}$	$- O_E^{\text{FS}}$
τ_5	$+ O_{C1}^{\text{FS}}$	$- O_{C2}^{\text{FS}}$	$- O_D^{\text{FS}}$	$- O_E^{\text{FS}}$
τ_{d5}	$+ O_{C1}^{\text{FS}}$	$+ O_{C2}^{\text{FS}}$	$+ O_D^{\text{FS}}$	$- O_E^{\text{FS}}$

Borici-Creutz fermions: construction part)

Graphene realizes a relativistic dispersion of two massless fermions on a hexagonal ($d=2$) lattice. Can this construction be generalized to $d=4$?

$$D_{BC}(x,y) = D(x,y) - \alpha r \sum_{n=1}^d i \gamma_n' \Delta_n(x,y)$$

$$\text{Key ingredient: } \Gamma = \frac{1}{\sqrt{d}} \sum_{n=1}^d \gamma_n = \Gamma^+, \quad \Gamma^2 = 1$$

$$\hookrightarrow \text{2nd set of } \gamma\text{-matrices: } \gamma_n' = \Gamma \gamma_n \Gamma = \frac{2}{\sqrt{d}} \Gamma - \gamma_n$$

$$\text{In } d=2: \gamma_1 \equiv \sigma_1, \gamma_2 \equiv \sigma_2 \rightsquigarrow \gamma_1' = \sigma_2, \gamma_2' = \sigma_1$$

$\hookrightarrow -\frac{\alpha r}{2} \gamma_2' \Delta_2(x,y)$ is extraterm vs KW fermions

In momentum space:

$$\begin{aligned} D_{BC}(\vec{p}) &= D(\vec{p}) + r \sum_{n=1}^d i \gamma_n' (1 - \cos(\gamma_n p_n)) \\ &= i \sum_{n=1}^d \gamma_n \bar{p}_n + \alpha r \sum_{n=1}^d i \gamma_n' \hat{p}_n^2 \end{aligned}$$

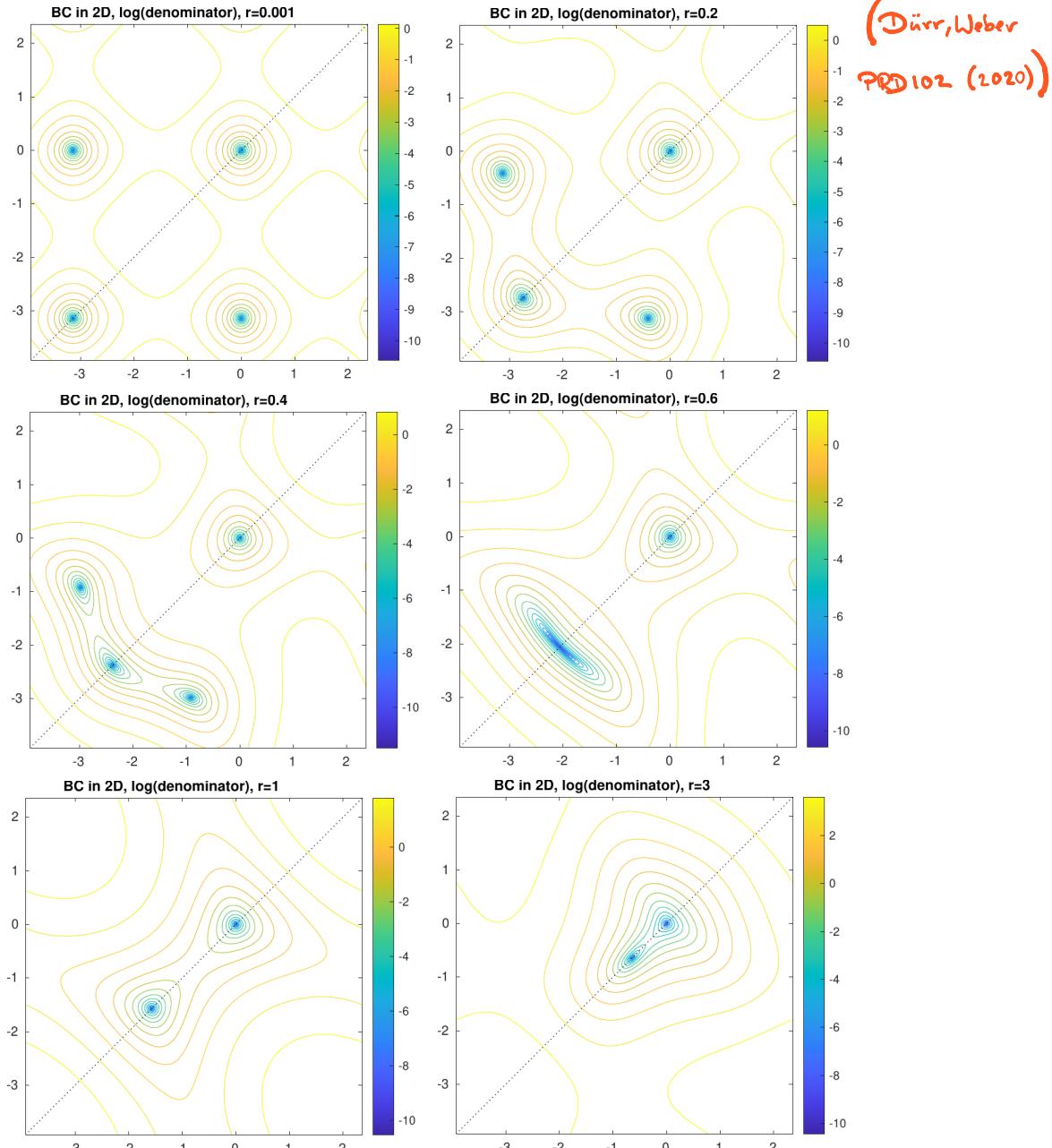
New term spreads the spectrum of $D(k)$ along the imaginary axis

$$|A_{BC}| \leq \frac{r}{2} (r + \sqrt{1+r^2})$$

CPT ; exchange of any axes ;
chiral: $\gamma_5 D_{BC} + D_{BC} \gamma_5 = 0$

M. Creutz
A. Borici
Pos LAT 08, JHEP 04 (2008)
Pos LAT 08, PRD 78 (2008)

The role of the r parameter for species reduction ($d=2$)



$r=1$ is special and has an extra symmetry!

Borici-Creutz fermions: construction part 2

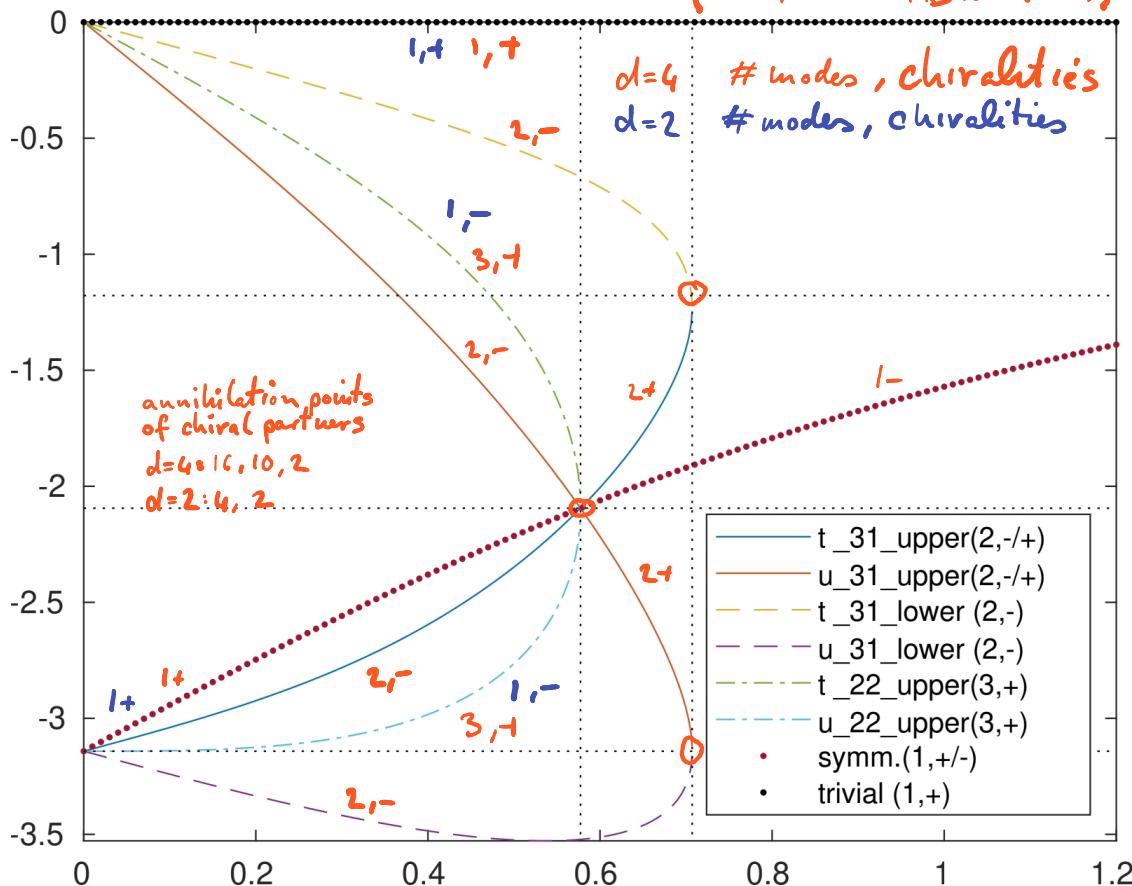
Axis exchange symmetry classes

Tastes : all k_μ equal, (two) pairs of unequal k_μ ,
or in two groupings of three vs one)

Why is $r=1$ special? (M. Creutz PoS LAT08 (2008))

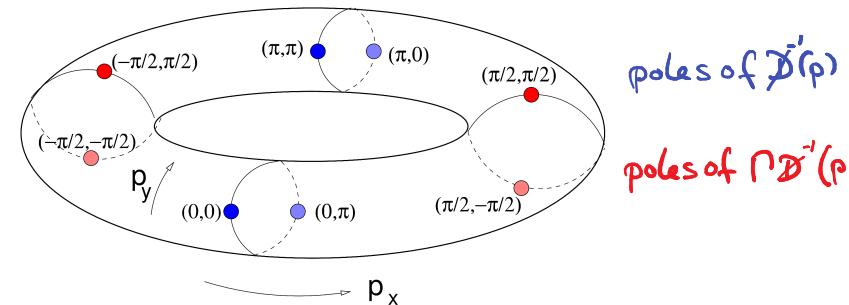
Take the naïve operator $\mathcal{D}(p)$, left-right multiply by Γ ,
then shift momentum by $\pm\pi/2a$ in all components.

Species chain of Borici-Creutz fermions, (Dürr,Weber PRD102 (2020))

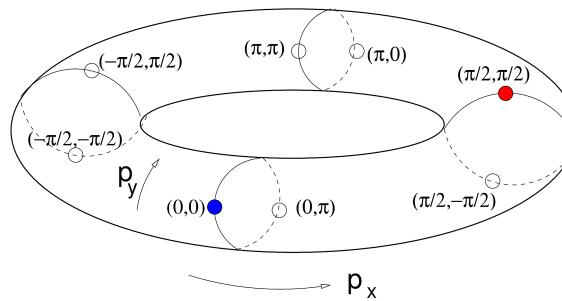


minimal doubling $k_\mu = 0 \neq n$ or $k_\mu = -2 \arctan(1/r)$

The second survivor changes its chirality in a brutal mass extinction event,
while its position as function of r pretends that nothing happened.



Choose one pole of $\mathcal{D}(p)$ such as $p=0$.
There: $\Gamma \mathcal{D}(p \pm \pi/2a) \Gamma = \pm \sqrt{d/a} i \Gamma$. Subtract this.



Then $\pm \sqrt{d/a} i \Gamma$ cancels the contribution from $\mathcal{D}(p)$ at exactly one pole of $\Gamma \mathcal{D}^{-1}(p \pm \pi/2a) \Gamma$. This is minimally doubled without reference to graphene.

We now have a symmetric expression:

$$\begin{aligned} \mathcal{D}_{BC}(p) &= \sum_{n=1}^d \left[\frac{i}{a} g_n \sin(\alpha p_n) \pm p g_n \Gamma \cos(\alpha p_n) \mp \Gamma \right] \\ &= \mathcal{D}(p) \mp \frac{1}{a} \sum_{n=1}^d g_n' (1 - \cos(\alpha p_n)) \end{aligned}$$

Borici-Creutz fermions: internal symmetries part 1

Unpublished!

Let us walk the Creutz construction backward in position space:

$$\begin{aligned} D_{BC}(x,y) &= D(x,y) - \frac{a}{2} \sum_{n=1}^d i \gamma_n' \Delta_n(x,y) + m_0 \delta_{x,y} = \sum_{n=1}^d \frac{1}{2} [\gamma_n (\partial_n^f + \partial_n^b) - i r \gamma_n' (\partial_n^f - \partial_n^b)](x,y) + m_0 \delta_{x,y} \\ &= \sum_{n=1}^d \frac{1}{2a} [(\gamma_n - r i \gamma_n') \delta_{x+\hat{n},y} + (-\gamma_n - r i \gamma_n') \delta_{x-\hat{n},y} + 2 i r \frac{\gamma_n}{\sqrt{a}} \delta_{x,y}] + m_0 \delta_{x,y} \end{aligned}$$

where we used $\sum_{n=1}^d \gamma_n' = \sum_{n=1}^d \gamma_n = \sqrt{a} \Gamma$.

First, we left-right multiply by $i P_{FS} = (\gamma_F \gamma_S)^+, (\gamma_F \gamma_S)^2 = 1$

$$i P_{FS} D_{BC}(x,y) i P_{FS} = \sum_{n=1}^d \frac{1}{2a} [(-\gamma_n' + r i \gamma_n) \delta_{x+\hat{n},y} + (+\gamma_n' + r i \gamma_n) \delta_{x-\hat{n},y} - 2 i r \gamma_n \delta_{x,y}] + m_0 \delta_{x,y}$$

We introduce a site-dependent phase $\xi_x = e^{+sign(r) i \pi/2 \bar{x}} = (+sign(r) i)^{\bar{x}}$, and transform $\psi(x) \rightarrow \xi_x \psi(x)$
 $\bar{\psi}(x) \rightarrow \bar{\psi}(x) \xi_x^*$

$$\begin{aligned} i P_{FS} \xi_x D_{BC}(x,y) i P_{FS} \xi_y^* &= \sum_{n=1}^d \frac{1}{2a} [(+i sign(r) \gamma_n' + i r \gamma_n) \delta_{x+\hat{n},y} + (+i sign(r) \gamma_n' - i r \gamma_n) \delta_{x-\hat{n},y} - 2 i r \gamma_n \delta_{x,y}] + m_0 \delta_{x,y} \\ &= |r| D(x,y) + \frac{a}{2} sign(r) \sum_{n=1}^d i \gamma_n' \Delta_n(x,y) + \underbrace{i \frac{a}{2} (sign(r)-r) \Gamma \delta_{x,y}}_{=0 \text{ for } r=\pm 1} + m_0 \delta_{x,y} \end{aligned}$$

For $r=\pm 1$:

the extra term cancels in this case.

the additional term changed sign, as it does under either charge conjugation or combined parity and time reflection.

NOTE: the transform with $\tau_P = i P_{FS} (+i sign(r))^{\bar{x}}$ does not just change $r \rightarrow -r$. It also swaps two terms!

Borici-Creutz fermions: internal symmetries part 2

Unpublished!

$$\tau_p D_{BC}(x,y) \tau_p^+ = \cancel{D}(x,y) + \text{sign}(r) \frac{a}{2} \sum_{n=1}^d i \gamma_n' \Delta_n(x,y) + m_0 \delta_{x,y} \text{ with } \tau_p = i \Gamma_5 (\pm \text{sign}(r))^{\frac{x}{a}}$$

for $r = \pm 1$.

The action has for $r = \pm 1$ has an additional internal non-singlet C or PT symmetry!

Is the non-hermitian, but unitary operator $\tau_p = i \Gamma_5 (\pm \text{sign}(r))^{\frac{x}{a}}$ part of a representation of a taste $\text{su}(2)$ algebra, where all operators have well-defined transformation patterns?

Yes! $\tau_S = \gamma_5 (-1)^{\frac{x}{a}}$ and $\tau_{PS} = \Gamma (\text{sign}(r))^{\frac{x}{a}}$ form an algebra $[\tau_{PS}, \tau_p] = 2i\tau_S$ and cyclic.

The hermitian conjugates correspond to the conjugated theory $r \rightarrow -r$.

$$\text{In momentum space, } D_{3C}(p) = \cancel{D}(p) + \text{sign}(r) \left(\underbrace{+i\sqrt{a}\Gamma}_{\equiv B} \underbrace{- \sum_{n=1}^d i \gamma_n' \cos(ap_n)}_{\equiv C} \right) + m_0 1 + m_3 M(p)$$

operator transform	\cancel{D}	\mathcal{B}	C	1	M
C	$+\cancel{D}$	$-B$	$-C$	$+1$	$+M$
PT	$+\cancel{D}$	$-B$	$-C$	$+1$	$+M$
τ_p	$-C$	$-B$	$+\cancel{D}$	$+1$	$-M$
γ_5	$-\cancel{D}$	$-B$	$-C$	$+1$	$+M$
τ_S	$+\cancel{D}$	$-B$	$+C$	$+1$	$+M$
τ_{PS}	$-C$	$+B$	$+\cancel{D}$	$+1$	$-M$

- a) Published results claiming the same taste structure for \mathcal{B} and C are wrong!
 (cf. P.Bedaque, et al. PLB 662 (2008))
- b) Taste non-singlet mass terms or chirality splitting terms could be realized as
- $1 - 2 \cos(ap_1) \cos(ap_2) \begin{cases} -1 & p_1 = p_2 = 0 \\ +1 & p_1 = p_2 = \pi/a \end{cases}$ for $d=2$
 - $1 - \frac{1}{2} (\cos(ap_1) + \cos(ap_2))^2 \begin{cases} -1 & p_1 = p_2 = 0 \\ +1 & p_1 = p_2 = \pi/a \end{cases}$ for arbitrary d
- problem: do not have uniform renormalization!
 or better:
 iii) $\frac{1}{d(d+1)} \sum_{n=1}^d \cos(\pi(p_n - p_0)) = \begin{cases} +1 & \text{all } p_n = 0, \pi/a \\ -1 & \text{all } p_n = \pi/a \end{cases}$ for arbitrary d
 has uniform renormalization

Borici-Creutz fermions in perturbation theory : part 1

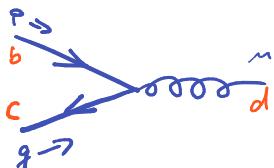
$$D_{BC}(x,y) = D(x,y) - \alpha \frac{r}{2} \sum_{n=1}^d i \gamma_n' \Delta_n(x,y) + m_0 \delta_{x,y} \quad \text{with minimal coupling to a gauge field}$$

propagator: $\frac{a}{p \rightarrow} \frac{b}{}$

$$D_{BC}^{-1}(p) = \frac{-i \left(\sum_{n=1}^d \gamma_n \bar{p}_n + \alpha r \sum_{n=1}^d \gamma_n' \hat{p}_n^2 \right) + m_0}{\sum_{n=1}^d \bar{p}_n^2 - \alpha r \underbrace{\bar{p}_n \hat{p}_n^2}_{-} + \frac{\alpha^2 r^2}{4} \hat{p}_n^4 + \underbrace{\frac{2\alpha r}{d} \sum_{n,v=1}^d \bar{p}_n \hat{p}_v^2}_{+} + m_0^2} g^{ab}$$

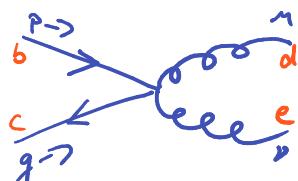
denominator is NOT EVEN in any p_n or r

ggg vertex:



$$V_1^\mu(p,q) = -i g_0 \frac{1}{2} \cdot \left\{ \gamma_n \cos\left(\alpha \frac{p_n + q_n}{2}\right) + r \gamma_n' \sin\left(\alpha \frac{p_n + q_n}{2}\right) \right\} (T^d)^{bc}$$

ggg² vertex:



$$V_2^{\mu\nu}(p,q) = +i g_0 \frac{g^2}{4} S^{\mu\nu} \left\{ \gamma_n \sin\left(\alpha \frac{p_n + q_n}{2}\right) - r \gamma_n' \cos\left(\alpha \frac{p_n + q_n}{2}\right) \right\} \left\{ T^d, T^e \right\}^{bc}$$

(Like vertices for Wilson fermions with $1 \rightarrow i \gamma_n'$) (\rightarrow see Stefano's talk)

Borici-Creutz fermions in perturbation theory: part 2

(Capitani,
Weber, Wittig
PLB 681 (2009))

Using standard gluon propagator (plaquette action) to compute self-energy

$$\begin{aligned}\Sigma(p, m_0; g_0^2, r, a) &= \left[\text{Diagram } 1 + \text{Diagram } 2 \right]_{\text{ext. prop. terms}} \\ &= \left[i g \cdot p \sum_1 (g_0^2, r^2, (ap)^2) + m_0 \sum_2 (g_0^2, r^2, (ap)^2) + i \int_a \prod c_3 (g_0^2, r^2) + i \int_p \sum_{n=1}^d p_n c_4 (g_0^2, r^2) + O(a, m_0^2) \right] \delta^{ab}\end{aligned}$$

↑
in fermion
action at tree-level

$$\Rightarrow \text{for } r=1: \quad \sum_1 = G_F^{\circ} (L + 6.80663) \quad \sum_2 = G_F^{\circ} (4L - 29.4872)$$

$$G_F^{\circ} = \frac{g_0^2 C_F}{16\pi^2}$$

$$L = \log (ap)^2$$

$$c_3 = G_F^{\circ} (-29.54170) \quad c_4 = G_F^{\circ} (+1.52766)$$

(also: Local currents & both Noether currents at 1-loop level)

and fermionic contribution to the vacuum polarization

$$\begin{aligned}\Pi_{\mu\nu}(p, m_0; g_0^2, r, a) &= \left[\text{Diagram } 3 + \text{Diagram } 4 \right]_{\text{ext. prop. terms}} \\ &= \left[(p_\mu p_\nu - \delta_{\mu\nu} p^2) \Pi_F^{\circ} (g_0^2, r^2, (ap)^2) + \underbrace{((p_\mu + p_\nu) \sum_{A=1}^d p_A - p^2 - \delta_{\mu\nu} (\sum_{A=1}^d p_A)^2) c_{4p} (g_0^2, r^2)}_{\equiv A_{\mu\nu}^{BC}} + O(a, m_0^2) \right] \delta^{\mu\nu}\end{aligned}$$

↑
present in fermion
action at tree-level,
NOT in gauge action

Ward identities etc: $p_\mu A_{\mu\nu}^{BC} = 0$.

$$G_A^{\circ} = \frac{g_0^2 C_A}{16\pi^2}$$

$$\Rightarrow \text{for } r=1: \quad \Pi_F^{\circ} = G_A^{\circ} (-\frac{8}{3}L + 23.6793) \quad c_{4p} = G_A^{\circ} (-0.9094)$$

(→ see Stefano's talk)

Due to broken reflection symmetries, odd powers of

indices cannot be ruled out in terms of naive arguments; e.g. $a D_\mu F_{\mu\nu} F_{\nu\rho}$ is not obviously forbidden...

Borici-Creutz fermions: implications

Unpublished!

$$D_{BC}(x,y) = D(x,y) - \alpha \sum_{n=1}^d i f_n' \Delta_n(x,y) + m_0 \delta_{x,y} \quad \text{with minimal coupling to a gauge field}$$

What to do about the actual counterterm operators on the lattice?

a) We suggested for $c_4(g^2, r^2) := i \Gamma \sum_{n=1}^d p_n \rightarrow X = +\frac{i}{2} \Gamma \sum_{n=1}^d (\partial_n^f - \partial_n^b)$. Unfortunately, this is wrong!

$$\begin{aligned} T_P + X T_P^+ &= +\text{sign}(r) \frac{i}{2} \Gamma \sum_{n=1}^d (\partial_n^f - \partial_n^b + 2) = -\text{sign}(r) Y \quad (Y \equiv -\frac{i}{2} \Gamma \sum_{n=1}^d (\partial_n^f - \partial_n^b + 2)) \\ T_P + Y T_P^+ &= +\text{sign}(r) i \Gamma \sum_{n=1}^d (\partial_n^f + \partial_n^b) = +\text{sign}(r) X \end{aligned} \quad \begin{array}{l} \text{(Note: the constant term must be added)} \\ \text{eventually to ensure the correct continuum limit!} \end{array}$$

Thus, $T_P^+ (X + \text{sign}(r) (Y - \frac{i}{2} \Gamma)) T_P = X - \text{sign}(r) (Y + \frac{i}{2} \Gamma)$. As Y is odd under C or \bar{PT} , it has the correct transformations under T_P, T_S, T_{PS} to restore the internal non-singlet C or \bar{PT} symmetry of the limit $r=\pm 1$.

For general r , the coefficients of X or $Y - \frac{i}{2} \Gamma$ will differ: $X + r(Y - \frac{i}{2} \Gamma)$ is an educated guess.

This implies that the two hopping parts of $\sum_{n=1}^d i f_n' \Delta_n(x,y) = \sum_{n=1}^d i (\frac{\gamma_n}{2} \Gamma - g_n) \Delta_n(x,y)$ renormalize differently!

Furthermore adjusting c_4 also implies adjusting the coefficient of K that receives contributions from both c_3 and c_4 .

b) Adding just $c_4 X$ must yield $c_4 \rightarrow 0$, as found by all groups to ever study BC fermions non-perturbatively!

(S.Basak, D.Chakrabarti, J.Goswami PRD 96 (2017)
R.Osunmaz, D.Xhako PoSLAT21 (2021))

c) Some authors suggest to include $2(1-r)X$ to shift the second pole from $-\frac{\pi}{2a} \arctan(\frac{1}{r})$ to $-\frac{\pi}{2a}$. Such an action has the wrong dispersion at the pole at $p_\mu \sim 0$.

d) There are also good news! The automatic $O(a)$ improvement of configurations carries over for any r .

Summary

Ist Symmetrie erst ruiniert,
dann simuliert man ungeniert.

- Minimally doubled fermions come in very different varieties.
- Some (Karsten-Wilczek, Borici-Creutz) have been shown to be consistent, renormalizable QFTs in perturbation theory.
- Spectra, dispersion, taste structure, cutoff effects, fermion determinant, topology are understood reasonably well. Parts of the community do not appreciate this yet!
- Taste structure has major implications for minimally doubled fermions and enables a wide range of symmetry arguments. Residual taste non-singlet symmetries are crucial!
- Dropping the dimension three term yields a radically different theory in terms of spin-taste structure and chiralities.
- $O(a)$ -improvement has been addressed (tree-level, Karsten-Wilczek fermions), too.
- For Borici-Creutz fermions much more work is needed. Taste structure is much more involved in this case!

Thanks for listening
till the end!